Repelling periodic points and landing of rays for post-singularly bounded exponential maps

A. M. Benini, M. Lyubich

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Abstract

We show that repelling periodic points are landing points of periodic rays for exponential maps whose singular value has bounded orbit. For polynomials with connected Julia sets, this is a celebrated theorem by Douady, for which we present a new proof. In both cases we also show that points in hyperbolic sets are accessible by at least one and at most finitely many rays. For exponentials this allows us to conclude that the singular value itself is accessible.

1 Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Then the dynamical plane $\mathbb{C}$ splits into two completely invariant subsets, the Fatou set $\mathcal{F}(f)$, on which the dynamics is stable, and its complement, the Julia set $J(f)$, on which the dynamics is chaotic. More precisely, the Fatou set is defined as

$$\mathcal{F}(f) := \{z \in \mathbb{C} : \{f^n\} \text{ is normal in a neighborhood of } z\}.$$

Another important role is played by the *escaping set*

$$I(f) := \{z \in \mathbb{C} : \ f^n(z) \to \infty \text{ as } n \to \infty\}.$$

In this paper, we will consider the case in which f is either a polynomial or a complex exponential map $e^z + c$. For polynomials, $I(f) \subset \mathcal{F}(f)$, while for exponentials $I(f) \subset J(f)$ (see [BR], [ELy]). However, in both cases the escaping set can be described as an uncountable collection of injective curves, called *dynamic rays* or just *rays*, which tend to infinity on one side and are equipped with some symbolic dynamics (see Sections 2 and 3).

For a polynomial of degree D with connected Julia set, $I(f)$ is an open topological disk centered at infinity, and the dynamics of f on $I(f)$ is conjugate to the dynamics of z^D on $\mathbb{C} \setminus \overline{\mathbb{D}}$ via the Böttcher map. In this case, dynamic rays are defined simply as preimages of straight rays under the Böttcher map, and the symbolic dynamics on them is inherited from the symbolic dynamics of z^D on the unit circle $\mathbb{S}^1$ (see e.g. [Mi]). For exponentials, we refer

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the reader to [SZ1] and to Section 3 of this paper. A ray is called *periodic* if it is mapped to itself under some iterate of the function.

It is important to understand the interplay between the rays and the set of non-escaping points. A ray $g_{\mathbf{s}}$ is said to *land* at a point z if $\overline{g_{\mathbf{s}}} \setminus g_{\mathbf{s}} = \{z\}$; conversely, a point is *accessible* if it is the landing point of at least one ray.

Ideally, like for hyperbolic maps (in both polynomial and exponential setting), every ray lands and every non-escaping point in the Julia set is accessible. One weaker, but very relevant, question to ask is whether all periodic rays land and whether all repelling/parabolic periodic points are accessible by periodic rays. By the Snail Lemma (see e.g. [Mi]) if a periodic ray lands it has to land at a repelling or parabolic periodic point.

Periodic rays are known to land in both the polynomial and the exponential case (for the latter, see [Re1]), unless one of their forward images contains the singular value.

The question whether repelling periodic points are accessible is harder and still open in the exponential case; in this paper, we give a positive answer to this problem for an exponential map $f(z) = e^z + c$ whose *postsingular set*

$$\mathcal{P}(f) := \overline{\bigcup_{n>0} f^n(c)}$$

is bounded. Observe that in this case the singular value is *non-recurrent*, i.e. $c \notin \mathcal{P}(f)$.

Theorem A. *Let f be either a poynomial or an exponential map, with bounded postsingular set; then any repelling periodic point is the landing point of at least one and at most finitely many dynamic rays, all of which are periodic of the same period.*

For polynomials with connected Julia set, all repelling periodic points are known to be accessible by a theorem due to Douady (see [Hu]). Another proof due to Eremenko and Levin can be found in [ELv]: their proof covers also the case in which the Julia set is disconnected. However, neither proof can be generalized to the exponential family, because both use in an essential way the fact that the basin of infinity is an open set. Our proof of Theorem A also gives a new proof in the polynomial setting (see Section 2).

Our second result is about accessibility of hyperbolic sets. A forward invariant compact set Λ is called *hyperbolic* (with respect to the Euclidean metric) if there exist $k \in \mathbb{N}$ and $\eta > 1$ such that $|(f^k)'(x)| > \eta$ for all $x \in \Lambda$, $k > \overline{k}$.

Theorem B. *Let f be either a polynomial or an exponential map, with bounded postsingular set. Then any point that belongs to a hyperbolic set is accessible.*

Theorem B for polynomials is a special case of Theorem C in [Pr]. Under an additional combinatorial assumption (only needed for polynomials) we also show that there are only finitely many rays landing at each point belonging to a hyperbolic set (see Propositions 2.11 and 4.13). This is a new result also in the polynomial case.

For an exponential map with bounded postsingular set contained in the Julia set, the postsingular set itself is hyperbolic (see [RvS], Theorem 1.2). We obtain hence as corollary of Theorem B that every point in the postsingular set, and the singular value itself, are accessible.

Part of the importance of Theorem A is that it gives indirect insight on the structure of the parameter plane. For example, for unicritical polynomials, it implies that there are no irrational subwakes attached to hyperbolic components (see Section I.4 in [Hu], Theorem 4.1 in [S1]). The proof in [S1] is combinatorial and can be applied to the exponential family (see Section 4.4). The results in this paper will also be used in [Be] to show rigidity for non-parabolic exponential parameters with bounded postsingular set.

The structure of this article is as follows: in Section 2 we introduce some background about polynomial dynamics; we then present the new proof of Douady's theorem about accessibility of repelling periodic points, followed by the proof of Theorem B in the polynomial case. The proof in this paper only uses quite weak information about the structure of dynamic rays, opening up this result to be generalized to other families of functions beyond the exponential family. In Section 3 we recollect some facts on exponential dynamics, including existence and properties of dynamic rays in this case. In Section 4 we state and prove Theorems A and B in the exponential setting. More precisely, in Section 4.2 we make some estimates about the geometry of rays near infinity that are needed to prove Theorems A and B for exponentials; the proofs themselves are presented in Section 4.3.

We denote by $\ell_{\text{eucl}}(\gamma)$ the Euclidean length of a curve γ and by $\ell_\Omega(\gamma)$ its hyperbolic length in a region Ω admitting the hyperbolic metric with density ρ_Ω. A ball of radius r centered at a point z is denoted by either $B_r(z)$ or $B(z,r)$.

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2 Accessibility of repelling periodic orbits for polynomials with connected Julia set

In this section we give a new proof of Douady's theorem for polynomials with connected Julia set, showing that any repelling periodic orbit is the landing point of finitely many periodic rays.

2.1 Setting

Let f be a polynomial of degree D with connected Julia set $J(f)$ and filled Julia set K. As K is full and contains more than one point, $\Omega = \mathbb{C} \setminus K$ is a domain that admits a hyperbolic metric with some density $\rho_\Omega(z)$. Since $f : \mathbb{C} \setminus K \to \mathbb{C} \setminus K$ is a covering map, it locally preserves the hyperbolic metric.

