POLY-TIME COMPUTABILITY OF THE FEIGENBAUM JULIA SET.

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Abstract. We present the first example of a poly-time computable Julia set with a recurrent critical point: we prove that the Julia set of the Feigenbaum map is computable in polynomial time.

1. Introduction.

Informally speaking, a compact set K in the plane is computable if there exists an algorithm to draw it on a computer screen with any desired precision. Any computer-generated picture is a finite collection of pixels. If we fix a specific pixel size (commonly taken to be 2^{-n} for some n) then to accurately draw the set within one pixel size, we should fill in the pixels which are close to the set (for instance, within distance 2^{-n} from it), and leave blank the pixels which are far from it (for instance, at least $2^{-(n-1)}$-far). Thus, for the set K to be computable, there has to exist an algorithm which for every square of size 2^{-n} with dyadic rational vertices correctly decides whether it should be filled in or not according to the above criteria. We say that a computable set has a polynomial time complexity (is *poly-time*) if there is an algorithm which does this in a time bounded by a polynomial function of the precision parameter n, independent of the choice of a pixel.

When we talk of computability of the Julia sets of a rational map, the algorithm drawing it is supposed to have access to the values of the coefficients of the map (again with an arbitrarily high precision). Computability of Julia sets has been explored in depth by M. Braverman and the second author (see monograph [6] and references therein). They have shown, in particular, that there exist quadratic polynomials $f_c(z) = z^2 + c$ with explicitly computable parameters c whose Julia sets J_c are not computable. Such parameters are rare, however; for almost every $c \in \mathbb{C}$ the set J_c is computable. In [1] it was shown that there exist computable quadratic Julia sets with an arbitrarily high time complexity. On the other hand, hyperbolic Julia sets are poly-time [4, 20]. The requirement of hyperbolicity may be weakened significantly. The first author has shown [11] that maps with non-recurrent critical orbits have poly-time Julia sets. However, even in the quadratic family f_c it is not at present known if J_c is poly-time for a typical value of c (see the discussion in [11]).

Until now, no examples of poly-time computable Julia sets with a recurrent critical point have been known. In this note we present the first such example. It is given by perhaps the most famous quadratic map of all – the Feigenbaum polynomial f_{c_*}. The Feigenbaum map is infinitely renormalizable under period-doubling, and its renormalizations converge to a fixed point of the renormalization operator. Historically, this is the first instance of renormalization in Complex Dynamics (see [16] for an overview of the history of the subject). As follows from [2], the Julia set of f_{c_*} is computable. However, infinite renormalizability

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implies, in particular, that we cannot expect to find any hyperbolicity in the dynamics of f_{c_*} to make the computation fast. We find a different hyperbolic dynamics, however, to speed up the computation – the dynamics of the renormalization operator itself. In a nutshell, this is the essense of the poly-time algorithm described in this paper.

The details, however, are quite technical and analytically involved. To simplify the exposition, we prove poly-time computability of the Julia set not of the map f_{c_*} itself, but of the Feigenbaum renormalization fixed point F. The map F is not a quadratic polynomial, but it is quadratic-like, it is conjugate to f_{c_*}, and its Julia set is homeomorphic to that of f_{c_*}. There are two advantages to working with F, as opposed to f_{c_*}. Firstly, the renormalization-induced self-similarity of the Julia set of F is exactly, rather than approximately linear. This allows us to streamline the arguments somewhat, making them easier to follow. More importantly, as we show, the map F itself is poly-time computable (an efficient algorithm for computing F is due to O. Lanford [14]). Hence, our main result – poly-time computability of the Julia set of F – can be stated without the use of an oracle for the map F.

We now proceed to give the detailed definitions and precise statements of our main results.

1.1. **Preliminaries on computability.** In this section we give a very brief review of computability and complexity of sets. For details we refer the reader to the monograph [7]. The notion of computability relies on the concept of a Turing Machine (TM) [22], which is a commonly accepted way of formalizing the definition of an algorithm. A precise description of a Turing Machine is quite technical and we do not give it here, instead referring the reader to any text on Computability Theory (e.g. [19] and [21]). The computational power of a Turing Machine is provably equivalent to that of a computer program running on a RAM computer with an unlimited memory.

Definition 1. *A function $f : \mathbb{N} \to \mathbb{N}$ is called computable, if there exists a TM which takes x as an input and outputs $f(x)$.*

Note that Definition 1 can be naturally extended to functions on arbitrary countable sets, using a convenient identification with $\mathbb{N}$. The following definition of a computable real number is due to Turing [22]:

Definition 2. *A real number α is called computable if there is a computable function $\phi : \mathbb{N} \to \mathbb{Q}$, such that for all n*

$$|\alpha - \phi(n)| < 2^{-n}.$$

The set of computable reals is denoted by $\mathbb{R}_\mathcal{C}$. Trivially, $\mathbb{Q} \subset \mathbb{R}_\mathcal{C}$. Irrational numbers such as e and π which can be computed with an arbitrary precision also belong to $\mathbb{R}_\mathcal{C}$. However, since there exist only countably many algorithms, the set $\mathbb{R}_\mathcal{C}$ is countable, and hence a typical real number is not computable.

The set of computable complex numbers is defined by $\mathbb{C}_\mathcal{C} = \mathbb{R}_\mathcal{C} + i\mathbb{R}_\mathcal{C}$. Note that $\mathbb{R}_\mathcal{C}$ (as well as $\mathbb{C}_\mathcal{C}$) considered with the usual arithmetic operation forms a field.

To define computability of functions of real or complex variable we need to introduce the concept of an oracle:

Definition 3. *A function $\phi : \mathbb{N} \to \mathbb{Q} + i\mathbb{Q}$ is an oracle for $c \in \mathbb{C}$ if for every $n \in \mathbb{N}$ we have*

$$|c - \phi(n)| < 2^{-n}.$$

A TM equipped with an oracle (or simply an *oracle TM*) may query the oracle by reading the value of $\phi(n)$ for an arbitrary n.

Definition 4. *Let $S \subset \mathbb{C}$. A function $f : S \to \mathbb{C}$ is called computable if there exists an oracle TM M^ϕ with a single natural input n such that if ϕ is an oracle for $z \in S$ then M^ϕ outputs $w \in \mathbb{Q} + i\mathbb{Q}$ such that*

$$|w - f(z)| < 2^{-n}.$$

We say that a function f is *poly-time computable* if in the above definition the algorithm M^ϕ can be made to run in time bounded by a polynomial in n, independently of the choice of a point $z \in S$ or an oracle representing this point. Note that when calculating the running time of M^ϕ, querying ϕ with precision 2^{-m} counts as m time units. In other words, it takes m ticks of the clock to read the argument of f with precision m (dyadic) digits.

Let $d(\cdot, \cdot)$ stand for Euclidean distance between points or sets in $\mathbb{R}^2$. Recall the definition of the *Hausdorff distance* between two sets:

$$d_H(S,T) = \inf\{r > 0 : S \subset U_r(T),\ T \subset U_r(S)\},$$

where $U_r(T)$ stands for the r-neighborhood of T:

$$U_r(T) = \{z \in \mathbb{R}^2 : d(z,T) \leqslant r\}.$$

We call a set T a 2^{-n} *approximation* of a bounded set S, if $d_H(S,T) \leqslant 2^{-n}$. When we try to draw a 2^{-n} approximation T of a set S using a computer program, it is convenient to let T be a finite collection of disks of radius 2^{-n-2} centered at points of the form $(i/2^{n+2}, j/2^{n+2})$ for $i, j \in \mathbb{Z}$. We will call such a set *dyadic*. A dyadic set T can be described using a function

$$h_S(n,z) = \begin{cases} 1, & \text{if } d(z,S) \leqslant 2^{-n-2}, \\ 0, & \text{if } d(z,S) \geqslant 2 \cdot 2^{-n-2}, \\ 0 \text{ or } 1 & \text{otherwise,} \end{cases} \tag{1}$$

where $n \in \mathbb{N}$ and $z = (i/2^{n+2}, j/2^{n+2}),\ i, j \in \mathbb{Z}$.

Using this function, we define computability and computational complexity of a set in $\mathbb{R}^2$ in the following way.

Definition 5. *A bounded set $S \subset \mathbb{R}^2$ is called computable in time $t(n)$ if there is a TM, which computes values of a function $h(n, \bullet)$ of the form (1) in time $t(n)$. We say that S is poly-time computable, if there exists a polynomial $p(n)$, such that S is computable in time $p(n)$.*

1.2. **Renormalization and the Feigenbaum map F.** In this section we recall some important notions from Renormalization Theory for quadratic like maps and introduce the Feigenbaum map F. We refer the reader to [10] and [17] for details on renormalization and to [8] for properties of the Feigenbaum map.

