ERROR DIFFUSION ON SIMPLICES: INVARIANT REGIONS, TESSELLATIONS AND ACUTENESS

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Abstract. The error diffusion algorithm can be considered as a time dependent dynamical system that transforms a sequence of *inputs* into a sequence of *inputs*. That dynamical system is a time dependent translation acting on a partition of the phase space $\mathbb{A}$, a finite dimensional real affine space, into the Voronoï regions of the set C of vertices of some polytope $\mathbf{P}$ where the inputs all belong.

Given a sequence $g(i)$ of inputs that are point in $\mathbb{A}$, $g(i)$ gets added to the *error vector* $e(i)$, the total vector accumulated so far, that belongs to the (Euclidean) vector space mofelling $\mathbb{A}$. The sum $g(i) + e(i)$ is then again in $\mathbb{A}$, thus in a well defined element of the partition of $\mathbb{A}$ that determines in turns one vertex $v(i)$. The point $v(i)$ of $\mathbb{A}$ is the i^{th} output, and the new error vector to be used next is $e(i+1) = g(i) + e(i) - v(i)$. The maps $e(i) \mapsto e(i+1)$ and $g(i) + e(i) \mapsto g(i+1) + e(i+1)$ are two form of error diffusion, respectively in the vector space and affine space. Long term behavior of the algorithm can be deduced from the asymptotic properties of invariant sets, especially from the absorbing ones that serve as traps to all orbits. The existence of invariant sets for arbitrary sequence of inputs has been established in full generality, but in such a context, the invariant sets that are shown to exist are arbitrarily large and only few examples of minimal invariant sets can be described. Since the case of constant input (that corresponds to a time independent translation) has its own interest, we study here the invariant set for constant input for special polytopes that contain the n-dimensional regular simplices.

In that restricted context of interest in number theory, we study the properties of the minimal absorbing invariant set and prove that typically those sets are bounded fundamental sets for a discrete lattice generated by the simplex and that the intersections of those sets with the elements of the partition are fundamental sets for specific derived lattices.

Error diffusion

Piecewise isometries. A wide family of algorithms in control use a simple method of feedback. These algorithms are used in digital printing (under the generic name of error diffusion), scheduling, resource management, game theory, signal processing including sigma-delta modulators and others fields. A simple form of the *error diffusion algorithm*, where the feedback is determined by the previous steps (only), is the example in this family that we will study here, indeed only in a very special case that will be easier to comprehend after some generalities on error diffusion. While from the dynamics viewpoint error diffusion acts by translations, like the model studied in [15] and [6], other models of piecewise isometries use rotation as in [8], in [9] and in [4] and [12] that treat so called *digital filters*, following there an abundant literature in theoretical electronics cited by these papers. Other

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significant mathematical studies in other element of the families comprise the study of *sigma-delta modulators* as in [7], [10]

Error diffusion can be viewed as a *time dependent dynamical system* that transforms a sequence of *inputs* (for instance the *real color*, *i.e.,* the color out of a set of up to millions of local values, of the successive pixels on half line, the theoretical first step before considering printing on a page or a screen) into a sequence of *outputs* (for instance a color out of a very small set of colors of ink or toner or light to be deposited on a page being printed upon or a screen). This dynamical systems viewpoint is over twenty years old as it goes back at least to [1] and was used also in [2], [3], [5], [6], [13], [14], [17], and [18].

Let C be a collection of points, finite or not, in an Euclidean affine space. The set of points not further away from one point v of C than from any other is a closed set that is classically called the Vononoï region of v (with respect to C): such objects form a covering with pairwise disjoint interiors. In the context of error diffusion, we use the same name of *Vononoï regions* for pieces that stay somewhere between the interior and the closure of what one most usually calls the Vononoï regions determined by the vertices of some polytope $\mathbf{P}$ in the phase space $\mathbb{A}$. As a consequence of using Vononoï regions in this new sense, we get pieces that form a *partition* of $\mathbb{A}$ (instead of a *covering* with overlap of the boundaries of the parts). The dynamical system of error diffusion is generated by piecewise isometries and more precisely acts as a time dependent translation chosen using a partition of the phase space $\mathbb{A}$, an affine space, into the pieces $\mathbf{V}_i$ that we call the Voronoï regions defined by the set C of vertices of some polytope $\mathbf{P}$. We notice that the partition into Vononoï regions as we use the term is unique once a rule to assign boundaries parts to each region has been chosen: we assume that such a choice has been made whenever we do not discussed it explicitly.

The elements of C can be referred to as *colors* by reference to a preferred application of a variation of what we describe here to digital printing, but we will rather call the elements v_j of C the *vertices (of* $\mathbf{P}$*)*, by habit but also in order to account for the fact that error diffusion has potentially applications way beyond digital printing. Indeed digital printing done that way is but one example of a *scheduling problem* and, for instance, generalizations of error diffusion to the case of a variable polytope are known to cover other applications (see [6], [18] and references therein).

Given a sequence $g(i)$ of inputs that are points in $\mathbf{P} \subset \mathbb{A}$, $g(i)$ gets added to the *error vector* $e(i)$. The error vector is in fact the total error vector accumulated so far by the algorithm, usually initialized as $e(0) = 0$ before taking step one; all $e(i)$'s belongs to the vector space upon which the affine space $\mathbb{A}$ is built. The sum $g(i) + e(i)$ is then again in $\mathbb{A}$, thus in a well defined element $\mathbf{V}_i$ of the partition of $\mathbb{A}$ into Vononoï regions ; this in turns determines one vertex $v(i)$ which is the only vertex of $\mathbf{P}$ that belongs to $\mathbf{V}_i$. The point $v(i)$ of $\mathbb{A}$ is the i^{th} output, and the new error vector to be used next is $e(i+1) = g(i) + e(i) - v(i)$.

In actual printing, *i.e.,* on a page or screen rather than on a half-line, instead of the (last) error vector one would rather use some linear combination of the errors vectors of neighboring pixels already treated by the algorithm: as recalled in [2] and references therein that reviews more of the printing issues, once one has bounds for the error in the simple case where only the last error is used at each step, the bounds for the case of linear combinations of former errors follows as long as one stay in the important class where only *convex combinations* of former errors are made (*i.e.,* the coefficients weighting formerly defined errors are nonnegative and have sum one; in practice, the non-zero coefficients are in finite number and special treatment is used for the borders and in particular for the corners of the pages).

Error diffusion is a *greedy algorithm* (*i.e.,* some norm of some form of error term is minimized at each step) that makes choices of outputs on the fly since also it does not comprise looking forward (otherwise speaking, error diffusion is an *online algorithm*). Its time performances are reasonable but of course not as good as in algorithms such as *halftoning* where the output depends only on the input and some fixed and relatively small data set kept in memory. At the other end of the spectrum, one finds algorithms such as the one in [16] that describes the best output (in the sup norm) for the problem of digital printing on the line, but with need of examination of a boundless future of inputs to come in order to decide on the output. Thus, too much knowledge is needed for running this best output quality algorithm for it to be practical.

The version of error diffusion studied as a problem in dynamics has permitted to find regions that are invariant for several polytopes at once (here the polytope is chosen once and for all in the finite collection being considered to build a common invariant region). In the case when the polytope is not fixed but can instead vary from any step to the next in a fixed finite family of polytopes, one can easily see that there is no invariant region [17] but it turns out that the set of error vectors remains bounded [18]. The carpool assignment problem (see [6] and references therein) can be recast as a problem of error diffusion on a family of polytopes (the faces of a simplex in all dimensions, including the simplex itself). Variable polytope can be used in printing, in order for instance to take into account the non-constance of the output as the ink or toner reserves evolve in time. Given a polytope, consider the whole polytope as set of inputs at each step and add the set of possible accumulated errors at each step (with initial error set to zero). Then, depending on the chosen polytope, either one reaches the minimal invariant region in a finite number of steps (*e.g.,* for a cube), or one needs to take the limit of this process repeated *ad infinitum* to get the minimal invariant region (as may arise even in dimension two, indeed for fairly simple convex polygons) [18].

There remains problems to be solved in error diffusion and rather natural modifications of it, both with the most general hypotheses and about the more precise issues in very particular cases such as the continued fraction algorithms defined by error diffusion with constant input in the d-dimensional standard simplex. For instance, considering translation by vector a on the torus defined by (doubling) a simplex as the time τ_a map of the geodesic flow, one sees immediately that the whole theory of error diffusion (as already developed or still to be worked on) calls for a parallel theory for the time τ_a maps of the geodesic flow on the surfaces defined by various Fuchsian Groups, be they co-compact or not. Of course this comprises the Modular Surface, a particularly important case since the quality of the ergodicity of its geodesic flow is deeply linked to the Riemann Hypothesis.

Error diffusion with constant input in special simplices. To be more specific, let us examine the case when one starts with the initial error $e(0) = 0$ in a special case. For an input $g(0)$ the output $v(0)$ is the vertex $v(0)$ such that the next error $e(1) = e(0) + (g(0) - v(0))$ is minimal (in some norm), then inductively at *step* t the output $v(t)$ is such that $e(t+1) = e(t) + (g(t) - v(t))$ is minimal. The inputs $g(t)$ are points in the affine d dimensional (codimension 1) subspace of the $d+1$ real space given by the condition that the sum of coordinates is 1. More precisely the inputs are probability vectors in the d dimensional standard simplex that is our chosen polytope, while the outputs are the vertices of this simplex, that is the standard unit vectors. Then the errors $e(t)$ belong to a d dimensional (codimension 1) subspace (of the $d+1$ dimensional real vector space on which the affine space is built) given by the condition that the sum of coordinates is 0. The time dependent dynamical system in the error subspace is given by the iterations of the map:

$$e(t) \mapsto e(t+1) = \mathcal{G}_{g(t)}(e(t)), \text{ with } \mathcal{G}_g(e) = e + (g - v)$$

where v is chosen to minimize $\mathcal{G}(e)$ so that v is the vertex closest to the point $e + g$. The minimization condition partitions the space into the regions $\mathbf{V}_i$ where the choice of v is v_i. The Voronoï regions $\mathbf{V}_i$ consist of points which are closer to v_i than to any other v_j, with some tie breaking rules to resolve the equality cases. This example of the error diffusion algorithm is thus an example of algorithm in which:

- The decision (*i.e.,* the output choice) has to be made knowing only the past and present (on-line algorithm).

- The decision has to be made by a simple minimization rule at each step (greedy algorithm).

We will be concerned with the case when we let the polytope be a more general simplex satisfying a geometric condition that will be discussed later, but we will only consider the case when the input is a constant g so that the dynamical system defined by the iteration of $\mathcal{G}_g$ is autonomous.

To be very concrete, in Example 3 in Part VII.3 we give a detailed description of a specific error diffusion in dimension 1, with constant input (the one-dimensional version of the case that this paper is devoted to, but in higher dimension).
It turns out, when contemplating all possible constant inputs $a \cdot (1, 0) + (1-a) \cdot (0, 1)$ in the standard simplex bounded by the points $(1, 0)$ and $(0, 1)$, that the dynamical system that we consider can be re-interpreted as a rigid rotation represented by a discontinuous map on an interval ([2]). Furthermore, the sequence of vertices that come up as outputs are Sturmian sequences ([2]).

The first question of interest to us about the dynamics of algorithms such as error diffusion is whether the algorithms that one considers are *stable*, that is if they produce bounded errors (which implies in particular that the algorithm is *trustworthy*), *i.e.,* that the average output will tend to average input. Of course, in case of printing, such questions only make sense on infinite half-lines and pages which is a needed idealization to raise essential asymptotic behavior issues. The answer to the stability question is positive for the most important examples: for error diffusion on polytopes see [2] and for sigma-delta modulators see [7].

Next the quality of the algorithm is investigated. That is, one can examine the properties of the invariant regions or invariant error sets [13, 10, 9] (*e.g.,* "how far is the invariant region or invariant error set from those for other methods?", "which is the algorithm that gives the smallest maximal error, or the smallest average error when tested on some test case that may be important for some application such as image quality which is why one sees about the same set of pictures and drawings repeated so many times in journals and books dealing with digital printing); the comparison can be made within the set of all known algorithms, or within a smaller set, *e.g.,* the set of algorithms with bounded look forward (with a specific bound or not). For instance, error diffusion is known to be the best algorithm in the class of online, greedy, algorithms on the standard simplex [6].

We will be interested in the asymptotic behavior of the error diffusion in the case when the input is constant. In particular we shall investigate the shape of the minimal *absorbing set*, that is a set such a set that every trajectory passes through and eventually stays in this set. We found that when the input is constant, then typically there is a connection between the shape of such a set and the lattice generated by the vertices of the simplex.

We shall work here only with a specific generalization of the standard simplex, that is with what we call *acute simplices*, *i.e.,* simplices whose all angles are acute. As already specified above, we shall limit our attention to the constant input situation, so that in particular we will consider autonomous (time-independent) dynamics since we will consider the inputs $g(t) = g$ to be constant. Part of the interest of that case lies in the fact that when the dimension is one:

- The resulting map is a rigid rotation by an angle determined by the constant input.
- The resulting symbolic systems represented by the sequence of outputs is as evenly distributed as possible with a proportion of each vertices given by the angle of the rotation.

We shall also generalize the error diffusion model to other partitions of the state space which are not necessarily Voronoï partitions, in order to include maps that are inverse to error diffusion on some subsets.

There is a natural way to generalize the issues presented here to error diffusion with an arbitrary polytope $\mathbf{P}$ which is a convex hulls of its vertices v_i, with Voronoï partition into the points closest to the vertices (with a tie-braking rule) and translations made of vectors from the input in the interior of $\mathbf{P}$ to the appropriate vertex. Such a generalization is still under investigation. Another generalization (that will be considered here in the constant input case) consists in replacing the Voronoï regions partition by another partition with one vertex per piece.

Results. We first announce the results in a somewhat imprecise way. We hope that this will convey at least the spirit of our results to readers who do not want to plunge into the intricacies of the problem considered here. Our complete results will be formulated precisely after we introduce the needed notions and notations. We say that *an input is ergodic* iff the dynamics it generates under constant input error diffusion is *ergodic* for Lebesgue measure (any invariant set has zero or full measure)

Result A (Ergodic Inputs)**.** *For acute simplices the minimal absorbing invariant set for the error diffusion with an ergodic constant input is a fundamental set for the lattice generated by the simplex.*

This is made more precise below as Theorem I.16. Under the same assumptions we know more about the shape of the minimal absorbing set, namely is is a finite union of polytopes all faces of which are orthogonal to the edges of the simplex.

We propose but cannot yet prove the following

Conjecture.
The ergodicity assumption in Theorem I.16 is redundant.

This conjecture is trivial in the case when $d = 1$. The case when $d = 2$, *i.e.,* the case of acute triangles was treated in [5], we give here an independent proof in Theorem VII.2. Moreover the invariant tile is a connected and simply connected finite union of polytopes see Theorem VII.1.

Result B (Sub-Tiles)**.** *If a bounded forward invariant set of a generalized (arbitrary partition) error diffusion on a simplex is fundamental for the simplex lattice, then each* part *of this invariant set (i.e., each intersection with the partition) is a fundamental domain for a derived lattice.*

This is made precise as Theorem I.19. The hypotheses of that theorem are technical conditions and we only know these assumptions to hold true in the acute case when $d = 1, 2$ and in the acute case with Ergodic Input when $d > 2$. The idea that there is some structure in the sub-tiles came in discussions with Marco Martens, many thanks Marco.

Strucure of the paper.
The paper is organized as follows:

In Part I we introduce the needed notions and notations to make the statement of the Theorems precise. At the end of that Part we state the Theorems and discuss the strategies that we use for their proofs.

In Part II using the technical results from Parts IV and V we prove Theorem I.16.

In Part III using the technical results from Parts IV and V we prove Theorem I.19.

In Part IV we prove all needed statements about the geometry of (acute) simplices. We include this detailed part to make the paper as self-contained as possible. The proofs are elementary, but (as is often the case in convexity matters) it took us an embarrassingly long time to figure some of them out, and we did not find a reference both expected to contain proofs of all these results and concise enough to let us check that this is indeed the case.

In Part V we describe some dynamical properties of piecewise translations coming from error diffusion.

In Part VI we discuss the shape of the minimal absorbing set.

In Part VII we prove that in the case od dimension $d = 2$ (acute triangle) the minimal absorbing invariant set for any input is a connected (and simply connected) continuous union of parallel intervals. We give also a proof, different from [5], of tiling properties for dimension 2 for any input. This proof uses Chinese Remainder Theorem in purely non-ergodic case. At the very end of this Part we illustrate

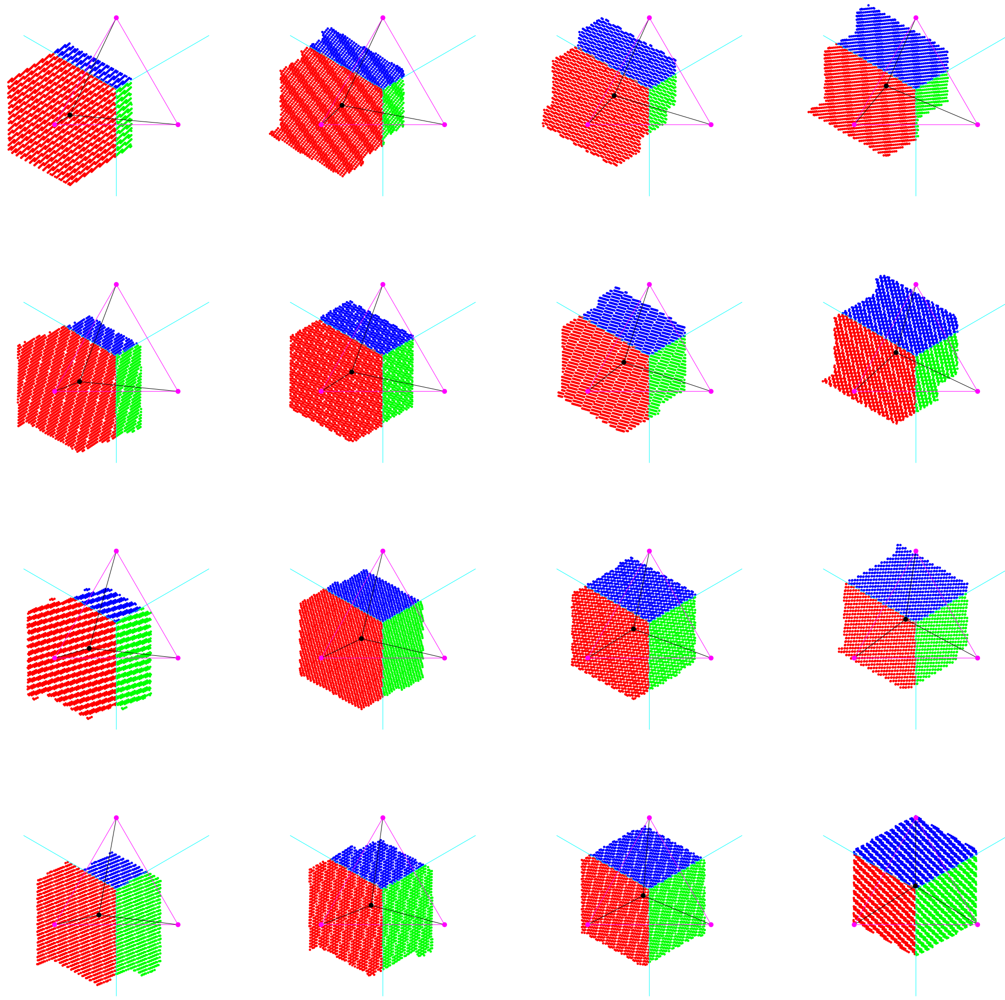

FIGURE 1. Simulation limit sets for several "inputs" g in an equilateral triangle. The parts ("sub-tiles") inside the three Voronoï regions are shaded differently. The boundaries should be made by segments, they are ragged due to numerical artifacts.

our results in the toy model of dimension $d = 1$ (interval) and we give an explicit example of error diffusion and a very simple proof of the tiling properties.

In Appendix A.I we investigate the properties of some candidates for absorbing sets, the construction of which, based of the notion of a Voronoï cell of a lattice, even if not generally valid, produces some insight into the problem that we judged worthwhile sharing with the interested readers. We hope that such constructions may help in proving tiling theorems in case on non-ergodic inputs. In particular we describe in detail the case of dimension $d = 3$.

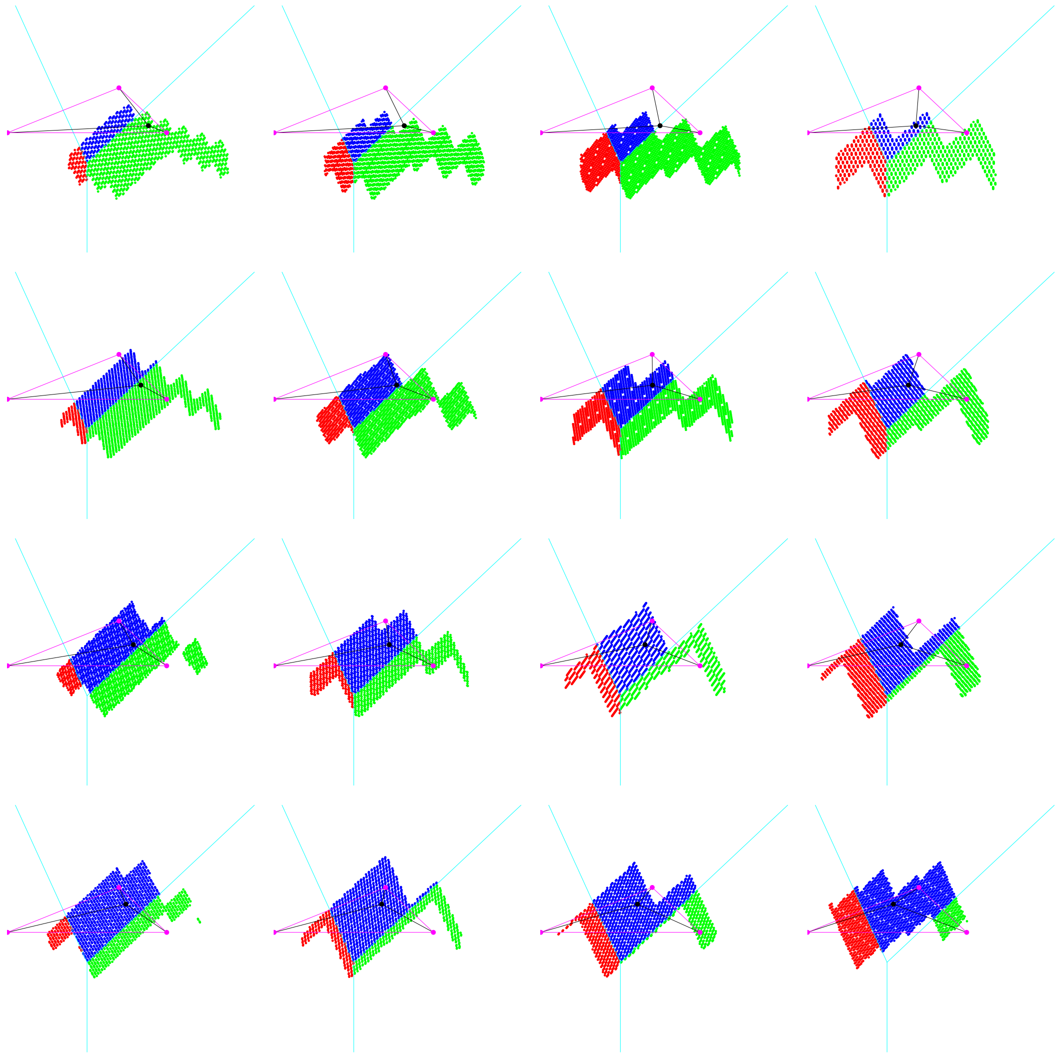

FIGURE 2. Simulation limit sets for several "inputs" g in an obtuse triangle. The parts ("sub-tiles") inside the three Voronoï regions are shaded differently. The boundaries should be made by segments, they are ragged due to numerical artifacts. The simulations show similar tiling and sub-tiling behaviour as for the equilateral triangle (even in the disconnected case, but we were not able to prove the two main theorems in a strictly non-acute case, even in dimension $d = 2$.

I. Notations and basic notions

I.1. **Notation conventions.** Points and sets are affine objects in a finite dimensional affine space $\mathbb{A}$, modeled on a real Euclidean vector space, equipped with the usual scalar product $s \cdot r$, hence with the ℓ^2 norm $||r||^2 = r^2 = r \cdot r$. The topological notions of *open* and *close* will refer to the metric topology derived from this norm and can be used as well in the affine space. The dimension of $\mathbb{A}$ is defined by the dimension of its model vector space.

We recall that in an *affine space* the difference of two points is a vector (of the vector space associated to the affine space) whence the sum of a point and a vector is a point.

A weighted sum of points in an affine space represents:

- A point when the sum of coefficients is equal to 1 (such a weighted sum is called an *affine combination*).

- Hence a vector (*i.e.,* an element of the associated vector space) when the sum of coefficients is equal to 0 (since the difference of two points is a vector).

Remark I.1. *Other weighted sums of points in an affine space are meaningless.*

It is in general improper but non-ambiguous to call *"a weighted sum of points in an affine space"* a *"linear combination"* combination of said points so that we do not hesitate to use this shorter expression.

Usually we will work in a minimal subspace of $\mathbb{A}$ which contains all points relevant to the discussion; the *dimension* of such space will be denoted by d. A typical model will be a subset of the space $\mathbb{R}^{d+1}$ with the condition that the sum of the coordinates equals one for the affine space, and this sum equals zero for an embodiment of the associated vector space. However we can talk about the affine space in general terms and only in examples will we refer to Cartesian coordinates. Otherwise, we shall instead use *barycentric coordinates* (see below in (I.7)).