The Böttcher function B conjugates the dynamics of f on $\mathbb{C} \setminus K$ to the dynamics of z^D on $\mathbb{C} \setminus \overline{\mathbb{D}}$. The preimage under B of the straight ray of angle $\mathbf{s}$ is called the *dynamic ray* of angle $\mathbf{s}$ and is denoted by $g_{\mathbf{s}}$. We parametrize $g_{\mathbf{s}}$ so that for a point with polar coordinates $(\mathbf{s}, e^t)$,

$g_{\mathbf{s}}(t) := B^{-1}(\mathbf{s}, e^t)$. The parameter t is called *potential*. Define the radial growth function as $F : t \mapsto Dt$. Then

$$f(g_{\mathbf{s}}(t)) = g_{\sigma\mathbf{s}}(F(t)), \tag{2.1}$$

where $\mathbf{s}$ is written in D-adic expansion and σ is the one-sided shift map. We put on the sequences $\mathbf{s} = s_0 s_1 s_2 \dots$ and $\mathbf{s}' = s_0' s_1' s_2' \dots$ over D symbols the metric

$$|\mathbf{s} - \mathbf{s}'|_D = \sum_i \frac{|s_i - s_i'|}{D^i}. \tag{2.2}$$

The shift map σ is locally expansive by a factor D with respect to this metric.

A *fundamental domain* starting at t for a ray $g_{\mathbf{s}}$ is the arc $g_{\mathbf{s}}([t, F(t)))$, and is denoted by $I_t(g_{\mathbf{s}})$.

The following lemma relates convergence of angles to convergence of dynamic rays, and follows directly from the uniform continuity of the Böttcher function on compact sets.

Lemma 2.1. *Let f be a polynomial of degree D. For each $t_*, t^* > 0$, the rays $g_{\mathbf{s}_n}(t)$ converge uniformly to the ray $g_{\mathbf{s}}(t)$ on $[t_*, t^*]$ as $\mathbf{s}_n \to \mathbf{s}$.*

The next lemma gives a sufficient condition to determine when the limit dynamic ray lands. Observe that the proof holds in both the polynomial and in the exponential cases.

Lemma 2.2. *Let $x_0 \in \mathbb{C}$, $t_0 > 0$, $t_m := F^{-m}(t_0)$. Also let $g_{\mathbf{s}_n}$ be a sequence of dynamic rays such that $\mathbf{s}_n \to \mathbf{s}$, and such that $I_{t_m}(g_{\mathbf{s}_n}) \subset B(x_0, \frac{A}{\nu^m})$ for some $A > 0, \nu > 1$ and for all $n > N_m$. Then $g_{\mathbf{s}}$ lands at x_0.*

Proof. To show that $g_{\mathbf{s}}$ lands at x_0 it is enough to show that for each m,

$$I_{t_m}(g_{\mathbf{s}}) \subset B\left(x_0, \frac{A}{\nu^m}\right).$$

For any fixed $m > 0$, $g_{\mathbf{s}_n} \to g_{\mathbf{s}}$ uniformly on $[t_m, t_{m-1}]$ by Lemma 2.1. As $I_{t_m}(g_{\mathbf{s}_n})$ is eventually contained in $B\left(x_0, \frac{A}{\nu^m}\right)$, taking the limit for $n \to \infty$ gives the claim. □

The following lemma is a consequence of the fact that for points tending to the boundary of a hyperbolic domain, the hyperbolic density tends to infinity.

Lemma 2.3. *Let $\Omega \subset \mathbb{C}$ be a hyperbolic region. Let $\gamma_n : [0,1] \to \Omega$ be a family of curves with uniformly bounded hyperbolic length and such that $\gamma_n(0) \to \partial\Omega$. Then $\ell_{eucl}(\gamma_n) \to 0$.*

2.2 Proof of Theorem A in the polynomial case

In this section we give a proof of Theorem A in the polynomial case. Up to taking an iterate of f, we can assume that the repelling periodic point in question is a repelling fixed point α. Let $\mu > 1$ be the modulus of its multiplier, and let L be a linearizing neighborhood for α.

For the branch ψ of f^{-1} fixing α, it is easy to show using linearizing coordinates that there exists a $C > 0$ such that

$$\frac{1}{C\mu^n} < |(\psi^n)'(x)| < \frac{C}{\mu^n}. \tag{2.3}$$

Before proceeding to the proof of Theorem A let us observe the following:

Proposition 2.4. *The Euclidean length $\ell_{eucl}(I_t(g_{\mathbf{s}}))$ tends to 0 uniformly in $\mathbf{s}$ as $t \to 0$.*

Proof. By definition of fundamental domains and uniform continuity of the inverse of the Böttcher map on compact sets, the Euclidean length $\ell_{\text{eucl}}(I_t(g_{\mathbf{s}}))$, and hence the hyperbolic length $\ell_\Omega(I_t(g_{\mathbf{s}}))$, are uniformly bounded for t in any compact interval $[t_-, t_+]$. So, by the Schwarz Lemma, for any t_* the arcs in the family $\{I_t(g_{\mathbf{s}})\}_{t<t_*}$ have uniformly bounded hyperbolic length. Since the inverse of the Böttcher map is proper, $\text{dist}(g_{\mathbf{s}}(t), J(f)) \to 0$ uniformly in $\mathbf{s}$ as $t \to 0$, hence by Lemma 2.3 $\ell_{\text{eucl}}\{I_t(g_{\mathbf{s}})\} \to 0$ as $t \to 0$. □

Theorem 2.5. *Let f be a polynomial with connected Julia set, and let α be a repelling fixed point for f. Then there is at least one dynamic ray $g_{\mathbf{s}}$ landing at α.*

Proof. Let $U' \subset L$ be a neighborhood of α, and let U be its preimage under the inverse branch ψ of f which fixes α. Let $\varepsilon = \inf_{x \in \partial U, x' \in \partial U'} |x - x'|$. By Proposition 2.4, there exists t_ε such that

$$\ell_{\text{eucl}}(I_t(g_{\mathbf{s}})) < \varepsilon \text{ for all } \mathbf{s} \in \mathbb{S}^1, t < t_\varepsilon. \tag{2.4}$$

As α is in the Julia set, it is approximated by escaping points with arbitrary small potential t, hence there exists a dynamic ray $g_{\mathbf{s}_0}$ such that $g_{\mathbf{s}_0}(t_0)$ belongs to U for some $t_0 < t_\varepsilon$. By (2.4), $\ell_{\text{eucl}}(I_{t_0}(g_{\mathbf{s}_0})) \leq \varepsilon$, hence $I_{t_0}(g_{\mathbf{s}_0}) \subset U'$ (See Figure 1).

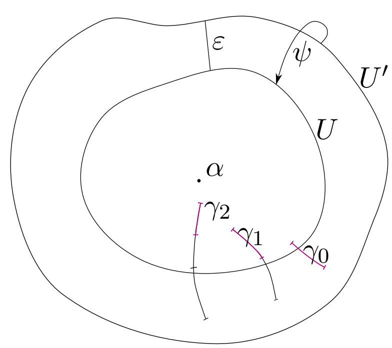

Figure 1: Construction of the curves γ_n in the proof of Theorem 2.5.

For any $n > 0$ let ψ^n be the branch of f^{-n} fixing α, and let $g_{\mathbf{s}_n}$ be the ray containing $\psi^n(I_{t_0}(g_{\mathbf{s}_0}))$. Let us define inductively a sequence of curves $\gamma_n \subset g_{\mathbf{s}_n}$ as

$$\gamma_0 := I_{t_0}(g_{\mathbf{s}_0})$$
$$\gamma_n := \psi(\gamma_{n-1}) \cup I_{t_0}(g_{\mathbf{s}_n}).$$

We show inductively that the curves γ_n are well defined and satisfy the following properties:

1. $\gamma_n = g_{\mathbf{s}_n}(t_n, F(t_0))$, where $t_n := F^{-n}(t_0)$;
2. $\gamma_n \subset U'$ for all n;
3. $I_{t_m}(g_{\mathbf{s}_n}) \subset B\left(\alpha, \frac{C \operatorname{diam} U'}{\mu^m}\right)$, for all $m \leq n$.