Definition 6. *A quadratic-like map is a ramified covering $f : U \to V$ of degree 2, where $U \Subset V$ are topological disks. For a quadratic-like map f we define its filled Julia set $K(f)$ and Julia set $J(f)$ as follows*

$$K(f) = \{z \in U : f^n(z) \in U \text{ for every } n \in \mathbb{N}\},\ \ J(f) = \partial K(f). \tag{2}$$

Without lost of generality, we will assume that the critical point of a quadratic-like map f is at the origin.

Let $P_c(z) = z^2 + c$. Then for R large enough the restriction of P_c onto the disk $D_R(0) = \{z : |z| < R\}$ is a quadratic-like map.

Definition 7. *Two quadratic-like maps f_1 and f_2 are said to be hybrid equivalent if there is a quasiconfromal map ψ between neighborhoods of $K(f_1)$ and $K(f_2)$ such that $\bar{\partial}\psi = 0$ almost everywhere on $K(f_1)$.*

Douady and Hubbard proved the following:

Theorem 8. *(Straightening Theorem) Every quadratic-like map f is hybrid equivalent to a quadratic map P_c. If the Julia set $J(f)$ is connected, then the map P_c is unique.*

Let f be a quadratic-like map with connected Julia set. The parameter c such that P_c is hybrid equivalent to f is called the *inner class* of f and is denoted by $I(f)$.

Recall the notion of renormalization.

Definition 9. *Let $f : U \to V$ be a quadratic-like map. Assume that there exists a number $n > 1$ and a topological disk $U' \ni 0$ such that $f^n_{|U'}$ is a quadratic-like map. Then f is called renormalizable with period n. The map $\mathcal{R}f := f^n_{|U'}$ is called a renormalization of f.*

Observe that the domain U' of $\mathcal{R}f$ from the definition above is not uniquely defined. Therefore, it is more natural to consider renormalization of germs rather than maps.

Definition 10. *We will say that two quadratic like maps f and g with connected Julia sets define the same germ $[f]$ of quadratic-like map if $J(f) = J(g)$ and $f \equiv g$ on a neighborhood of the Julia set.*

We define the renormalization operator $\mathcal{R}_2$ of period 2 as follows.

Definition 11. *Let $[f]$ be a germ of a quadratic-like map renormalizable with period 2. Let $U' \supset 0$ be such that $g := f^2_{|U'}$ is a quadratic-like map. We set*

$$\mathcal{R}_2[f] = [\alpha^{-1} \circ g \circ \alpha],$$

where $\alpha(z) = g(0)z$.

We have introduced the normalization $\alpha(z)$ in order to have that the critical value of the renormalized germ is at 1.

We recall, that the Feigenbaum parameter value $c_{Feig} \in \mathbb{R}$ is defined as the limit of the parameters $c_n \in \mathbb{R}$ for which the critical point 0 of the quadratic polynomial is periodic with period 2^n. The Feigenbaum polynomial is the map

$$P_{Feig}(z) = z^2 + c_{Feig}.$$

The next theorem follows from the celebrated work of Sullivan (see [9]):

Theorem 12. *The sequence of germs $\mathcal{R}_2([P_{Feig}])$ converges to a point $[F]$. The germ $[F]$ is a unique fixed point of the renormalization operator $\mathcal{R}_2$ and is hybrid equivalent to $[P_{Feig}]$.*

1.3. **The main result.** Note that the germ $[F]$ from Theorem 12 has a well-defined quadratic-like Julia set J_F. We state:

Main Theorem. *The Julia set J_F is poly-time computable.*

2. The structure of the Feigenbaum map F.

In this section we show how to compute the coefficients of the map F and discuss the combinatorial structure of F.

2.1. **The combinatorial structure of F.** Recall that the Feigenbaum map F is a solution of Cvitanović-Feigenbaum equation:

$$\begin{cases} F(z) &= -\frac{1}{\lambda}F^2(\lambda z), \\ F(0) &= 1, \\ F(z) &= H(z^2), \text{ with } H^{-1}(z) \text{ univalent in } \mathbb{C}_\lambda, \end{cases} \tag{3}$$

where $\mathbb{C}_\lambda := \mathbb{C} \setminus ((-\infty, -\frac{1}{\lambda}] \cup [\frac{1}{\lambda^2}, \infty))$ and $\frac{1}{\lambda} = 2.5029\ldots$ is one of the Feigenbaum constants. From (3) we immediately obtain:

$$F^{2^m}(z) = (-\lambda)^m F(\tfrac{z}{\lambda^m}) \tag{4}$$

whenever both sides of the equation are defined. Another corollary of (3) is the following (cf. Epstein [13]):

Proposition 13. *Let x_0 be the first positive preimage of 0 by F. Then*

$$F(\lambda x_0) = x_0,\ F(1) = -\lambda,\ F(\tfrac{x_0}{\lambda}) = -\tfrac{1}{\lambda}$$

and $\frac{x_0}{\lambda}$ is the first positive critical point of F.

A map $g : U_g \to \mathbb{C}$ is called an analytic extension of a map $f : U_f \to \mathbb{C}$ if f and g are equal on some open set. An extension $\hat{f} : S \supset U \to \mathbb{C}$ of f is called a maximal analytic extension if every analytic extension of f is a restriction of $\hat{f}$. The following crucial observation is also due to H. Epstein (cf. [13, 8]):

Theorem 14. *The map F has a maximal analytic extension $\hat{F} : \hat{W} \to \mathbb{C}$, where $\hat{W} \supset \mathbb{R}$ is an open simply connected set.*

For simplicity of notation, in what follows we will routinely identify F with its maximal analytic extension $\hat{F}$.

Set $\mathbb{H}_+ = \{z : \mathrm{Im} z > 0\}$ and $\mathbb{H}_- = \{z : \mathrm{Im} z < 0\}$. For a proof of the following, see [13, 8]:

Theorem 15. *All critical points of F are simple. The critical values of F are contained in real axis. Moreover, for any $z \in \hat{W}$ such that $F(z) \notin \mathbb{R}$ there exists a bounded open set $U(z) \ni z$ such that F is one-to-one on $U(z)$ and $F(U(z)) = \mathbb{H}_\pm$.*

We illustrate the statement of the theorem in Figure 1 (a very similar figure appears in X. Buff's paper [8]). The lighter and darker "tiles" are the bounded connected components of the preimage of $\mathbb{H}_+$ and $\mathbb{H}_-$ respectively. Black tree is the boundary of the domain $\hat{W}$. In the gray region, colors cannot be effectively rendered at the given resolution.

Following [8], we introduce the following combinatorial partition of $\hat{W}$.

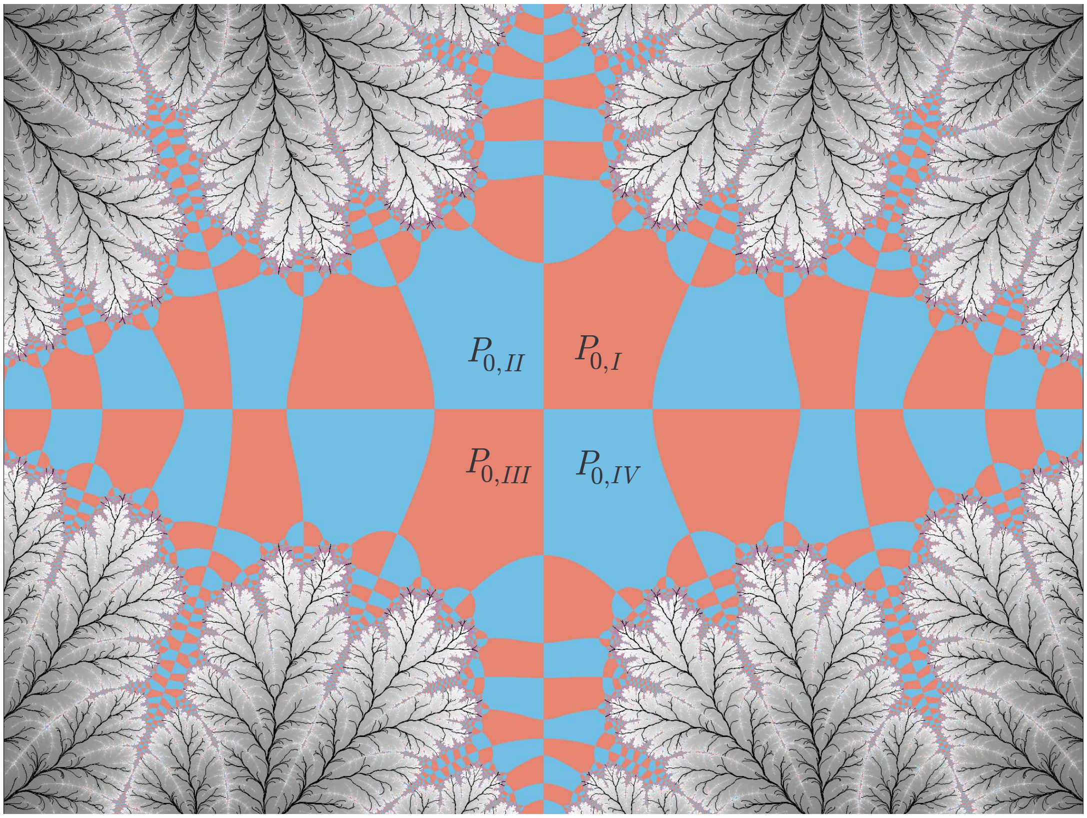

FIGURE 1. Illustration to Theorem 2.1. We thank Scott Sutherland for computing this image for us.