The point sets and vector sets will be denoted by boldface characters such as $\mathbf{V}$ or $\mathbf{F}$ or $\mathbf{K}$. The points will be denoted by small Latin letters $u, v, w, x, y \in \mathbb{A}$, the points p, q will lie in a special (usually invariant) set $\mathbf{Q}$. The center of the ball circumscribed to a simplex will be denoted by $\mathcal{O}$ (IV.1). The letter g will be dedicated to the input point (I.3) that is constant after the background part the Introduction except if otherwise specified. Real coefficients of linear combinations as in convex hulls, cones and barycentric coordinates of the points and integer coefficients in lattices will use corresponding small Greek letters with upper indices such as ξ^i for the point x. For the vectors we shall use the letters r, s, t; in particular the letter e will be used for the error vector and the letter s for external normal vectors.

Lower indices will refer to specific objects in finite collections that usually consist of $d{+}1$ objects. The letters i, j, k, l can be used for both kinds of indices, superscript and subscript, running from 0 to d. Calligraphic letters will represent maps; $\mathcal{F}$ will always stand for the error diffusion map $\mathcal{F}_g$ (where in most of the paper the input g is constant) while the meaning of $\mathcal{G}$ will change depending on the context. In some formulas that have outcomes depending on index choices, we will use the Kronecker delta $\delta_{i,j}$ that is 1 if $i = j$ and zero otherwise.

I.2. **Error diffusion on polytopes.** Let the polytope $\mathbf{P}$ be a convex hull of its vertices v_i, with $v_i \in \mathbb{A}$, that is $\mathbf{P} = \{x : x = \sum \xi^i v_i,\ \xi^i \geq 0,\ \sum \xi^i = 1\}$; the coordinates ξ^i representing x are not unique for general polytopes but are unique for the simplices and (in particular) for simplicial faces (I.3), including the vertices v_i. Let $\mathbf{A}^\circ$ stand for the interior of the set $\mathbf{A}$, and $\overline{\mathbf{A}}$ for its closure. We define the Voronoï partition of the affine space $\mathbb{A}$ by the sets $\mathbf{V}_k$ where:

1:

$$\overline{\mathbf{V}}_k = \{x \,|\, \forall j : (x - v_k)^2 \leq (x - v_j)^2\}\,. \tag{I.1}$$

2: Some tie-breaking rules designates the chosen k in the cases when I.1 results in an equality.

We shall be mainly concerned with the interiors of the Voronoï regions for which we always have $\mathbf{V}_k^\circ = \overline{\mathbf{V}}_k^\circ$.

Define the *vertex assignment*:

$$v(x) = v_i \ \Leftrightarrow\ x \in \mathbf{V}_i \,. \tag{I.2}$$

This assignment is unambiguous for any x in the interiors of $\mathbf{V}_i$,*i.e.,* in $\cup_i \mathbf{V}_i^\circ$, by direct application of I.1 and on the boundaries of the $\mathbf{V}_i$'s by using the tie-breaking rule chosen for that effect, hence unambiguous everywhere. Among the multitude of such tie-breaking-rules, one consists in assigning the lowest possible index in (I.2). This rule has the benefit of being quite simple and general but it is not fair as some piece of the partiction ends up being closed while one ends up being open, with a variety of boundary behavior of the rule for the other pieces, yet fairness is not necessarily helping when discussing some questions.

Let the constant input g be a point in the interior of $\mathbf{P}$. Suppose that at some moment we have the accumulated error vector e and we receive the input g. We want to chose v_i as an output to minimize the next error, that is the difference between $(e+g)$ and v_i. By definition the choice must satisfy $||(e+g)-v_i||^2 \le ||(e+g)-v_j||^2$ for all j, hence (using the tie-breaking rules) $v_i = v(e+g)$. This defines a piecewise linear map $\mathcal{G}(e) = (e+g) - v(e+g) = e + (g - v(g+e))$.

I.3. **The Error Diffusion Map $\mathcal{F}$.** Since we are only concerned with constant inputs, we shall always assume that the input g is in the interior of the polytope. Indeed the boundary cases can be reduced to a lower dimensional situation. We have:

$$g = \sum_{i=0}^{d} \gamma^i v_i, \text{ such that } \sum_{i=0}^{d} \gamma^i = 1, \gamma^i > 0\,. \tag{I.3}$$

For technical and ideological reasons it is sometimes better to work in the affine space of g and $e+g$ than in the vector space of e and $\mathcal{G}(e)$, at least as long as one considers only one polytope at once. It so happens that one *has to* work in the vector space (the error space) when the polytope may vary from one step to the next ([17], [18]) but we shall not consider variable polytopes in the present work.

Defining $x = e + g$ and $\mathcal{F}_g(x) = \mathcal{G}_g(x-g) + g$ we get the following dynamical system:

$$x \mapsto \mathcal{F}(x) = \mathcal{F}_g(x) = x + (g - v(x)), \text{ with } v(x) = v_i \text{ iff } x \in \mathbf{V}_i\,. \tag{I.4}$$

This is a *piecewise translation* in $\mathbb{A}$, whose pieces are the Voronoï regions (after the tie-breaking rule is chosen) with $g - v(x)$ the translation vector associated to the piece $\mathbf{V}_i$. The simpler pieces that one gets in the affine version of error diffusion make it reasonable to choose that representation whenever the polytope is fixed, as in our case where the input g is fixed as well. The iterates $\mathcal{F}^N$ are defined by the compositions of the map when setting $\mathcal{F}^0 \equiv \mathrm{id}$, $\mathcal{F}^1 \equiv \mathcal{F}$, and $\mathcal{F}^{N+1} = \mathcal{F} \circ \mathcal{F}^N$ for any $N \ge 0$.

Digression 1. *As a side note we remark that the time dependent system can be represented as an orbit of a time independent system on the much larger space*

$\mathbb{A} \times \mathbf{P}^{\mathbb{Z}}$, *with*

$$\overline{\mathcal{F}}(x, \overline{g}) = (x + \overline{g}_0 - v(x), \sigma(\overline{g}))\,,$$

where σ is the shift operator on the sequence $\overline{g}$. It is worth mentioning that in the case of the non constant input the g in the formulae for $\overline{\mathcal{G}}$ and $\overline{\mathcal{F}}$ are shifted in time by one time unit in the sequence $\overline{g}$. However, as we work with the constant input we will keep the things simple by using the formula in its simplest of the forms that are correct when the input is constant.

I.4. **Simplices and barycentric coordinates.** Besides the **standard simplex**

$$\triangle = \{(\xi^0, \dots, \xi^d) \in \mathbb{R}^{d+1} : \sum_{i=0}^{d} \xi^i = 1, \xi^i \geq 0\}$$

we will also consider more general simplices spanned by arbitrary collections of $d+1$ independent points (to be defined next) in a d dimensional real affine space $\mathbb{A}$.

Definition I.2 (Independent points)**.** *We say that the points $v_0, \dots, v_d$ in the affine space $\mathbb{A}$ are **independent** if for any k the vectors $(v_i - v_k)$, $i = 0, \dots d, i \neq k$ are linearly independent.*

It is an easy exercise to check that the independence condition does not depend on the choice of k.

We can also speak about independent points when their number is less than $d + 1$. In particular any nonempty subset of independent points is independent from which it follows that a small collection of points is independent if and only if it can be completed into a collection of $d + 1$ independent points. In fact, $n + 1$ points in the d-dimensional affine space $\mathbb{A}$ are independent iff their convex hull is n-dimensional (which in particular implies that $n \leq d$). The dimension of the set is here understood as the dimension of the minimal affine subspace containing this set.

From independence it follows that for any k and any collection of d numbers β^i with $i \neq k$, the system of d equations with unknown the vector r:

$$r \cdot (v_i - v_k) = \beta^i,\ i \in \{0, 1, \dots, d\} \setminus \{k\}, \tag{I.5}$$

has a unique solution r that is of course a vector (this being the manifestation with vectors seen as pairs of points in an affine space of the basic linear algebra fact that a vector is determined uniquely by the collection of its projections on a basis).

Definition I.3 (Simplex)**.** *A (closed) **simplex** in $\mathbb{A}$ is the convex hull of a collection of independent points v_i*

$$\triangle(v) = \{x = \sum \xi^i v_i : \quad \sum \xi^i = 1, \quad \xi^i \geq 0\}\,. \tag{I.6}$$

Remark I.4 (**Shortcut notation**)**.** *To lighten the notation we shall restrict the use of the indices by adopting some conventions on notations.*
The summation symbol without the indices such as in $\sum b_i$ will always stand for the summation over all indices $i = 0, \dots, d$, that is $\sum_{i=0}^{d} b_i$ unless a limiting condition such as $i \neq k$ is specified as in $\sum_{i \neq k} b_i$.
In contexts that are clear and reasonably unambiguous, we shall use the symbol b to indicate the collection of all the objects b_i (or all the objects b^i as will be used later on) when the index (or the superscript) set is $\{0, 1, \dots, d\}$. Instead of a collection, the symbol b may as well represent an ordered collection that can indeed also be

understood as a vector. Thus, for instance, by $\triangle(v)$ *we shall mean* $\triangle(\{v_0, \ldots, v_d\})$ *as a set but it will often be important to have a fixed indexation in which case by* $\triangle(v)$ *we shall mean* $\triangle(\,(v_0, \ldots, v_d)\,)$*: the context should tell.*

From now on we assume that $\mathbb{A}$ is a minimal affine space containing all the independent points v, that is the associated vector space has the basis $(v_i - v_k)$, $i = 0, \ldots, d$, $i \neq k$ and the points v determine a simplex $\triangle(v)$. Thus, we say that the points v_i are the *vertices* of the simplex the dimension of which is one less than the number of its vertices.

From the independence of the points and the comparison of the dimensions it follows that any point x in the (minimal) affine space $\mathbb{A}$ can be uniquely represented by its **barycentric coordinates** ξ^i (*derived from the simplex* $\triangle(v)$).

$$x = \sum_{i=0}^{d} \xi^i v_i, \quad \sum_{i=0}^{d} \xi^i = 1\,, \tag{I.7}$$

because for any point v we have: $x = v + \sum_{i=0}^{d} \xi^i (v_i - v)$. In particular for any index k the point x can be represented as $x = v_k + \sum_{i=0, i\neq k}^{d} \xi^i (v_i - v_k)$, a representation that is unique by the independence of the vectors (basis) $(v_i - v_k)_{i \neq k}$.
It is easy to check that in the space $\mathbb{A} = \{x = (\xi^0, \ldots, \xi^d) \in \mathbb{R}^{d+1} : \sum_{i=0}^{d} \xi^i = 1\}$ the barycentric coordinates derived from the standard simplex are the same as the standard Cartesian ones.

If $\mathbb{A}_1 \subset \mathbb{A}_2$ are affine spaces with respective dimensions $d_1 \leq d_2$, and Q is a polytope in $\mathbb{A}_1$ not contained in any smaller affine space, then $d_2 - d_1$ is the *codimension of* Q *in* $\mathbb{A}_2$, or simply the *codimension of* Q when the space $\mathbb{A}_2$ is unambiguous.

I.5. **Face-wise acuteness.**

Definition I.5 (Face)**.** *The **face*** $\mathbf{F}_k$ *of the simplex opposite to the vertex* v_k *is the (codimension one) simplex*

$$\mathbf{F}_k = \{x = \sum \xi^i v_i : \quad \xi^k = 0, \quad \xi^i \geq 0, \quad \sum \xi^i = 1\} \tag{I.8}$$

The face $\mathbf{F}_k$ lies in an affine subspace

$$\mathbb{A}_k = \{x = \sum \xi^i v_i,\ \xi^k = 0,\ \sum \xi^i = 1\} \tag{I.9}$$

Definition I.6 (External normal vector)**.** *The **external normal** vector* s_k *to the face* $\mathbf{F}_k$ *is the unique vector* (I.5) *such that the scalar products satisfy:*

$$s_k \cdot (v_j - v_k) = 1 \text{ for any } j \neq k\,, \tag{I.10}$$

which can be written in a more obscure way to cover all the index combinations in one formula by:

$$s_k \cdot (v_j - v_i) = (1 - \delta_{ij})(\delta_{ik} - \delta_{jk})\,.$$

For existence and uniqueness of the collection s see Remark IV.3.

Definition I.7 (Face-wise acuteness)**.** *We say that a simplex is face-wise **acute** if*

$$i \neq j \ \Rightarrow\ s_i \cdot s_j < 0\,. \tag{I.11}$$

We now describe a picture that may be more intuitively clear for some of the readers. To this effect, consider any two faces and the union of the two hyperplanes containing them. Then intersect this union of hyperplanes with the plane determined by the normals to these hyperplanes: one gets an angle in a two dimensional subspace. The angles constructed this way are acute for all pairs of faces iff the simplex is face-wise acute.

We leave to the reader to verify that the inequality $<$ may be weakened to $\leq$ for face-wise "non-obtuse" simplices, with all the results holding *mutatis mutandis*.

I.6. **Forward invariance.**

Definition I.8 (Invariant sets)**.** *We say that a set is (forward)* ***invariant*** *with respect to the transformation $\mathcal{F}$ if*

$$\mathcal{F}(\mathbf{Q}) \subset \mathbf{Q} \tag{I.12}$$

More precisely the statement $\mathcal{F}(\mathbf{Q}) \subset \mathbf{Q}$ should correspond to the property of $\mathcal{F}$ being *forward inclusive*, but here and whenever we judge it acceptable, we will stick to what we understand as being the denominations most used, at least in the context of dynamical systems theory.

Definition I.9 (Absorbing sets)**.** *We say that the set $\mathbf{Q}$ is* ***absorbing*** *if for every point x in $\mathbb{A}$ there is a $N\,(= N(x))$ such that for every $M \geq N$*

$$\mathcal{F}^M(x) \in \mathbf{Q}$$

We say that the invariant absorbing set is **minimal** if it does not contain a proper invariant absorbing subset.

Digression 2. *A less restrictive version of absorption is that $\mathbf{Q}$ contains the ω-limit set of every x. However then the minimal such $\mathbf{Q}$ must be closed. In our case this leads to some unnecessary complications when considering the tiling.*

I.7. **Lattices and Tiles.**

Definition I.10 (Lattice)**.** *For the points $v_i \in \mathbb{A}$ the* ***lattice*** $\mathbf{L} = \mathbf{L}(v)$ *is a subgroup of the vector space modeling $\mathbb{A}$, generated by the vectors $v_i - v_j$, that is*

$$\mathbf{L} = \{r = \sum_{i \neq k} n^i (v_i - v_k), n^i \in \mathbb{Z}\}\,, \text{ for any } k. \tag{I.13}$$

In this definition the lattice does not depend on the choice of k and can be expressed in a symmetric way as

$$\mathbf{L} = \{r = \sum n^i v_i \,:\, n^i \in \mathbb{Z}, \sum n^i = 0\}\,.$$

When the points v_i are independent this group is discrete in the metric topology.

Definition I.11. *Given a set $\mathbf{Q}$ and a lattice $\mathbf{L}$*

1. *We will say that $\mathbf{Q} + \mathbf{L}$ is* "onto" *if the map $\mathbf{Q} \times \mathbf{L} \ni (q, r) \mapsto q + r \in \mathbb{A}$ is* onto *(or* surjective*, i.e., every point of the target space has a pre-image).*
2. *It is* "into" *if the map $\mathbf{Q} \times \mathbf{L} \ni (q, r) \mapsto q + r \in \mathbb{A}$ is* into *(or* injective*, i.e., distinct points in the source space have distinct images).*
3. *It is* "$1 - 1$" *if this map is $1 - 1$ (read and also written* one-to-one *)or* bijective *(i.e., both surjective and injective).*

Definition I.12. *We say that a set* $\mathbf{Q}$ *is* ***a tile*** *for the lattice* $\mathbf{L}$ *if* $\mathbf{Q}+\mathbf{L}$ *is* "$1-1$"*.*

In other words all lattice translates of $\mathbf{Q}$ cover the space $\mathbb{A}$ without overlaps. If $\mathbf{Q}$ is a tile that has some minimal topological regularity it is sometimes called a *fundamental domain* of the group $\mathbf{L}$.

We say that the points x and y are **equivalent** (with respect to the group $\mathbf{L}$) if $y - x \in \mathbf{L}$, that is

$$y - x = \sum n^i v_i \text{ with } n^i \in \mathbb{Z} \text{ and } \sum n^i = 0\,. \tag{I.14}$$

Remark I.13. *In the case of an independent and full dimensional collection of points the quotient space* $\mathbb{A}/\mathbf{L}$ *of classes of* ***equivalent*** *points* $[x]$ *is a* d*-dimensional torus. The map* $\mathcal{F}$ *projects on the rotation (in exponential model on the product of unit circles) or translation (in cube model) on this torus (Lemma V.1).*
A tile cannot be open *because its translates would not cover (the "onto" property would necessarily fail to hold true). On the other hand,* a tile cannot be closed*, because it would then have equivalent points on the boundary (the "into" property would necessarily fail to hold true).*

I.8. **Ergodicity.**

Definition I.14 (Ergodicity)**.** *Given* $d+1$ *fixed points* v_i *we say that an input* $g = \sum_{i=0}^{d} \gamma^i v_i \in \triangle(v) \subset \mathbb{A}$ *is* ***ergodic*** *it the projection* $[\mathcal{F}_g]$ *of* $\mathcal{F}_g$ *on the torus is an ergodic rotation.*

For more information on the properties of ergodic rotations on the torus see for instance [11] Chapters 1.4 and 4.2.a, in particular the Propositions 1.4.1 and 4.2.2.; we just assemble in Remark I.15 some properties that are important for us and/or that we found noticeable.

Remark I.15 (Properties of ergodic rotations)**.** *An ergodic rotation on the torus has the following properties:*

1. *Every (forward and backward) trajectory passes through any open set (infinitely many times).*
2. *In particular if an invariant set contains an open set, it contains also the whole torus.*
3. *For every open set* U *there is an* N *such that for every point* x *one of its first* N *iterates passes through* U *(*N *depends on* U *but is uniform in* x*).*
4. *Ergodic inputs form a set of full Lebesgue measure, and hence are typical from the measure theoretical properties (they are indeed also typical from the topological point of view, meaning that this property is generic, i.e., holds true in a residual subset of the set of possible inputs).*
5. *The algebraic characteristic condition of ergodicity of the input states that there is no non-zero rational (or integer) linear combination of its coefficients* γ^i *which produces 0:*

$$\sum_{i=0}^{d} \gamma^i n^i \neq 0, \text{ for any } n \equiv (n^0, n^1, \ldots, n^d) \in \mathbb{Z}^{d+1} \setminus \{0\}\,. \tag{I.15}$$

6. *In particular an ergodic input cannot lie on the (hyperplanes of the) faces of* $\triangle(v)$*.*

The rotation on the d dimensional torus in $\mathbb{R}^d$ is defined by a vector $(\gamma^1, \ldots, \gamma^d)$ and the ergodicity condition is expressed by the algebraic independence of γ^i on $\mathbb{Q}$ otherwise speaking the fact that there is no $m \in \mathbb{Z}^d \setminus \{0\}$ such that $\sum_{i=1}^d m^i\gamma^i \in \mathbb{Z}$, a condition that is equivalent to (I.15).

Theorem of Ergodic Inputs in Acute Simplices. We have now all the notions needed to reformulate Result A in a precise form.

Theorem I.16 (Ergodic Inputs in Acute Simplices)**.** *If the simplex is face-wise acute and the input g is ergodic then the minimal absorbing invariant set for $\mathcal{F}_g$ is a bounded tile for $\mathbf{L}$.*

I.8. **Parts of sets and derived lattices.** Denote

$$J_d = \{0, 1, \ldots, d\}\,, \tag{I.16}$$

and let J denote a nonempty subset of indices J_d. These sets may sometimes be considered as ordered sets, hence as vectors (the context will tell). We shall always assume $0 \in J$. In this context, except otherwise stipulated, we will use j for the indices in J and k for the indices from its complement $J_d \setminus J$.

Definition I.17 (Parts)**.** *Given a partition $\{\mathbf{V}_j\}_{j\in J_d}$, of $\mathbb{A}$ and a set $\mathbf{Q} \subset \mathbb{A}$ we define the **parts** $\mathbf{Q}_J$ of $\mathbf{Q}$ (relative to said partition of $\mathbb{A}$) by:*

$$\mathbf{Q}_J = \mathbf{Q} \cap \bigcup_{j\in J} \mathbf{V}_j\,. \tag{I.17}$$

Definition I.18 (Derived Lattices)**.** *Given:*

1. *a collection of points $v_j \in \mathbb{A}$ indexed by J_d and defining a lattice $\mathbf{L}$,*
2. *a point $g \in \mathbb{A}$,*
3. *and a nonempty set of indices $J \subseteq J_d$,*

*we define the **derived lattice** $\mathbf{L}_J$ (relative to the given data) by one of the following equivalent representations (indexed by $i \in J$):*

$$\text{for any } i \in J, \quad \mathbf{L}_J = \{\sum_{j\in J} n^j(v_j - v_i) + \sum_{k\notin J} n^k(v_k - g), n \in \mathbb{Z}^{d+1}\}\,.$$

Note that we have $\mathbf{L}_{J_d} = \mathbf{L}$ as in (I.13).

Example 1. *When $d = 2$ we have three points to choose from v_0, v_1, v_2. Then $\mathbf{L}_{\{0\}}$ is generated by the two vectors $v_1 - g$ and $v_2 - g$, while $\mathbf{L}_{\{0,1\}}$ is generated by the two vectors $v_1 - v_0$ and $v_2 - g$.*

I.9. **A generalization of error diffusion.** Let $d+1$ subsets $\mathbf{W}_i$ form an arbitrary partition of $\mathbb{A}$. Moreover let $v_0, v_1, \ldots, v_d$ be $d+1$ independent points and let the input $g \in \mathbb{A}$ be given. We define a **generalized error diffusion map** $\mathcal{G}_g : \mathbb{A} \to \mathbb{A}$

$$x \mapsto \mathcal{G}_g(x) = x + g - v(x), \text{where } [\, v(x) = v_i \;\Leftrightarrow\; x \in \mathbf{W}_i\,]\,.$$

I.10. **Theorem on Sub-Tiles.** We can now state the reformulation of Result B.

Theorem I.19 (Sub-Tiles)**.** *Suppose that $\mathbf{Q}$ is a bounded tile for $\mathbf{L}$, invariant under a generalized error diffusion map $\mathcal{G}_g$. Then for any nonempty subset of indices $J \subset \{0, 1, \ldots, d\}$ the set $\mathbf{Q}_J$ is a tile with respect to the lattice $\mathbf{L}_J$.*

I.11. **The strategy of the proofs.** For the Theorem on Ergodic Inputs we construct a large set, in fact a simplex, which, due to face-wise acuteness, is forward invariant and absorbing. Inside this large simplex there lies a smaller open set (also a simplex) which does not have equivalent points in the large one. We project the sets and the trajectories onto the torus and due to ergodicity of the input we conclude that there is indeed an absorbing invariant set which projects 1-1 on the the torus hence it is an invariant tile, absorbing and minimal.

The proof of the Sub-Tiles Theorem is purely computational: it requires some transpositions of terms in summations. We use two technical propositions which assume very specific properties of forward and backward trajectories. These properties are satisfied due to the boundedness and invariance of the original tile.

II. Proof of Theorem I.16 on Ergodic Inputs

Proof.
We begin with a preview of some notations and definitions from Part IV.

- For a simplex $\triangle(v)$, $\mathcal{O}$ stands for the center of its circumscribed sphere (IV.1), whence:

$$\forall i, \forall j \qquad (v_i - \mathcal{O})^2 = (v_j - \mathcal{O})^2 \,.$$

- Given a $g \in \mathbb{A}$ we define the points in $w \equiv w(v)$ by $w_k = \mathcal{O} + g - v_k$ and the simplex they generate (IV.9):

$$\triangle_R \equiv \triangle(w) = \{\sum_{i=0}^{d} \xi^i w_i,\ \xi^i \geq 0,\ \sum \xi^i = 1\}\,,$$

which is called the *Inverted Simplex*.

- With the affine half-spaces (IV.11):

$$\mathbb{W}_k = \mathbb{A}_k^-(w_k) = \{x : (x - w_k) \cdot s_k \leq 0\}\,.$$

we define next the *Big Simplex* (IV.12):

$$\triangle_B = \bigcap_{i=0}^{d} \mathbb{W}_i\,.$$

In the proof of of Theorem I.16, we use the following results from Part IV

(1) The Big Simplex $\triangle_B$ is invariant and absorbing under $\mathcal{F}_g$, Proposition V.7.
(2) The Inverted Simplex is contained in the Big Simplex, Corollary IV.10.
(3) The interior of the Inverted Simplex does not contain points equivalent to points in the Big Simplex, Proposition V.8).

Construction of the invariant set Q. Consider $\mathbf{Q}_0 = \triangle_R^\circ \subset \triangle_B$, the interior of the inverted simplex. Let $\mathbf{Q}_{n+1} = \mathbf{Q}_0 \cup \mathcal{F}(\mathbf{Q}_n)$. Then $\{\mathbf{Q}_n\}$ is an increasing family of sets that has a limit $\mathbf{Q} = \bigcup_{i=0}^{\infty} \mathbf{Q}_i$. Due to face-wise acuteness we have invariance of $\triangle_B$ under $\mathcal{F}$, hence $\mathbf{Q} \subset \triangle_B$. By construction $\mathcal{F}(\mathbf{Q}) \subset \mathbf{Q}$, which makes $\mathbf{Q}$ invariant. From Corollary V.9 the set $\mathbf{Q}$ does not contain equivalent points.