All properties are true for γ_0, so let us suppose that they hold for γ_{n-1} and show that they also hold for γ_n. We have that

$$\psi(\gamma_{n-1}) = \psi(g_{\mathbf{s}_{n-1}}(t_{n-1}, F(t_0))) = g_{\mathbf{s}_n}(t_n, t_0)$$

by the functional equation (2.1) and by the definition of $g_{\mathbf{s}_n}$. Also, $\psi(\gamma_{n-1}) \subset U$ because by the inductive assumption $\gamma_{n-1} \subset U'$ and $\psi(U') = U$. As $\ell_{\text{eucl}}(I_{t_0}(g_{\mathbf{s}_n})) \leq \varepsilon$, and $g_{\mathbf{s}_n}(t_n, t_0) \subset U$, we have that $\gamma_n \subset U'$.

If $x \in I_{t_m}(g_{\mathbf{s}_n})$ for $m \leq n$, then $x = \psi^m y$ for some $y \in I_{t_0}(g_{\mathbf{s}_{n-m}}) \subset U'$, hence by (2.3) we have

$$|x - \alpha| \leq \frac{C|y - \alpha|}{\mu^m} \leq \frac{C \operatorname{diam} U'}{\mu^m},$$

proving Property 3.

As the sequence $\{\mathbf{s}_n\}$ of angles of the rays $g_{\mathbf{s}_n}$ is contained in $\mathbb{S}^1$, there is a subsequence converging to some angle $\mathbf{s}$. As the Julia set is connected, no singular value is escaping, hence the ray $g_{\mathbf{s}}$ of angle $\mathbf{s}$ is well defined for all potentials $t > 0$. Landing of $g_{\mathbf{s}}$ at α follows from Property 3 together with Lemma 2.2. $\square$

To prove periodicity of the landing ray constructed in Theorem 2.5 we use the following lemma about rotation sets.

Lemma 2.6 (Rotation sets)**.** *Let $\mathcal{A} \subset \mathbb{S}^1$ be closed and forward invariant under the shift map $\sigma : \theta \mapsto D\theta$. If $\sigma|_{\mathcal{A}}$ is homeomorphism, then $\mathcal{A}$ is finite.*

Proof. Note that $\sigma^{-1}|_{\mathcal{A}}$ is a locally contracting homeomorphism and that such homeomorphisms do not exist on infinite compact spaces. $\square$

Proposition 2.7. *Any dynamic ray $g_{\mathbf{s}}$ obtained from the construction of Theorem 2.5 is periodic.*

Proof. Let $\mathcal{B} := \{\mathbf{s}_n\}$ be the set of addresses of the rays $\psi^n(g_{\mathbf{s}_0})$ constructed in the proof of Theorem 2.5, and $\mathcal{A}$ be their limit set defined as

$$\mathcal{A} := \{\mathbf{s} \in \mathbb{S}^1 : \mathbf{s}_{n_k} \to \mathbf{s} \text{ for } \mathbf{s}_{n_k} \in \mathcal{B}, n_k \to \infty\}.$$

The set $\mathcal{A}$ is closed and forward invariant by definition. Also $\sigma|_{\mathcal{A}}$ is injective because the local dynamics near α is injective, and by Theorem 2.5 any of the limiting rays $g_{\mathbf{s}}$ lands at α. Surjectivity follows from the fact that if $\mathbf{s} \in \mathcal{A}$, there is a sequence $\mathbf{s}_{n_k} \to \mathbf{s}$, hence for any limit point $\tilde{\mathbf{s}}$ of the sequence $\mathbf{s}_{n_k+1}$, $\sigma\tilde{\mathbf{s}} = \mathbf{s}$. The claim then follows from Lemma 2.6. $\square$

The next lemma can be found in [Mi], Lemma 18.12; the proof holds also in the exponential case thanks to Remark 3.3.

Lemma 2.8. *If a periodic ray lands at a repelling periodic point z_0, then only finitely many rays land at z_0, and these rays are all periodic of the same period.*

Corollary 2.9. *All the rays landing at a repelling fixed point are periodic.*

This concludes the proof of Theorem A in the polynomial case.

2.3 Proof of Theorem B in the polynomial case

In this subsection we prove Theorem B in the polynomial case and we show that under an additional combinatorial condition, every point in a hyperbolic set is the landing point of only finitely many dynamic rays.

Proof of Theorem B. Up to taking an iterate of f, we can assume that there is a δ-neighborhood U of Λ such that $|f'(x)| > \eta > 1$ for all $x \in U$.

Fix some x_0 in Λ, and let us construct a dynamic ray landing at x_0. Let $x_n := f^n(x_0)$, $B'_n := B_\delta(x_n)$, $B_n := B_{\delta/\eta}(x_n)$. Observe that for each n there is a branch ψ of f^{-1} such that $\psi(B'_n) \subset B_{n-1}$. We refer to ψ^m as the composition of such branches, mapping x_n to x_{n-m}. Let $\varepsilon := \delta - \delta/\eta$, and let t_ε be such that the length of fundamental domains starting at $t < t_\varepsilon$ is smaller than ε (see Proposition 2.4). Let us define a family of rays to which we will apply the construction of Theorem 2.5.

Let $t_0 < t_\varepsilon$ be such that each B_n contains a point of potential t_0. For each n, let $\mathcal{A}_{x_n}$ be the family of angles $\mathbf{s}$ such that $g_\mathbf{s}(t_0) \in B_n$. Observe that by Proposition 2.4, and because $\operatorname{dist}(\partial B_n, \partial B'_n) > \varepsilon$, $I_{t_0}(g_\mathbf{s}) \subset B'_n$ for any $\mathbf{s} \in \mathcal{A}_{x_n}$. For each $\mathbf{s} \in \mathcal{A}_{x_n}$, denote by $\psi^m_* g_\mathbf{s}$ the ray to which $\psi^m(g_\mathbf{s})(t_0)$ belongs to (see Figure 2). Let $t_n := F^{-n}(t_0)$; following the construction of Theorem 2.5, we obtain that

$$(\psi^m_* g_\mathbf{s})(t_m, F(t_0)) \subset B'_{n-m} \text{ for any } \mathbf{s} \in \mathcal{A}_{x_n}.$$

Also,

$$I_{t_m}(\psi^n_* g_\mathbf{s}) \subset B\left(x_0, \frac{\delta}{\eta^m}\right) \text{ for } m \leq n,\ \mathbf{s} \in \mathcal{A}_{x_n}. \tag{2.5}$$

By (2.5) and Lemma 2.2, any sequence of rays $g_{\mathbf{s}_n}$ such that $\sigma^n \mathbf{s}_n \in \mathcal{A}_{x_n}$ has a subsequence that converges to a ray landing at x_0. □

If we assume f to be a unicritical polynomial satisfying some combinatorial conditions, we can show that there are only finitely many rays landing at each $x \in \Lambda$. We say that two points z_1, z_2 are *combinatorially separated* if there is a curve Γ formed by two dynamic rays landing together such that z_1, z_2 belong to different components of $\mathbb{C} \setminus \Gamma$.

Remark 2.10. By the cyclic order at infinity, two dynamic rays of angles $\mathbf{s}, \mathbf{s}'$ can land together only if $|\mathbf{s} - \mathbf{s}'|_D \leq 1/D$; otherwise, the two dynamic rays of angles $\mathbf{s} + 1/D, \mathbf{s}' + 1/D$ (obtained from the dynamic rays of angles $\mathbf{s}, \mathbf{s}'$ through a rotation of angle $2\pi/D$) would intersect them, giving a contradiction.