Definition 16. *Denote by $\mathcal{P}$ the set of all connected components of $F^{-1}(\mathbb{C}\setminus\mathbb{R})$. Set*

$$\mathcal{P}^{(n)} = \{\lambda^n P : P \in \mathcal{P}.\}$$

Thus, for any $P \in \mathcal{P}$ the map F sends P one-to-one either onto $\mathbb{H}_+$ or onto $\mathbb{H}_-$. Notice that $\mathcal{P}$ is invariant under multiplication by -1 and under complex conjugation. Using Cvitanović-Feigenbaum equation we obtain the following:

Lemma 17. *For any n, the partition $\mathcal{P}^{(n)}$ coincides with the set of connected componets of the preimage of $\mathbb{C}\setminus\mathbb{R}$ under F^{2^n}. Moreover, one has:*

1) *for any $m \in \mathbb{N}\cup\{0\}$, $m > n$ and any $P \in \mathcal{P}^n$ the iterate $F^{2^n-2^m}$ maps P bijectively onto some $Q \in \mathcal{P}^{(m)}$;*
2) *for any $n \in \mathbb{N}, s \in \mathbb{N}, s \leqslant 2^{n-1}$ and any $P \in \mathcal{P}^{(n)}$ there exists $Q_0 \in \mathcal{P}^{(n)}, Q_1 \in \mathcal{P}^{(n-1)}$ such that $Q_0 \subset F^s(P) \subset Q_1$.*

Let us describe the structure of F on the real line near the origin. Since F maps $[1, \frac{x_0}{\lambda}]$ homeomorphically onto $[-\frac{1}{\lambda}, -\lambda]$ there exists a unique $a \in (1, \frac{x_0}{\lambda})$ such that $F(a) = -\frac{x_0}{\lambda}$.

Lemma 18. *The first three positive critical points of F counting from the origin are $\frac{x_0}{\lambda}, \frac{a}{\lambda}, \frac{x_0}{\lambda^2}$. One has:*

$$F(\tfrac{x_0}{\lambda}) = -\tfrac{1}{\lambda},\ F(\tfrac{a}{\lambda}) = \tfrac{1}{\lambda^2}.$$

Proof. By Cvitanović-Feigenbaum equation, $F'(z) = -F'(\lambda z)F'(F(\lambda z))$. One has:

$$F'(\tfrac{a}{\lambda}) = -F'(a)F'(F(a)) = -F'(a)F'(-\tfrac{x_0}{\lambda}) = 0,$$
$$F'(\tfrac{x_0}{\lambda^2}) = -F'(\tfrac{x_0}{\lambda})F'(F(\tfrac{x_0}{\lambda})) = 0.$$

Assume that there is another critical point on $(0, \frac{x_0}{\lambda^2})$. Let b be the minimal such critical point. Then

$$F'(b) = 0, F'(\lambda b) \neq 0 \text{ therefore } F'(F(\lambda b)) = 0.$$

Since $\lambda b \in (0, \frac{x_0}{\lambda})$, using Proposition 13 we get that $F(\lambda b) \in (-\frac{1}{\lambda}, 1)$. Then one of the following three possibilities holds:

- $F(\lambda b) = 0 \ \Rightarrow\ \lambda b = x_0,\ b = \frac{x_0}{\lambda}$;
- $F(\lambda b) = -\frac{x_0}{\lambda} \ \Rightarrow\ \lambda b = a,\ b = \frac{a}{\lambda}$;
- $F(\lambda b) \in (-\frac{1}{\lambda}, -\frac{x_0}{\lambda}) \ \Rightarrow\ \lambda b \in (a, \frac{x_0}{\lambda})$ and hence $b > \frac{a}{\lambda} > \frac{1}{\lambda} > -F(\lambda b)$.

Each of the possibilities above contradicts the choice of b. □

Now, Theorem 15 together with Lemma 18 imply that for each of the segments

$$[0, \tfrac{x_0}{\lambda}], [\tfrac{x_0}{\lambda}, \tfrac{a}{\lambda}], [\tfrac{a}{\lambda}, \tfrac{x_0}{\lambda^2}]$$

there exists exactly one tile $P \in \mathcal{P}$ in the first quadrant which contain this segment in its boundary. Denote these tiles by $P_{0,I}, P_{1,I}$ and $P_{2,I}$ correspondingly. For a quadrant $J \neq I$ (that is, $J = II, III$ or IV) and $k \in \{0, 1, 2\}$ let $P_{k,J} \in P$ be the tile in quadrant J which is symmetric to $P_{k,I}$ with respect to one of the axis or the origin. For any set P and any n set $P^{(n)} = \lambda^n P$.

Proposition 19. *The Feigenbaum map F satisfies the following:*

1) $F(P^{(2)}_{0,I}) \subset P^{(1)}_{1,IV}$;
2) $P^{(2)}_{1,IV} \cup P^{(2)}_{2,IV} \subset F(P^{(2)}_{1,I}) \subset P^{(1)}_{0,IV}$;
3) $F(P^{(2)}_{2,I}) \subset P^{(1)}_{0,IV}$;
4) $F(P^{(1)}_{0,I}) = P^{(0)}_{0,IV}$, $F(P^{(1)}_{1,I}) = P^{(0)}_{0,III}$.

Proof. Since

$$P(0) = 1 \in P^{(1)}_{1,IV} \text{ and } F(P^{(2)}_{0,I}) \subset F(P^{(0)}_{0,I}) = \mathbb{H}_-,$$

by Lemma 17 2) we get that $F(P^{(2)}_{0,I}) \subset P^{(1)}_{1,IV}$.

Further, we have:

$$F(\lambda x_0) = x_0, \text{ and } F(F(\lambda a)) = -\lambda F(a) = x_0.$$

Since F is one-to-one on $[0, \frac{x_0}{\lambda}]$, it follows that $F(\lambda a) = \lambda x_0$. Thus,

$$F((\lambda x_0, \lambda a)) = (\lambda x_0, x_0).$$

By Lemma 17 2) we obtain the property 2) of Proposition 19. The properties 3) and 4) can be proven in a similar fashion. □

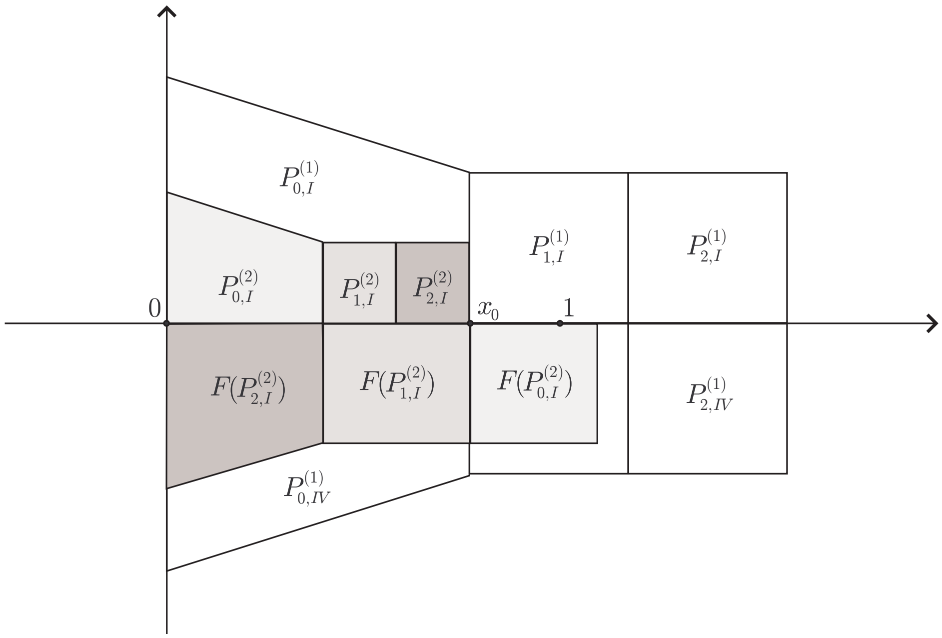

FIGURE 2. Illustration to Proposition 19.

2.2. **Computing the Feigenbaum map F.** Let us set

$$W = \mathrm{Int}\overline{P_{0,I} \cup P_{0,II} \cup P_{0,III} \cup P_{0,IV}}.$$

Consistently with our previous notation, let us define $W^{(0)} = W$ and $W^{(n)} = \lambda^n W$.