Now we will use ergodicity and the properties of ergodic toral rotations (*cf.* Remark I.15 below Definition I.14).

The set Q contains a fundamental set. When we project $\mathbf{Q}$ onto the torus $\mathbb{A}/\mathbf{L}$ as in Remark (I.13), we see that the projection is an invariant set containing an open set, hence by ergodicity the projection must contain the whole torus. That

means that $\mathbf{Q}$ itself must contain a fundamental set.
We proved that $\mathbf{Q}$ contains a fundamental set but nothing else, thus it is a fundamental set for the lattice $\mathbf{L}$, invariant with respect to $\mathcal{F}$.
The set $\mathbf{Q}$ absorbs all trajectories. Any trajectory must enter the set $\triangle_B$. After projection, because of ergodicity, every trajectory must pass through the projection of the interior of $\triangle_R$. Because there are no points in $\triangle_B$ which project there other than $\triangle_R$ itself, we conclude that every trajectory from $\triangle_B$ must pass through the interior of $\triangle_R$, and once there, it remains in $\mathbf{Q}$. □

III. Proof of the Theorem I.19 on Sub Tiles

III.1. **Technical Lemmata.** As we shall see, Theorem I.19 is a consequence of the two following Propositions:

Proposition III.1 ("onto")**.** *Suppose that the set $\mathbf{Q}+\mathbf{L}$ is "onto" and fulfills the following conditions:*

$$\forall_{q\in\mathbf{Q}}\forall_N > 0\,\exists_{N^+\geq N}\,\exists_{q^+\in\mathbf{Q}_J} \quad | \quad \mathcal{F}_g^{N^+}(q) = q^+ \tag{III.1}$$

$$\forall_{q\in\mathbf{Q}}\forall_N > 0\,\exists_{N^-\geq N}\,\exists_{q^-\in\mathbf{Q}_J} \quad | \quad \mathcal{F}_g^{N^-}(q^-) = q \tag{III.2}$$

Then $\mathbf{Q}_J+\mathbf{L}_J$ is "onto".

The hypotheses of Proposition III.1, that we shall refer to as *the technical conditions*, say that every trajectory starting in $\mathbf{Q}$ visits each part of it infinitely many times both for forward and backward iterates. As we shall see, luckily in our case the backward iterates are well defined.

Proposition III.2 ("into")**.** *Suppose that the set $\mathbf{Q}+\mathbf{L}$ is "into" and fulfills the following condition:*

$$\forall_{N\geq 0}\,\mathcal{F}_g^N(\mathbf{Q}_J)\subset\mathbf{Q}$$

Then $\mathbf{Q}_J+\mathbf{L}_J$ is "into".

We notice that the condition in this theorem states that the forward iterates of the part $\mathbf{Q}_J$ all belong to $\mathbf{Q}$ (we also say that the forward iterates of the part $\mathbf{Q}_J$ "cannot escape $\mathbf{Q}$" or that "they are trapped in $\mathbf{Q}$".

Before we prove these two propositions, let us extract from them Theorem I.19. In all that follows the point g (*cf.* (I.3)) from the interior of the simplex $\triangle(v)^\circ$ (*cf.* (I.6)) is fixed and to simplify the notation we will write $\mathcal{F}$ for $\mathcal{F}_g$.

III.2. **Proof of the Sub-Tile Theorem I.19.**

Theorem I.19 follows from Proposition III.1 and III.2.

1. If $\mathbf{Q}$ is a tile for the lattice $\mathbf{L}$, then by definition $\mathbf{Q}+\mathbf{L}$ is both "into" and "onto" $\mathbb{R}^d$.
2. The technical conditions of Proposition III.1 follows from Corollary V.15 which states that any bounded trajectory visits each Voronoï region with the asymptotic frequency equal to the corresponding barycentric coordinate of the input.
3. Because $\mathbf{Q}$ is a tile there exists a map $\mathcal{G}$ inverse to $\mathcal{F}$ on $\mathbf{Q}$ (Corollary V.3), which is a piecewise translation with the same input and the Frequency Lemma V.15 applies as well.

(4) The assumption of Proposition III.2 follows from the invariance of $\mathbf{Q}$, because by definition $\mathbf{Q}_J \subset \mathbf{Q}$.
(5) Both Propositions together yield $\mathbf{Q}_J + \mathbf{L}_J$ both "onto" and "into", hence "1-1". Therefore $\mathbf{Q}_J$ is a tile for the lattice $\mathbf{L}_J$.

□

III.3. **Proofs of Proposition III.1 and III.2.**

Proof of Proposition III.1. Recall that since J is nonempty we can assume without loss of generality that $0 \in J$, and that indeed we always make this simplifying assumption. Let $x \in \mathbb{A} = \mathbf{Q} + \mathbf{L}$ and $q \in \mathbf{Q}$ be such that for some $n \in \mathbb{Z}^{d+1}$ we have:

$$x = q + \sum_i n^i (v_i - v_0) = q + \sum_{0 \neq j \in J} n^j (v_j - v_0) + \sum_{k \notin J} n^k (v_k - v_0) \,.$$

We want to show that $x \in \mathbf{Q}_J + \mathbf{L}_J$, *i.e.*, we want to represent x as

$$x = p + \sum_{0 \neq j \in J} \nu^j (v_j - v_0) + \sum_{k \notin J} \nu^k (v_k - g) \text{ for some } p \in \mathbf{Q}_J, \text{ and } \nu \in \mathbb{Z}^{d+1} \,.$$

Suppose first that :

$$\sum_{k \notin J} n^k \geq 0 \,.$$

Let then $N \geq 0$ be maximal such that

$$\#\{0 \leq l < N \,:\, \mathcal{F}^l(q) \in \mathbf{Q}_J\} \leq \sum_{k \notin J} n^k \,.$$

Such N exists by the technical assumptions. Define $q_J = \mathcal{F}^N(q)$. By maximality of N we have $q_J \in \mathbf{Q}_J$ and $\#\{0 \leq l < N \,:\, \mathcal{F}^l(q) \in \mathbf{Q}_J\} = \sum_{k \notin J} n^k$.
Thus

$$q_J = \mathcal{F}^N(q) = q + \sum_{j \in J} m^j (g - v_j) + \sum_{k \notin J} m^k (g - v_k)$$

for some $m \in (\mathbb{Z}^+)^{d+1}$ with $\sum_{j \in J} m^j = \sum_{k \notin J} n^k$.

$$\begin{aligned}
x &= q + \sum_{0 \neq j \in J} n^j (v_j - v_0) + \sum_{k \notin J} n^k (v_k - v_0) \\
&= q_J - \left(\sum_{j \in J} m^j (g - v_j) + \sum_{k \notin J} m^k (g - v_k) \right) + \left(\sum_{j \in J} n^j (v_j - v_0) + \sum_{k \notin J} n^k (v_k - v_0) \right) \\
&= q_J + \left(\sum_{j \in J} m^j (v_j - v_0) + \sum_{j \in J} n^j (v_j - v_0) \right) + \left(\sum_{k \notin J} m^k (v_k - g) + \sum_{k \notin J} n^k (v_k - g) \right) \\
&\quad + \left(\sum_{j \in J} m^j - \sum_{k \notin J} n^k \right) (v_0 - g) \\
&= q_J + \sum_{0 \neq j \in J} (m^j + n^j)(v_j - v_0) + \sum_{j \notin J} (m^k + n^k)(v_k - g) \in \mathbf{Q}_J + \mathbf{L}_J
\end{aligned}$$

Remark that the value of n^0 does not matter and can be set so that $\sum n^i = 0$.

When
$$\sum_{k\notin J} n^k < 0\,,$$
we do the same trick with $\mathcal{G}$, where $\mathcal{G}=\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$ on the tile $\mathbf{Q}$, see Corollary V.3. To be more explicit let N be maximal such that (notice the change of the sharpness of the inequalities):
$$\#\{0\le l<N\,:\,\mathcal{G}^l(q)\in\mathbf{Q}_J\} < -\sum_{k\notin J} n^k$$

Let $q_J=\mathcal{G}^N(q)$. Then again
$$q_J\in\mathbf{Q}_J \text{ and } \#\{0<l\le N\,:\,\mathcal{G}^l(q)\in\mathbf{Q}_J\} = -\sum_{k\notin J} n^k$$
(and again notice the inequalities). It follows that also
$$\#\{0\le l<N\,:\,\mathcal{F}^l(q_J)\in\mathbf{Q}_J\} = -\sum_{k\notin J} n^k\,.$$
Thus
$$q=\mathcal{F}^N(q_J)=q_J+\sum_{j\in J} m^j(g-v_j)+\sum_{k\notin J} m^k(g-v_k)$$
for some $0\le m^i\in\mathbb{Z}$ with $\sum_{j\in J} m^j=-\sum_{k\notin J} n^k$. It follows that for any $x\in\mathbb{A}=\mathbf{Q}+\mathbf{L}$:
$$\begin{aligned}
x &= q+\sum_{0\neq j\in J} n^j(v_j-v_0)+\sum_{k\notin J} n^k(v_k-v_0)\\
&= q_J+\left(\sum_{j\in J} m^j(g-v_j)+\sum_{k\notin J} m^k(g-v_k)\right)+\left(\sum_{j\in J} n^j(v_j-v_0)+\sum_{k\notin J} n^k(v_k-v_0)\right)\\
&= q_J+\left(-\sum_{j\in J} m^j(v_j-v_0)+\sum_{j\in J} n^j(v_j-v_0)\right)+\left(-\sum_{k\notin J} m^k(v_k-g)+\sum_{k\notin J} n^k(v_k-g)\right)\\
&\qquad+\left(-\sum_{j\in J} m^j-\sum_{k\notin J} n^k\right)(v_0-g)\\
&= q_J+\sum_{0\neq j\in J}(-m^j+n^j)(v_j-v_0)+\sum_{j\notin J}(-m^k+n^k)(v_k-g)\in\mathbf{Q}_J+\mathbf{L}_J
\end{aligned}$$
□

Proof of Proposition III.2. We know that $q,q'\in\mathbf{Q}$ and $q-q'\in\mathbf{L}$ imply together $q=q'$ (the "into" part of the "1-1" property of $\mathbf{Q}+\mathbf{L}$). In order to prove that $\mathbf{Q}_J+\mathbf{L}_J$ is "into" it is enough to prove that if for some $q_J,q'_J\in\mathbf{Q}_J$ and $w_J,w'_J\in\mathbf{L}_J$ the relation $q_J+w_J=q'_J+w'_J$ holds true then $q_J=q'_J$ and (hence) $w_J=w'_J$, so that it suffices to prove
$$q_J-q'_J\in\mathbf{L}_J \;\Rightarrow\; q_J=q'_J \text{ and } w_J=w'_J\,.$$
Let then $n\in\mathbb{Z}^{d+1}$ with $\sum n^i=0$ be such that
$$q_J-q'_J=\sum_{j\in J} n^j(v_j-v_0)+\sum_{k\notin J} n^k(v_k-g)\in\mathbf{L}_J\,.$$

We may assume that $N = \sum_{k \notin J} n^k \geq 0$ since otherwise we exchange q_J and q'_J. Again the value of n^0 does not matter.

Next let $q = \mathcal{F}^N(q'_J)$. Then for some $m \in (\mathbb{Z}^+)^{d+1}$ with $\sum_i m^i = N = \sum_{k \notin J} n^k$

$$q = q'_J + \sum_{j \in J} m^j (g - v_j) + \sum_{k \notin J} m^k (g - v_k)\,,$$

By invariance we have $q \in \mathbf{Q}$. Consider now the difference $q - q_J$, with both $q, q_J \in \mathbf{Q}$.

$$\begin{aligned}
q - q_J \quad &= \quad \left(q'_J + \sum_{j \in J} m^j (g - v_j) + \sum_{k \notin J} m^k (g - v_k) \right) - q_J \\
&= \quad q'_J - q_J + \sum m^i (g - v_i) \\
&= \quad \left(\sum_{j \in J} n^j (v_j - v_0) + \sum_{k \notin J} n^k (v_k - g) \right) + \left(\sum_{j \in J} m^j (g - v_j) + \sum_{k \notin J} m^k (g - v_k) \right) \\
&= \quad \sum_{j \in J} (n^j - m^j)(v_j - v_0) + \sum_{k \notin J} (n^k - m^k)(v_k - v_0) \\
&\qquad - \sum_{j \in J} m^j (v_0 - g) + \sum_{k \notin J} (n^k - m^k)(v_0 - g) \\
&= \quad \sum_i (n^i - m^i)(v_i - v_0) + \left(\sum_{k \notin J} n^k - \sum_i m^i \right) (v_0 - g) \\
&= \quad \sum_i (n^i - m^i)(v_i - v_0) \in \mathbf{L}\,,
\end{aligned}$$

hence $q = q_J$, that is $q_J = \mathcal{F}^N(q'_J)$ and $n^i = m^i$ for all i.

But then

$$\sum_i n^i = \sum_i m^i = \sum_{k \notin J} n^k$$

and therefore $\sum_{j \in J} n^j = 0$. Thus $\sum_{j \in J} m^j = 0$ and hence $N = 0$ because

$$N > 0 \text{ implies } \sum_{j \in J} m^j > 0\,,$$

since $q'_J \in \mathbf{Q}_J$ which means that $q'_J \in \mathbf{V}_j$ for some $j \in J$. However if $N = 0$ then $q_J = q'_J$. □

IV. Geometry of an acute simplex

IV.1. **Preliminaries to a discussion of the Geometry of an acute simplex.** In what follows we abandon the convention about $k \in J$ and $j \notin J$. We assume that the points v_i, $i \in J_d$ are independent (Definition I.2), $J_d = \{0, \ldots, d\}$ as in (I.16).

Definition IV.1 (The Center)**.** *For a simplex $\triangle(v)$, denote by $\mathcal{O}$ the center of its circumscribed sphere.*

Thus
$$(v_i - \mathcal{O})^2 = (v_j - \mathcal{O})^2 \, i, j \in J_d.$$
where for a vector p, $p^2 = p \cdot p$.

Digression 3. *When the points $v_0, \ldots, v_d$ are independent, the point $\mathcal{O}(v)$ exists and is unique in the affine space spanned by the points of v. The center $\mathcal{O}(v)$ fulfills the system of linear equations:*

$$0 = (\mathcal{O} - v_i)^2 - (\mathcal{O} - v_j)^2 = (v_j - v_i) \cdot (2\mathcal{O} - (v_j + v_i)) \tag{IV.1}$$

of full rank d since for any j the vectors $v_i - v_j$, $i \in J_d \backslash \{j\}$ are linearly independent.

IV.1.1. *Voronoï regions.* (c.f. I.1). Recall that the (closed) Voronoï regions are defined by

$$\overline{\mathbf{V}}_k = \{x \,|\, \forall j : (x - v_k)^2 \leq (x - v_j)^2\} \,. \tag{IV.2}$$

We have the following alternate characterization of the closures of the $\mathbf{V}_k$'s:

Lemma IV.2.
$$x \in \overline{\mathbf{V}}_k \quad \Leftrightarrow \quad \forall i \; (x - \mathcal{O}) \cdot (v_i - v_k) \leq 0 \,.$$

Proof. For any two vectors p, q we have $p^2 - q^2 = (p + q) \cdot (p - q)$. Thus, from
$$(v_k - \mathcal{O})^2 = (v_i - \mathcal{O})^2$$
we have
$$(v_k + v_i - 2\mathcal{O}) \cdot (v_k - v_i) = 2(\frac{v_k + v_i}{2} - \mathcal{O}) \cdot (v_k - v_i) = 0$$
i.e., the interval joining the center and the midpoint of the edge is orthogonal to the edge. Next we similarly also have
$$\begin{aligned}(x - v_k)^2 - (x - v_i)^2 &= (2x - (v_k + v_i)) \cdot (v_i - v_k) \\ &= (2x - 2\mathcal{O} + 2\mathcal{O} - (v_k + v_i))(v_i - v_k) = 2(x - \mathcal{O})(v_i - v_k) \,,\end{aligned}$$
and the signs of the two expressions at the end of this chain of equalities are the same. □

The proof shows that Lemma IV.2 is nothing but an algebraic reformulation of Definition I.1. In some sense, while the defining equation I.1 is a distances statement (for all polytopes), Lemma IV.2 is an equivalent angular statement for simplices: we shall see that the Voronoï regions of simplices are cones (*cf.* Lemma IV.4) and Lemma IV.2 just tells us which of these cones x has to stand in for it to be in $\overline{\mathbf{V}}_k$.

Recall from Definition I.5 that $\mathbf{F}_k$ is the face of the simplex $\triangle(v)$ opposite to the vertex v_k and also recall:

Remark IV.3. *The* ***external normal vectors*** s_k *(cf.* (I.10)*) to the faces* $\mathbf{F}_k$ *(cf.* (I.8)*) are uniquely given by the normalization condition: $s_k \cdot (v_i - v_k) = 1$, for $i \neq k$, which implies the normality $s_k \cdot (v_i - v_j) = 0$ for $i, j \neq k$.*

The existence of a normal vector follows from the existence of a Gram-Schmidt orthonormalization. Any two distinct points, equidistant to all vertices v_i, $i \neq k$ will provide a vector normal to $\mathbf{F}_k$, forming a one dimensional vector space. We can choose any normalization and since once for a given k we know one scalar product $s_k \cdot (v_i - v_k)$ we know they are equal for any i. Typically we can take as the orthogonal vector to the face $\mathbf{F}_k$ the vector $\mathcal{O} - \mathcal{O}_k$, where $\mathcal{O}_k$ is the center of the codimension one sphere circumscribed to $\mathbf{F}_k$ (except of course in the untypical cases when the two centers $\mathcal{O}$ and $\mathcal{O}_k$ coincide as when the polytope $\mathbf{P}$ is a right triangle and $\mathcal{O}_k$ is the center of is the hypothenuse $\mathbf{F}_k$ of $\mathbf{P}$: we let to the reader to check that similar phenomena happen, albeit as exceptional cases, in any higher dimension).

Digression 4.

$$\sum \sigma^i s_i = 0 \textit{ iff all } \sigma^i \textit{ are equal.}$$

We have $\sum \sigma^i s_i \cdot (v_k - v_j) = \sigma^j - \sigma^k$. *If the sum is zero then so are the scalar products, thus* $\sigma^j = \sigma^k$ *for any* $k, j \in J_d$. *If all* σ*'s are equal then the sum annihilates a basis hence must be a zero vector.*

Lemma IV.4 (Voronoï regions as normal cones)**.**

$$\overline{\mathbf{V}}_k = \{\mathcal{O} + \sum_{i \neq k} \lambda^i s_i : \lambda^i \geq 0\}$$

Proof. Let $J = J_d \setminus \{k\}$ and suppose that $x = \mathcal{O} + \sum_{i \in J} \lambda^i s_i$. Then, for any $j \in J$

$$\begin{aligned} (x - v_k)^2 - (x - v_j)^2 &= 2(x - \mathcal{O}) \cdot (v_j - v_k) \\ &= 2(\sum_{i \in J} \lambda^i s_i) \cdot (v_j - v_k) = -2\lambda^j \end{aligned}$$

since, by the definition of s (*cf.* I.10), we both get that all terms $(v_j - v_k) \cdot s_i$ with furthermore $j \neq k$ vanish and $(v_k - v_j) \cdot s_j = 1$.

If all $\lambda^j \geq 0$ then the point x is closer to v_k than to any other v_j. If the point is in $\mathbf{V}_k$ then the difference $(x - v_k)^2 - (x - v_j)^2$ is negative for all j and all λ^j are positive. □

IV.2. **Affine subspaces parallel to the faces.** We shall denote by $\mathbb{A}_k$ the affine subspace spanned by the face $\mathbf{F}_k$:

(IV.3) with $J = J_d \setminus k$ $$\mathbb{A}_k = \{x = \sum_{i \in J} \xi^i v_i \;:\; \sum_{i \in J} = 1\xi^i = 1\} = \{x \;:\; \xi^k = 0\}$$

This subspace (in effect an hypersurface since it has codimension 1) is also fully characterized by

(IV.4) $$\mathbb{A}_k = \{x : \forall j \neq k\,,\, (x - v_j) \cdot s_k = 0\}\,.$$

We shall also consider the half-spaces:

(IV.5) $$\mathbb{A}_k^-(z) = \{x : (x - z) \cdot s_k \leq 0\} \text{ and } \mathbb{A}_k^+(z) = \{x : (x - z) \cdot s_k \geq 0\}$$

The half-spaces $\mathbb{A}_k^-(z)$ for $z \in \mathbf{F}_k$ are all equal, which we express as $\mathbb{A}_k^-(\mathbf{F}_k)$. The half-space $\mathbb{A}_k^-(\mathbf{F}_k)$ contains the simplex $\triangle(v)$, and its boundary is the subspace $\mathbb{A}_k$. In particular,

$$\forall j \neq k \;\; \mathbb{A}_k^-(v_j) = \mathbb{A}_k^-(\mathbf{F}_k)\,.$$

IV.3. **Acuteness.** We say that the simplex is *acute* (*face-wise acute*) if for $i \neq j$

(IV.6) $$s_i \cdot s_j < 0\,.$$

Acuteness is important because it is equivalent to the Supporting Half-space condition expressed in the following Lemma.

Lemma IV.5 (Supporting Half-spaces)**.** *The simplex is acute if and only if for every k*

$$\mathbf{V}_k \subset \mathbb{A}_k^-(\mathcal{O})\,.$$

It follows readily from this Lemma that strict acuteness requires for each $k \in J_d$ the intersection $\overline{\mathbf{V}}_k \cap \mathbb{A}_k^+(\mathcal{O})$ to be the singleton $\{\mathcal{O}\}$.

Proof. We are using the representation of the Voronoï regions from Lemma IV.4. Let $x = \mathcal{O} + \sum_{i\neq k} \lambda^i s_i$. Then $(x-\mathcal{O})\cdot s_k = \sum_{i\neq k} \lambda^i s_i \cdot s_k$. If the point x is in $\mathbf{V}_k$ and the simplex is acute, then all λ^i's are positive and all $s_i \cdot s_k$'s are negative so that by Definition (IV.5) $x \in \mathbb{A}_k^-(\mathcal{O})$. If on the other hand we have $x \in \mathbb{A}_k^-(\mathcal{O})$ for any choice of positive λ^i's (which constitutes the Vornonoï region $\mathbf{V}_k$), then in particular $x = \mathcal{O} + s_i$ satisfies the condition $(x-\mathcal{O})\cdot s_k = s_i \cdot s_k \leq 0$. But by Definition IV.6, the last inequality is indeed strict whence the strict inclusions $\forall\, k\,,\ \mathbf{V}_k \subset \mathbb{A}_k^-(\mathcal{O})$ stated by the lemma. □

We shall often use a weaker condition that we call *edge-wise acuteness*:

$$\text{(IV.7)} \qquad \forall i \in J_d \forall k, l \in J_d \setminus \{i\} \qquad (v_j - v_i)\cdot(v_k - v_i) > 0\,,$$

and which is motivated by the next Proposition.

Proposition IV.6 (Acuteness is hereditary)**.** *If the simplex $\triangle(v)$ is acute then each of its face $\mathbf{F}_k$ viewed as a simplex is acute as well.*

Proof. We construct explicitly the external normal vectors to the faces of an $\mathbf{F}_k$. For $j \neq k$ denote by $\mathbf{F}_{k,j}$ the simplex

$$\text{with } J = J_d \backslash k, j \qquad \mathbf{F}_{k,j} = \{x = \sum_{i\in J} \xi^i v_i \ :\ \sum_{i\in J} \xi^i = 1, \xi^i \geq 0\} = \{x : \xi^j = 0, \xi^k = 0\}\,.$$

It is the subface of $\mathbf{F}_k$ opposite to the vertex v_j with $j \neq k$. Incidentally it is also the subface of $\mathbf{F}_j$ opposite to the vertex v_k. Define:

$$r_{k,j} = s_j - \frac{s_j \cdot s_k}{s_k^2} s_k,$$

the orthogonal projection of the vector s_j on the vector space modeling $\mathbb{A}_k$. Indeed we have:

$$r_{k,j}\cdot s_k = s_j \cdot s_k - \frac{s_j \cdot s_k}{s_k^2}(s_k \cdot s_k) = 0$$

and $r_{k,j}$ is a vector parallel to $\mathbb{A}_k$, since it is orthogonal to s_k which defines the direction of this subspace. The edges of $\mathbf{F}_{k,j}$ are given by all differences $v_i - v_l$ where both i and l belong to $J_d \setminus \{k, j\}$, and we have:

$$r_{k,j}\cdot(v_i - v_l) = s_j \cdot (v_i - v_l) - \frac{s_j \cdot s_k}{s_k^2} s_k \cdot (v_i - v_l) = 0$$

The remaining edges of the simplex $\mathbf{F}_k$ share v_j as an end point and are given by the differences $v_i - v_j$, with $i, j \neq k$. Then:

$$\forall i \in J,\ \ r_{k,j}\cdot(v_i - v_j) = s_j \cdot (v_i - v_j) - \frac{s_j \cdot s_k}{s_k^2} s_k \cdot (v_i - v_j) = 1 - 0 = 1\,,$$

which confirms the normalization of $r_{k,j}$ and proves that it is indeed the external normal vector to the face $\mathbf{F}_{k,i}$ of the simplex $\mathbf{F}_k$.

It is worth mentioning that even if the sub-faces $\mathbf{F}_{k,j}$ and $\mathbf{F}_{j,k}$ are equal as sets, the vectors $r_{k,j}$ and $r_{j,k}$ are not equal since they are respectively parallel to the non parallel hypersurfaces $\mathbb{A}_k$ and $\mathbb{A}_j$.