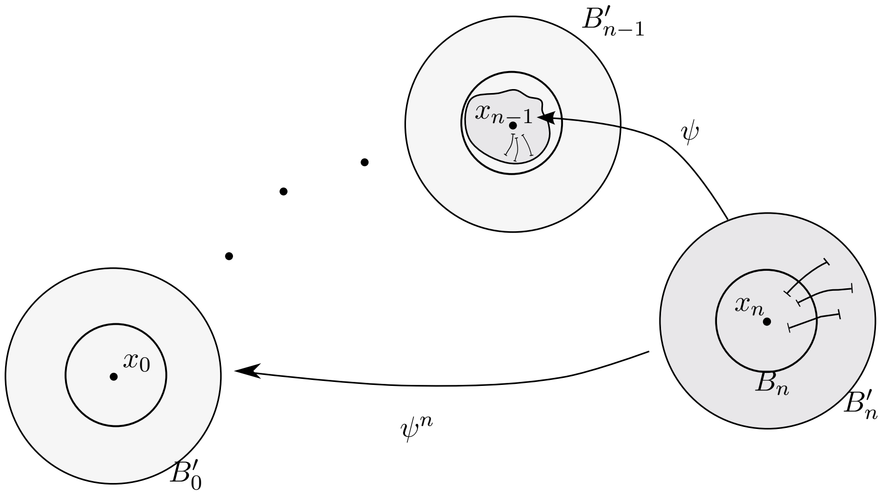

Figure 2: Construction of landing rays for the hyperbolic set Λ.

Proposition 2.11. *Let $f(z) = z^D + c$ be a unicritical polynomial of degree D, and let Λ be a hyperbolic set. Suppose moreover that either 0 is accessible or that any $x \in \Lambda$ is combinatorially separated from 0. Then there are only finitely many dynamic rays landing at each $x \in \Lambda$.*

Proof. For $x \in \Lambda$, let $\mathcal{A}_x$ be the set of addresses of the rays landing at x. By Theorem B, each $\mathcal{A}_x$ is non empty. Near x, f is locally a homeomorphism, so the set $\mathcal{A}_x$ is mapped bijectively to the set $\mathcal{A}_{f(x)}$ by the shift map σ and there is a well defined inverse $\sigma^{-1} : \mathcal{A}_{f(x)} \to \mathcal{A}_x$. Because σ is locally expansive by the factor D, uniform continuity of σ^{-1} would give local contraction for σ^{-1} by the factor $1/D$. Suppose that σ^{-1} is not uniformly continuous on $\{\mathcal{A}_x\}_{x\in\Lambda}$. Then there is a sequence of points $x_n \in \Lambda$, and two sequences of angles $\mathbf{a}_n, \mathbf{a}_n{}' \in \mathcal{A}_{f(x_n)}$ such that $|\mathbf{a}_n - \mathbf{a}_n{}'|_D \to 0$, but $|\sigma^{-1}\mathbf{a}_n - \sigma^{-1}\mathbf{a}_n{}'|_D \to k/D$ with some integer $k \in [1, D-1]$. In fact, by Remark 2.10, $k = 1$. Call $\mathbf{s}_n, \mathbf{s}_n{}'$ the angles $\sigma^{-1}\mathbf{a}_n, \sigma^{-1}\mathbf{a}_n{}'$, and assume for definiteness that $\mathbf{s}_n < \mathbf{s}_n{}'$. By the D-fold symmetry of the Julia set, the rays of angles $\mathbf{s}_n + j/D, \mathbf{s}_n{}' + j/D$ for $j = 1 \dots D-1$ also land together at the points $e^{2\pi i/D}x_n$ (see Figure 3).

Altogether, these D pairs of dynamic rays divide $\mathbb{C}$ into $D+1$ connected component. Let V_n be the one that contains the critical point 0. Since for each i we have:

$$(\mathbf{s}_n{}' + j/D) - (\mathbf{s}_n + j/D) \to 1/D \quad \text{as } n \to \infty,$$

we conclude that

$$(\mathbf{s}_n + (j+1)/D) - (\mathbf{s}_n{}' + j/D) \to 0 \quad \text{as } n \to \infty. \tag{2.6}$$

Let $Q = \bigcap \overline{V_n}$. Then $0 \in Q$ by definition, and $Q \cap \Lambda \neq \emptyset$ because Λ is compact and $\overline{V_n} \cap \Lambda \neq \emptyset$ for all n. Let $z \in Q \cap \Lambda$. By (2.6), there are exactly D limiting rays entering Q. By symmetry, if these rays land, they either all land at 0, or they land at D different points belonging to some orbit of the $2\pi/D$-rotation. This gives an immediate contradiction with either of our assumptions:

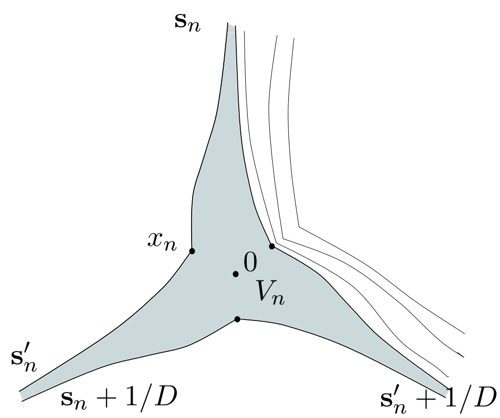

Figure 3: Illustration to the proof of Proposition 2.11 for $D = 3$; the region V_n is shaded in gray.

• Since there are no pairs of rays which could separate 0 from points of Q, the point $z \in \Lambda$ is not combinatorially separated from 0, giving a contradiction with the first assumption;

• In the case when 0 is accessible, this implies that no other accessible point can belong to Q; in particular $z \in \Lambda$ is not accessible, contradicting Theorem B.

So, there exists an $\varepsilon > 0$ such that if $\mathbf{a}, \mathbf{a}' \in \mathcal{A}_{f(x)}$ and $|\mathbf{a} - \mathbf{a}'|_D < \varepsilon$ then

$$|\sigma^{-1}\mathbf{a} - \sigma^{-1}\mathbf{a}'|_D < \frac{1}{D}\,|\mathbf{a} - \mathbf{a}'|_D. \tag{2.7}$$

Now fix a point $x_0 \in \Lambda$, and consider a finite cover of $\mathbb{S}^1$ by ε-balls, say N balls. In particular, for any n, $\mathcal{A}_{x_n}$ is covered by N balls of radius ε. By (2.7), their n-th preimages have diameter ε/D^n and cover $\mathcal{A}_x$. Passing to the limit, we conclude that $\mathcal{A}_x$ contains at most N points. □

3 Dynamic rays in the exponential family

For the remaining two sections, $f(z) = e^z + c$ will be an exponential function, and $J(f)$ will be its Julia set. Any two exponential maps whose singular values differ by $2\pi i$ are conformally conjugate, so we can assume $-\pi \le \operatorname{Im} c < \pi$.

Let $\arg(z)$ be defined on $\mathbb{C} \setminus \mathbb{R}_-$ so as to take values in $(-\pi, \pi)$, and let

$$R := \{z \in \mathbb{C} : \ \operatorname{Im} z = \operatorname{Im} c, \ \operatorname{Re} z \le \operatorname{Re} c\}.$$

.

We define a family $\{L_n\}$ of inverse branches for $f(z) = e^z + c$ on $\mathbb{C} \setminus R$ as

$$L_n(w) := \log|w - c| + i \arg(w - c) + 2\pi i n.$$

Observe that L_n maps $\mathbb{C} \setminus R$ biholomorphically to the strip

$$S_n := \{z \in \mathbb{C} : 2\pi n - \pi < \operatorname{Im} z < 2\pi n + \pi\}.$$

Observe also that

$$|(L_n)'(w)| = \frac{1}{|w-c|}.$$

Dynamic rays have been first introduced by Devaney and Krych in [DK] (with the name *hairs*) and studied for example in [BDGHHR]. A full classification of the set of escaping points in terms of dynamic rays has been then completed by Schleicher and Zimmer in [SZ1]. For points whose iterates never belong to R, we can consider itineraries with respect to the partition of the plane into the strips $\{S_n\}$, i.e.

$$\text{itin}(z) = s_0 s_1 s_2 \dots \text{ if and only if } f^j(z) \in S_{s_j}.$$

For a point z whose itinerary with respect to this partition exists, we refer to it as the *address* of z. By construction, addresses are sequences in $\mathbb{Z}^{\mathbb{N}}$.