Let us fix a rational number $r > 0$ and a dyadic set U such that

$$U_r(W^{(1)}) \subset U \text{ and } U_r(U) \subset W^{(0)}. \tag{5}$$

We state:

Proposition 20. *The restriction of F onto U is poly-time computable.*

The proof of Proposition 20 will occupy the rest of the section.

Let us begin by defining some functional spaces. For a topological disk $W \subset \mathbb{C}$ we will denote $\mathcal{A}_W$ the Banach space of bounded analytic functions in W equipped with the sup norm. In the case when the domain W is the disk $\mathbb{D}_\rho$ of radius $\rho > 0$ centered at the origin, we will denote $\mathcal{A}_{\mathbb{D}_\rho} \equiv \mathcal{A}_\rho$.

For each $\rho > 0$ we will also consider the collection $\mathcal{L}^1_\rho$ of analytic functions $f(z)$ defined on $\mathbb{D}_\rho$, equipped with the weighted l_1 norm on the coefficients of the Maclaurin's series:

$$\|f\|_\rho = \sum_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!}\rho^n. \tag{6}$$

The proof of the following elementary statement is left to the reader:

Lemma 21.

1) *Let* $f \in \mathcal{L}^1_\rho$*, then* $\sup_{\mathbb{D}_\rho} |f(z)| \leqslant \|f\|_\rho$*;*
2) *Let* $f \in \mathcal{A}_{\rho'}$ *and* $\rho' > \rho$*, then* $\|f\|_\rho \leqslant \frac{\rho}{\rho'-\rho} \sup_{\mathbb{D}_{\rho'}} |f(z)|$*.*

As an immediate consequence, we have:

Corollary 22. $\mathcal{L}^1_\rho$ *is a Banach space.*

To compute the Feigenbaum map F we will recall a rigorous computer-assisted approach of Lanford [14], based on an approximate Newton's method for $\mathcal{R}_2$. Note that Lanford also proved hyperbolicity of period-doubling renormalization at F (but not uniqueness of F) using the same approach before Sullivan's work which we have quoted above.

Lanford used the Contraction Mapping Principle to find F. Since $\mathcal{R}_2$ is not a contraction – as it has an unstable eigenvalue at F – he replaced the fixed point problem for $\mathcal{R}_2$ with the fixed point problem for the approximate Newton's Method

$$g \mapsto g + (I - \Gamma)^{-1}(\mathcal{R}_2(g) - g),$$

where Γ is a high-precision finite approximation of $D\mathcal{R}_2|_F$. Formally, his results can be summarized as follows:

Theorem 23. [14] *There exist rational numbers* $\rho > 0$ *and* $\Delta > 0$*, a polynomial* $p(z)$ *with rational coefficients, and an explicit linear operator* Γ *on* $\mathcal{L}^1_\rho$ *(which is given by a finite rational matrix in the canonical basis of* $\mathcal{L}^1_\rho$*) such that the following properties hold.*

The operator $I - \Gamma$ *is invertible. Denoting* $\mathcal{D}$ *the Banach ball in* $\mathcal{L}^1_\rho$ *given by*

$$||g - p||_\rho < \Delta,$$

and

$$\Phi : g \mapsto g + (I - \Gamma)^{-1}(\mathcal{R}_2(g) - g),$$

we have:

- $\Phi(\mathcal{D}) \Subset \mathcal{D}$*;*
- *moreover, there exists* $\rho' > \rho$ *such that for every* $g \in \mathcal{D}$ *the image* $\Phi(g) \in \mathcal{L}^1_{\rho'}$*;*
- *the Feigenbaum map* $F \in \mathcal{D}$ *(note that it immediately follows that* $\Phi(F) = F$*);*
- *finally, there exists a positive* $\epsilon < 1$ *such that*
$$||\Phi(g) - F||_\rho < \epsilon ||g - F||_\rho$$
for all $g \in \mathcal{D}$*.*

Note that by Cvitanović-Feigenbaum equation (3) to prove Proposition 20, it is sufficient to compute F in polynomial time in a disk $\mathcal{D}_r$ for some $r > 0$. Let ρ and ρ' be as above. Fix m, where 2^{-m} is the desired precision for F. For a decimal number b denote $\lfloor b \rceil_l$ its round-off to the l-th decimal digit. For

$$f = \sum_{k=1}^{\infty} b_k z^k \text{ set } \lfloor f \rceil_l \equiv \sum_{k=1}^{\infty} \lfloor b_k \rceil_l z^k.$$

Fix $l = O(m)$ such that for all $g \in \mathcal{D}$

$$||g - \lfloor g \rceil_l||_\rho < 2^{-(m+2)}.$$

Further, for

$$f=\sum_{k=1}^{\infty} b_k z^k \text{ set } \mathrm{Poly}_n(f)\equiv\sum_{k=1}^{n} b_k z^k.$$

Applying Cauchy derivative estimate to the remainder term in Maclaurin series, we see that there exists $n=O(m)$ such that for all $g\in\mathcal{D}\cap\mathcal{L}^1_{\rho'}$

$$||g-\mathrm{Poly}_n(g)||_\rho<2^{-(m+2)}.$$

Now let

$$p_0=\lfloor p_0\rfloor_l\sum_{n=0}^{n} a_k^0 z^k\in\mathcal{D}$$

be a polynomial with rational coefficients. The binomial formula implies that $\Phi(p_0)$ can be computed in time $O(m^4)$. Define

$$p_1=\mathrm{Poly}_n(\lfloor\Phi(p_0)\rfloor_l).$$

Note that

$$||p_1-F||_\rho<\epsilon||p_0-F||_\rho+2^{-(m+1)}.$$

Iterating the procedure $O(m)$ times (total computing time $O(m^5)$), we obtain a $2^{-(m+1)}$-approximation of F in $||\cdot||_\rho$. This, and Lemma 21 imply the desired statement.

3. Computing long iterations.

Introduce the following notations:

$$Q_J=\mathrm{Int}\overline{P_{0,J}\cup P_{1,J}},\quad R_J=\mathrm{Int}\overline{P_{1,J}\cup P_{2,J}},\quad S_J=\mathrm{Int}\overline{P_{0,J}\cup P_{1,J}\cup P_{2,J}},$$

where $J\in\{I,II,II,IV\}$ and Int stands for the interior of a set. Let

$$P_k=\mathrm{Int}\overline{P_{k,I}\cup P_{k,II}\cup P_{k,III}\cup P_{k,IV}}.$$

Similarly define Q,R and S. Note that $W^{(0)}=P_0$. Recall that for a set P we defined $P^{(n)}=\lambda^n P$.

Observe that $J_f\Subset Q^{(1)}\Subset W^{(0)}=W$. For any $w\in W$ let $m(w)$ be such that $w\in W^{(m)}\setminus W^{(m+1)}$. Fix a point $z_0=z\in Q^{(1)}$. Introduce inductively a sequence $\{z_k\}$ of iterates of z under F as follows:

$$z_{k+1}=\begin{cases} F(z_k), & \text{if } z_k\in Q^{(1)}\setminus W^{(1)},\\ F^{2^{m_k-1}}(z_k), & \text{if } m_k=m(z_k)\geqslant 1.\end{cases}\tag{7}$$

If $z_k\notin Q^{(1)}$ then the sequence terminates at the index $k_{term}:=k$. For every k let m_k be the number such that $z_k\in W^{(m_k)}\setminus W^{(m_k+1)}$. By (4) if $m_k\geqslant 1$ then one has:

$$z_{k+1}=(-\lambda)^{(m_k-1)}F(z_k/\lambda^{m_k-1}).$$

In particular, this implies that

$$m_{k+1}\geqslant m_k-1 \text{ for all } k.\tag{8}$$

Define inductively a sequence of indexes s_k such that $z_k=f^{s_k}(z)$:

$$s_0=0,\quad s_{k+1}=\begin{cases} s_k+1, & \text{if } m_k=0,\\ s_k+2^{m_k-1}, & \text{if } m_k\geqslant 1.\end{cases}\tag{9}$$

Set
$$\epsilon = \lambda d\big(W \setminus S^{(1)}, \mathbb{R}\big).$$
If there exists i such that $z_i \in W^{(1)}$ then denote $j_{term} = \max\left\{i : z_i \in W^{(1)}\right\}$. Otherwise set $j_{term} = \infty$.