With k fixed, consider $r_{k,i}, r_{k,j}$, $i \neq j$ and $i, j \neq k$.

$$\begin{aligned} r_{k,i} \cdot r_{k,j} &= (s_i - \frac{s_i \cdot s_k}{s_k^2} s_k) \cdot (s_j - \frac{s_j \cdot s_k}{s_k^2} s_k) \\ &= s_i \cdot s_j - \frac{s_i \cdot s_k}{s_k^2} s_k \cdot s_j - \frac{s_j \cdot s_k}{s_k^2} s_i \cdot s_k + \frac{s_i \cdot s_k}{s_k^2} \frac{s_j \cdot s_k}{s_k^2} s_k^2 \\ &= s_i \cdot s_j - \frac{(s_j \cdot s_k)(s_i \cdot s_k)}{s_k^2} . \end{aligned}$$

If we know that the scalar products of the pairs of different external normal vectors in s are all negative then so is the product of any two different external normal vectors in r (to two sub-faces of the same face), which concludes the proof. □

In particular, inductively descending to dimension one sub-faces, *i.e.,* edges, we deduce that if the simplex is face-wise acute, then each triangle formed from three different vertices is acute; a property that we call *edge-wise acuteness*, otherwise speaking the following corollary holds true:

Corollary IV.7 (Edge-wise acuteness)**.** *Any face-wise acute simplex is edge-wise acute, i.e.,*

$$i, j \in J_d \setminus \{k\} \;\Rightarrow\; (v_i - v_k) \cdot (v_j - v_k) > 0 . \tag{IV.8}$$

There are simplices (in any dimension greater than 2) which are edge-wise acute and not face-wise acute.

Example 2. *For simplicity we do not work in an hypersurface of* $\mathbb{R}^4$ *but in* $\mathbb{R}^3$*. We will construct a family of full dimensional simplices. For that effect consider the fixed vertices* $v_0 = (1,0,0)$*,* $v_1 = (-1,0,0)$*, and* ϵ*-dependent vertices* $v_{2,\epsilon} = (0, 1 + \epsilon, 0)$*, and* $v_{3,\epsilon} = (0,0,1+\epsilon)$*. Then* $v_{2,\epsilon} - v_0 = (-1, 1+\epsilon, 0)$*,* $v_{3,\epsilon} - v_0 = (-1,0,1+\epsilon)$*,* $v_{2,\epsilon} - v_1 = (1, 1+\epsilon, 0)$*,* $v_{3,\epsilon} - v_1 = (1, 0, 1+\epsilon)$*. With* $\epsilon = 0$*:*

- Two faces with edge (v_0, v_1) *are isosceles right triangles and the angle between these faces is also right.*

- The two other faces, i.e., $(v_{2,0} - v_0, v_{3,0} - v_0) = ((-1,1,0), (-1,0,1))$ *and* $(v_{2,0} - v_1, v_{3,0} - v_1) = ((1,1,0),(1,0,1))$ *are equilateral triangles, respectively with outward normal vectors* $(1,1,1)$ *and* $(-1,1,1)$ *so that they meet at the obtuse angle (the angle between the normals has cosine equal to* $\frac{1}{3}$*).*

With small positive ϵ *we get that all edge-wise angles are acute, but it is plain that the obtuse angle between faces that we observed at* $\epsilon = 0$ *remains obtuse.*

Digression 5. *Edge-wise acuteness, even weak, provides a geometric insight. For example:*

Edge-wise Acute and Inverted Vertex Cones.

A simplex is *edge-wise acute* iff every inverted vertex cone fits in the Voronoï cone of the same vertex. The vertex cone is given by $\mathbf{K}_k = \{v_k + \sum_{i \neq k} \mu^i (v_i - v_k) : \mu^i \geq 0\}$.

$$v_k - \mathbf{K}_k \subset \overline{\mathbf{V}_k - \mathcal{O}} .$$

The closure is needed, since due to the tie-breaking rules (and more generally because we need a partition to have deterministic output deffinition) $\mathcal{O}$ *belongs to only one* *element* $\mathbf{V}_k$ *of the partition by the* $\mathbf{V}_i$*'s. It follows that* 0 *is not in all other sets* $\mathbf{V}_k - \mathcal{O}$. Proof*: Let* $x = v_k + \sum_{i\neq k} \mu^k (v_i - v_k)$ *and consider the point* $y = v_k - x + \mathcal{O}$*.*

$$\begin{aligned} (y - v_k)^2 \quad - &\quad (y - v_j)^2 = (2y - v_k - v_j)\cdot(v_j - v_k) \\ = &\quad (2\mathcal{O} - 2\sum_{i\neq k} \mu^i (v_i - v_k) - v_k - v_j)\cdot(v_j - v_k) \\ = &\quad -\sum_{i\neq k} \mu^i (v_i - v_k)\cdot(v_j - v_k) \end{aligned}$$

If the point x *were in the vertex cone then all* $\mu^i \geq 0$ *and by edge-wise acuteness* $y \in \mathbf{V}_k$*, that is* $v_k - x \in \mathbf{V}_k - \mathcal{O}$*. On the other hand if* $y \in \mathbf{V}_k$ *for any choice of positive* μ^i*, then choosing all of them zero except for* $\mu^l = 1$ *we get* $0 \geq (y - v_k)^2 - (y - v_j)^2 = -(v_l - v_k)\cdot(v_j - v_k)$*, which is the edge-wise acuteness.*

Digression 6. *We remark that the vertex cone and the Voronoï cone are dual, that is each one dimensional boundary of one cone is orthogonal to a corresponding codimension 1 face of the other. The one dimensional boundary of* $\mathbf{K}_k$ *is given by the ray* $v_k + \mu^j (v_j - v_k)$ *and the corresponding codimension 1 boundary of* $\mathbf{V}_k$ *is given by* $\mathcal{O} + \sum_{i\neq j,k} \lambda^i s_i$*. Similarly the one dimensional boundary of* $\mathbf{V}_k$ *is given by* $\mathcal{O} + \lambda^j s_j$ *and the corresponding codimension 1 boundary of* $\mathbf{K}_k$ *is given by* $v_k + \sum_{i\neq j,k} \mu^i (v_i - v_k)$*.*

IV.4. **Big simplex and Inverted simplex.** Given a point $g \in \mathbb{A}$ we define the points of the collection $w \equiv w(v)$ and the simplex they generate by

$$w_k = \mathcal{O} + g - v_k, \quad \triangle_R = \triangle(w) = \{\sum_{i=0}^{d} \xi^i w_i :\ \xi^i \geq 0,\ \sum \xi^i = 1\}\,. \tag{IV.9}$$

As $w_k - w_j = v_j - v_k$, the points w are independent and that the simplex $\triangle_R$ is non-degenerate; we call $\triangle_R$ the *Inverted Simplex*.

The simplex $\triangle(w)$ is symmetric to $\triangle(v)$ with respect to the point $(\mathcal{O}+g)/2$. The two simplices can be disjoint. The inverted simplex will be the minimal founding block in the construction of the invariant set.

Digression 7. *There is another simplex symmetric to* $\triangle_R$*, a translation of* $\triangle(v)$ *namely* $\triangle(m)$ *where*

$$m_i = -\mathcal{O} + g + v_i$$

and the center of symmetry is g*. This simplex plays an important role the construction of the set* $\mathcal{H}$*, see further* (VI.1)*. We have*

$$m_i \in \overline{\mathbf{V}_i} \tag{IV.10}$$

as

$$\begin{aligned} (m_i - v_i)^2 - (m_i - v_j)^2 \quad &= \quad (2m_i - v_i - v_j)\cdot(v_j - v_i) \\ &= \quad (-2\mathcal{O} - 2g + v_i + v_j - 2v_j)\cdot(v_j - v_i) = 2(g - v_j)\cdot(v_j - v_i) \\ &= \quad 2\sum \gamma^k (v_k - v_j)\cdot(v_j - v_i) \leq 0\,. \end{aligned}$$

Using the same points we shall construct (as will turn out, around $\triangle_R$) a large simplex which will be proved to be invariant, but which is by no means minimal.

Consider the affine half-spaces

$$\mathbb{W}_k = \mathbb{A}^-_k(w_k) = \{x : (x - w_k)\cdot s_k \leq 0\}\,, \tag{IV.11}$$

Then:

Lemma IV.8. *If $g \in \triangle(v)$ then:*

$$\mathbf{V}_k \subset \mathbb{A}_k^-(\mathcal{O}) \subset \mathbb{W}_k \,.$$

Proof. The first inclusion repeats Lemma (IV.5). If $x \in \mathbb{A}_k^-$ then:

$$\begin{aligned} (x - w_k)\cdot s_k &= (x-\mathcal{O})\cdot s_k + (\mathcal{O} - w_k)\cdot s_k \\ &\leq (v_k - g)\cdot s_k = \sum \gamma^i (v_k - v_i)\cdot s_k = -(1-\gamma^k) \leq 0\,, \end{aligned}$$

where we used the normalization of s_k and $g \in \triangle(v)$. □

Lemma IV.9 (Inverted simplex in $\mathbb{W}$)**.** *For every k:*

$$\triangle_R \subset \mathbb{W}_k$$

Proof. By convexity it is enough to prove that for any j and k, we have $w_j \in \mathbb{W}_k$. But we have

$$(w_j - w_k)\cdot s_k = (v_k - v_j)\cdot s_k = -1 + \delta_{jk} \leq 0\,.$$

□

Define the *Big Simplex*:

$$\triangle_B = \bigcap_{i=0}^{d} \mathbb{W}_i\,. \tag{IV.12}$$

Corollary IV.10.

$$\triangle_R \subset \triangle_B\,.$$

Let

$$u_k = w_k + \sum_{i=0}^{d} (v_k - v_i) \tag{IV.13}$$

Observe that $u_k - u_j = d(v_k - v_j)$, hence the points u are independent. Then

Lemma IV.11.

$$\triangle_B = \triangle(u)$$

In particular $\triangle_B$ is a scaled up version of $\triangle(v)$.

Proof. Consider $x = \sum \xi^j u_j = \sum_j \xi^j (w_j + \sum_i (v_j - v_i))$, with $\sum \xi^j = 1$, and the difference $x - w_k = \sum_j \xi^j (w_j - w_k) + \sum_j \xi^j \sum_i (v_j - v_i)$, where $w_j - w_k = v_k - v_j$. We shall see that the condition of $x \in \triangle_B = \cap_k \mathbb{W}_k$, which depends on the signs of $(x - w_k)\cdot s_k$ is the same as the condition of $x \in \triangle(u)$ which depends on the signs of ξ^j. In the following we use the normality $(v_j - v_i)\cdot s_k = 0$ whenever both $i, j \neq k$ or when $i = j$. By normalization the remaining cases give $(v_j - v_k)\cdot s_k = 1$.

$$\begin{aligned} (x - w_k)\cdot s_k &= \sum \xi^j (v_k - v_j)\cdot s_k + \sum_j \sum_i \xi^j (v_j - v_i)\cdot s_k \\ &= \sum \xi^j (v_k - v_j)\cdot s_k + \sum_{j\neq k}\sum_i \xi^j (v_j - v_i)\cdot s_k + \xi^k \sum_i (v_k - v_i)\cdot s_k \\ &= \sum_{j\neq k} \xi^j (v_k - v_j)\cdot s_k + \sum_{j\neq k} \xi^j (v_j - v_k)\cdot s_k + \xi^k \sum_{i\neq k} (v_k - v_i)\cdot s_k \\ &= \xi^k \sum_{i\neq k} (v_k - v_i)\cdot s_k = -d\xi^k \end{aligned}$$

All $\xi^k \geq 0$ that is $x \in \triangle(u)$ if and only if all scalar products $(u_k - w_k) \cdot s_k \leq 0$ that is $x \in \cap_k \mathbb{A}^-_k(w_k) = \triangle_B$. □

Digression 8. *In case $x = u_j$ we have $\xi^j = 1$ and all other $\xi^i = 0$, which proves that u_j lies on boundaries of d different $\mathbb{W}_k$, and in the interior of $\mathbb{W}_j$. That shows in a different way that u_j are the vertices of $\triangle_B$.*

Digression 9 (Edge-wise acute big simplex has corners in the right places)**.** *If the simplex $\triangle(v)$ is acute and $g \in \triangle(v)$ then $u_j \in \mathbf{V}_j$.*

Proof. We use Lemma IV.2 and $g = \sum_i \gamma^i v_i$, $\sum_i \gamma^i = 1$, $0 < \gamma^i < 1$.

$$\begin{aligned}(u_j - \mathcal{O}) \cdot (v_k - v_j) &= \left(g - v_j + \sum_i (v_j - v_i)\right) \cdot (v_k - v_j) \\ &= \left(\sum_i \gamma^i v_i - v_j + \sum_i (v_j - v_i)\right) \cdot (v_k - v_j) \\ &= \left(\sum_i \gamma^i (v_i - v_j) - \sum_k (v_i - v_j)\right) \cdot (v_k - v_j) \\ &= -\sum_i (1 - \gamma^i)(v_i - v_j) \cdot (v_k - v_j) < 0\end{aligned}$$

which proves $(u_j - v_j)^2 < (u_j - v_k)^2$, and thus $u_j \in \mathbf{V}_j$. □

V. Dynamics of the Error Diffusion on acute simplices

Recall (I.4): $\mathcal{F}(x) = \mathcal{F}_g(x) = x + g - v(x)$.

V.1. **Consequences of tiling.**

Lemma V.1 (Equivalence of points is an invariant of motion)**.**

$$y - x \in \mathbf{L} \ \Leftrightarrow\ \mathcal{F}(y) - \mathcal{F}(x) \in \mathbf{L}\,.$$

Proof. $\mathcal{F}(y) - \mathcal{F}(x) = (y + g - v(y)) - (x + g - v(x)) = (y - x) - (v(y) - v(x))$ and as $v(y) - v(x) \in \mathbf{L}$ the results follows from the group properties of $\mathbf{L}$. □

Lemma V.2 ($\mathcal{F}$ acting on tiles)**.** *For any tile $\mathbf{Q}$ of $\mathbf{L}$ the set $\mathcal{F}(\mathbf{Q})$ is a tile, and the map $\mathcal{F}_{|\mathbf{Q}}$ is 1-1.*

Proof. "into": We have to prove that for $x, y \in \mathcal{F}(\mathbf{Q})$ and $p, q \in \mathbf{L}$ we have $(y+q)-(x+p) = 0$ we have $x = y$. Let $x', y' \in \mathbf{Q}$ such that $x = \mathcal{F}(x') = x' + g - v(x')$ and $y = \mathcal{F}(y') + g - v(y')$. Then with $q' = q + v(x') - v(y') \in \mathbf{L}$ we have $(y'+q')-(x'+p) = (y' + q + v(x') - v(y') - (x' + p) = (y + q) - (x + p) = 0$, which, from the "into" property for $\mathbf{Q}$, implies that $x' = y'$ and thus $x = y$.

"onto": Let $z \in \mathbb{A}$. As $\mathbf{Q}$ is a tile there are $x' \in \mathbf{Q}$ and $r' \in \mathbf{L}$ such that the point $z' = z - g + v(z)$ is covered by $z' = x' + r'$. Let $r = r' - v(z) + v(x') \in \mathbf{L}$ and $x = \mathcal{F}(x') = x' + g - v(x') \in \mathcal{F}(\mathbf{Q})$ then $z = z' + g - v(z) = x' + r' + g - v(z) = x - g + v(x') + r' + g - v(z) = x + r$ is the required coverage of z by $x \in \mathcal{F}(\mathbf{Q})$ and $r \in \mathbf{L}$.

Finally: If for $x, y \in \mathbf{Q}$, we have $\mathcal{F}(x) = \mathcal{F}(y)$, then x and y are equivalent, hence equal. □

Corollary V.3 (Invariant tiles admit inverse maps)**.** *If* $\mathbf{Q}$ *is a tile and* $\mathcal{F}(\mathbf{Q}) \subset \mathbf{Q}$ *then there exists the inverse map* $\mathcal{G} : \mathbf{Q} \to \mathbf{Q}$ *such* $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathbf{Q}}$ *and* $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathbf{Q}}$

Proof. $\mathcal{F}(\mathbf{Q}) \subset \mathbf{Q}$ being a tile by Lemma V.2 it must be equal to $\mathbf{Q}$. Any map which is into and onto has an inverse. □

Digression 10. *If* $\mathcal{F}$ *is "onto"* $\mathbb{A}$ *the map* $\mathcal{G}$ *can be extended to a piecewise translation on* $\mathbb{A}$ *by the vectors* $w_k - g$ *on a partition* $\mathbf{U}_k$*, with* $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{|\mathbb{A}}$*. For every* y *pick any* $x(y)$ *with* $\mathcal{F}(x) = y$*. Then* $x = y - g + v(x)$ *and we can define*

$$\mathbf{U}_k = \{y : x(y) \in \mathbf{V}_k\}$$

with $w(y) = v_k$ *on* $\mathbf{U}_k$*. Then* $\mathcal{G}(y) = y - g + w(y) = y - g + v(x(y)) = x(y)$ *with* $\mathcal{F}(x(y)) = y$*.*

Digression 11. *As we have seen before the map* $\mathcal{F}$ *projects onto a map on the torus* $\mathbb{T} = \mathbb{A}/\mathbf{L}$ *as a rotation (i.e., a group translation). Any tile will cover all the points of the torus. Now it is enough to see that for any* $x, y \in \mathbb{A}$ *with* $[x] = [y]$ *(that is with* $x - y \in \mathbf{L}$*) we have* $\mathcal{F}(y) - \mathcal{F}(x) = (y + g - v(y)) - (x + g - v(x)) = (x - y) + (v(x) - v(y)) \in \mathbf{L}$*, which means* $[\mathcal{F}(x)] = [\mathcal{F}(y)]$*. In other words the translations by all the vectors* $g - v_k$ *project on the torus to the translation by the vector* $[g - v_0] = [g - v_k]$*, where the whole lattice* $v_0 + \mathbf{L}$ *projects on the origin* 0*,* $[\mathbf{L}] = 0$*. That means that* $[\mathcal{F}(y)]$ *does not depend on* $y \in [x]$*. One may denote the projection of* $\mathcal{F}$ *by* $[\mathcal{F}]$*.*

V.2. **Dynamics of the Big Simplex.** The next Lemma will allow us to show the invariance of the Big Simplex.

Lemma V.4 (Shifted Voronoïs stay in subspaces)**.** *If the simplex* $\triangle(v)$ *is face-wise acute and* $g \in \triangle(v)$ *then for any* k *and* j*:*

$$\mathbf{V}_k + g - v_k \quad \subset \quad \mathbb{W}_k \tag{V.1}$$

$$\mathbf{V}_k \cap \mathbb{W}_j + g - v_k \quad \subset \quad \mathbb{W}_j \tag{V.2}$$

For $j = k$ inclusion (V.1) is equivalent to (V.2) by Lemma IV.8.

Proof. Let $x = \mathcal{O} + \sum_{i \neq k} \lambda^i s_i \in \mathbf{V}_k$, with $\lambda^i \geq 0$ and $y = x + g - v_k$ then $y - w_j = v_j - v_k + \sum_{i \neq k} \lambda^i s_i$ and:

$$(y - w_k) \cdot s_k = \sum_{i \neq k} \lambda^i s_i \cdot s_k < 0$$

as $s_i \cdot s_k < 0$, but that means $y \in \mathbb{W}_k$. Suppose now that additionally $x \in \mathbb{W}_j$ that is $(x - w_j) \cdot s_j < 0$ for some $j \neq k$. Then:

$$(y - w_j) \cdot s_j = (x - w_j) \cdot s_j + (g - v_k) \cdot s_j < \sum_i \gamma^i (v_i - v_k) \cdot s_j = \gamma^j (v_j - v_k) \cdot s_j = -\gamma^j < 0$$

by normality and normalization of s_j and positivity of γ^i for all i. Again that means that $y \in \mathbb{W}_j$. □

Corollary V.5 (The half spaces in $\mathbb{W}$ are invariant)**.**

$$\mathcal{F}(\mathbb{W}_j) \subset \mathbb{W}_j \, .$$

Proof. Let $x \in \mathbb{W}_j$, $v(x) = v_k$, then by previous Lemma $\mathcal{F}(x) = x + g - v_k \in \mathbb{W}_j$. □

Lemma V.6 (The half spaces in $\mathbb{W}$ are absorbing)**.** *Assume $g \in \triangle(v)$. For every $x \in \mathbb{A}$ and every k there is an N such that for all $M \geq N$ we have*

$$\mathcal{F}^M(x) \in \mathbb{W}_k\,.$$

Proof. By previous Corollary it is enough to drive x into this half-space once. Let $N \geq 0$ be such that $\mathcal{F}^N(x) \notin \mathbb{W}_k$. Then

$$\mathcal{F}^N(x) = x + Ng - \sum_{0\leq K<N} v(\mathcal{F}^K(x)) = x + \sum_{i=0}^{d} n^i(g - v_i),$$

where $n^i = \#\{K < N : v(\mathcal{F}^K(x)) = v_i$. Because $\mathbf{V}_k \subset \mathbb{W}_k$ by (IV.8) none of the $v(\mathcal{F}^K(x))$'s equals v_k, which means $n^k = 0$. Denoting $L_k = (x - \mathcal{O} + v_k - g)\cdot s_k$, we have:

$$\begin{aligned}
(\mathcal{F}^N(x) - w_k)\cdot s_k &= (x - \mathcal{O} + v_k - g)\cdot s_k + \sum_{i\neq k} n^i \sum_l \gamma^l (v_l - v_i)\cdot s_k \\
&= L_k + \sum_{i\neq k} n^i \gamma^k (v_k - v_i)\cdot s_k = L_k - \gamma^k \sum_{i\neq k} n^i = L_k - \gamma^k N\,.
\end{aligned}$$

That means that $\mathcal{F}^N(x)$ stays away from $\mathbb{W}_k$ for at most $\frac{L_k}{\gamma^k}$ steps, where L_k depends on x, k and the geometry of the simplex. □

Observe that we needed $\gamma^k > 0$, since otherwise the statement would not be true; then indeed the trajectory may stay on a plane parallel to $\mathbb{A}_k$.

Proposition V.7 (The Big simplex is invariant and absorbing)**.** *We have $\mathcal{F}_g(\triangle_B) \subset \triangle_B$ and for every x there is an N such that for all $M \geq N$, $\mathcal{F}^M(x) \in \triangle_B$.*

Proof. Using L_k from the proof above, after $N = \max(L_k/\gamma^k)$ steps the trajectory lands in all $\mathbb{W}_k$ and stays there, that is it lands in $\triangle_B$. □

V.3. **Dynamics of the Inverted Simplex.**

Proposition V.8 (No equivalences in the Inverted Simplex)**.** *No point in the interior of $\triangle_R$ has an equivalent point in the big simplex $\triangle_B$.*

Proof. We recall that y and x are equivalent (I.14) if $y - x \in \mathbf{L}$ or $y - x = \sum n^i v_i$, for all collections of n^i's in $\mathbb{Z}$ with $\sum n^i = 0$. Let $x \in \triangle_R^\circ$; then for some collection of ξ^i's greater than 0 with $\sum_i \xi^i = 1$:

$$x = \sum \xi^i w_i = \mathcal{O} + g - \sum \xi^i v_i\,.$$

Let $y \in \triangle_B$; then for some collection of nonnegative η^i's with $\sum_i \eta^i = 1$:

$$y = \sum \eta^i u_i = \mathcal{O} + g - \sum_i \eta^i v_i + \sum_i \eta^i \sum_j (v_i - v_j)\,.$$

Using the formulas that we just got for x and y, we have:

$$\begin{aligned}
y - x &= \sum_i \xi^i v_i - \sum_i \eta^i v_i + \sum_i \eta^i \sum_j (v_i - v_j) \\
&= \sum_i (\xi^i + d\eta^i - 1) v_i,
\end{aligned}$$

where we used

$$\begin{aligned} & \sum_i \eta^i \sum_j (v_i - v_j) = \sum_i \eta^i \sum_j (v_i - v_0) - \sum_i \eta^i \sum_j (v_j - v_0) \\ = \quad & \sum_j \sum_i \eta^i (v_i - v_0) - \sum_j (v_j - v_0) = (d+1) \sum_i \eta^i (v_i - v_0) - \sum_i (v_i - v_0) \\ = \quad & \sum_i ((d+1)\eta^i - 1)(v_i - v_0) = \sum_i ((d+1)\eta^i - 1) v_i, \text{ as } \sum_i ((d+1)\eta^i - 1) = 0 \,. \end{aligned}$$

Since the ξ^i's are strictly positive and the η^i's non negative, we have $\xi^i + d\eta^i - 1 > -1$. When $y - x \in \mathbf{L}$, all coefficients $\xi^i + d\eta^i - 1$ are integers and being larger than -1 they have to be non-negative. But they sum up to 0, and therefore $\xi^i + d\eta^i - 1 = 0$ for all i, that is $y - x = 0$. Thus $y - x \in \mathbf{L}$, *i.e.*, the equivalence of x and y, cannot happen except in the trivial case when $y = x$. □

Digression 12. *In the above proof we established that for any point x the barycentric coordinates with respect to the inverted simplex ξ and the big simplex η fulfill the condition $\xi = 1 - d\eta$.*

Corollary V.9 (No equivalences in forward images of the Inverted Simplex)**.** *Suppose that for some $0 \le K \le N$, two points $x \in \mathcal{F}^K(\triangle_R^\circ)$ and $y \in \mathcal{F}^N(\triangle_B)$ are equivalent. Then there exists a unique $x' \in \triangle_R^\circ$ such that for any $y' \in \triangle_B$ with $\mathcal{F}^N(y') = y$ we have:*

$$x' = \mathcal{F}^{N-K}(y') \quad \text{ and } \quad x = \mathcal{F}^K(x') = \mathcal{F}^N(y') = y \,.$$

Proof. By assumption there is an $x' \in \triangle_R^\circ$ with $\mathcal{F}^K(x') = x$. Such point is unique in $\triangle_B$. If there were another one then their images would be equal, hence equivalent, which means the points themselves would be equivalent and we can apply Proposition V.8. Let $y' \in \triangle_B$ be any point with $\mathcal{F}^N(y') = y$ and denote $x'' = \mathcal{F}^{N-K}(y')$. Then $x'' \in \triangle_B$ by the invariance part of Proposition V.7 Furthermore x'' is equivalent to x' by Lemma V.1 since x and y, their respective images under $\mathcal{F}^K$ are equivalent. Thus using Proposition V.8 we conclude that $x'' = x'$, and $y = x$. □

The following two corollaries are straightforward:

Corollary V.10. *$\mathcal{F}^K(\triangle_R^\circ)$ and $\mathcal{F}^N(\triangle_R^\circ)$ have no distinct equivalent points.*

Corollary V.11. *$\mathbf{Q} = \bigcup_{N=0}^\infty \mathcal{F}^N(\triangle_R^\circ)$ has no distinct equivalent points.*

V.4. **Consequences of boundedness.**

Lemma V.12. *The distance of a point $x = \sum_{i=0}^d \xi^i v_i$, $\sum \xi^i = 1$ to the affine subspace $\mathbb{A}_k$ spanned by the face $\mathbf{F}_k$ is equal to $|\xi^k|/|s_k|$, where s_k is the external normal vector to this face* c.f. (I.10)*.*

Proof. We can represent the point x as a point from $\mathbb{A}_k$ plus a vector parallel to s_k, that is as $x = \sum_{j=0, j\neq k}^d \eta^j v_j - \beta s_k$, $\sum_{j \neq k} \eta^j = 1$ where we chose $\beta > 0$ for the halfspace including v_k. By orthogonality of s_k to $\mathbb{A}_k$ the distance we are looking for equals $|\beta s_k|$. Because $1 = \sum_i \xi^i = \sum_{j\neq k} \eta^j$ we have $v_k = \sum_i \xi^i v_k = \sum_{j \neq k} \eta^j v_k$

and:

$$\begin{aligned}0 &= x - x = \sum_{i=0}^{d} \xi^i v_i - \sum_{j=0, j\neq k}^{d} \eta^j v_j + \beta s_k \\ &= \sum_{i\neq k} \xi^i (v_i - v_k) + \xi^k (v_k - v_k) - \sum_{j\neq k} \eta^j (v_j - v_k) + \beta s_k \,, \\ \beta s_k \cdot s_k &= \sum_{j\neq k} \eta^j - \sum_{i\neq k} \xi^i = 1 - (1 - \xi^k) = \xi^k,\end{aligned}$$

as for $i \neq k$, $s_k \cdot (v_i - v_k) = 1$. It follows that $\beta = \xi^k / s_k^2$ and hence $|\beta s_k| = |\xi^k|/|s_k|$. □

In particular the length of the altitude drawn from the vertex v_k is equal to $1/|s_k|$. The altitude is the distance from the vertex to the opposite face, but for $x = v_k$ we have $\xi^k = 1$.