We use the function $F : t \mapsto e^t - 1$ to model real exponential growth. According to the construction, addresses of points cannot have entries growing faster than iterates of the exponential function. A sequence $\mathbf{s} = s_0 s_1 s_2 \dots$ is called *exponentially bounded* if there exists $x \in \mathbb{R}$ such that $|2\pi s_j| < F^j(x)\ \forall j \geq 0$.

This growth condition turns out to be not only necessary but also sufficient (see [SZ1]), so that any sequence $\mathbf{s}$ contained in the set

$$\mathcal{S} := \{\mathbf{s} \in \mathbb{Z}^{\mathbb{N}} : \mathbf{s} \text{ is exponentially bounded}\}$$

is realized as address of some point z (see Theorem 3.1 below).

An address is called *periodic* if it is a periodic sequence, *preperiodic* if it is a preperiodic sequence. We consider the set $\mathcal{S}$ with the lexicographic order and the weak* topology.
If $\mathbf{s} = s_0 s_1 s_2 \dots$, let $\|\mathbf{s}\|_\infty = \sup_i |s_i|/2\pi$. We call $\mathbf{s}$ *bounded* if $\|\mathbf{s}\|_\infty < \infty$. For $j \in \mathbb{Z}$ and $\mathbf{s} \in \mathcal{S}$, $j\mathbf{s}$ stands for the sequence $j s_0 s_1 s_2 \dots$.

Given an external address $\mathbf{s}$ we define its *minimal potential*

$$t_{\mathbf{s}} := \inf\left\{ t > 0 : \limsup_{k\to\infty} \frac{|s_k|}{F^k(t)} = 0 \right\}.$$

Observe that if $\mathbf{s}$ is bounded, $t_{\mathbf{s}} = 0$.

Definition, existence and properties of *dynamic rays* for the exponential family are summarized in the following theorem ([SZ1], Proposition 3.2 and Theorem 4.2; the quantitative estimates are taken from Proposition 3.4).

Theorem 3.1 (Existence of dynamic rays)**.** *Let $f(z) = e^z + c$ be an exponential map such that c is non-escaping, and K be a constant such that $|c| \leq K$. Let $\mathbf{s} = s_0 s_1 s_2 \dots \in \mathcal{S}$ be an address satisfying the condition $|s_k| < AF^{(k)}(x)$ for any $k \geq 1$, with $A \geq 1/2\pi$, $x \geq 0$. Then there exists a unique maximal injective curve $g_{\mathbf{s}} : (t_{\mathbf{s}}, \infty) \to \mathbb{C}$ consisting of escaping points such that*

(1) $g_{\mathbf{s}}(t)$ has address $\mathbf{s}$ for $t > x + 2\log(K+3)$;

(2) $f(g_{\mathbf{s}}(t)) = g_{\sigma\mathbf{s}}(F(t))$;

(3) $|g_{\mathbf{s}}(t) - 2\pi i s_0 - t - t_{\mathbf{s}}| \leq 2e^{-t}(K + 2 + 2\pi|s_1| + 2\pi AC)$ *for large* t*,*
and for a universal constant C*.*

The curve $g_{\mathbf{s}}$ is called the *dynamic ray of address* $\mathbf{s}$. As for polynomials, a dynamic ray is periodic or preperiodic if and only if its address is a periodic or preperiodic sequence respectively. Like before, the *fundamental domain* starting at t for $g_{\mathbf{s}}$ is defined as the arc $I_t(g_{\mathbf{s}}) := g_{\mathbf{s}}(t, F(t))$.

Remark 3.2. If $\mathbf{s}$ is bounded, property (1) holds for $t > 2\log(K+3)$. Moreover, $g_{\mathbf{s}}(t)$ is approximately straight, i.e. $|g'_{\mathbf{s}}(t) - 1| < C(K, t_{\mathbf{s}})$ for large t, so $C(K, t_{\mathbf{s}})$ does not depend on $\mathbf{s}$ if $\mathbf{s}$ is bounded (see Proposition 4.6 in [FS]). It then follows from the asymptotic estimates in Theorem 3.1 that for any t there exists $B(t)$ such that

$$\ell_{\text{eucl}}(I_t(g_{\mathbf{s}})) \leq B(t, \|\mathbf{s}\|)) \sim e^t - t. \tag{3.1}$$

Remark 3.3. The addresses of two rays $g_{\mathbf{s}}, g_{\mathbf{s}'}$ landing together cannot differ by more than one in any entry. Otherwise, by the asymptotic estimates in Theorem 3.1, the curve $\Gamma = \overline{g_{\mathbf{s}} \cup g_{\mathbf{s}'}}$ would intersect its translate by $2\pi i$.

The following continuity lemma holds, and will play the role of Lemma 2.1 for the exponential family. A proof can be found in [Re3]; we will use the formulation from [Re2], Lemma 4.7).

Lemma 3.4 (Convergence of rays)**.** *Let* f *be an exponential map,* $\{\mathbf{s}_n\}$ *be a sequence of exponentially bounded addresses,* $\mathbf{s}_n \to \mathbf{s} \in \mathcal{S}$ *such that* $t_{\mathbf{s}_n} \to t_{\mathbf{s}}$*. Then* $g_{\mathbf{s}_n} \to g_{\mathbf{s}}$ *uniformly on* $[t_*, \infty)$ *for all* $t_* > t_{\mathbf{s}}$*.*

Remark 3.5. A family of addresses which is uniformly bounded by a constant M can be equivariantly embedded in $\mathbb{S}^1$ by identifying angles with their $2M+1$-adic expansion. The embedding is monotonic and the dynamics of the shift map on the addresses is conjugate to the dynamics of z^{2M+1} on $\mathbb{S}^1$. If a repelling periodic point z_0 is the landing point of a periodic dynamic ray $g_{\mathbf{s}}$ with $\|\mathbf{s}\| < M$, by Remark 3.3 the address of any other ray landing at z_0 is bounded by $M+1$. The proof of Lemma 2.8 only uses local dynamics and the fact that the set of angles of rays landing at z_0 is equivariantly embedded in $\mathbb{S}^1$, hence holds also in the exponential case.

4 Accessibility for exponential parameters with bounded postsingular set

4.1 Statement of theorems and some basic facts

From now on we will consider an exponential function f with bounded postsingular set, which implies that the singular value is non-recurrent; this excludes the presence of Siegel disks (see Corollary 2.10, [RvS]). The strategy used for the proof of Theorem A and B for polynomials can be extended to the exponential family to prove the following theorems.

Theorem 4.1 (Accessibility of periodic orbits for non-recurrent parameters)**.** *Let f be an exponential map with bounded postsingular set; then any repelling periodic point is the landing point of at least one and at most finitely many dynamic rays, all of which are periodic of the same period.*

Corollary 4.2. *For Misiurewicz parameters, the postsingular periodic orbit and hence the singular value are accessible.*

For hyperbolic, parabolic and Misiurewicz parameters Theorem 4.1 has been previously proven in [SZ2].