The main result of this section is the following:

Proposition 24. *There exist constants $A, B > 0$ such that if $d(z, J_f) \geqslant 2^{-n}$ then the sequence $\{z_k\}$ terminates at some index $k = k_{term} \leqslant An + B$.*

The rest of Section 3 is devoted to the proof of Proposition 24. But first let us state an important

Corollary 25. *If $d(z, J_f) \geqslant 2^{-n}$ then $m(F^s(z)) \leqslant An + B$ for all $s \leqslant s_{k_{term}}$.*

Proof. Proposition 24 and (8) imply that $m_j \leqslant An + B$ for all j. Let $s_j < s < s_{j+1}$. Then
$$m_j > 1 \ \text{ and } \ s_{j+1} = s_j + 2^{m_j - 1}.$$
We have: $F^{s_j}(z) = z_j \in W^{(m_j)}$. Since the first landing map from $W^{(m_j)}$ to $W^{(m_j-1)}$ is $F^{2^{m_j-1}} = F^{s_{j+1}-s_j}$ we get
$$m(F^s(z)) \leqslant m_j - 1,$$
which finishes the proof. □

Lemma 26. *Let $z \in W^{(0)} \setminus J_F$, then $\{z_k\}$ is finite. Moreover, if $j_{term} < \infty$ then $z_{j_{term}} \notin S^{(2)}$ and hence $|Im(z_{j_{term}})| > \epsilon$.*

Proof. Let $z \in Q^{(1)} \setminus J_F$. Assume that $\{z_k\}$ is infinite. Then $z_k \in Q^{(1)}$ for every k. Since $F_{|Q^{(1)}}$ is quadratic-like, there exists l such that $F^l(z) \notin Q^{(1)}$. Let k be the maximal index such that $s_k < l$. Two cases possible:
a) $m_k = 0$ or $m_k = 1$. Then $z_{k+1} = F(z_k)$ and $s_{k+1} = s_k + 1$. It follows that $l = s_{k+1} = s_k + 1$ and $F^l(z) = z_{k+1} \in Q^{(1)}$. We arrive at a contradiction.
b) $m_k \geqslant 2$. Since $z_k \notin J_f \supset \mathbb{R} \cup \mathrm{i}\mathbb{R}$ we obtain that $z_k \in P_{0,J}^{(m_k)}$ for some J. Observe that $s_{k+1} = s_k + 2^{m_k - 1} \geqslant l$. Lemma 17 implies that
$$F^{l-s_k}(P_{0,J}^{(m_k)}) \subset T, \ \text{ where } \ T \in \mathcal{P}^{(m_k-1)}.$$
Clearly, $T \cap J_F \neq \varnothing$. Thus, $F^l(z)$ belongs to a tile T of level $m_k - 1 \geqslant 1$ which intersects J_F. This implies that $F^l(z) \in Q^{(1)}$. We arrive at a contradiction. This shows that the sequence z_j is finite.

Further, assume that $j = j_{term} < \infty$. Set $k = k_{term}$. Observe that $z_k \notin Q^{(1)}$. It follows from Proposition 19 that $z_{k-1} \notin S^{(2)}$. Thus, if $z_{k-1} \in W^{(1)}$ then $j = k - 1$ and
$$|\mathrm{Im}(z_j)| \geqslant d(W^{(1)} \setminus S^{(2)}, \mathbb{R}) = \epsilon.$$
Otherwise, $k \geqslant j + 2, z_{j+1} \in Q^{(1)} \setminus W^{(1)}$ and $z_{j+2} \notin W^{(1)}$. Since
$$F^2(W^{(2)}) \subset W^{(1)}, \ \ F\big(P_1^{(2)}\big) \subset W^{(1)} \ \text{ and } \ F\big(P_2^{(2)}\big) \subset W^{(1)}$$
we obtain that $z_j \notin S^{(2)}$. It follows that $|\mathrm{Im} z_j| \geqslant \epsilon$. □

3.1. **Expansion in the hyperbolic metrics.** Fix $z_0 \in Q^{(1)} \setminus J_f$. Let $\{z_k\}$ be the sequence defined above. Set $r_k = \max\{m_k - 1, 0\}$, so that $z_{k+1} = F^{2^{r_k}}(z_k)$ for all $k < k_{term}$. Introduce an auxiliary sequence $w_k = z_k/\lambda^{r_k}$. From (4) we obtain that

$$w_{k+1} = (-1)^{r_k} \lambda^{r_k - r_{k+1}} F(w_k).$$

Observe that $w_k \subset Q^{(1)} \setminus W^{(2)}$. Therefore, $F(w_k) \subset W$. It follows that for all $k < k_{term}$ we have: $r_{k+1} \geqslant r_k - 1$. For convenience, set

$$H_k(w) = \lambda^{r_k - r_{k+1}} F(w), \ \ H_{k,l} = H_{l-1} \circ H_{l-2} \circ \ldots \circ H_{k+1} \circ H_k, \ \ k < l$$

so that $w_{k+1} = \pm H_k(w_k)$, $\quad w_l = \pm H_{k,l}(w_k)$, $k < l$. For a point z such that $F(z) \notin \mathbb{R}$ define by $\|DF(z)\|_{\mathbb{H}}$ the norm of the differential of z in the hyperbolic metrics on either $\mathbb{H}_+ = \{z : \mathrm{Im} z > 0\}$ or $\mathbb{H}_- = \{z : \mathrm{Im} z < 0\}$. Since F is even and one-to-one from $P^{(0)}_{0,I}$ onto $\mathbb{H}_+$, from Schwarz-Pick Theorem we obtain the following:

Lemma 27. *For all $w \in W \setminus (\mathbb{R} \cup \mathrm{i}\mathbb{R})$ one has $\|DF(w)\|_{\mathbb{H}} > 1$. Moreover, there exists $\lambda_1 > 1$ such that $\|DF(w)\|_{\mathbb{H}} > \lambda_1$, assuming that $w \in W^{(1)} \setminus S^{(2)}$.*

Let $N = N(z)$ be the number of indexes k for which $m_k \geqslant 1$ and $z_k \notin S^{(m_k+1)}$. Since the hyperbolic metric on $\mathbb{H}_\pm$ is scaling invariant, using (4) we get:

Proposition 28. *If $m_k \geqslant 1$ and $z_k \notin S^{(m_k+1)} \cup \mathrm{i}\mathbb{R}$, then*

$$\left\|DF^{2^{m_k-1}}(z_k)\right\|_{\mathbb{H}} > \lambda_1.$$

Moreover, there is a universal constant C_1 (independent from z) such that

$$\left\|DF^{s_{j_{term}}}(z)\right\|_{\mathbb{H}} > \lambda_1^{N-1}, \ \ \left|DF^{s_{j_{term}}}(z)\right| > C_1 \lambda_1^N,$$

assuming that $j_{term} < \infty$.

Proof. If $z_k \notin S^{(m_k+1)}$ then $z_k/\lambda^{m_k-1} \notin S^{(2)}$. By Lemma 27 we obtain:

$$\left\|DF^{2^{m_k-1}}(z_k)\right\|_{\mathbb{H}} = \left\|DF(z_k/\lambda^{m_k-1})\right\|_{\mathbb{H}} > \lambda_1.$$

It follows that $\left\|DF^{s_{j_{term}}}(z)\right\|_{\mathbb{H}} > \lambda_1^{N-1}$. Since the hyperbolic metric of $\mathbb{H}_\pm$ is equivalent to the Euclidean metric on any compact subset of $\mathbb{H}_\pm$, using Lemma 26 we obtain the last inequality of Proposition 28. $\square$

Set

$$R_+ = \mathrm{Int}\overline{R_I \cup R_{IV}} = R \cap \{z : \mathrm{Re} z > 0\},$$

$$W_+ = \mathrm{Int}\overline{P_{0,I} \cup P_{0,IV}} = W \cap \{z : \mathrm{Re} z > 0\}, \ \ F_+ = F_{|W_+}.$$

Observe that

$$F(W_+) = \mathbb{C} \setminus ((-\infty, -\tfrac{1}{\lambda}] \cup [1, +\infty)) \Supset W_+^{(1)} \supset R_+^{(2)}.$$

Introduce sets

$$V = \mathbb{C} \setminus ((-\infty, -\tfrac{\lambda}{2}] \cup [1, +\infty)), \ \ V' = F_+^{-1}(V) = W_+ \setminus (t, \tfrac{x_0}{\lambda}),$$

where $t = F_+^{-1}(-\frac{\lambda}{2}) \subset (x_0, 1)$, since $F_+^{-1}(-\lambda) = 1$ and $F_+^{-1}(0) = x_0$. Notice that

$$R_+^{(2)} \Subset V' \subset V.$$

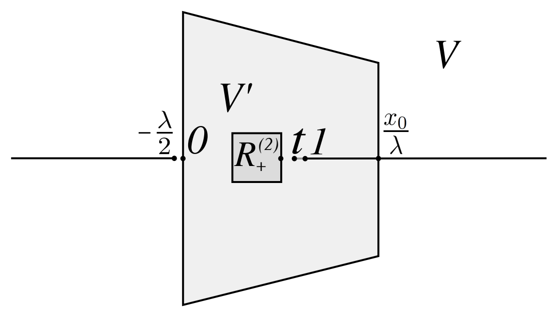

FIGURE 3. The sets V, V' and $P_+^{(2)}$.

Remark 29. *Let k be an index such that $w_k \in R_+^{(2)}$ and $m_k \geqslant 1$. Then, by definition,*

$$H_k(w_k) = \lambda^{r_k - r_{k+1}} F(w_k) = \pm w_{k+1} \subset W_+^{(1)} \Subset V.$$

Therefore, the norm of the derivative $DH_k(w_k)$ in the hyperbolic metric on V is well defined.