Corollary V.13. *If the distance between two points $x = \sum \xi^i v_i$ and $y = \sum \eta^i v_i$ is smaller than $\frac{|\xi^k|}{2|s_k|}$ then $|\eta^k| > |\xi^k/2|$, and η^k and ξ^k have the same signs.*

Proof. The point x is at the distance $|\xi^k|/|s_k|$ from $\mathbb{A}_k$, therefore the point y is at the distance at least $|\xi^k|/2|s_k|$ from this plane, so that $|\eta^k|/|s_k| > |\xi^k|/2|s_k|$. The sign condition means that they are on the same side of $\mathbb{A}_k$. □

Lemma V.14. *If the trajectory $\mathcal{F}^K(x)$ of x is bounded by L then there exists an N depending only on L and g such that in any time segment of length N the trajectory visits each Voronoï region.*

Proof. Define $n^k(N, x) = \#\{0 \leq K < N : \mathcal{F}^K(x) \in \mathbf{V}_k\}$, the number of visits in a Voronoï region k. Let $D = \min_i \frac{\gamma^i}{|s_i|} > 0$ be the distance from the point g to the exterior of the simplex, or the minimal distance to its faces. Take N large enough so that $\frac{2L}{N} < \frac{D}{2}$. Then

$$2L > |\mathcal{F}^N(x) - x| = |\sum_{n=0}^{N-1} (g - \mathbf{V}(\mathcal{F}^n(x)))| = N|g - \sum_{i=0}^{d} \frac{n^i(N,x)}{N} v_i| \tag{V.3}$$

This means that the distance between the point $z = \sum \zeta^i v_i$ with $\zeta^i = \frac{n^i}{N}$ and the point g is smaller than half of the distance from g to $\mathbb{A}_k$. It follows by Corollary V.13 that for all i $\zeta^i > 0$ and in particular all n^i's satisfy $n^i > 0$. This implies in turns that the trajectory segment so prescribed visits each Voronoï region at least once. □

In the limit process we have a more precise estimate:

Corollary V.15 (Frequency Lemma)**.** *For any bounded trajectory*

$$\lim_{N\to\infty} \frac{n^i(N,x)}{N} = \gamma^i$$

Proof. By taking the limit in (V.3) we obtain

$$\sum_{i=0}^{d} \frac{n^i(N,x)}{N} v_i \to g = \sum_{i=0}^{d} \gamma^i v_i$$

and the result follows from the uniqueness of barycentric coordinates. □

Corollary V.16. *If $\mathbf{Q}$ is a bounded invariant tile then there exists an N such that both for the map $\mathcal{F}$ and the map $\mathcal{G} = \mathcal{F}^{-1}$ which is the inverse of $\mathcal{F}$ on $\mathbf{Q}$, the trajectory of length N of any point visits each Voronoï region. In particular for any $x \in \mathbf{Q}$ and any i there exists an infinite sequence of iterates N_i such that $\mathcal{F}^{N_i}(x) \in \mathbf{Q} \cap \mathbf{V}_i$. As by Corollary V.3 $\mathcal{G}$ is well defined on $\mathbf{Q}$, for any x and any i there exists and infinite sequence of N_i such that $\mathcal{G}^{N_i}(x) \in \mathbf{Q} \cap \mathbf{V}_i$, or in other words there exist points $x_{N_i} \in \mathbf{Q} \cap \mathbf{V}_i$ such that $F^{N_i}(x_{N_i}) = x$.*

VI. Additional properties of invariant sets

VI.1. **The shape of the Q.**

Proposition VI.1. *The closure of the minimal invariant absorbing set for ergodic constant input error diffusion on face-wise acute simplices is a finite union of compact of convex polytopes. Each codimension 1 face of each of those polytopes is orthogonal to some edge $v_j - v_k$, $j \neq k$.*

Before we prove this Proposition we need some technical facts. Let us consider all potential preimages of the points w_k. Define:

$$w_{jk} = w_k - g + v_j = \mathcal{O} + g - v_k - g + v_j = \mathcal{O} + v_j - v_k \,.$$

Lemma VI.2 (w_k has all the preimages)**.** *In an edge-wise acute simplex for any j and k we have $w_{jk} \in \overline{\mathbf{V}}_j$, and thus $\mathcal{F}_g(w_{jk}) = w_k$ for every j.*

Proof. We use the characterization of $\mathbf{V}_j$ from Lemma IV.2:

$$(w_{jk} - \mathcal{O})(v_i - v_j) = (v_j - v_k)(v_i - v_j) \leq 0$$

by edge-wise acuteness. Thus $w_{jk} \in \overline{\mathbf{V}}_j$. □

Corollary VI.3. *For any j the open simplex generated by the points w_{jk}, (where $k = 0, \dots, d$) lies in $\mathbf{V}_j$ and is translated by $\mathcal{F}$ onto $\triangle_R$. In particular $\triangle_R^\circ \subset \mathcal{F}(V_j)$.*

Proof. This follows from the previous Lemma by convexity. □

Let

$$\mathcal{H} = \bigcap \mathcal{F}(\mathbf{V}_i)\,. \tag{VI.1}$$

We have:

Corollary VI.4.

$$\triangle_R^\circ \subset \mathcal{H} \subset \triangle_B\,.$$

Proof. The first inclusion follows from Corollary VI.3. By the definition of $\mathcal{H}$, Corollary V.5 and the definition (IV.12) of $\triangle_B$ we have

$$\mathcal{H} = \bigcap \mathcal{F}(\mathbf{V}_i) \subset \bigcap \mathbb{W}_i = \triangle_B\,.$$

□

Proof. **of Proposition VI.1**

(1) For an ergodic rotation on the torus any open set will cover the whole torus in finitely many steps, in particular only a finite number M of iterations are needed for $\triangle_R^\circ$ to cover a tile $\mathbf{Q}$ in $\mathbb{A}$. This means that the minimal absorbing set can be constructed in finitely many steps.

(2) For an ergodic rotation on the torus the trajectory of any point will enter a given open set after a bounded number of steps, where the bound depends on the set, rotation but is uniform in starting points. Hence, there is an N such that for any $x \in \triangle_B$ there is an $n < N$ with $\mathcal{F}^n(x) \in \triangle_R$, but that means $\mathcal{F}^N(x) \in \mathbf{Q}$, or $\mathcal{F}^N(\triangle_B) \subset \mathbf{Q}$

(3) For any $\triangle_R \subset \mathcal{R} \subset \triangle_B$, $\bigcup_{N \leq n \leq N+M} \mathcal{F}^n(\mathcal{R}) = \mathbf{Q}$. The inclusion follows from the properties of N and the coverage from the properties of M as stated above.

(4) Suppose we have a set $\mathcal{R}$ which is a finite union of convex polytopes with every codimension 1 face parallel to one of the codimension 1 faces of some $\mathbf{V}_i$. That is each face of each component is orthogonal to one of the edges $(v_j - v_k)_{j \neq k}$. Then the image $\mathcal{F}(\mathcal{R})$ has the same property, it is also an finite union of convex polytopes with every codimension 1 face orthogonal to some edge.

    The image under $\mathcal{F}$ of a component polytope $\mathbf{P}$ is a union of translations of $\mathbf{P} \cap \mathbf{V}_i$. The boundary of each component is either an image of a piece of the boundary or an image of a piece of the boundary of some Voronoï region.

(5) Take $\mathcal{R} = \bigcap \mathcal{F}(\mathbf{V}_i)$, it is a convex, bounded polytope and by Corollary VI.4$\triangle_R \subset \mathcal{R} \subset \triangle_B$. The $d(d+1)$ codimension 1 faces of $\mathcal{R}$ are parallel to the boundaries of $\mathbf{V}_i$, that is to $d(d+1)/2$ hyperplanes orthogonal to the edges. By previous points after $N + M$ iterates we obtain the minimal invariant set $\mathbf{Q}$ and each of its boundaries is parallel to some boundary of some Voronoï region.

□

VI.2. **The size of $\mathbf{Q}_I$ and some remarks on the volume.**

VI.2.1. *Measures of sub-tiles.* Given a discrete lattice $\mathbf{L}$ the volume of the standard tile, the parallelepiped $\mathbf{P} = \{v_0 + \sum \xi^i (v_i - v_0), 0 \leq \xi^i < 1\}$, equals $|\det(v_i - v_0)_{i \neq 0}|$. The same is the volume of any measurable tile $\mathbf{Q}$, as can be proven by partitioning this tile into $\mathbf{Q}_w = \mathbf{Q} \cap (\mathbf{P} + w)$ with $w \in \mathbf{L}$. Then we have $\mathbf{Q} = \bigcup_w \mathbf{Q}_w$, where the union is disjoint, but also $\mathbf{P} = \bigcup_w \mathbf{P}_w$, a disjoint union, where $\mathbf{P}_w = \mathbf{Q}_w - w$. Therefore $|\mathbf{Q}| = \sum_w |\mathbf{Q}_w| = \sum_w |(\mathbf{Q}_w - w)| = \sum_w |\mathbf{P}_w| = |\mathbf{P}|$. It follows that the sub-tiles from the Theorem I.19 have volumes:

$$|\mathbf{Q}_I| = |\det(v_i - v_0, v_j - g)|_{i \in I, i \neq 0, j \notin I} = (\sum_{r \in I} \gamma^r) \cdot |\det(v_i - v_0)_{i \neq 0}|$$

If $I = \{0, \ldots, d\}$ there is nothing to prove. Otherwise assume there is a $k \notin I$. Then, assuming $0 \neq i \in I$ and $k \neq j \notin I$:

$$\begin{aligned}
& |\det(v_i - v_0, v_j - g)| = |\det(v_i - v_0, v_j - v_k, v_k - g)| \\
= \quad & |\det(v_i - v_0, v_j - v_k, \sum_{r \in I} \gamma^r (v_k - v_i) + \sum_{s \notin I} \gamma^s (v_k - v_s))| \\
= \quad & |\det(v_i - v_0, v_j - v_k, \sum_{r \in I} \gamma^r (v_k - v_r))| = \sum_{r \in I} \gamma^r |\det(v_i - v_0, v_j - v_k, v_k - v_r)| \\
= \quad & \sum_{r \in I} \gamma^r |\det(v_i - v_0, v_j - v_k, v_k - v_0)| = \sum_{r \in I} \gamma^r |\det(v_i - v_0, v_j - v_0, v_k - v_0)|
\end{aligned}$$

In the case of ergodic inputs this result is consistent with the Frequency Lemma, because the Ergodic Theorem states that in ergodic dynamical systems the measure of a set (space average) is equal to its limit frequency (time average).

VI.2.2. *Non ergodic inputs.* The volume of the inverted simplex $\triangle_R$ is equal to the volume of the original simplex and is a small $1/d!$ fraction of the volume of the standard (and hence of any) tile. If the input g is purely rational, then the dynamics is periodic for the translation on the torus and eventually periodic for the error diffusion in the affine space. For each point the period in the affine space is a multiple of the period on the torus, which equals to the smallest common denominator of the γ^i's. However for the points in $\triangle_R$ the period is exactly equal to the period on the torus, as each torus-period produces equivalent point in the affine space. For small periods (the smallest possible is $1/(d+1)$) the images of $\triangle_R$ cannot cover the tile. Therefore there are points in $\mathbf{Q}$ which cannot be controlled by specific properties of $\triangle_R$ and we need different arguments to prove that $\mathbf{Q}$ is a tile for non-ergodic inputs. We produced such arguments for $d = 1$ and $d = 2$, but for $d > 2$ the tiling Theorem is still a Conjecture.It is worth mentioning that if the input is such that each trajectory on the torus must enter the projection of the simplex $\triangle_R$ the proof of Theorem I.16 is still valid.

VII. The invariant absorbing tile in dimension 2

VII.1. **In dimension $d = 2$ the tile $\mathbf{Q}$ is simply connected.** In this section we shall prove:

Theorem VII.1. *For acute triangles the minimal absorbing invariant tile* $\mathbf{Q}$ *in dimension* $d = 2$ *is simply connected and its closure is a polygon.*

Proof. Recall (VI.1) that $\mathcal{H} = \bigcap \mathcal{F}(\mathbf{V})$. We shall assume that $\triangle(v)$ is strictly acute so that $\mathcal{H}$ is indeed hexagonal. In the right triangle case $\mathcal{H}$ becomes rectangular, and we leave to the reader to make the needed straight forward adjustments to the following proof.

It is convenient to label the maps

$$\mathcal{F}_i = \mathcal{F}|_{V_i} : x \mapsto x + g - v_i.$$

Then

$$\mathcal{F}_i(x) = \mathcal{F}_j(x) + v_j - v_i. \tag{VII.1}$$

Also we introduce the notation

$$\mathcal{H}^{(k)} = \mathcal{F}^k(\mathcal{H}), \quad \mathcal{H}_0 = \mathcal{H}.$$

We have

$$\overline{\mathcal{H}^{(k)}\mathcal{F}^k(\mathcal{H})} = \bigcap \mathcal{F}_i(\overline{\mathcal{H}}_{k-1}) \cap \overline{\mathbf{V}}_i$$

For the rest of the section *equivalence* will mean $\mathbf{L}$-*equivalence*. The extreme points of $\overline{\mathcal{H}}$ are $w_i = \mathcal{F}_i\mathcal{O}$ and their symmetrical partners $m_i = 2g - \mathcal{F}_i\mathcal{O}$. The three w_i points are equivalent and so are the three m_i. Furthermore, each side $[w_i, m_j]$ of the boundary of $\mathcal{H}$ is equivalent to its symmetrical opposite $[w_j, m_i]$. The difference between $\mathcal{H}$ and $\overline{\mathcal{H}}$ is that the boundary of $\mathcal{H}$ is missing some points due to the tie-breaking rules. However each of these missing boundary points has lattice equivalent another boundary point in $\mathcal{H}$. As we shall see the same is true for $\mathcal{H}^{(k)}$ and $\overline{\mathcal{H}}^{(k)}k$. We shall also see that like $\overline{\mathcal{H}}$ each $\overline{\mathcal{H}}^{(k)}k$ is a polygon by induction.

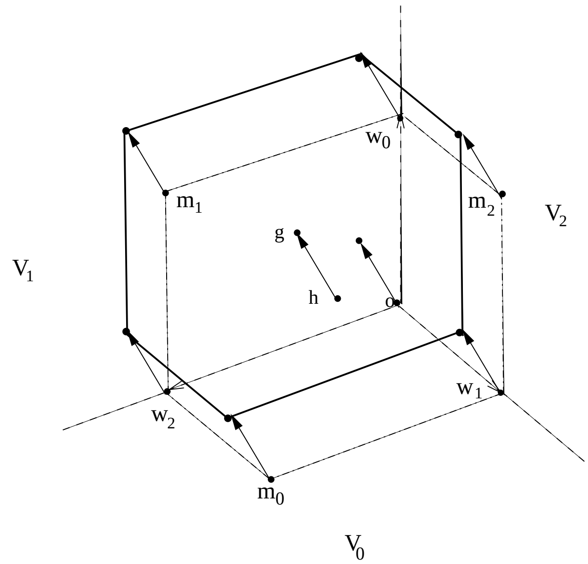

FIGURE 3. The Hexagon

We know that $\overline{\mathcal{H}}$ contains a tile and which of course turns out to be $\mathcal{H}$ (In higher dimensions this will not be true in general). Since $\mathcal{H}$ is a tile, we can conclude that every $\mathcal{H}^{(k)}$ is a tile. Furthermore, since $\mathcal{H} \subset \triangle_B$, we can conclude that $\mathbf{Q} = \mathcal{H}^{(n)}$ for some finite n.

From (IV.10) we have that $m_i \in \mathbf{V}_i^o$, but something quite different holds for w_i. Let h be the orthocenter of $\triangle(v)$, that is the intersection of altitudes. For acute triangles the orthocenter lies in the interior of the triangle (for right triangles on an edge). If we take $g = h$, then each w_i lies on a boundary of a Voronoï region and all Voronoï interiors are free from such points. See Fig. 3. If $g \in \mathbf{V}_i$ then $\overline{\mathcal{H}}$ is translated so that $\mathbf{V}_i^o$ contains two of three w_i. Since there are three Voronoï regions, the interior of one of them will not contain any w_i.

We shall orient $\triangle_R(v$ so that the Voronoï regions appear as in Fig. 3 with the base of the triangle horizontal. We label its vertices so that $w_0, w_1 \in V_1$ and $w_2 \in V_2$. We can refer to the boundaries of the three Voronoï regions as the *vertical*one, the *left* one and the *right*. We can refer to parts of the boundary of $\mathcal{H}^{(k)}$ as the *left side*, the *right side*, the *upper boundary* and the *lower boundary*.

We single out five important vertical lines: the *left side* and the *right*–namely the vertical lines through w_2 and w_1 respectively, the *vertical Voronï boundary* through $\mathcal{O}$, the *vertical line through* m_0, and finally the *vertical line through* $\overline{r}$ where $\overline{r} = r + v_0 - v_2$. The vertical line through m_0 will contain the points $m_0^{(k)}$. The vertical line through $\overline{r}$ will contain the points $\overline{r}^{(k)}$ Since the left side is equivalent to the right, it will perhaps be helpful to visualize the $\mathcal{H}^{(k)}$ as lying on a cylinder and the action of $\mathcal{F}$ on this cylinder by VII.1 as a rotation.

From the preceding section we know that points of $\triangle(w_0w_1w_2)$ are not equivalent to any other point in $\triangle_B$. So $\mathcal{O} \in \triangle(w_0w_1w_2) \subset \mathcal{H}_k$ for all k. Also this means the boundary of $\mathcal{H}^{(k)}$ does not penetrate the interior of $\triangle(w_0w_1w_2)$.

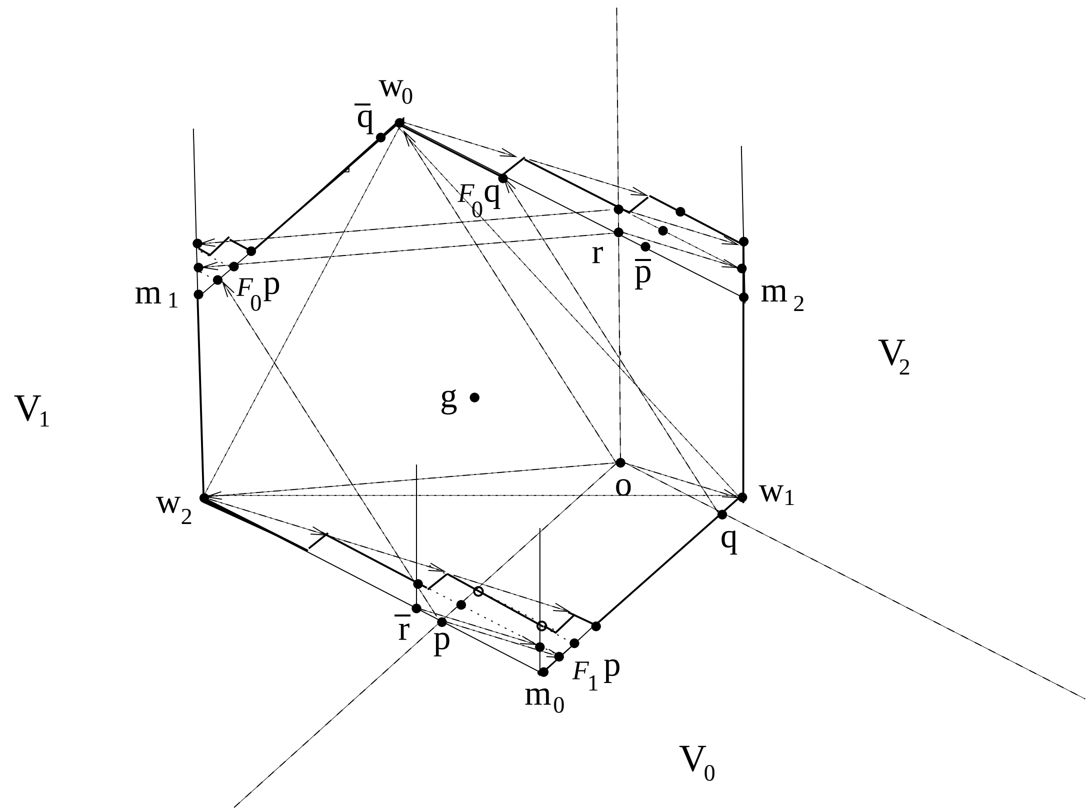

FIGURE 4. Iterates of $\overline{\mathcal{H}}$

We shall specify polygonal paths by their endpoints and sometimes intermediate points. For example we refer to the upper boundary of $\mathcal{H}$ as the polygonal path $[m_1 \dots w_0 \dots m_2]$. We can also use the convention of brackets and parenthesis to indicate whether the endpoints are included.

Inductive Hypotheses

(1) The left side boundary is a vertical line segment $[w_2, m_1^{(k)}] \subset V_1$, the right a vertical line segment $w_1, m_2^{(k)}] \subset V_2$. Initially $m_1^{(0)} = m_1$ and $m_2^{(0)} = m_2$. These two pieces of boundary are equivalent, their difference being $v_2 - v_1$
(2) The lower boundary of $\mathcal{H}^{(k)}$ is a polygonal path from w_2 to w_1. It consists of line segments parallel to either of the lower Voronoï boundaries. It intersects the left Voronoï boundary either at a unique point $p^{(k)}$ or a line segment starting at $p^{(k)}$, and the right at either a unique point $q^{(k)}$ or a line segment ending at $q^{(k)}$. The lower boundary consist of two sides: the right is the polygonal path from w_2 to $m_0^{(k)}$, which lies on the vertical line through m_0, and the left the polygonal path from $m_0^{(k)}$ to w_2.
(3) The upper boundary of $\mathcal{H}^{(k)}$ is a polygonal path from $m_1^{(k)}$ to $[m_2^{(k)}$. Likewise, it consists of line segments parallel to either of the lower Voronoï boundaries. The upper boundary consists of two sides: the right is the

polygonal path from $m_1^{(k)}$ to $[w_0$ and the left the polygonal path form w_0 to $m_2^{(k)}$.

(4) The left side of the lower boundary is equivalent to the right side of the upper, and the right side of the lower is equivalent to the left side of the upper.

In figure 4 all $p^{(k)}$ are equal, but this need not be the case for other choices of g. We label the following points equivalent to $p^{(k)}$ and $q^{(k)}$

$$\overline{p}^{(k)} = p^{(k)} + v_2 - v_0 \text{ with } \overline{p}^{(0)} = p$$

$$\overline{q}^{(k)} = q^{(k)} + v_1 - v_0 \text{ with } \overline{q}^{(0)} = q.$$

Let $r^{(k)}$ with $r^{(0)} = r$ be the point where the upper boundary of $\mathcal{H}^{(k)}$ intersects the vertical Voronoï boundary, unique because the upper boundary does not contain any vertical line segments. The equivalent point to $r^{(k)}$ on the lower boundary is $\overline{r}^{(k)}$.