Theorem 4.3 (Accessibility of hyperbolic sets)**.** *Let f be an exponential map with bounded postsingular set, and Λ be a hyperbolic set. Then any point in Λ is accessible; moreover, the dynamic rays landing at $x \in \Lambda$ all have uniformly bounded addresses.*

Remark 4.4. The family of rays constructed in Theorem 4.3 form a lamination. Continuity of the family of rays on compact sets is a consequence of Lemma 3.4. Continuity up to the endpoints follows from the estimates in (4.7).

We also prove that there are only finitely many rays landing at each $x \in \Lambda$ (see Proposition 4.13).

Proposition 4.5. *Let f be an exponential map with bounded postsingular set, and Λ be a hyperbolic set. Then there are only finitely many dynamic rays landing at each $x \in \Lambda$, and this number of rays is bounded.*

In the case in which the postsingular set is bounded and contained in the Julia set, Rempe and van Strien ([RvS], Theorem 1.2) have shown that it is hyperbolic (see also [MS], Theorem 3, for a different perspective). Together with Theorem 4.3, this implies accessibility of the postsingular set.

Corollary 4.6. *Let f be an exponential map with bounded postsingular set; then any point in the postsingular set is accessible.*

Like in the polynomial case, the strategy is to first prove a uniform bound on the length of fundamental domains $I_T(g_{\mathbf{a}})$ for some fixed T and some specific family of addresses, then to translate this into a uniform shrinking for fundamental domains $I_t(g_{\mathbf{a}})$ as $t \to 0$, and finally to study the local dynamics near a repelling periodic orbit. The main difficulties compared to the polynomial case are to find an analogue of Proposition 2.4, and to show that the dynamic rays obtained by pullbacks near the repelling fixed point have uniformly bounded addresses.

In the following, let $\mathcal{P}(f)$ be the postsingular set, and $\Omega := \mathbb{C} \setminus \mathcal{P}(f)$. As $\mathcal{P}(f)$ is forward invariant, Ω is backward invariant, i.e. $f^{-1}(\Omega) \subset \Omega$.

Proposition 4.7. *If $\mathcal{P}(f)$ is bounded, Ω is connected.*

Proof. As $\mathcal{P}(f)$ is bounded, there are no Siegel disks, hence either $J(f) = \mathbb{C}$ or f is parabolic or hyperbolic. In the last two cases $\mathcal{P}(f)$ is a totally disconnected set and the claim follow. If $J(f) = \mathbb{C}$, consider the connected components V_i of Ω; as $\mathcal{P}(f)$ is bounded, there is only one unbounded V_i. On the other side, by density of escaping points, each V_i contains escaping points; as dynamic rays are connected sets, each V_i has to be unbounded, hence there is a unique connected component. □

As Ω is connected, and omits at least three points because c cannot be a fixed point, it admits a well defined hyperbolic density ρ_Ω. As $\mathcal{P}(f)$ is bounded, we have

$$\lim_{|z|\to\infty} \frac{\rho_\Omega(z)}{\rho_{\text{eucl}}(z)} = 0. \tag{4.1}$$

4.2 Bounds on fundamental domains for exponentials

In this section we prove a uniform bound on the length of fundamental domains for an appropriate family of dynamic rays (Proposition 4.11).

Proposition 4.8 (Bounded fundamental domains for exponentials)**.** *Let $g_{\widetilde{\mathbf{s}}}$ be a dynamic ray with bounded address and let C be a positive constant. Let*

$$\mathcal{A} := \{\mathbf{a} \in \mathcal{S} : \sigma^m(\mathbf{a}) = \widetilde{\mathbf{s}} \textit{ for some } m \geq 0\},$$

and let $\{g_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{A}}$ be the collection of pullbacks of $g_{\widetilde{\mathbf{s}}}$. Then there exists T such that for all $t > T, \mathbf{a} \in \mathcal{A}$:

(P1) If $\mathbf{a} = a_m ... a_2 a_1 \widetilde{\mathbf{s}}$, then $g_{\mathbf{a}}(t) = L_{a_m} \circ \cdots \circ L_{a_1} g_{\widetilde{\mathbf{s}}}(F^m(t))$;

(P2) $\operatorname{Re} g_{\mathbf{a}}(t) > C$;

(P3) $\ell_{eucl}(I_t(g_{\mathbf{a}})) \leq B(t)$, with $B(t)$ independent of $\mathbf{a}$.

Proposition 4.8 is a consequence of the following proposition.

Proposition 4.9 (Branches of the logarithm)**.** *Fix ε, $C > 0$. Let $g_{\widetilde{\mathbf{s}}}$ be a dynamic ray with bounded address. Then there exists $T > 0$ such that for any $m > 0$, for any $z = g_{\widetilde{\mathbf{s}}}(t)$ with $t > F^m(T)$ and for any finite sequence $a_m \dots a_1$ we have the following two properties:*

(1_m) $\operatorname{Re} L_{a_m} \circ \cdots \circ L_{a_1}(z) - \operatorname{Re} c \geq \operatorname{Re} L_0^m(z) - \operatorname{Re} c - \varepsilon/2 > C$

(2_m) $|L_{a_m} \circ \cdots \circ L_{a_1}(z) - c| \geq |L_0^m(z) - c| - \varepsilon.$

Moreover,

$$|(L_{a_m} \circ \cdots \circ L_{a_1})'(z)| \leq |(L_0^m)'(z)| + 2\varepsilon. \tag{4.2}$$

Proof. Denote by 0^m the finite sequence formed by m zeroes, and let

$$\mathcal{A}' = \{\mathbf{a} \in \mathcal{A} : \mathbf{a} = 0^m \widetilde{\mathbf{s}},\, m \in \mathbb{N}\}.$$

As $\|\widetilde{\mathbf{s}}\|_\infty < M$ for some M, we have $\|\mathbf{a}\|_\infty < M$ for any $\mathbf{a} \in \mathcal{A}'$. It follows from (3) in Theorem 3.1 that for any ε there exists T_ε such that

$$|g_{\mathbf{a}}(t) - t| < \varepsilon \tag{4.3}$$

for any $\mathbf{a} \in \mathcal{A}'$, $t > T_\varepsilon$. By Remark 3.2, we have that

$$g_{0^m\widetilde{\mathbf{s}}}(t) = L_0^m g_{\widetilde{\mathbf{s}}}(F^m(t)) \quad \text{for } t \geq 2\log(K+3) \tag{4.4}$$

and that

$$\ell_{\text{eucl}}(I_t(g_{\mathbf{a}})) < B(t) \sim e^t - t \ \forall a \in \mathcal{A}'. \tag{4.5}$$

We first show that (1_m) implies (2_m), and then that (2_m) implies (1_{m+1}).

Let T be large enough, ε small. By (4.3) for $t > T$,

$$\operatorname{Re} L_0^m(g_{\widetilde{\mathbf{s}}}(F^m(t))) > T - \varepsilon > C + \operatorname{Re} c \ \text{ for all } m.$$

For $m = 1$, (1_1) holds because all preimages of a point are $2\pi i$ translate of each other, and by (4.3).