Proposition 30. *There exists $\lambda_2 > 1$ such that if $w_k \in R_+^{(2)}$ and $m_k \geqslant 1$ then*

$$\|DH_k(w_k)\|_V \geqslant \lambda_2.$$

Let us introduce an auxiliary function

$$a(r) = \frac{1 - x(r)^2}{2|x(r) \log x(r)|}, \text{ where } x(r) = \frac{e^r - 1}{e^r + 1}.$$

Observe that $a(r)$ is decreasing on $[0, \infty)$, $a(r) \to \infty$ when $r \to 0+$ and $a(r) \to 1$ when $r \to \infty$. The proof of Proposition 30 relies on the following consequence of the Schwarz-Pick Theorem:

Lemma 31. *Let $U \subset V$ be domains in $\mathbb{C}$, $G : U \to V$ be a conformal map, $z \in U$ and $r = dist_V(z, V \setminus U)$. Then $\|DG(z)\|_V \geqslant a(r)$.*

Proof. Since $\|DG(z)\|_{U,V} = 1$ we obtain:

$$\|DG(z)\|_V = \|D\mathrm{Id}(z)\|_{V,U},$$

where Id is the identity map. Let $\zeta \in V \setminus U$ and $R = \mathrm{dist}_V(\zeta, z)$. Set $\widetilde{V} = V \setminus \{\zeta\}$. By the Schwarz-Pick Theorem,

$$\|D\mathrm{Id}(z)\|_{V,U} = \|D\mathrm{Id}(z)\|_{V,\widetilde{V}} \|D\mathrm{Id}(z)\|_{\widetilde{V},U} \geqslant \|D\mathrm{Id}(z)\|_{V,\widetilde{V}}.$$

Let $\phi : V \to \mathbb{U}$ be the conformal map such that $\phi(\zeta) = 0$ and $w = \phi(z) > 0$. Then

$$\|D\mathrm{Id}(z)\|_{V,\widetilde{V}} = \|D\mathrm{Id}(w)\|_{\mathbb{U},\mathbb{U}\setminus\{0\}}, \text{ and } R = \mathrm{dist}_V(z, \zeta) = \mathrm{dist}_{\mathbb{U}}(w, 0).$$

The value of $\|D\mathrm{Id}(w)\|_{\mathbb{U},\mathbb{U}\setminus\{0\}}$ can be computed explicitly and is equal to $a(R)$. Since ζ is any point in $V \setminus U$ we obtain that $\|DG(z)\| \geqslant a(r)$. □

Proof of Proposition 30. Let $w_k \in R_+^{(2)}$. Then

$$r_{k+1} \geqslant r_k \ \text{ and } \ \lambda^{r_{k+1}-r_k} V \subset V.$$

Set $A = F_+^{-1}(\lambda^{r_{k+1}-r_k} V)$. Then $H_k : A \to V$ is a conformal isomorphism and $w_k \in A$. Since

$$R_+^{(2)} \Subset V' \subset V \ \text{ and } \ A \subset V'$$

we obtain that $R = \mathrm{dist}_V(R_+^{(2)}, V \setminus V') + \mathrm{diam}_V(R_+^{(2)})$ is finite and

$$\mathrm{dist}_V(w_k, V \setminus A) \leqslant R.$$

By Lemma 27 we obtain that $\|DH_k(w_k)\|_V \geqslant a(R)$. □

Proposition 32. *There exists $1 > C_2 > 0$ such that the following is true. Let k, l be such that $w_j \in R^{(2)}$ for $j = k, k+1, \dots, k+l-1$. Then*

$$\left|DH_{k,k+l}(w_k)\right| \geqslant C_2 \lambda_2^l.$$

Proof. Let k, l be as in the conditions of the proposition. Without loss of generality we may assume that $w_k \in R_+^{(2)}$. Let $\rho(z)\mathrm{d}z$ be the hyperbolic metric on V. Since $R_+^{(2)} \Subset V$ there exists a constant $M > 0$ such that

$$\tfrac{1}{M} < |\rho(z)| < M \ \text{ and } \ |DF(z)| > \tfrac{1}{M} \ \text{ for all } \ z \in R_+^{(2)}.$$

Notice that for all $k \leqslant j \leqslant k+l-1$ we have:

$$F(w_j) \subset W_+^{(1)}, \ \ \lambda^{r_j - r_{j+1}} F(w_j) = H_j(w_j) = \pm w_{j+1} \subset W_+^{(1)} \setminus W_+^{(2)},$$

therefore $r_j - r_{j+1} \leqslant 0$ and $|DH_j(z)| = \lambda^{r_j - r_{j+1}} |DF(z)| \geqslant \frac{1}{M}$ for all $z \in R_+^{(2)}$. By Proposition 30 we obtain:

$$|DH_{k,k+l}(w_k)| = |DH_{k+l-1}(w_{k+l-1})| \cdot |DH_{k,k+l-1}(w_k)| \geqslant \tfrac{1}{M^3} \|DH_{k,k+l-1}(w_k)\|_V \geqslant \tfrac{\lambda_2^{l-1}}{M^3},$$

which finishes the proof. □

Further, set

$$P_{1,+} = \mathrm{Int}\overline{P_{1,I} \cup P_{1,IV}}, \ W_- = -W_+.$$

Then $F : P_{1,+}^{(1)} \to W_-$ is a conformal isomorphism. Notice that $P_{1,+}^{(1)} \Subset W_+$. Similarly to Proposition 32 we obtain:

Proposition 33. *There exists $\lambda_3 > 1, 1 > C_3 > 0$ such that the following is true. Let k, l be such that $w_j \in P_1^{(1)}$ for $j = k, k+1, \dots, k+l-1$. Then*

$$\left|DH_{k,k+l}(w_k)\right| \geqslant C_3 \lambda_3^l.$$

3.2. **Proof of Proposition 24.**

Lemma 34. *There exists a constant $C > 1$ such that if one of the following is true*
1) $F^j(z) \in W$ *for* $j = 0, 1, \ldots, s-1$ *and* $F^s(z) \in W^{(1)} \setminus S^{(2)}$,
2) $z \in Q^{(1)} \setminus W^{(2)}$, *and* $F^j(z) \in Q^{(1)} \setminus W^{(1)}$ *for* $j = 1, \ldots, s-1$ *then*

$$d(z, J_f) \leqslant \frac{C d(F^s(z), J_f)}{|DF^s(z)|}.$$

Proof. Assume that the first condition is true: $F^j(z) \in W$ for $j = 0, 1, \ldots, s-1$ and $F^s(z) \in W^{(1)} \setminus S^{(2)}$. Without loss of generality let $\mathrm{Im} F^s(z) > 0$. Fix two convex domains $V_1 \Subset V_2 \subset \mathbb{H}_+$ such that

$$V_1 \Supset \left(W^{(1)} \setminus S^{(2)}\right) \cap \mathbb{H}_+, \text{ and } V_1 \cap J_f \neq \varnothing.$$

Since the postcritical set of F belongs to $\mathbb{R}$ there exists $\phi : V_2 \to \mathbb{C}$ such that

$$F^s \circ \phi = \mathrm{Id} \text{ and } \phi(F^s(z)) = z.$$

Notice that $\phi(V_2) \subset W$. By Koebe Distortion Theorem there exists a constant M_1 (independent from s or z) such that

$$|\phi'(u)| \leqslant M_1 |\phi'(F^s(z))| = \frac{M_1}{|DF^s(z)|} \text{ for all } u \in V_1.$$

The condition $F^j(z) \in W$ for $j = 0, 1, \ldots, s$ implies that $\phi(V_1 \cap J_F) \subset J_F$. Set

$$M_2 = \sup_{u \in V_1} \frac{d(u, \overline{V}_1 \cap J_F)}{d(u, J_F)}.$$

Then

$$d(z, J_f) \leqslant \frac{M_1 d(F^s(z), \overline{V}_1 \cap J_f)}{|DF^s(z)|} \leqslant \frac{M_1 M_2 d(F^s(z), J_f)}{|DF^s(z)|}.$$