The points of the boundary of $\mathcal{H}^{(k+1)}$ will be images of either points on the boundary of $\mathcal{H}^{(k)}$ or points on the boundary of Voronoï regions. Since one of the two sources of boundary points of $\mathcal{H}^{(k+1)}$ are images of boundary points of $\mathcal{H}^{(k)}$, we shall account for which images of boundary points of $\mathcal{H}^{(k)}$ are part of the boundary of $\mathcal{H}^{(k+1)}$ and which are not. For example, the $\mathcal{F}_0$-image of $(p^{(k)} \dots m_0^{(k)})$ of the lower boundary of $\mathcal{H}^{(k)}$ equals the $\mathcal{F}_2$-image of $(\overline{p}^{(k)} \dots m_2^{(k)})$ of the upper boundary. Since there are no other equivalent points of these two images in $\overline{\mathcal{H}^{(k+1)}}$, they contain no boundary points. Similarly the $\mathcal{F}_0$-image of $(m^{(k)_0} \dots q^{(k)})$ equals the $\mathcal{F}_1$-image of $(m^{(k)_1} \dots \overline{q^{(k)}}$ and this image disappears from the boundary of $\mathcal{H}^{(k+1)}$ for the same reason. Finally, the $\mathcal{F}_1$-image of $(w_2, m_1^{(k)}]$ equals the $\mathcal{F}_2$-image of $(w_2, m_1^{(k)}]$. and this image also disappears into the interior of $\mathcal{H}^{(k+1)}$.

(1) The left side boundary of $\mathcal{H}^{(k+1)}$ is the vertical segment $[w_2, m_1^{(k+1)}]$ which is the $\mathcal{F}_2$ image of the vertical Voronoï boundary segment $[\mathcal{O}, r^{(k)}]$ and the right the vertical segment $[w_1, m_2^{(k+1)}]$ which is the $\mathcal{F}_2$ image of the vertical Voronoï boundary segment $[\mathcal{O}, r^{(k)}]$. Clearly the left side boundary is equivalent to the right. We have that the $\mathcal{F}_1$ image of $[w_2, m_1^{(k)}] = \mathcal{F}_2([w_1, m_2^{(k)}])$. No points of this image has an equivalent point in $\overline{\mathcal{H}^{(k+1)}}$, and therefore cannot be the boundary point of $\mathcal{H}^{(k+1)}$.

(2) The lower boundary of $\mathcal{H}^{(k+1)}$ consists of the polygonal path $\mathcal{F}_1[w_2 \dots p^{(k)} \dots \mathcal{O}]$ joined to the polygonal path $\mathcal{F}_2[\mathcal{O} \dots q^{(k)} \dots w_1]$ at $\mathcal{F}_1 w_2 = \mathcal{F}_2 w_1$.
We single out two sides of this boundary, the left the polygonal path from w_2 to $\mathcal{F}_1\overline{r} = m_0^{(k+1)}$, and the right the polygonal path from $m_0^{(k+1)}$ to w_1. Note that $m_i^{(k+1)}$ are all equivalent.

(3) The upper boundary of $\mathcal{H}^{(k+1)}$ consists of the polygonal path $\mathcal{F}_2[r^{(k)} \dots \overline{p}^{(k)}] = [m_1^{(k+1)} \dots \mathcal{F}_2\overline{p}^{(k)}]$ joined at $\mathcal{F}_2\overline{p}^{(k)} = \mathcal{F}_0 P^{(k)}$ to the polygonal path $\mathcal{F}_0[p^{(k)} \dots \mathcal{O} \dots q^{(k)}]$ which is joined at $\mathcal{F}_0 q^{(k)} = \mathcal{F}_1\overline{q}^{(k)}$ to the polygonal path $\mathcal{F}_1[\overline{q}^{(k)} \dots w_0 \dots \mathcal{F}_0 q \dots r] = [\mathcal{F}_1\overline{q}^{(k)} \dots m_2^{(k+1)}]$.
We single out two sides of this boundary, the left the polygonal path from $m_1^{(k+1)}$ to w_0, and the right the polygonal path from w_0 to $m_2^{(k+1)}$.

(4) The left side of the lower boundary of $\mathcal{H}^{(k+1)}$ is equivalent to the right side of the upper, and the right side of the lower is equivalent to the left side

of the upper. This is a consequence of the fact that the two parts that are $\mathcal{F}_)$-images of Voronoï boundaries are obviously equivalent. The parts that are $\mathcal{F}_1$ or $\mathcal{F}_2$ images of the previous boundaries which were equivalent stay equivalent.

This induction leads to the fact that the boundary of the tile **Q** is a closed polygonal path implying that the tile is a polygon. □

VII.2. **In dimension** $d = 2$ **the set Q is a tile.** This proof is different from [5], it gives a stronger version of Theorem VI.1

Theorem VII.2. *Let* $\triangle(v) = \triangle(v_0, v_1, v_2)$ *be a triangle with vertices* v_0, v_1, v_2 *which satisfies the non-sharp acuteness condition–i.e., is either an acute or a right triangle, and let* $g \in \triangle(v)^\circ$*. Then there exists an integer* $n > 0$ *such that* $\mathbf{Q} = \mathcal{F}_g^n \triangle_B$ *is a minimal absorbing invariant tile.*

Note: no assumption of ergodicity is made.

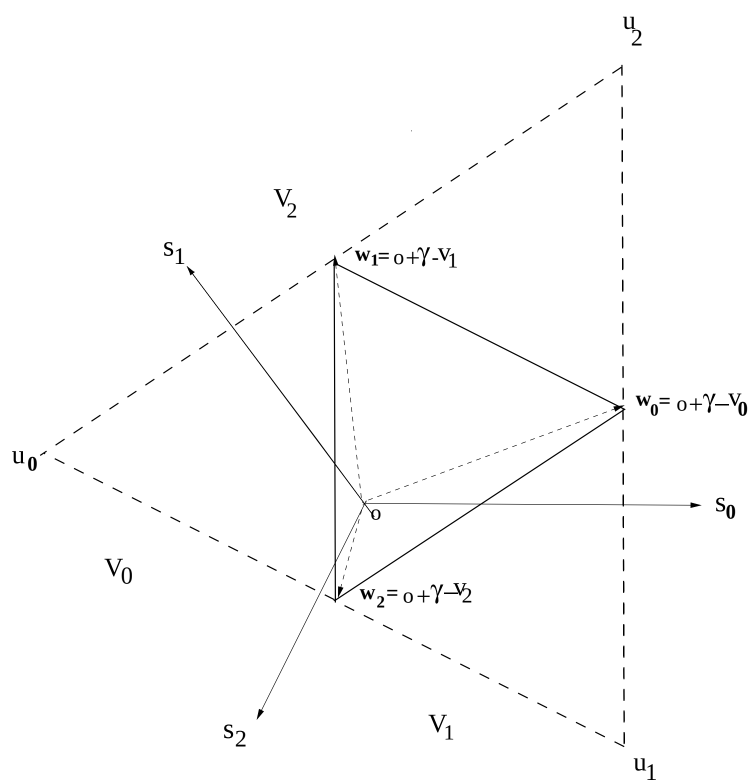

FIGURE 5. $\triangle_{R^*}$ in invariant $\triangle_{B^*}$

Proof. Recall that $\triangle_R = \triangle(w_0 w_1 w_2)$ where $w_k = \mathcal{O} + g - v_k$ is given by equation IV.9 and $\triangle_B = \triangle(u_0 u_1 u_2)$ where

$$u_k = w_k + \sum_{i=0}^{d} (v_k - v_i)$$

is given by equation IV.13. Let $w_g = \mathcal{F}_g(\mathcal{O})$ and

$$\triangle_R^* = (\triangle_R \backslash \{w_0, w_1, w_2\}) \cup \{w_g\}.$$

Note that w_g equals either w_0, w_1, or w_2 depending on the tie-breaking rules. Consider the following translates of $\triangle(v)$:

$$\triangle(u_i) \equiv \triangle(u_i w_j w_k) = \triangle(v) + g - v_j - v_k + \mathcal{O}$$

with no pair of indices equal. Note that

$$\triangle(u_i) = \triangle(u_j) + (v_i - v_j)$$

and

$$\triangle_B = \triangle(u_0) \cup \triangle(u_1) \cup \triangle(u_2) \cup \triangle_R.$$

Let

$$\triangle_B^* = \triangle(u_0)^\circ \cup \triangle(u_1)^\circ \cup \triangle(u_2)^\circ \cup \triangle_R^*.$$

Each of the three sets $\triangle(u_i)^\circ \cup \triangle_R^*$ forms a tile and no point of $\triangle_R^*$ is equivalent to a point of $\bigcup_i \triangle(u_i)^\circ$. In addition $\mathcal{F}\triangle_B \subset \triangle_B^*$.

Since $\triangle_B$ contains a tile, by Lemma V.2 we have $\mathcal{F}^n\triangle_B$ contains a tile for any integer n. All that remains to prove:

Claim 1. *There exists an integer $n > 0$ such that $\mathcal{F}^n\triangle_B$ does not contain two equivalent points.*

Once that is achieved we get as that $\mathcal{F}^n\triangle$ not only contains a fundamental set but also is itself one and that $\mathcal{F}^n\triangle_B$ isn't merely contained in its image $\mathcal{F}^{n+1}\triangle_B$ but is equal to it by virtue of the fact that both are fundamental sets.

Claim 1 is a corollary of the following:

Claim 2. *There exists an integer $n > 0$ such for each $x \in \triangle_B$ there exists $k \leq n$ for which $\mathcal{F}_g^k(x) \in \triangle_{R^*}$.*

Recall Digression 11. In this section $\mathbb{T} = \mathbb{A}/\mathbf{L}$ is the two dimensional torus: *i.e.*, $\mathbb{A}$ is the affine plane with points with barycentric coordinates (see (I.7))

$$x = \xi^0 v_0 + \xi^1 v_1 + \xi^2 v_2, \quad \sum_{i=0}^{2} \xi^i = 1\,, \tag{VII.2}$$

and $\mathbf{L}$ is the lattice generated by the vectors $v_1 - v_0,\ v_2 - v_0$. The map $\mathcal{F} = \mathcal{F}_g$ on $\mathbb{A}$ consists of piece-wise translation by vectors $g - v_i$, where g is the input given by I.3:

$$g = \gamma^0 v_0 + \gamma^1 v_1 + \gamma^2 v_2, \quad \sum_{i=0}^{2} \gamma^i = 1, \quad 0 < \gamma^i < 1.$$

These vectors are all projected to the translation by $[g - v_0] = [g - v_i]$ on $\mathbb{T}$.

It will be convenient to transfer the problem to a different setting–namely, to the standard torus $\hat{\mathbb{T}} = \hat{\mathbb{A}}/\mathbb{Z}^2$ where $\hat{\mathbb{A}}$ is the affine plane with points represented by

$$x = \xi^0(0,0) + \xi^1(1,0) + \xi^2(0,1), \quad \sum_{i=0}^{2} \xi^i = 1\,.$$

and where $\mathbb{Z}^2$ is the lattice generated by the vectors $e_1 = (1,0) - (0,0)$ and $e_2 = (0,1) - (0,0)$. To do this we shall introduce another barycentric coordinate system for $\mathbb{A}$: namely one based on the points u_0, w_1, w_2. The coordinates of a point in the new system differ from those in the old by the coordinates of the translation vector $u_0 - v_0$. The coordinates of the translation vectors $g - v_i$ remain unchanged, and the lattice $\mathbf{L}$ is still generated by the same pair of vectors $w_2 - u_0 = v_1 - v_0$ and $w_1 - u_0 = v_2 - v_0$.

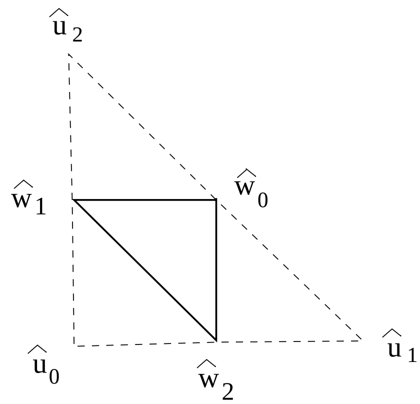

FIGURE 6. $\hat{\triangle}_{R^*}$ and $\hat{\triangle}_{B^*}$

We conjugate the map $\mathcal{F}$ on $\mathbb{A}$ to $\hat{\mathcal{F}}$ on $\hat{\mathbb{A}}$ by the affine transformation of $\mathbb{A}$ to $\hat{\mathbb{A}}$ that takes the points $u_0,\ w_2,\ w_1$ to $\hat{u_0} = (0,0),\ \hat{w_2} = (1,0),\ \hat{w_1} = (0,1)$ respectively and induces the linear map of $\mathbf{L}$ onto $\mathbb{Z}^2$ that takes vectors $v_1 - v_0,\ v_2 - v_0$ to $e_1,\ e_2$. Objects which are images by the conjugating transformation will be denoted with a hat. For example, $[\mathcal{F}]$ on $\mathbb{T}$ is conjugated to $[\hat{\mathcal{F}}]$ on $\hat{\mathbb{T}}$. This transformation also takes the points $u_1,\ u_2,\ w_0$ to $\hat{u_1} = (2,0),\ \hat{u_2} = (0,2).\ \hat{w_o} = (1,1)$. In addition, it maps the vector $g - v_0$ to $\hat{g} - (0,0)$. Since g lies in the interior of $\triangle(v_0v_1v_2)$, we get $\hat{g}$ lies in the interior of $\triangle(\hat{u_0}\hat{w_2}\hat{w_1}$ which means

$$\hat{g} = \hat{\gamma_0}(0,0) + \hat{\gamma_1}(1,0) + \hat{\gamma_2}(0,1).$$

where

$$0 < \hat{\gamma_1} + \hat{\gamma_2} < 1, \quad 0 < \hat{\gamma_i} < 1.$$

Furthermore, the map $\hat{\mathcal{F}}$ consists of piece-wise translation by vectors

$$\hat{g} - (0,0),\ \hat{g} - (1,0),\ \hat{g} - (0,1).$$

The orbit of any point in $\triangle_{B^*}$ under $\mathcal{F}$ has a unique $\mathbf{L}-$equivalent orbit in the tile $\mathbf{P}^*(u_0w_2w_0w_1) = \triangle(u_0)^\circ \cup \triangle_R^*$, and these orbits project to orbits under the translation by $[\mathcal{F}]$ in $\mathbb{T}$.

Likewise, the orbit $x_k = \hat{\mathcal{F}}^k$ of any point $x = (x_1, x_2) \in \hat{\triangle}_B^*$ has a unique $\mathbb{Z}^2-$equivalent orbit $\hat{x}_k$ in the tile $\hat{\mathbf{P}}^* = \hat{\mathbf{P}}^*(u_0w_2w_0w_1) = \hat{\triangle}(u_0)^\circ \cup \hat{\triangle}_R^*$, which is the unit square less certain boundary points. Furthermore,as before, the orbits in the tile $\hat{\mathbf{P}}^*$ project to orbits under the translation by $[\hat{\mathcal{F}}]$ in $\hat{\mathbb{T}}$. Now we must prove:

Claim 3. *There exists an integer $n > 0$ such for each $x \in \hat{\mathbf{P}}^*$ there exists $k \leq n$. such that $\hat{x_k} \in \hat{\triangle}^*_R$.*

Let $\hat{\phi}$ be the map of $[0,1)^2$ onto itself defined by

$$\hat{\phi}: \ (x_1, \ x_2) \to (x_1 + \hat{\gamma_1} \mod 1, \ x_2 + \hat{\gamma_2} \mod 1)$$

The orbits in $\hat{\mathbf{P}}^*$ have unique equivalent ones in $[0,1)^2$. and these orbits are images under the iterations of the map $\hat{\phi}$.

Now it all comes down to proving:

Claim 4. *There exists an integer $n > 0$ such for each $x \in [0,1)^2$. there exists $k \leq n$ such that*

$$y_1 + y_2 \geq 1$$

where $y = \hat{\phi}^k x$.

There are four cases to consider–one ergodic and three non-ergodic, one of which is periodic.

Case 1.: Ergodic: $\hat{\gamma}_1, \ \hat{\gamma}_2$ rationally independent irrationals.
Already proved

Case 2.: $\hat{\gamma}_1, \hat{\gamma}_2$ rationally dependent.
Rational dependence means $\hat{\gamma}_2/\hat{\gamma}_1 = p/q$ where p and q are positive integers with $\gcd(p,q) = 1$.
The orbit $\{\hat{\phi}^k x : \ k \in \mathbb{Z}\}$ of a point x is confined to a closed path (a topological circle) represented on the square $[0,1)^2$ by q equally spaced parallel straight line segments with slope p/q. The intersection of any such closed path with $\triangle((1,0),(1,1),(0,1))$ contains a segment L of length at least $1/(q+1)$. This is essentially the one dimensional case of ergodic rotation on the circle: a finite number n of pre-images of the segment L under $\hat{\phi}$ covers the whole path containing x. The number n is independent of x. So for every $x \in [0,1)^2$, we have $\hat{\phi} k x \in \triangle((1,0),(1,1),(0,1))$ for some $k \leq n$.

Case 3.: Partially periodic: one $\hat{\gamma}_i$ irrational, the other rational Say $\hat{\gamma}_1 = p/q < 1$ where p, q are positive integers and $\hat{\gamma}_2$ is irrational. Here the orbit $\{\hat{\phi}^k x : \ k \in \mathbb{Z}\}$ of a point x is confined to a set of q closed paths (topological circles) represented in$[0,1)^2$ by q equally spaced vertical lines going from top to bottom. On this structure $\hat{\phi}$ acts as the cartesian product of a cyclic permutation of the vertical lines with an ergodic rotation. At least one of the vertical lines intersects $\triangle((1,0),(1,1),(0,1))$ in an interval L of length $\geq 1/q$. So like the previous case, independent of x, a finite number n of pre-images of some segment L under $\hat{\phi}$ covers the whole path containing x; and we have the same conclusion as the previous case.
When $\hat{\gamma} = p/q$, the argument is the same: only verticals have become horizontals and the order of transformations in the cartesian product is reversed.

Case 4.: Periodic: both $\hat{\gamma}_i$ rational.
Let $\hat{\gamma}_1 = p_1/q_1, \ \hat{\gamma}_2 = p_2/q_2$ where $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$ and

$$\gcd(q_1, q_2) = D\,.$$

What remains to show is that there exists n such that for each x and some $k \le n$ we get:

$$y = x + k\hat{\gamma} \mod 1$$

where

$$y_1 + y_2 \ge 1.$$

(1) $D \neq q_1, q_2$.
We shall work with iterates of the D^{th} power of $\hat{\phi}$. Consider $D\hat{\gamma} = (Dp_1/q_1, Dp_2/q_2) = (p_1/q_1', p_2/q_2')$. We have $\gcd(q_1', q_2') = 1$ and $q_1', q_2' \ge 2$. We can assume that $x_1 + x_2 < 1$, in which case choose positive integers m_1, m_2 such that:

$$\begin{aligned} 1 - 1/q_1' &\le (x_1 + m_1(p_1/q_1')) \bmod 1 < 1\,, \\ 1 - 1/q_2' &\le (x_2 + m_2(p_2/q_2')) \bmod 1 < 1\,. \end{aligned}$$

By the Chinese Remainder theorem the pair of congruences

$$\begin{aligned} m &= m_1 \mod q_1' \\ m &= m_2 \mod q_2' \end{aligned}$$

can be solved for an integer m. Thus for $y = x + mD\gamma' \mod 1$ we have

$$\begin{aligned} y_1 &= x_1 + m(p_1/q_1') \mod 1 \\ y_2 &= x_2 + m(p_2/q_2') \mod 1 \quad \text{and} \end{aligned}$$
$$y_1 + y_2 \ge 1 - 1/q_1' + 1 - 1/q_2' \ge 1$$

There are $q_1'q_2'$ choices of m $mod(q_1'q_2')$. So the bound for k is $n = Dq_1'q_2'$.

(2) $D = q_1$ or q_2, say $D = q_2$.
Here $\hat{\gamma} = (p_1/q_1q_2, p_2/q_2) = (p_1/q_1q_2, q_1p_2/q_1q_2))$ where $\gcd(p_1, q_1q_2) = 1$ and

$$0 < p_1/q_1q_2 + q_1p_2/q_1q_2 < 1.$$

So we have

$$p_1 + q_1p_2 \neq 0 \mod q_1q_2.$$

We can choose m such that $\gcd(m, q_1q_2) = 1$ and $mp_1 = 1 \mod (q_1q_2)$: whereupon

$$1 + mq_1p_2 \neq 0 \mod (q_1q_2).$$

In other words

$$mq_1p_2 \mod (q_1q_2) \le q_1q_2 - 2.$$

We now work with the j^{th} iterate of

$$\hat{\phi}^m : (x_1, x_2) \to (x_1 + 1/q_1q_2,\ x_2 + q)$$

where $q = mq_1p_2/q_1q_2$ with $0 < q \le 1 - 2/q_1q_2$.
Consider $y = \hat{\phi}^{jm}$. Let

$$\begin{aligned} y_1 &= x_1 + j/q_1q_2 \mod 1 \\ y_2 &= x_2 + jq \mod 1 \end{aligned}$$

We can choose $j < q_1 q_2$ such that

$$1 - 1/q_1 q_2 \leq y_1 < 1.$$

Suppose

$$y_1 + y_2 < 1 \mod 1.$$

In which case consider $z = \hat{\phi}^{(j-1)m}$ where we get, because $y_2 < 1 - y_1 < 1/q_1 q_2$ and $q \geq 1/q_1 q_2$,

$$\begin{aligned} z_1 &= y_1 - 1/q_1 q_2 \\ z_2 &= y_2 - q + 1. \quad \text{Thus:} \\ z_1 &\geq 1 - 2/q_1 q_2, \\ z_2 &\geq y_2 + 2/q_1 q_2. \quad \text{Therefore:} \\ z_1 + z_2 &\geq y_2 + 1\,. \end{aligned}$$

The bound on k is $n = m q_1 q_2,$.

□

VII.3. **Error Diffusion in dimension 1, $\triangle(v) = [v_0, v_1]$.**

Example 3. *Let* $(1,0)$ *represent a red dot and* $(0,1)$ *a yellow dot. We want to print a sequence of red an yellow dots to emulate the constant sequence of orange inputs* $(0.3, 0.7)$*, i.e., the color corresponding to* 30% *of red and* 70% *of yellow.*

error	input	modified input	output	new error
e	g	$e+g$	v	$e+g-v$
(0.0,0.0)	(0.3, 0.7)	(0.3, 0.7)	(0,1) (y)	(0.3,-0.3)
(0.3, -0.3)	(0.3, 0.7)	(0.6, 0.4)	(1,0) (r)	(-0.4, 0.4)
(-0.4, 0.4)	(0.3, 0.7)	(-0.1, 1.1)	(0,1) (y)	(-0.1, 0.1)
(-0.1, 0.1)	(0.3, 0.7)	(0.2, 0.8)	(0,1) (y)	(0.2, -0.2)
(0.2, -0.2)	(0.3, 0.7)	(0.5, 0.5)	?(0,1) (y)	(0.5, -0.5)
(0.5, -0.5)	(0.3, 0.7)	(0.8, 0.2)	(1,0) (r)	(-0.2, 0.2)
(-0.2, 0.2)	(0.3, 0.7)	(0.1, 0.9)	(0,1) (y)	(0.1, -0.1)
(0.1, -0.1)	(0.3, 0.7)	(0.4, 0.6)	(0,1) (y)	(0.4, -0.4)
(0.4, -0.4)	(0.3, 0.7)	(0.7, 0.3)	(1,0) (r)	(-0.3, 0.3)
(-0.3, 0.3)	(0.3, 0.7)	(0.0, 1.0)	(0,1) (y)	(0.0, 0.0)

The cyclic nature of the output is due to the rational character of the input. We see the need of a tie-breaking rule in line five (symbolized by the question mark sign "?"). If the choice there was different (i.e., r instead of y) then the sequence would be (yryy r yyyry) instead of (yryy y ryyry), again with the same proportion of r's and y's. The reader may check that simple fact about the tie-breaking dependence of symbolic sequences by noticing that the modified input $(0.8, 0.2)$ *in the next line would be replaced by* $(0.8, 1.2)$ *since the new error* $(0.5, -0.5)$ *would be replaced by a new error* $(0.5, 0.5)$.