Now let us show that (1_m) implies (2_m).

$$\begin{aligned}
|L_0^m(z) - c| &= \sqrt{|\operatorname{Re} L_0^m(z) - \operatorname{Re} c|^2 + |\operatorname{Im} L_0^m(z) - \operatorname{Im} c|^2} \leq \\
&\leq (\operatorname{Re} L_0^m(z) - \operatorname{Re} c)\sqrt{1 + \frac{\varepsilon}{|\operatorname{Re} L_0^m(z) - \operatorname{Re} c|}} \leq \\
&\leq \operatorname{Re} L_0^m(z) - \operatorname{Re} c + \frac{\varepsilon}{2} \leq \operatorname{Re} L_{a_m} \circ \cdots \circ L_{a_1}(z) - \operatorname{Re} c + \varepsilon \leq \\
&\leq |L_{a_m} \circ \cdots \circ L_{a_1}(z) - c| + \varepsilon.
\end{aligned}$$

To show that (2_m) implies (1_{m+1}) for any m, observe that

$$\begin{aligned}
&\operatorname{Re} L_{a_{m+1}} \circ \cdots \circ L_{a_1}(z) = \log|L_{a_m} \circ \cdots \circ L_{a_1}(z) - c| \geq \\
&\geq \log(|L_0^m(z) - c| - \varepsilon) \geq \log|L_0^m(z) - c| - \frac{2\varepsilon}{|L_0^m(z) - c|} \geq \\
&\geq \log|L_0^m(z) - c| - \varepsilon/2 = \operatorname{Re} L_0^{m+1}(z) - \varepsilon/2 > C + \operatorname{Re} c.
\end{aligned}$$

Equation (4.2) can be checked by direct computation using Property (2_m). □

Proof of Proposition 4.8. All addresses in $\mathcal{A}$ are bounded, so by Remark 3.2, $g_{\mathbf{a}}(t)$ has address $\mathbf{a}$ for $t > 2\log(K+3)$, proving Property $(P1)$. Take T such that Proposition 4.9 holds. From Property (1_m), $\operatorname{Re} g_{\mathbf{a}}(t) \geq \operatorname{Re} L_0^m(g_{\mathbf{s}}(F^m(t))) - \varepsilon/2 > C + \operatorname{Re} c$ proving Property $(P2)$.

Finally, $\ell_{\text{eucl}}(I_t(g_{\mathbf{a}})) \leq B(t)$ for all $\mathbf{a} \in \mathcal{A}'$ by (4.5), so $\ell_{\text{eucl}}(I_t(g_{\mathbf{a}})) \leq B'(t)$ for all $\mathbf{a} \in \mathcal{A}$ and for some B' by (4.2); property $(P3)$ follows. □

Let us now show that the length of fundamental domains $I_t(g_{\mathbf{a}})$ shrinks as $t \to 0$ for this family of pullbacks. The next proposition follows from the classical shrinking lemma, see e.g. [Ly], Proposition 3.

Proposition 4.10 (Shrinking under inverse iterates). *Let f be an exponential map, and let $U \subset J(f)$ be a simply connected domain not intersecting the postsingular set. Also, let $L \subset U$, and $K \subset \mathbb{C}$ be compact sets, and let $\{f_\lambda^{-m}\}$ be the family of branches of f^{-m} such that $f_\lambda^{-m}(L) \cap K \neq \emptyset$. Then*

$$\operatorname{diam}(f_\lambda^{-m}(L)) \to 0 \text{ as } m \to \infty$$

uniformly in λ.

Proof. As U is simply connected, and does not intersect the postsingular set, inverse branches are well defined on U. Suppose by contradiction that there is $\varepsilon > 0$, $m_k \to \infty$, and branches $f_{\lambda_k}^{-m_k}$ of f^{-m_k} such that

(1) $\operatorname{diam}_{\text{eucl}}(f_{\lambda_k}^{-m_k}(L)) > \varepsilon$ for any m_k, λ_k;

(2) $f_{\lambda_k}^{-m_k}(L) \cap K \neq \emptyset$ for any m_k, λ_k.

By normality of inverse branches, there is a subsequence converging to a univalent function ϕ, which is non-constant by (1). By (2) there is a sequence of points $\{x_k\} \in L$ such that $f_{\lambda_k}^{-m_k}(x_k) \in K$, and by compactness of K, the $f_{\lambda_k}^{-m_k}(x_k)$ accumulate on some point $y \in K$. As ϕ is not constant, there is a neighborhood V of y and infinitely many m_k such that $f^{m_k}(V) \subset L$, contradicting the fact that $y \in J(f)$. □

Proposition 4.11 (Fundamental domains shrinking for exponentials). *Let $g_{\underline{\mathbf{s}}}$ be a dynamic ray with bounded address, $\{g_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{A}}$ be its family of pullbacks as defined in Proposition 4.8. Given an $\varepsilon > 0$ and a compact set K there exists $t_\varepsilon = t_\varepsilon(K)$ such that $\ell_{eucl}(I_t(g_{\mathbf{a}})) < \varepsilon$ whenever $t < t_\varepsilon$ and $I_t(g_{\mathbf{a}}) \cap K \neq \emptyset$.*

Proof. Let C be such that the postsingular set does not intersect the right half plane

$$\mathbb{H}_C := \{z \in \mathbb{C} : \operatorname{Re} z > C\}.$$

Let T be as in Proposition 4.8 so that $I_T(g_{\mathbf{a}}) \subset \mathbb{H}_C$ for any $\mathbf{a} \in \mathcal{A}$. Inverse branches of f^n are well defined on the family $\{I_T(g_{\mathbf{a}})\}$ because it is contained in a right half plane not intersecting the postsingular set. The arcs $\{g_{\mathbf{a}}(T, F^(T))\}$ have uniformly bounded Euclidean length by $(P3)$ in Proposition 4.8, so they have uniformly bounded hyperbolic length because they are contained in $\mathbb{H}_C$ and $\mathbb{H}_C \cap \mathcal{P}(f) = \emptyset$. Then by the Schwarz Lemma the hyperbolic length of the arcs in the families $\{I_t(g_{\mathbf{a}})\}_{t<T}$ is bounded uniformly as well; so there exists a compact set K' such that $I_t(g_{\mathbf{a}}) \subset K'$ whenever $I_t(g_{\mathbf{a}}) \cap K \neq \emptyset$. By compactness of K' there exists ε' such that $\ell_{\text{eucl}}(\gamma) < \varepsilon$ for any curve $\gamma \subset (K' \cap \Omega)$ with $\ell_\Omega(\gamma) < \varepsilon'$.

Let D be a closed disk of sufficiently large radius such that $\ell_\Omega(I_T(g_{\mathbf{a}})) < \varepsilon'$ for any fundamental domain with $I_T(g_{\mathbf{a}}) \not\subset D$. Such a disk exists because $\ell_{\text{eucl}}(I_T(g_{\mathbf{a}}))$ is bounded for every $\mathbf{a} \in \mathcal{A}$ by Proposition 4.8, and by the asymptotic estimates in (4.1). For any $\mathbf{a} \in \mathcal{A}$ such that $I_T(g_{\mathbf{a}}) \subset (D \cap \mathbb{H}_C)$, by Proposition 4.10 there is n_ε such that $\operatorname{diam}_{\text{eucl}} f^{-n}(L) < \varepsilon$ for any $n > n_\varepsilon$, and the claim holds for any $t < t_\varepsilon = F^{-n_\varepsilon}(T)$. For any $\mathbf{a} \in \mathcal{A}$ such that $I_T(g_{\mathbf{a}})$ is not contained in D, $\ell_\Omega(f^{-n}(I_T(g_{\mathbf{a}}))) < \varepsilon'$ for any $n \geq 0$ by the Schwarz Lemma, hence for any such $\mathbf{a}$, $\ell_{\text{eucl}}(f^{-n}(I_T(g_{\mathbf{a}})))) < \varepsilon$ whenever $f^{-n}(I_T(g_{\mathbf{a}}))) \cap K \neq \emptyset$. □

4.3 Proof of Accessibility Theorems

In this Section we prove Theorems 4.1 and 4.3. We start by proving Theorem 4.1.