The second case can be treated similarly. □

Proof of Proposition 24. 1) First assume that $j_{term} < \infty$. Set $r = \inf\{|F'(z)| : z \in W^{(1)} \setminus S^{(2)}\}$. Clearly, $r > 0$. Let k be such that $w_k \in W^{(1)} \setminus S^{(2)}$. Since $r_{k+1} \geqslant r_k - 1$ we have:

$$|DH_k(w_k)| = \lambda^{r_k - r_{k+1}} |DF(w_k)| \geqslant \lambda r.$$

Further, let I_0 be the set of indexes i from $0, 1, \ldots, j_{term}$ such that $w_i \in W^{(1)} \setminus S^{(2)}$. Notice that $j_{term} \in I_0$. Set $I = I_0 \cup \{0\}$. let $j_1 < j_2$ be two consecutive indexes from I. Observe that if $w_k \in R^{(2)}$ for some k then $F(w_k) \subset W_+^{(1)}$, $m_{k+1} \geqslant 1$ and thus either $w_{k+1} \in R^{(2)}$ or $k+1 \in I$. It follows that there exists $j_1 < j \leqslant j_2$ such that $m_k = 0$ for $k = j_1 + 1, j_1 + 2, \ldots, j-1$ and $w_k \in R^{(2)}$ for $k = j, j+1, \ldots, j_2 - 1$. Using Propositions 32 and 33 we get:

$$\left|DH_{j_1, j_2}(w_{j_1})\right| = \left|DH_{j_1}(w_{j_1})\right| \cdot \left|DH_{j_1+1, j}(w_{j_1+1})\right| \cdot \left|DH_{j, j_2}(w_j)\right| \geqslant$$
$$\lambda r C_3 \lambda_3^{j - j_1 - 1} C_2 \lambda_2^{j_2 - j} \geqslant C_4 \lambda_4^{j_2 - j_1}, \text{ where } C_4 = \lambda r C_2 C_3 / \lambda_3, \lambda_4 = \min\{\lambda_2, \lambda_3\}.$$

As before, let N be the number of indexes k such that $m_k \geqslant 1$ and $w_k \in W^{(1)} \setminus S^{(2)}$. We obtain:

$$|DH_{0, j_{term}}(w_0)| \geqslant C_4^{N+1} \lambda_4^{j_{term}}.$$

Notice that $m_{j_{term}} = 1, z_{j_{term}} = w_{j_{term}}$ and hence

$$H_{0,j_{term}}(w) = F^{s_{j_{term}}}(\lambda^{r_0} w).$$

It follows that

$$|DF^{s_{j_{term}}}(z)| \geqslant C_4^{N+1}\lambda_4^{j_{term}}.$$

Since $r_j = 0$ for $j \geqslant j_{term}$ we have $H_j = F$ for $j \geqslant j_{term}$. By Proposition 33,

$$|DF^{s_{k_{term}}-s_{j_{term}}}(z_{j_{term}})| \geqslant C_3 r \lambda_3^{k_{term}-j_{term}-1} \geqslant C_4\lambda_4^{k_{term}-j_{term}}.$$

Therefore

$$|DF^{s_{k_{term}}}(z)| \geqslant C_4^{N+2}\lambda_4^{k_{term}}.$$

Assume that $d(z, J_f) \geqslant 2^{-n}$. Then by Lemma 34,

$$|DF^{s_{j_{term}}}(z)| \leqslant 2^n C d(z_{j_{term}}, J_f) \text{ and } |DF^{s_{k_{term}}-s_{j_{term}}}(z_{j_{term}})| \leqslant C\frac{d(z_{k_{term}}, J_f)}{d(z_{j_{term}}, J_f)}.$$

Thus, $|DF^{s_{k_{term}}}(z)| \leqslant 2^n M$, where $M = C^2\mathrm{diam}(W)$. Therefore, $k_{term} \leqslant A_1 n + A_2 N + A_3$ where

$$A_1 = \log_{\lambda_4} 2, A_2 = -\log_{\lambda_4} C_4, A_3 = \log_{\lambda_4} M - 2\log_{\lambda_4} C_4.$$

On the other hand, using Proposition 28 we obtain:

$$N \leqslant n\log_{\lambda_1} 2 + \log_{\lambda_1}(M/C_1).$$

Thus, $k_{term} \leqslant An + B$, where

$$A = A_1 + A_2\log_{\lambda_1} 2, B = A_3 + A_2\log_{\lambda_1}(M/C_1).$$

2) Assume now that $j_{term} = \infty$, that is $z_j \in Q^{(1)} \setminus W^{(1)}$ for all $j < k_{term}$. Then $m_j = 0$, $z_j = w_j = F^j(z)$ and $H_j = F$ for all $j < k_{term}$. Using Proposition 33 we get:

$$|DF^{k_{term}}(z)| \geqslant C_3\lambda_3^{k_{term}}.$$

If $d(z, J_f) \geqslant 2^{-n}$ then

$$|DF^{k_{term}}(z)| \leqslant M2^n \text{ and } k_{term} \leqslant n\log_{\lambda_3} 2 + \log_{\lambda_3}(M/C_3),$$

which finishes the proof. □

4. The algorithm.

Fix a dyadic number $\delta > 0$ such that

$$\delta < \tfrac{1}{2} d(\mathbb{C}\setminus W, W^{(1)}) \text{ and } F(U_\delta(J_f)) \subset Q^{(1)}.$$

Proposition 35. *There exist constants $K_1, K_2 > 0$ such that for any $z \in U_\delta(J_F)$ and any $k \in \mathbb{N}$ if $F^k(z) \in U_\delta(Q^{(1)}) \setminus U_\delta(J_F)$ then one has*

$$\frac{K_1}{|DF^k(z)|} \leqslant d(z, J_F) \leqslant \frac{K_2}{|DF^k(z)|}.$$

Proof. There exist a finite number of pairs of simply connected sets $W_j \Subset U_j$ such that the following is true

1) $W_j \cap J_F \neq \varnothing$ for any j;
2) $\bigcup W_j \supset U_\delta(Q^{(1)}) \setminus U_\delta(J_F)$;

3) $\bigcup U_j \subset W \setminus [-1,1]$.

Assume now that z, k satisfy the conditions of the proposition. Then $F^k(z) \in W_j$ for some j. Since the postcritical set of $F_{|W}$ belongs to $[-1,1]$ the map F^k admits an inverse on U_j. Let $\phi : U_j \to U(z)$ be the branch of $(F^k)^{-1}$ such that $\phi(F^k(z)) = z$. By Koebe Distortion Theorem there exists $R_j > 0$ independent of z or k such that $\phi(W_j)$ is contained in the disk of radius $R_j |D\phi(F^k(z))|$ centered at z. It follows that

$$d(z, J_F) \leqslant \frac{R_j}{|DF^k(z)|}.$$

Set $K_2 = \max\{R_j\}$. Construction of K_1 is similar. □

Fix dyadic sets U_1, U_2 such that

$$Q^{(1)} \subset U_1 \subset U_{\delta/2}(Q^{(1)}), \;\; U_{\delta/4}(J_f) \subset U_2 \subset U_{\delta/2}(J_f).$$

Assume that we would like to verify that a dyadic point z is 2^{-n} close to J_f. Consider first points z which lie outside U_2. Fix a dyadic set U_3 such that

$$J_F \subset U_3 \Subset U_2.$$

Then we can approximate the distance from a point $z \notin U_2$ to J_F by the distance form z to U_3 up to a constant factor.

Now assume $z \in U_2$. The key tool of the algorithm is a sequence of approximations of the iterates $z_k = F^{s_k}(z)$. However, the numbers s_k depend on the levels m_k such that $z_k \in W^{(m_k)} \setminus W^{(m_k+1)}$. These levels cannot be computed exactly. Because of this, we inductively define approximations

$$p_j \approx \tilde{z}_j = F^{\tilde{s}_j}(z)$$

of, possibly, different iterates closely related to $\{z_k\}$.

Construction of $\{p_j\}$. Set $p_0 = z, \tilde{s}_0 = 0$. Assume that p_j is constructed. If $p_j \in U_1$ let $\tilde{m}_j$ be such that

$$U_{\delta/4}\big(\lambda^{-\tilde{m}_j} p_j\big) \subset W, \;\text{ but }\; U_{\delta/2}\big(\lambda^{-\tilde{m}_j} p_j\big) \not\subseteq W^{(1)}.$$

Notice that for some p_j there are two choices of $\tilde{m}_j$. We fix one of them arbitrarily. Set

$$\tilde{r}_j = \max\{0, \tilde{m}_j - 1\}, \;\; \tilde{s}_{j+1} = \tilde{s}_j + 2^{\tilde{r}_j}.$$

Let p_{j+1} be an approximation of $F^{\tilde{s}_{j+1}}(z)$ with precision at least $\lambda^{An+B+3}\delta$, with A, B as in Proposition 24.

If $p_j \notin U_1$ the sequence $\{p_i\}$ terminates at the index $i = j$.

Observe that $m(F^{\tilde{s}_j}(z)) - 1 \leqslant \tilde{m}_j \leqslant m(F^{\tilde{s}_j}(z))$.