Let $\mathbf{P}$ be the closed interval $[v_0, v_1]$ and $g \in (v_0, v_1)$. We have the following closed Voronoï regions:

$$\overline{\mathbf{V}}_0 = (-\infty, \mathcal{O}] \text{ and } \overline{\mathbf{V}}_1 = [\mathcal{O}, \infty)$$

where $\mathcal{O} = (v_0 + v_1)/2$. In this case there are only two tie-breaking rules depending on to which interval of the Voronoï partition $\mathcal{O}$ belongs. Repeating the definition of $\mathcal{F}_g$ we have $\mathcal{F}_g(x) = x + g - v(x)$. In this dimension for either tie-breaking rule

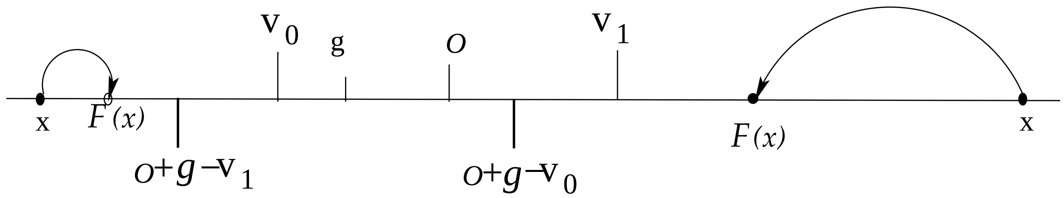

Figure 7. Interval case

$\mathcal{H} = \triangle_R = \triangle_B = [\mathcal{O}+g-v_1, \mathcal{O}+g-v_0]$. Say the tie-breaking rule is $\mathbf{V}_0 = (\infty, \mathcal{O}]$ and $\mathbf{V}_1 = (\mathcal{O}, \infty)$. Then the minimal invariant absorbing set $\mathbf{Q} = \mathcal{F}_g\mathbf{Q}$ is

$$\mathbf{Q} = (\mathcal{O}+g-v_1, \mathcal{O}+g-v_0],$$

which is a tile for the lattice $\mathbf{L}$ generated by the vector $v_1 - v_0$. We have two nontrivial subsets of $J_1 = \{0,1\}$ which are $J = \{0\}$ and $K = \{1\}$. Then $\mathbf{Q}_J = (\mathcal{O}+g-v_1, \mathcal{O}]$ of size $v_1 - g$, which is a tile for the lattice $\mathbf{L}_J = \{n(v_1-g),\, n \in \mathbb{Z}\}$. Similarly $\mathbf{Q}_K = (\mathcal{O}, \mathcal{O}+g-v_0]$, which is a tile for the lattice $\mathbf{L}_K = \{n(v_0-g),\, n \in \mathbb{Z}\}$. For the other tie-breaking rule we have

$$\mathbf{Q} = [\mathcal{O}+g-v_0, \mathcal{O}+g-v_1),$$

and similar pieces and sub-tiles with endpoints adjusted accordingly.

A.I. Properties of the intersection of the images of the Voronoï regions

A.I.1. The Voronoï cell of a lattice. Given a lattice $\mathbf{L}$ we call the Voronoï cell of a point g the set of all points which are closer or equidistant to g than to any of its lattice translates.

$$\mathcal{V} = \mathcal{V}(g) = \{x : (x-g)^2 \leq (x-(g+r))^2\, \forall r \in \mathbf{L}\} = \{x : (x-(g+\frac{1}{2}r))\cdot r \leq 0 \forall r \in \mathbf{L}\}\,.$$

The cell $\mathcal{V}$ is closed and convex and it is a closure of a tile for $\mathbf{L}$ with which it shares the interior, in other words the tile differs from $\mathcal{V}$ by some boundary points.

A.I.2. Arbitrary dimension d. We recall the definition (VI.1) of $\mathcal{H} = \cap\mathcal{F}(\mathbf{V}_i)$. As $\overline{\mathbf{V}_i} = \{x : (x-v_i)^2 - (x-v_j)^2 = 2(x-\mathcal{O})\cdot(v_j - v_i) \leq 0, \forall j\}$, we have $\mathcal{F}(\mathbf{V}_i) = \{x : ((\mathcal{F}_{|\mathbf{V}_i})^{-1}(x) - \mathcal{O})\cdot(v_j-v_i) = (x-g+v_i-\mathcal{O})\cdot(v_j-v_i) \leq 0, \forall j\}$ and finally

$$\overline{\mathcal{H}} = \{x : (x-w_i)\cdot(v_j-v_i) \leq 0, \forall j, i\}\,. \tag{A.I.1}$$

Lemma A.I.1.

$$\overline{\mathcal{H}} = \{x : (x-g)^2 \leq (x+(v_i-v_j)-g)^2, \forall i,j\}$$

Proof.

$$(x-g)^2-(x+(v_i-v_j)-g)^2 = (2(x-g)-v_i-v_j+2v_i)\cdot(v_j-v_i) = 2(x-g-\mathcal{O}+v_i)\cdot(v_j-v_i)$$

□

Corollary A.I.2. *The set $\overline{\mathcal{H}}$ contains a tile, which is a Voronoï cell of the point in the lattice $g+\mathbf{L}$.*

Proof. The set $\mathcal{H}$ is contained in a set defined by the same inequalities as $\mathcal{V}$ but with r restricted to the edges $r = v_i - v_j$. □

Each $\mathbf{V}_k$ is a cone with vertex $\mathcal{O}$ and edges $s_i, i \neq k$ and has d co-dimension 1 faces with external normal vectors $v_i - v_k,\ i \neq k$. Each $\mathcal{F}(\mathbf{V}_k)$ is an affine cone with the vertex at $w_k = \mathcal{O} + g - v_k$ and as $w_k \in \mathcal{F}(\mathbf{V}_i)$ for all i, by Corollary VI.3, and therefore each w_k is a vertex of the set $\mathcal{H}$.

Lemma A.I.3 ($\mathcal{H}$ is symmetric)**.** *The convex set $\mathcal{H}$ is centrally symmetric with respect to the point g. In particular $g \in \mathcal{H}$.*

Proof. Convexity is trivial. If $x \in \overline{\mathcal{H}}$ then $(x - w_i) \cdot (v_i - v_j) \leq 0$, for all i, j. For the symmetric point $2g - x$ we have

$$\begin{aligned}(2g - x - w_i) \cdot (v_j - v_i) &= (g + \mathcal{O} - v_j - x - 2\mathcal{O} + v_j + v_i) \cdot (v_j - v_i) \\ &= (w_j - x) \cdot (v_j - v_i) = (x - w_j) \cdot (v_i - v_j) \leq 0\end{aligned}$$

because we can switch the indices i and j in the condition for $x \in \overline{\mathcal{H}}$. □

By symmetry each point $m_k = -\mathcal{O} + g + v_k$ is also a vertex of $\mathcal{H}$. Let us formulate it prove directl.

Lemma A.I.4. *In an edgewise acute simplex every point w_l and every point m_i is a vertex of the set $\mathcal{H}$.*

Proof. We have $(w_l - w_j) \cdot (v_k - v_j) = (v_j - v_l) \cdot (v_k - v_j) \leq 0$ and $(m_i - w_j) \cdot (v_j - v_k) = (v_i + v_j - 2\mathcal{O} + v_k - v_k) \cdot (v_j - v_k) = (v_i - v_k) \cdot (v_k - v_j) \leq 0$. □

In dimension $d = 1$ we have $\mathcal{H} = [m_0, m_1] = [w_1, w_0]$, which is an invariant tile. In dimension $d = 2$ of $\mathcal{H}$ is a centrally symmetric hexagon with alternating vertices w and m (a degeneration to a rectangle is possible if the triangle is not strictly acute). It is a tile but usually not invariant. In higher dimensions there are many more vertices of $\mathcal{H}$ and we shall describe the shape of it in some details.

For fixed two indices

$$k, j \in J = \{0, \ldots, d\} \text{ define } J_{kj} = J \setminus \{k, j\}$$

Proposition A.I.5 (Faces of $\mathcal{H}$)**.** *In edgewise acute simplices each co-dimension 1 face of $\mathcal{H}$ is uniquely determined by a pair of points w_k and m_j with $k \neq j$ and is included in a co-dimension 1 parallelepiped P_{kj} given by the intersection of two cones:*

$$\begin{aligned}P_{kj} &= \{w_k + \sum_{i \in J_{kj}} \lambda^i s_i, \lambda_i \geq 0\} \cap \{m_j - \sum_{i \in J_{kj}} \mu^i s_i, \mu_i \geq 0\} \\ &= \{w_k + \sum_{i \in J_{kj}} \lambda^i s_i, 0 \leq \lambda^i \leq \Lambda^i_{kj} = (v_k - v_i) \cdot (v_j - v_i)\}\,.\end{aligned}$$

We shall call the face of $\mathcal{H}$ contained in P_{kj} by $\mathcal{H}_{kj}$.

Proof. The faces of $\mathcal{H}$ are contained in the translates of the faces of $\mathbf{V}_k$. For each k there are d such faces, with an external normal vectors $v_j - v_k,\ j \neq k$. More precisely, following the characterization in Lemma IV.4 such a face of $\mathbf{V}_k$ is a cone $\{\mathcal{O} + \sum_{i \in J_{kj}} \lambda^i s_i, \lambda^i \geq 0\}$. Hence the co-dimension 1 faces of $\mathcal{H}$ are contained in the cones $\{w_k + \sum_{i \in J_{kj}} \lambda^i s_i, \lambda^i \geq 0\}$. There are no other co-dimension 1 faces created by the intersections of $\mathcal{F}(\mathbf{V}_k)$. By symmetry the faces are also included in the cones $\{m_j - \sum_{i \neq j,k} \mu^i s_i, \mu^i \geq 0\}$. As $w_k - m_j = 2\mathcal{O} - v_k - v_j$ is orthogonal to $v_j - v_k$ it follows that both w_k and m_j lie on the same face which is included in the intersection of the two cones. If $x \in P_{kj}$ then the two representations of x give rise to the equation $\sum_{i\in J_{kj}}(\lambda^i(x)+\mu^i(x))s_i = m_j - w_k = (v_k + v_j - 2\mathcal{O})$. After multiplying both sides by $v_k - v_l$ we get $\lambda^l + \mu^l = (v_j - v_l + v_l + v_k - 2\mathcal{O})\cdot(v_k - v_l) = (v_j - v_l)\cdot(v_k - v_l) = \Lambda^l_{kj}$, and the estimate on λ (and μ) follows from $\lambda, \mu \geq 0$. $\square$

Corollary A.I.6. *The 2^{d-1} vertices of P_{kj} are given by all subsets of indices $I \subset J_{kj}$, namely:*

$$P^I_{kj} = w_k + \sum_{i\in I}\Lambda^i_{kj}s_i\,.$$

If we denote by $I^{\mathbf{c}} = J_{kj}\setminus I$ then also:

$$P^I_{kj} = m_k - \sum_{i\in I^{\mathbf{c}}}\Lambda^i_{kj}s_i\,.$$

In particular $P^{\emptyset}_{kj} = w_k$ and $P^{J_{kj}}_{kj} = P^{\emptyset^{\mathbf{c}}}_{kj} = m_j$. The vertices adjacent to w_j are $P^{\{i\}}_{kj} = w_k + \Lambda^i_{kj}s_i$ and the vertices adjacent to m_j are $P^{\{i\}^{\mathbf{c}}}_{kj} = m_j - \Lambda^i_{kj}s_i$. Not for all subsets I the points P^I_{kj} are the vertices of $\mathcal{H}$.

Lemma A.I.7. *The vertices $P^{\{i\}}_{kj}$ adjacent to w_k belong to $\mathcal{H}$ if and only if $\Lambda^i_{kj} = \min_l \Lambda^i_{kl}$. Similarly the vertices $P^{\{i\}^{\mathbf{c}}}_{kj}$ adjacent to m_j belong to $\mathcal{H}$ if and only if $\Lambda^i_{kj} = \min_l \Lambda^i_{lj}$. In particular $P^{\{i\}}_{kj} \notin \mathcal{H}$, $i \in J_{kj} = J\setminus\{k,j\}$ if and only if for some $l \in J\setminus\{k,j,i\}$, $P^{\{l\}^{\mathbf{c}}}_{kj} \notin \mathcal{H}$*

Proof. The set $\mathcal{H}$ near w_k has d edges in the directions of s_i, $i \neq k$. If $P = w_k + \lambda s_i \in \mathcal{H}$, each edge belonging to $d-1$ faces P_{kj}. Then $P \in \cap_j P_{kj}$ and therefore $0 \leq \lambda \leq \Lambda^i_{kj}$ for all $j \in J_{ki}$. On the other hand the maximal such λ produces a point lying below (or on) all the faces and hence in $\mathcal{H}$. The statement for m_j follows from symmetry. Hence if $P^{\{i\}}_{kj} \notin \mathcal{H}$ then for some l:

$$\begin{aligned}
0 &< \Lambda^i_{kj} - \Lambda^l_{kj} = (v_k - v_i)\cdot(v_j - v_i) - (v_k - v_i)\cdot(v_l - v_i)\\
&= (v_k - v_i)\cdot(v_j - v_l) = (v_j - v_l)\cdot(v_k - v_l) - (v_j - v_l)\cdot(v_i - v_l)\\
&= \Lambda^l_{kj} - \Lambda^l_{ij}\,.
\end{aligned}$$

which means that Λ^l_{kj} was not minimal hence $P^{\{i\}}_{kj} = m_j - \Lambda^l_{kj}s_l \notin \mathcal{H}$. $\square$

A.I.3. **Dimension** $d = 3$**.** In dimension 3 previous Lemma can be checked by calculations.

Corollary A.I.8. *When $d = 3$ the point $P^{\{i\}}_{kj}$ is a vertex of $\mathcal{H}$ if and only if $(v_k - v_i)(v_j - v_l) \leq 0$.*

Proof. We will check the conditions (A.I.1) for arbitrary indices $a \neq b$:

$$\begin{aligned}
(P^i_{kj} - w_a)(v_b - v_a) &= (w_k - w_a + \Lambda^i_{kj}s_i)(v_b - v_a)\\
&= (v_a - v_k)(v_b - v_a) + \begin{cases} \Lambda^i_{kj} & \text{when} \quad a = i\\ -\Lambda^i_{kj} & \qquad\quad b = i\\ 0 & \qquad\quad a, b \neq i\end{cases}
\end{aligned}$$

If $a \neq i$ then both terms are non positive. In case $a = i$, and then $b \neq i$, we have $(v_i - v_k)(v_b - v_i) + (v_k - v_i)(v_j - v_i) = (v_i - v_k)(v_b - v_j)$ which maybe positive only when $b = l$ which means that $(v_i - v_k)(v_l - v_j) > 0$ is the only condition when $P^{\{i\}}_{kj} \notin \mathcal{H}$. $\square$

Digression 13. *Previous Lemma has the following geometric meaning in dimension $d = 3$. The twelve faces $\mathcal{H}_{kj}$ lie on the parallelograms P_{kj} with edges in directions s_i and s_l, with $\{i, j, k, l\} = \{0, 1, 2, 3\} = J$. If $\mathcal{H}_{kj} \neq P_{kj}$ then one of the points $P_{kj}^{\{i\}}$ or $P_{kj}^{\{l\}}$ was cut off, suppose it was the former. But then we have also $P_{kj}^{\{l\}^\mathsf{c}} \notin \mathcal{H}$ (as there is no other choice of index left). Incidentally in dimension $d = 3$ we have $P_{kj}^{\{i\}} = P_{kj}^{\{l\}^\mathsf{c}}$. That means that this corner of P_{kj} was cut off and an additional edge of $\mathcal{H}$ was created. But we know the endpoints of this edge, those are $P_{kl}^{\{i\}}$ and $P_{ij}^{\{l\}^\mathsf{c}}$.*

We see that the condition of the face $\mathcal{H}_{kj}$ to have an extra edge is expressed as either by

- $(v_k - v_i) \cdot (v_l - v_j) < 0$,
  in which case the vertex $P_{kj}^{\{i\}} = P_{kj}^{\{l\}^\mathsf{c}}$ of P_{kj} is cut off by the additional edge $[P_{kl}^{\{i\}}, P_{ij}^{\{l\}^\mathsf{c}}]$, or by
- $(v_k - v_l) \cdot (v_i - v_j) < 0$,
  in which case the vertex $P_{kj}^{\{l\}} = P_{kj}^{\{i\}\mathbf{c}}$ of P_{kj} is cut off by the additional edge $[P_{ki}^{\{l\}}, P_{kj}^{\{i\}^\mathsf{c}}]$.

In dimension $d = 3$ for any k there are three products which indicate the length of edges in the direction of s_k, namely $(v_i - v_k) \cdot (v_j - v_k)$, $(v_j - v_k) \cdot (v_l - v_k)$ and $(v_l - v_k) \cdot (v_i - v_k)$, depending on the order of those numbers they determine the number of the edges of the faces of $\mathcal{H}$. Suppose that $(v_i - v_k) \cdot (v_j - v_k) < (v_j - v_k) \cdot (v_l - v_k) < (v_l - v_k) \cdot (v_i - v_k)$. Then the edge in the direction of s_k between the faces $\mathcal{H}_{ij}$ and $\mathcal{H}_{il}$ has length $(v_i - v_k) \cdot (v_j - v_k)$ and hence the face $\mathcal{H}_{ij}$ contains the point $P_{ij}^{\{k\}}$ and no additional edges at this side, while the face $\mathcal{H}_{il}$ contains an additional edge and this side.

Digression 14. *In dimension $d = 3$ the face $\mathcal{H}_{ij}$ of $\mathcal{H}$ is a tetragon (a quadrilateral, in fact a parallelogram) iff for $k, l \neq i, j$ if for any $a \neq b$*

$$(v_i - v_k)\cdot(v_j - v_k) = \min_{a,b\neq k}(v_a - v_k)\cdot(v_b - v_k) \text{ and } (v_i - v_l)\cdot(v_j - v_l) = \min_{a,b\neq l}(v_a - v_l)\cdot(v_b - v_l)$$

The face is a hexagon when we exchange the min *by the* max *and a pentagon if the products are both the middle numbers in of the corresponding three products order.*

Proof. Below we consider only the edges in the direction of s_k. Consider the edge s_k on the face $\mathcal{H}_{ij}$ from the point w_i and the parallel one from the edge m_j, then the second one coincides (at least partially) with the edge of the face $\mathcal{H}_{lj}$ which has a parallel edge at point w_l a partial common to the face $\mathcal{H}_{li}$. Suppose that $(v_i - v_k) \cdot (v_j - v_k) < (v_j - v_k) \cdot (v_l - v_k) < (v_l - v_k) \cdot (v_i - v_k)$. Then the first inequality does not produce the additional edge on the face $\mathcal{H}_{ij}$ but does on the face $\mathcal{H}_{lj}$. Similarly the second inequality does not produce the additional edge on $\mathcal{H}_{lj}$ but does on $\mathcal{H}_{il}$. That makes the face $\mathcal{H}_{lj}$ a pentagon, while using the symmetric argument to the faces $\mathcal{H}_{ij}$, $\mathcal{H}_{il}$ and $\mathcal{H}_{jl}$ we deduce that the face $\mathcal{H}_{ij}$ has no additional edges and thus is a tetragon. By central symmetry (or by exchanging the role of i and j the face $\mathcal{H}_{ji}$ is a tetragon, the face $\mathcal{H}_{jl}$ is a pentagon and the faces $\mathcal{H}_{il}$ and $\mathcal{H}_{li}$ acquire additional edges from both directions, that is they are both hexagonal.

It is interesting to see that one condition is enough, in fact if

$$(v_i - v_k) \cdot (v_j - v_k) < (v_j - v_k) \cdot (v_l - v_k) < (v_l - v_k) \cdot (v_i - v_k)$$

then

$$(v_i - v_l) \cdot (v_j - v_l) < (v_k - v_l) \cdot (v_i - v_l) < (v_j - v_l) \cdot (v_k - v_l)$$

First inequality of the top chain is equivalent to $(v_i - v_l) \cdot (v_j - v_k) < 0$ the second one is equivalent to $(v_l - v_k)(v_j - v_i) < 0$. First inequality of the bottom chain is equivalent to $(v_i - v_l) \cdot (v_j - v_k) < 0$ and the second one to $(v_k - v_l) \cdot (v_i - v_j) < 0$. That means that if $(v_i - v_k) \cdot (v_j - v_k)$ is minimal so is $(v_i - v_l) \cdot (v_j - v_l)$ and $\mathcal{H}_i j$ is a tetragon. Similarly if we reversed the inequalities we would have $\mathcal{H}_i j$ a hexagon. That implies that if $(v_i - v_k) \cdot (v_j - v_k)$ were in the middle so must have been also $(v_i - v_l) \cdot (v_j - v_l)$, and the face would be a pentagon. □

The product condition involves a pair of edges of the original simplex that have no point in common. In dimension $d = 3$ there are three such pairs, and their products are not independent.

Lemma A.I.9 (Opposite edges condition)**.**

$$(v_i - v_j)(v_k - v_l) + (v_i - v_k)(v_l - v_j) + (v_i - v_l)(v_j - v_k) = 0$$

Proof. Adding $0 = v_j - v_j$ to each first factor we obtain $(v_i - v_j)(v_k - v_l + v_l - v_j + v_j - v_k) + (v_j - v_k)(v_l - v_j) + (v_j - v_l)(v_j - v_k) = 0$ □

Digression 15 (Opposite edges convention OE)**.** *After permuting the indices we shall always assume the following opposite edge conditions.*

$$\begin{array}{rccc} OE1: & (v_0 - v_1)(v_2 - v_3) & \geq & 0 \\ OE2: & (v_0 - v_2)(v_3 - v_1) & \geq & 0 \\ OE3: & (v_0 - v_3)(v_1 - v_2) & \leq & 0 \end{array}$$

By Lemma A.I.9 either all three products are zero or the last one is negative.

Digression 16. *The following gives a geometric interpretation of the orthogonality of opposite edges in dimension $d = 3$:*
Let h_i denote the altitude of a simplex from the vertex v_i, that is a segment from v_i to its orthogonal projection onto the affine subspace containing the face $\mathbf{F}_i$. If $(v_0 - v_1)(v_2 - v_3) = 0$ then $h_0 \cap h_1 \neq \emptyset$ and $h_2 \cap h_2 \neq \emptyset$. If two of the conditions (and thus all three) are zero then all the altitudes meet at one point.

Proof. Let M be a two dimensional plane orthogonal to the edge (v_2, v_3) and passing through the point v_0. Then M contains all the segments orthogonal to this edge and passing through v_0 in particular it contains the edge (v_0, v_1) and the altitude h_0 which is orthogonal to the face $\mathbf{F}_0$ containing (v_2, v_3). It contains also h_1 which is orthogonal to $\mathbf{F}_1$ containing (v_2, v_3). The intersection of this plane with the simplex form a triangle, whose two altitudes are h_0 and h_1 meet at one point. The statement about h_2 and h_3 is proven in a similar way. □

From previous considerations there follows:

Proposition A.I.10 (Structure of the faces of $\mathcal{H}$ in case $d = 3$)**.**

(1) *When all three opposite edge OE conditions are not zero the twelve faces of $\mathcal{H}$ consist of:*
four tetragons (quadrilaterals, more precisely parallelograms)ruled by $OE2$ and $OE3$: $\mathcal{H}_{01}, \mathcal{H}_{23}, \mathcal{H}_{10}, \mathcal{H}_{32}$,
four hexagons ruled by $OE1$ and $OE3$: $\mathcal{H}_{02}, \mathcal{H}_{13}, \mathcal{H}_{20}, \mathcal{H}_{31}$, and
four pentagons ruled by $OE2$ and $OE1$:$\mathcal{H}_{03}, \mathcal{H}_{12}, \mathcal{H}_{30}, \mathcal{H}_{21}$.
In this case the eight points (from the tetragons) with lower indices 01 *and* 23 *(in both orders and both possible upper indices) and the four points (from pentagons) $P_{03}^{\{1\}}, P_{21}^{\{3\}}, P_{30}^{\{2\}}, P_{12}^{\{0\}}$ are the remaining vertices of $\mathcal{H}$.*
There are six edges (additional to the 24 edges in the direction of s vectors attached to the points w and m): two common faces of two pairs of pentagons $(P_{01}^{\{2\}}, P_{23}^{\{2\}})$ and $(P_{10}^{\{3\}}, P_{32}^{\{1\}})$ (ruled by $OE2$), and four edges common to pairs of hexagons: $(P_{30}^{\{2\}}, P_{21}^{\{3\}}), (P_{12}^{\{0\}}, P_{03}^{\{1\}})$ (ruled by $OE1$), and $(P_{32}^{\{0\}}, P_{01}^{\{3\}}), (P_{10}^{\{2\}}, P_{23}^{\{1\}})$ (ruled by $OE3$). Of the additional edges of the hexagons the first and the second connect the tetragons, while the third and the fourth connect the pentagons.
We have 12 faces, 12 edges from the points w in the direction of the vectors s, 12 edges from the points m in the direction of the vectors $-s$ and 6 additional edges, making 30 edges total, and we have 4 w vertices, 4 m vertices and 12 P vertices making 20 vertices total. Each vertex has three edges, the P points have each one s edge, one $-s$ edge and one edge to another P point.

(2) *A typical bifurcation can occur uniquely by changing the sign of either OE1 and then the hexagons become pentagons and pentagons become hexagons or by changing the sign of $OE2$ and then the pentagons become tetragons and tetragons become pentagons.*

(3) *When the condition $OE1$ is zero then the hexagons become pentagons resulting in four tetragons and eight pentagons.*
The third and fourth hexagonal edges collapse to one point each, producing two vertices with four edges collecting four pentagons around such a vertex. That gives 12 faces, 28 edges and 18 vertices. It is not a tile.

(4) *When the condition $OE2$ is zero then the pentagons become tetragons resulting in eight tetragons and four hexagons.*
The pentagonal edges collapse to one point each, producing two vertices with four edges collecting four tetragons.
That gives 12 faces, 28 edges and 18 vertices. Then the set $\mathcal{H}$ is a tile. It is a hexa-rhombic dodecahedron.

(5) *When all three condition are zero then each additional edge collapses to a point, leaving six vertices with four edges.*
Each face becomes a tetragon. That gives 12 faces, 24 edges (no additional ones) and 14 vertices (6 points P). The set $\mathcal{H}$ is a tile. This is a rhombic dodecahedron.

A.I.4. **The tile $\mathcal{T}$, the set $\mathcal{H}$ cut by two additional half-spaces.** We are in dimension $d = 3$.

A.I.5. **Definition of $\mathcal{T}$.** For $k \neq i$ define

$$w_i^k = w_i + \lambda^k s_k \text{ and } m_i^k = m_i - \lambda^k s_k$$

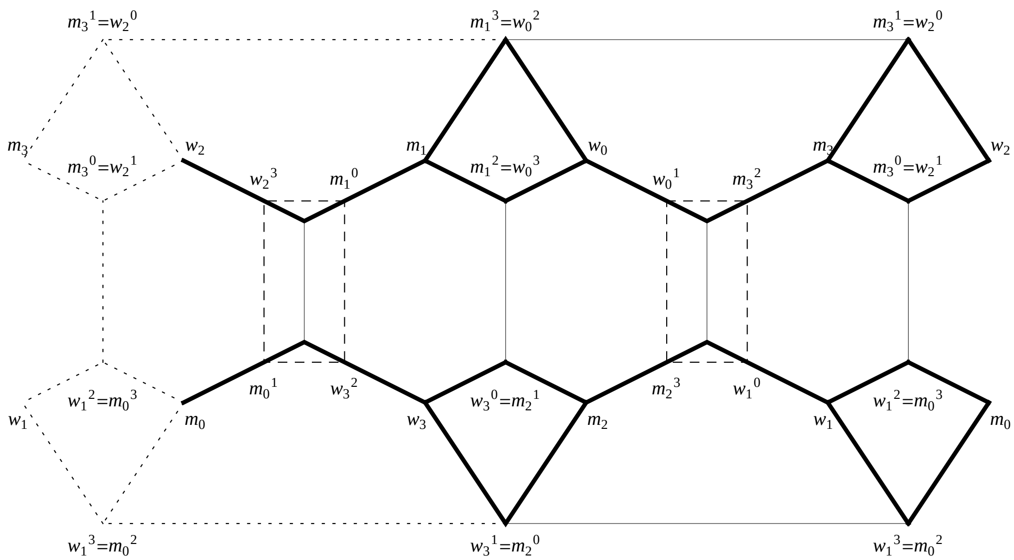

FIGURE 8. The combinatorial structure of the set $\mathcal{T}$. The dotted lines are glued (identified) with corresponding continuous ones. The thick lines correspond to the edges in the direction of one of the s vectors. The thin lines are the remaining edges of $\mathcal{H}$. The dashed boxes show the edges of $\mathcal{T}$ which are not the edges of $\mathcal{H}$, they represent the two additional tetragones (hence the edges inside the boxes are cut off). The proportions are distorted: all horizontal edges of $\mathcal{T}$ have equal length and all vertical edges of $\mathcal{T}$ have equal length.

Lemma A.I.11. *If* $\lambda_k = \min_{ab}(v_a - v_k) \cdot (v_b - v_k) = (v_i - v_k) \cdot (v_j - v_k)$ *then*

$$\begin{array}{ll} m_j^l = w_i^k & \qquad m_l^j = w_i^k \\ m_j^k = w_i^l & \qquad m_l^k = w_i^l \\ w_j^l = m_i^k & \qquad w_l^j = m_i^k \\ w_j^k = m_i^l & \qquad w_l^k = m_i^l \,. \end{array}$$

Proof. Under the assumption by Digression 14 the face $\mathcal{H}_{ij}$ is a tetragon hence $w_i^k = w_i + \lambda^k s_k = P_{ij}^{\{k\}} = P_{ij}^{\{l\}^{\mathsf{c}}} = m_j - \lambda_l s_l = m_j^l$. The other equalities follow similarly. $\square$

Lemma A.I.12. *Under Assumtions OE in dimesnion $d = 3$ we have:*

$$\begin{aligned}
\lambda^0 &= \lambda^0_{23} = (v_2 - v_0)\cdot(v_3 - v_0)\\
\lambda^1 &= \lambda^1_{23} = (v_2 - v_1)\cdot(v_3 - v_1)\\
\lambda^2 &= \lambda^2_{01} = (v_0 - v_2)\cdot(v_1 - v_2)\\
\lambda^3 &= \lambda^3_{01} = (v_0 - v_3)\cdot(v_1 - v_3)\,.
\end{aligned}$$

Proof. Direct computations from OE. Remark that those are the coefficients for the edges of two pairs of tetragonal faces $\mathcal{H}_{23}$ with $\mathcal{H}_{32}$ and $\mathcal{H}_{01}$ with $\mathcal{H}_{10}$. □

Definition A.I.13 (Separated indices)**.** *Under Assuption OE partition the set of indices $\{0,1,2,3\}$ into $\{0,1\}\cup\{2,3\}$. An upper index i and a lower index j are said* separated *if they belong to two different parts of the partition.*

Lemma A.I.14. *The points w_i^j (and by symmetry m_i^j belong to tetragonal faces of $\mathcal{H}$ if they have separated indices and are additional vertices of $\mathcal{T}$ if their indices are not separated.*

Proof. By inspection. □

Lemma A.I.15. *Under Assumptions OE1 and OE2, in the following four groups of all four points within the group are equivalent with respect to* $\mathbf{L}$.

$$\begin{aligned}
&w^0_3 \sim w^0_1 \sim w^0_2 = m^1_3 \sim m^1_0 \sim m^1_2 (= w^0_3)\\
&w^1_2 \sim w^1_0 \sim w^1_3 = m^0_2 \sim m^0_1 \sim m^0_3 (= w^1_2)\\
&w^2_1 \sim w^2_3 \sim w^2_0 = m^3_1 \sim m^3_2 \sim m^3_0 (= w^2_1)\\
&w^3_0 \sim w^2_3 \sim w^3_1 = m^2_0 \sim m^2_3 \sim m^2_1 (= w^3_0)\,.
\end{aligned}$$

Remark that the middle points w and m do not have their equal counterparts.

Proof. As $w_i - w_j = v_j - v_i$ all the w points are $\mathbf{L}$-equivalent, similarly $m_i - m_j = v_i - v_j$. Therefore all the three points $w_i^k = w_i + \lambda^k s_k$ with the same upper index are equivalent. Similarly all three m_j^l points with the same upper index are equivalent. Lemma A.I.11 merges the groups. □

Lemma A.I.16 (All vertical and all horizontal segments are equal)**.** *Under Assumptions OE1 and OE2 there are following two groups of six equal vectors:*

Vertical in Figure A.I.4

$$\begin{aligned}
&m^0_3 - w^2_1 \boxed{=} w^1_2 - m^3_0 = w^3_2 - m^1_0 = w^1_0 - m^3_2 = w^3_0 - m^1_2\\
&m^0_3 - w^2_1 = m^2_3 - w^0_1 = m^0_1 - w^2_3 = m^2_1 - w^0_3 \boxed{=} w^3_0 - m^1_2
\end{aligned}$$

Horizontal in Figure A.I.4

$$\begin{aligned}
&m^1_3 - w^2_0 \boxed{=} w^0_2 - m^3_1 = w^3_2 - m^0_1 = w^0_1 - m^3_2 = w^3_1 - m^0_2\\
&m^1_3 - w^2_0 = m^2_3 - w^1_0 = m^1_0 - w^2_3 = m^2_0 - w^1_3 \boxed{=} w^3_1 - m^0_2
\end{aligned}$$

Remark that compared to Figure A.I.4 some vectors seem to have reversed order. This is due to the fact that after gluing the solid together these vectors will go "behind" $\mathcal{T}$. The boxed equality $\boxed{=}$ refers to the fact that the endpoints are the same, for example in $m^0_3 - w^2_1 \boxed{=} w^1_2 - m^3_0$ we have $m^0_3 = w^1_2$ and $w^2_1 = m^3_0$, that is the equality happens in the affine space as well as in vector space.

Proof. Each of these vectors is written as a difference of an m and a w point. Each can be expressed in a symmetric way. The "vertical" non boxed equalities follows from:

$$2\mathcal{O} - v_2 - v_0 + \lambda^1 s_1 + \lambda^3 s_3 = w_2^1 - m_0^3 = w_2^3 - m_0^1 = w_0^3 - m_2^1 = w_0^1 - m_2^3$$

and similar expressions with lower indices $1, 3$ and upper $0, 2$. The link between the two is given by the boxed equalities which result from Lemma A.I.11. Similarly one calculates the "horizontal" chain of equalities. $\square$

Corollary A.I.17. *Assume OE. The points with no separated indices: w_2^3, m_0^1, m_1^0, w_3^2 form a parallelogram and hence lie on the same two-dimensional plane. They form a face of $\mathcal{T}$. The same statement holds for the points $w_0^1, m_2^3, m_3^2, w_1^0$. For the first four points the plane is given by*

$$\{x : -(x-g)^2 = (x-(g+r))^2\} \quad r = v_2 + v_3 - v_1 - v_0\,,$$

for the second four points take $r = v_0 + v_1 - v_2 - v_3$.

Proof. The parallelogram statement was proven in the previous Lemma. Using the properties of $\mathcal{O}, s_1$ and λ^1 we get for w_0^1:

$$\begin{aligned}
& (w_0^1 - g)^2 - (w_0^1 - g - r)^2 = (2\mathcal{O} - 2v_0 + 2\lambda^1 s_1 - r) \cdot r \\
= \quad & 2\lambda^1 s_1 \cdot r + (2\mathcal{O} - 2v_0) \cdot (v_0 - v_2) + (2\mathcal{O} - 2v_0) \cdot (v_1 - v_3) + r^2 \\
= \quad & -2\lambda^1 + (v_2 - v_0) \cdot (v_0 - v_2) + 2(v_1 - v_0) \cdot (v_1 - v_3) + (v_3 - v_1) \cdot (v_1 - v_3) + r^2 \\
= \quad & -2(v_2 - v_1) \cdot (v_3 - v_1) + 2(v_1 - v_0) \cdot (v_1 - v_3) + 2(v_0 - v_2) \cdot (v_1 - v_3) \\
= \quad & 2(v_2 - v_1 + v_1 - v_0 + v_0 - v_2) \cdot (v_1 - v_3) = 0\,.
\end{aligned}$$

The computation for all other points is similar and will be skipped. $\square$

Given $g \in \mathbb{A}$, for any $r \in \mathbf{L}$ let $\mathbf{M}(r)$ be the (closed) half space of points closer to g than to $g + r$.

$$\mathbf{M} = \mathbf{M}(r) = \{x : (x-g)^2 \le (x-(g+r))^2\} = \{x : (2x - 2g - r) \cdot r \le 0\}\,.$$

Digression 17. *The set*

$$\bigcap_{r \in \mathbf{L}} \mathbf{M}(r)$$

consists of points which are closer to g then to any of it lattice translates. It is called a Voronoï cell of the point g with respect to the lattice $\mathbf{L}$. It is a closed, convex, bounded set. It is a closure of a (Voronoï) tile with which it shares the interior.

Lemma A.I.18. *In any dimension d:*

$$\mathcal{H} = \bigcap_{ij} \mathbf{M}(v_i - v_j)$$

Proof. This is geometrically well understood. Each $\mathbf{V}_i$ is the cone, an intersection of half spaces of points closer to v_i than to any other vertex. After translation by a vector $g-v_i$ an intersecting all such translates we recover $\mathcal{H}$. Computation follows:

$$\begin{aligned}\mathcal{H} &= \bigcap_i \mathcal{F}(\mathbf{V}_i) = \bigcap_i (\mathbf{V}_i + g - v_i) \\ &= \bigcap_i \left(\bigcap_j \big(\{y : (y-v_i)^2 - (y-v_j)^2 \leq 0\}\big) + g - v_i \right) \\ &= \bigcap_i \left(\bigcap_j (\{y : (2y - v_i - v_j)\cdot(v_j - v_i) \leq 0\}) + g - v_i \right) \\ &= \bigcap_i \left(\bigcap_j \{x = y + g - v_i : (2(x-g+v_i) - v_i - v_j)\cdot(v_j - v_i) \leq 0\} \right) \\ &= \bigcap_{ij} \{x : (2x - 2g - (v_j - v_i))\cdot(v_j - v_i) \leq 0 = \bigcap_{ij} \mathbf{M}(v_j - v_i)\,.\end{aligned}$$

□

Lemma A.I.19. *In dimension $d=3$ under assumptions OE the two additional (as compared to $\mathcal{H}$) faces of $\mathcal{T}$ lie on the boundaries of $\mathbf{M}(r)$ and $\mathbf{M}(-r)$ with $r = v_3 + v_2 - v_1 - v_0$.*

Proof. This was proven in Corollary A.I.17. □

Corollary A.I.20. *The points with no separated indices belong to $\mathbf{M}(r) \cap \mathbf{M}(-r)$*

Proof. By geometry. The two groups of points are symmetric to each other with respect to g, and so are $\mathbf{M}(r)$ and $\mathbf{M}(-r)$. But both half planes containg therefore $\mathbf{M}(-r)$ contains the symmetric image of $\partial\mathbf{M}(r)$ and hence the first group of four points with separated indices and $\mathbf{M}(r)$ contains $\partial\mathbf{M}(-r)$ and hence the second group. Interested reader is welcome to perform the computation on inequalities. □

Digression 18. *Remark that the faces of $\mathcal{H}$ which are adjacent to the additional face w_0^1,m_2^3,w_1^0,m_3^2 with external normal vector $r = v_3+v_2-v_1-v_0$ are $\mathcal{H}_{12}, \mathcal{H}_{13}, \mathcal{H}_{03}, \mathcal{H}_{02}$ with external normal vectors $v_2 - v_1, v_3 - v_1, v_3 - v_1, v_2 - v_1$.*

From geometrical point of view we have just proven that the two additional half spaces $\mathbf{M}(r)$ and $\mathbf{M}(-r)$ cut off the edge of $\mathcal{H}$ joining two pentagonal faces along hexagonal ones. But to be sure that we do not rely to muich on the intuition we provide an algebraic proof.

Lemma A.I.21 (w and m inside additional cuts)**.** *If $r = v_i + v_j - v_k - v_l$, then, in an edge-wise acute simplex, for every $t \in i, j, k, l$ we have $w_t, m_t \in \mathbf{M}$.*

Proof. First note that by symmetry for every t: $w_t - g = \mathcal{O} - v_t = g - m_t$ and the lengths of these two vectors are equal (and equal for all t). Moreover $(m_t - (g-r))^2 = (r - (g - m_t))^2 = (-(w_t - g) + r)^2 = (w_t - (g+r))^2$. Thus the statement about m_t and r follows from the statement about w_t and $-r$. We represent the inequality defining $\mathbf{M}$ as $0 \leq (x-g+r)^2 - (x-g)^2 = (2(x-g)+r)\cdot r$. As $w_t = \mathcal{O} + g - v_t$, we have:

$$\begin{aligned}
& (2(w_t-g)+r)\cdot r \\
= \; & (2\mathcal{O}-v_i-v_k+v_i+v_k-2v_t+r)\cdot(v_i-v_k) \\
& +(2\mathcal{O}-v_j-v_l+v_j+v_l-2v_t+r)\cdot(v_j-v_l) \\
= \; & (v_i+v_k-2v_t+r)\cdot(v_i-v_k)+(v_j+v_l-2v_t+r)\cdot(v_j-v_l) \\
= \; & 2(v_i-v_t)\cdot(v_i-v_k)+(v_j-v_l)\cdot(v_i-v_k)+2(v_j-v_t)\cdot(v_j-v_l) \\
& +(v_i-v_k)\cdot(v_j-v_l) \\
= \; & 2\left((v_i-v_t)\cdot(v_i-v_k)+(v_j-v_l)\cdot(v_i-v_k)+(v_j-v_t)\cdot(v_j-v_l)\right) \\
= \; & 2\left\{\begin{array}{cl} t=i & (v_j-v_l)\cdot(v_j-v_k) \\ t=j & (v_i-v_l)\cdot(v_i-v_k) \\ t=k & (v_i-v_l)\cdot(v_i-v_k)+(v_j-v_k)\cdot(v_i-v_k)+(v_j-v_k)\cdot(v_j-v_l) \\ t=l & (v_i-v_l)\cdot(v_i-v_k)+(v_j-v_l)\cdot(v_i-v_l)+(v_j-v_l)\cdot(v_j-v_k) \end{array}\right\} \geq 0\,,
\end{aligned}$$

by edgewise acuteness. □

Lemma A.I.22. *Under Assumption OE set $r=v_0+v_1-v_2-v_3$. The points w_i^j (and by symmetry m_i^j) with separated indices belong to $\mathbf{M}(r)$ and to $\mathbf{M}(-r)$.*

Proof. Let us calculate for w_0^3 and r, other calculations are similar (or simpler).

$$\begin{aligned}
& (w_0^3-g)^2-(w_0^3-(g+r))^2=(2\mathcal{O}-2v_0+2\lambda^3 s_3-r)\cdot r \\
= \; & 2\lambda^3 s_3\cdot r+(2\mathcal{O}-2v_0)\cdot r-r^2=2\lambda^3+(2\mathcal{O}-v_0-v_2)\cdot(v_0-v_2)-(v_0-v_2)^2 \\
& +(2\mathcal{O}-v_1-v_3)\cdot(v_1-v_3)+(v_1+v_3-2v_0)\cdot(v_1-v_3)-((v_0-v_2)+(v_1-v_3))^2 \\
= \; & 2(v_1-v_3)\cdot(v_0-v_3)-2(v_0-v_2)^2+(v_1-v_3)\cdot(v_1+v_3-2v_0-(v_1-v_3)-2(v_0-v_2)) \\
= \; & 2(v_1-v_3)(v_2-v_0)-2(v_0-v_2)^2=2((v_1-v_0+v_0-v_3)(v_2-v_0)-(v_0-v_2)^2) \\
= \; & 2((v_0-v_2)\cdot(-v_1+v_0-v_0+v_2)+(v_0-v_2)\cdot(-v_0+v_3)) \\
= \; & 2((v_0-v_2)\cdot(v_2-v_1)+(v_0-v_2)\cdot(v_3-v_0))\leq 0 \text{ by edgewise acuteness.}
\end{aligned}$$

□

Digression 19. *In the last lines of the previous proof we used the following trick: In an edgewise simplex*

$$\text{If } p=v_a-v_b, q=v_c-v_d \text{ then } p\cdot q-p^2\leq 0$$

which follows from writing $q=v_c-v_b+v_b-v_d$:

$$\begin{aligned}
& (v_c-v_b)\cdot(v_a-v_b)+(v_b-v_d)(v_a-v_b)-(v_a-v_b)\cdot(v_a-v_b) \\
= \; & (v_c-v_a)\cdot(v_a-v_b)+(v_b-v_d)(v_a-v_b)\leq 0
\end{aligned}$$

Proposition A.I.23. *In dimension $d=3$:*

$$\mathcal{T}=\bigcap_{ijkl}\mathbf{M}(v_i+v_j-v_k-v_l)$$

In fact under the assumptions OE, with $r=v_0+v_1-v_2-v_3$:

$$\mathcal{T}=\mathcal{H}\cap\mathbf{M}(r)\cap\mathbf{M}(-r)$$

Proof. We have proven that all the vertices belong to the convex intersection $\cap\mathbf{M}(r)$ for appropriate subset of vectors r. Also we have proven that the faces of $\mathcal{T}$ belong to the boundaries $\partial\mathbf{M}(r)$ for appropriate r. In particular $r=v_i-v_j$ for the faces of $\mathcal{T}$ which are parts of the faces or $\mathcal{H}$ and the two additional faces are on the boundaries of $\mathbf{M}(r)$ for $\pm r = v_0 + v_1 - v_2 - v_3$. □

Theorem A.I.24. *For a face wise acute simplex in dimension $d = 3$ the set $\mathcal{T}$ is a closure with shared interior of a tile for the simplex lattice $\mathbf{L}$.*

Proof. It is enough to prove that the translations by lattice vectors of $\mathcal{T}$ fill the space around each vertex, with fitting edges and faces. We work under Assumption OE. There are six groups of four equivalent vertices:

(1) Two groups of: four w_i and four m_i points.
(2) Four groups of: two vertices with separated indices $w_i^k = m_j^l$ and $w_j^k = m_i^l$ together with two veritces with no separated indices w_l^k and m_k^l, were the choice is determined by the the direction of an edge (upper index k) of a point w, and the points m are determined by completion.

In general to a fixed (original) vertex of each group we translate three other equivalent vertices. Each of the translated points brings an adjacent face symmetric (with opposite external normals) to on of the three faces of the original vertex. Those paired faces share two equal (vector) edges. The remaining faces of translated points fit pairwise with each other around a “sticking out” edge which is common to all translated vertices but absent at the original one. In case of the points w and m all the edges are s_i edges of the same length λ^i with the “sticking out” edge being the s vector with the same index as the original point. In case of the doubly indexed points they share two s directions and a “horizontal” and a “vertical” one.

- Consider w_0 which lattice equivalent to any w_i and translate each such vertex w_i to w_0 by $v_i - v_0$. Then each face $\mathcal{T}_{0i}$ adjacent to w_0 will be matched with the translated face $\mathcal{T}_{i0}$. Remark the change of order in the indices, which shows that the faces are matched with opposite external vectors. Each edge starting at w_0 in the direction s_j, $j \neq 0$ will be common with two such translated edges from the faces $\mathcal{T}_{i0}$, $i \neq j$, and they all share the length λ^j. There will be an additional edge, common to all three translated faces in the direction of s_0, which “sticks out” from $\mathcal{T}$. We recover the partition of $\mathbb{A}$ near $\mathcal{O}$ translated by $g - v_0$. Similar argument works for all other points w and by symmetry m.
- Consider now the point $w_0^3 = m_1^0$ with separated indices. It is adjacent to the faces $\mathcal{T}_{01}, \mathcal{T}_{31}, \mathcal{T}_{02}$, with three edges $\lambda^2 s_2$, $-\lambda_3 s_3$ and the additional “vertical” edge $(v_0 - v_3) + \lambda^0 s_0 - \lambda^3 s_3$. The points equivalent to $w_0^3 = w_0 + \lambda^3 s_3$ are all points w_i^3 and m_j^0. There are three such points in addition to w_0^3 itself. Each of such point is adjacent to a face symmetric to one of the faces adjacent to w_0^3 with the pair of adjacent edges equal (due to common upper index in case of s edges and an equal “vertical” edge). There is an additional edges “sticking out” common to all the other points which is “horizontal”. Remark how the symmetric additional two faces stick together with a non s edge of $\mathcal{H}$ filling a wedge between them.

□

References

[1] R. Adler, B. Kitchens, M. Martens, A. Nogueira, C. Tresser, C.W. Wu, *Error bounds for error diffusion and other mathematical problems arising in digital halftoning*, S&T/SPIE's Int. Symp. Elec. Imaging, San Jose, CA **3963**, (2000) 437–43.

[2] R. Adler, B. Kitchens, M. Martens, C.Pugh M. Shub, C.Tresser, *Convex Dynamics and Its Applications Ergodic Theory and Dynamical Systems*, Erg.Th.&Dyn.Syst. **25**, (2005) 321–352.

[3] R.L. Adler, B. P. Kitchens, M. Martens, C. P. Tresser, C. W. Wu, *The mathematics of halftoning*, IBM J. Res. & Dev., **47-1**, (2003) 5–15.

[4] R. Adler, B. Kitchens, C. Tresser, *Dynamics of nonergodic piecewise affine maps of the torus*, Erg.Th.&Dyn.Syst. **21**, (2001) 959–999.

[5] R. L. Adler, T. Nowicki, G. Świrszcz, C. Tresser, *Convex dynamics with constant input*, Erg.Th.&Dyn.Syst. **30**, (2010) 957–972.

[6] D. Coppersmith, T. Nowicki, G. Paleologo,C. Tresser, C.W. Wu, *The Optimality of the Online Greedy Algorithm in Carpool and Chairman Assignment Problems*, ACM Trans. Algorithms, **7-3**, (2011) 1–22.

[7] I. Daubechies, R. DeVore, *Approximating a bandlimited function using very coarsely quantized data: A family of stable sigma-delta modulators of arbitrary order*, Annals of Math. **158-2**, (2003) 679–710.

[8] A. Goetz, *Stability of piecewise rotations and affine maps,* Nonlinearity **14**, (2001) 205–219.

[9] A. Goetz, P.Ashwin, *Invariant curves and explosion of periodic islands in systems of piecewise rotations*, Siam Journal of Applied Dynamical Systems, **4**, (2005) 437–458.

[10] C. S.Gunturk, N. T.Thao, *Ergodic dynamics in sigma-delta quantization: tiling invariant sets and spectral analysis of error*, Advances in Applied Mathematics **34**, (2005) 523–560.

[11] A. Katok, B. Hasselblatt Introduction to the Modern Theory of Dynamical Systems Cambridge University Press, (1995) ISBN 0-521-57557-6

[12] J. H. Lowenstein, K. L. Kouptsov, F. Vivaldi, *Recursive tiling and geometry of piecewise rotations by $\frac{\pi}{7}$,* Nonlinearity **17**, (2004), 371–395.

[13] T. Nowicki, C. Tresser, *Convex dynamics: properties of invariant sets*, Nonlinearity **17**, (2004) 1645–1676.

[14] T. Nowicki, C. Tresser, *Convex dynamics: unavoidable difficulties in bounding some greedy algorithms*, CHAOS **14**, (2004) 55–71.

[15] C. Sparrow , S. van Strien , C. Harris, *Fictitious Play in 3*x *3 Games: The Transition between Periodic and Chaotic Behaviour*, Games and Economic Behaviour, **63**, (2008), 259–291.

[16] R. Tijdeman, *The chairman assignment problem*, Discrete Mathematics, **32**, (1980) 323–330.

[17] C. Tresser, *Dynamique de la diffusion de l'erreur sur plusieurs polytopes*, C. R. Mathématiques Acad. Sc. Paris **338**, (2004) 793–798.

[18] C. Tresser, *Bounding the errors for convex dynamics on one or more polytopes*, CHAOS **17**, (2007) 33–49.

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