Lemma 36. *If $d(z, J_f) \geqslant 2^{-n}$ then there exists a finite sequence $j_0 = 0 < j_1 < \ldots < j_{k_{term}}$ such that $\tilde{s}_{j_i} = s_i$ and $j_{i+1} \leqslant j_i + An + B$ for all i.*

Proof. We will prove existence of j_i by induction. The base is obvious: $\tilde{s}_0 = s_0 = 0$. Assume that $\tilde{s}_{j_i} = s_i$. Recall that by Corollary 25

$$m_i \leqslant An + B.$$

By definition of p_{j_i} we have
$$m_i - 1 \leqslant \tilde{m}_{j_i} \leqslant m_i.$$
Thus, if $m_i \leqslant 1$ then $\tilde{r}_{j_i} = r_i = 0$ and $\tilde{s}_{j_i+1} = s_{i+1}$. Let $m_i \geqslant 2$. Set
$$k = \max\{l : \tilde{s}_l \leqslant s_{i+1}\}.$$
Assume that $\tilde{s}_k < s_{i+1}$. We have:
$$s_{i+1} - \tilde{s}_k < 2^{\tilde{r}_k}.$$
Set $x = F^{\tilde{s}_k}(z)$. Since the first return map from $W^{(m_i)}$ to $W^{(m_i-1)}$ is $F^{2^{m_i-1}} = F^{s_{i+1}-s_i}$ we have
$$m(x) = m(F^{\tilde{s}_k - s_i}(z_i)) \leqslant m_i - 2.$$
Notice that $m(x)-1 \leqslant \tilde{m}_k \leqslant m(x)$. On the other hand, $F^{s_{i+1}-\tilde{s}_k}(x) \in W^{(m_i-1)} \subset W^{(m(x)-1)}$, therefore,
$$s_{i+1} - \tilde{s}_k \geqslant 2^{m(x)-1} \geqslant 2^{\tilde{r}_k}.$$
This contradiction shows that $\tilde{s}_k = s_{i+1}$ and finishes the proof of existence of j_i.

Further, fix $i < k_{term}$. Clearly, if $\tilde{r}_{j_i} = r_i$ then $j_{i+1} = j_i + 1$. Assume that $\tilde{r}_{j_i} \neq r_i$. Then $\tilde{r}_{j_i} = r_i - 1$. If
$$\tilde{r}_{k+1} = \tilde{r}_k - 1 \text{ for all } k \geqslant j_i$$
then $\{p_l\}$ terminates at an index $l \leqslant j_i + m_i \leqslant j_i + An + B$, and so $j_{i+1} \leqslant j_i + An + B$. Otherwise, let
$$k = \min\{l \geqslant j_i : \tilde{r}_{k+1} \neq \tilde{r}_k - 1\}.$$
For simplicity, set $r = \tilde{r}_k$. Observe that
$$\tilde{z}_{j_i} = z_i \in W^{(r_i+1)}, \text{ therefore } \tilde{z}_{k+1} \subset F^{2^{r_i-1}+2^{r_i-2}+\ldots+2^r}(W^{(r_i+1)}) \subset \pm\lambda^r F^{2^{r_i-r}-1}(W^{(r_i-r+1)}).$$
Notice that $F^{2^{r_i-r}}(W^{(r_i-r+1)}) \subset W$, and thus $F^{2^{r_i-r}-1}(W^{(r_i-r+1)})$ lies inside the connected component of $F^{-1}(W)$ containing 0. This connected component is $Q^{(1)}$. We obtain:
$$\tilde{z}_{k+1} \subset Q^{(r+1)}, \text{ therefore } \tilde{m}_{k+1} \geqslant r \text{ and } \tilde{r}_{k+1} \geqslant r - 1.$$
From definition of k we conclude that $\tilde{r}_{k+1} \geqslant r$. Thus,
$$2^{\tilde{r}_{j_i}} + 2^{\tilde{r}_{j_i+1}} + \ldots + 2^{\tilde{r}_k} + 2^{\tilde{r}_{k+1}} = 2^{r_i-1} + 2^{r_i-2} + \ldots + 2^r + 2^{\tilde{r}_{k+1}} \geqslant 2^{r_i} = s_{i+1} - s_i.$$
It follows that $j_{i+1} \leqslant k + 1 \leqslant j_i + m_i \leqslant j_i + An + B$. □

Consider the following

Main subprogram:

$i := 1$
while $i \leqslant (An+B)^2 + 2$ **do**
(1) Compute dyadic approximations
$$p_i \approx \tilde{z}_i = F^{\tilde{s}_i}(z) \text{ introduced above and } d_i \approx \left|DF^{\tilde{s}_i}(z)\right|;$$

(2) Check the inclusion $p_i \in U_1$:
- if $p_i \in U_1$, go to step (5);
- if $p_i \notin U_1$, proceed to step (3);

(3) Check the inequality $d_i \geqslant K_2 2^n + 1$. If true, output 0 and exit the subprogram, otherwise
(4) output 1 and exit the subprogram.
(5) $i \to i+1$
end while
(6) Output 0 end exit.
end

Observe that for all i

$$\tilde{z}_{i+1} = F^{\tilde{s}_{i+1}}(z) = F^{2^{\tilde{r}_i}}(\tilde{z}_i) = (-\lambda)^{\tilde{r}_i} F(\tilde{z}_i/\lambda^{\tilde{r}_i}), \text{ and}$$
$$DF^{\tilde{s}_{i+1}}(z) = DF^{2^{\tilde{r}_i}}(\tilde{z}_i) DF^{\tilde{s}_i}(z) = DF(\tilde{z}_i/\lambda^{\tilde{r}_i}) DF^{\tilde{s}_i}(z).$$

Thus, the subprogram runs for at most $(An+B)^2+2$ number of while-cycles each of which consists of $O(n)$ arithmetic operations and evaluations of F and F' with precision $O(n)$ dyadic bits. Hence the running time of the subprogram can be bounded by a polynomial.

Proposition 37. *Let $h(n,z)$ be the output of the subprogram. Then*

$$h(n,z) = \begin{cases} 1, & \text{if } d(z,J_f) > 2^{-n}, \\ 0, & \text{if } d(z,J_f) < K2^{-n}, \\ \text{either } 0 \text{ or } 1, & \text{otherwise,} \end{cases} \tag{10}$$

where $K = \frac{K_1}{K_2+1}$,

Proof. Suppose first that the subprogram runs the while-cycle $(An+B)^2+2$ times and exits at the step (6). This means that $p_i \in U_1$ and $\tilde{z}_i \in W$ for $i = 1, \ldots, (An+B)^2+1$. Since $F(W \setminus Q^{(1)}) \cap W = \varnothing$ we get that $\tilde{z}_i \in Q^{(1)}$ for all $i = 1, \ldots, (An+B)^2$. By Proposition 24 and Lemma 36 we obtain that $d(z, J_F) \leqslant 2^{-n}$. Thus if $d(z, J_F) > 2^{-n}$, then the subprogram exits at a step other than (6).

Now assume that for some $i \leqslant (An+B)^2+2$ the subprogram falls into the step (3). Then

$$p_{i-1} \in U_1 \text{ and } p_i \notin U_1.$$

By the choice of δ we get $\tilde{z}_{i-1} \in U_\delta(Q^{(1)}) \setminus U_\delta(J_F)$. Further, if $d_{i-1} \geqslant K_2 2^n + 1$, then $|DF^{\tilde{s}_{i-1}}(z)| \geqslant K_2 2^n$. By Proposition 35,

$$d(z, J_F) \leqslant 2^{-n}.$$

Otherwise, $|Df^{\tilde{s}_{i-1}}(z)| \leqslant K_2 2^n + 2 \leqslant (K_2+1)2^n$. In this case Proposition 34 implies that

$$d(z, J_f) \geqslant \frac{K_1}{K_2+1} 2^{-n}.$$

□

Now, to distinguish the case when $d(z, J_f) < 2^{-n-1}$ from the case when $d(z, J_f) > 2^{-n}$ we can partition each pixel of size $2^{-n} \times 2^{-n}$ into pixels of size $(2^{-n}/K) \times (2^{-n}/K)$ and run the subprogram for the center of each subpixel. This would increase the running time at most by a constant factor.

5. Remarks and open questions.

We remark that our approach should carry over from the Feigenbaum map F to the Feigenbaum polynomial f_{c_*} (with an oracle for c_*) in a straightforward fashion, however, at the cost of further complicating what already is a rather technical proof. It should also work for other infinitely renormalizable real quadratic polynomials with bounded combinatorics. Unbounded combinatorics for real quadratics is created either by small perturbations of parabolic or of Misiurewicz dynamics (see [15]). Both of these cases is computationally "tame" (see [11]), hence, we conjecture:

Conjecture. *For every infinitely renormalizable real quadratic polynomial f_c its Julia set is poly-time computable with an oracle for c.*

Many open questions on computational complexity of quadratic Julia sets remain open. Let us conclude by mentioning two foremost ones. The first is complexity bounds on Cremer quadratic Julia sets: it is known that all of them are computable [2], but no informative pictures have ever been produced. Nothing is known about their computational complexity, in particular, it is not known if any of them are computationally hard. New ideas and techniques are likely required to make progress here.

The second question has already been formulated in [11]:

Question. *Is the Julia set of a typical real quadratic map poly-time?*

Weak hyperbolicity is typical in the real quadratic family – however, it is not clear to us whether it is sufficient for poly-time computability (see the discussion in [11]). Our renormalization-based approach developed in the present work may also prove useful in tackling this problem.

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