Proof of Theorem 4.1. Let us first assume that the repelling periodic point under consideration is a fixed point α. Let U, U', ψ and ε be as in the proof of Theorem 2.5. Let $g_{\tilde{\mathbf{s}}}$ be a dynamic ray with bounded address, $\{g_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{A}}$ be its family of pullbacks and T be such that Proposition 4.8 holds for C such that $\mathbb{H}_C \cap \mathcal{P}(f) = \emptyset$. By Proposition 4.11, for any $t < t_\varepsilon$, and for any $I_t(g_{\mathbf{a}})$ intersecting $K := \overline{U'}$, we have that $\ell(I_t(g_{\mathbf{a}})) < \varepsilon$.

Consider a dynamic ray $g_{\mathbf{s}_0}$ such that $g_{\mathbf{s}_0}(t_0) \in U$ for some $t_0 < t_\varepsilon$. To show that such a point exists, observe that for sufficiently large N_0, $f^{N_0}(U) \supset I_{t_\varepsilon}(g_{\tilde{\mathbf{s}}}) \neq \emptyset$, so there exists $y \in I_{t_\varepsilon}(g_{\tilde{\mathbf{s}}})$ such that $f^{-N_0}(y) \in U$ giving the desired $g_{\mathbf{s}_0}(t_0)$.

Now we can use the inductive construction from Theorem 2.5 to obtain a sequence of dynamic rays $g_{\mathbf{s}_n} = \psi_*^n(g_{\mathbf{s}_0})$ such that the arcs $\gamma_n := g_{\mathbf{s}_n}(t_n, F(t_0))$ are well defined and satisfy properties *1-3* in the proof of Theorem 2.5. In view of Lemma 2.2 it is only left to show that the addresses $\mathbf{s}_n$ obtained from the construction are uniformly bounded and hence have a convergent subsequence. By construction, $g_{\mathbf{s}_n}(t_0) \in U$ for each n. Let N be such that $F^N(t_0) > T$. The set $f^N(U)$ is bounded, so there is some constant M such that $|\operatorname{Im} z| < 2\pi M$ for any $z \in f^N(U)$. Let $n > N$; because $g_{\mathbf{s}_n}(t_0) \in U$, $g_{\mathbf{s}_{n-N}}(F^N(t_0)) \in f^N(U)$, and the arc $g_{\mathbf{s}_{n-N}}(F^N(t_0), \infty)$ is contained in a finite number of the strips S_n from the partition in Section 3. So, by claim *(P1)* of Proposition 4.8, the first entry of the address $\sigma^N \mathbf{s}_n$ is bounded by M for all $n > N$, and $\|\mathbf{s}_n\|_\infty$ is uniformly bounded.

In the case of a repelling periodic point of period $p > 1$, the construction can be repeated using as fundamental domains the arcs between potential t_0 and $F^p(t_0)$.

The proof of periodicity of the landing rays is the same as the proof for polynomials, using the fact that the family of addresses $\{\mathbf{s}_n\}$ is uniformly bounded hence Lemma 2.8 holds by Remark 3.5. □

In order to prove Theorem 4.3, we need the proposition below.

Proposition 4.12. *Let K be a compact set, $g_{\mathbf{s}_0}$ be a dynamic ray with bounded address, $\mathcal{A}$ as in Proposition 4.8, and $\mathcal{A}' \subset \mathcal{A}$ such that for any $\mathbf{a} \in \mathcal{A}'$:*

(1) $g_{\mathbf{a}}(t_0) \in K$ for some t_0, $\forall \mathbf{a} \in \mathcal{A}'$;

(2) If $\mathbf{a} \neq \mathbf{s}_0$, $\mathbf{a} \in \mathcal{A}'$, then $\sigma\mathbf{a} \in \mathcal{A}'$.

Then there exists $M > 0$ such that for all $\mathbf{a} \in \mathcal{A}'$, $\|\mathbf{a}\|_\infty < M$.

Proof. Let T be as in Proposition 4.8, and let N be such that $F^N(t) > T$. The set $f^N(K)$ is compact, so there exists $M' > \|\mathbf{s}_0\|_\infty$ such that $|\operatorname{Im} z| < 2\pi M'$ for all $z \in f^N(K)$. For any $\mathbf{a} \in \mathcal{A}'$, $g_{\mathbf{a}}(t_0) \in K$, so $g_{\sigma^N \mathbf{a}}(F^N(t_0)) \in f^N(K)$ and by Property *(P1)* in Proposition 4.8, the first entry of $\sigma^N \mathbf{a}$ is bounded by M'. From (2), we get that $\|\sigma^N \mathbf{a}\|_\infty < M'$ for any $\mathbf{a} \in \mathcal{A}$.

So any sequence $\{\sigma^N \mathbf{a}_n\}$ with $\mathbf{a}_n \in \mathcal{A}'$ admits a convergent subsequence, and by Lemma 3.4, the corresponding rays converge uniformly on compact sets. Using (3.1) it follows that $\ell_{\text{eucl}}(g_{\sigma^N \mathbf{a}}(F^N(t_0), F^{2N}(t_0))) \leq B$ for some $B > 0$, and for all $\mathbf{a} \in \mathcal{A}'$. Note that $|(f^N)'|$ is bounded on K, so there exists $B' > 0$ such that $\ell_{\text{eucl}}(g_{\mathbf{a}}(t_0, F^N(t_0))) \leq B'$ for all $\mathbf{a} \in \mathcal{A}'$, and the points $g_{\mathbf{a}}(F^N(t_0))$ are contained in finitely many of the strips S_n. By Property *(P1)* in Proposition 4.8, the first entry of $\mathbf{a}$ is bounded by some M for all $\mathbf{a} \in \mathcal{A}$, and the claim follows by (2). □

Proof of Theorem 4.3. Up to taking an iterate of f, we can assume that there is a δ-neighborhood U of Λ such that $|f'(x)| > \eta > 1$ for all $x \in U$. Let $g_{\tilde{\mathbf{s}}}$ be a dynamic ray with bounded address, $\{g_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{A}}$ its the family of pullbacks. For $\varepsilon := \delta - \delta/\eta$, let t_ε be such that $\ell_{\text{eucl}}(I_t(g_{\mathbf{a}})) < \varepsilon$ for $\mathbf{a} \in \mathcal{A}$ and $t < t_\varepsilon$ whenever $I_t(g_{\mathbf{a}}) \cap \overline{U} \neq \emptyset$ (see Proposition 4.11).

Let $t_0 < t_\varepsilon$ be such that for any $x \in \Lambda$, $B_{\delta/\eta}(x)$ contains $g_{\mathbf{a}}(t_0)$ for some $\mathbf{a} \in \mathcal{A}$ depending on x. To show that such a t_0 exists, consider a finite covering $\mathcal{D}$ of Λ by balls of radius $\frac{\delta}{3\eta}$. For any $D \in \mathcal{D}$, there is N_D such that $f^{N_D}(D) \supset g_{\tilde{\mathbf{s}}}(0, t_\varepsilon)$, hence for each D there is some $\mathbf{a} \in \mathcal{A}$ such that $g_{\mathbf{a}}(0, F^{-N_D}(t_\varepsilon)) \subset D$. By letting $N = \max N_D$, we have that $g_{\mathbf{a}}(0, F^{-N}t_\varepsilon) \subset D$ for any D, hence that each $B_{\delta/\eta}(x)$ contains $g_{\mathbf{a}}(F^{-N}(t_\varepsilon))$ for some $\mathbf{a} \in \mathcal{A}$.

Now fix x_0 in Λ, and let us construct a dynamic ray landing at x_0. Let $x_n := f^n(x_0)$, $B'_n := B_\delta(x_n)$, $B_n := B_{\delta/\eta}(x_n)$. For each n there is a branch ψ of f^{-1} such that $\psi(B'_n) \subset B_{n-1}$. Take a disk $D \subset \mathcal{D}$ containing some subsequence $\{x_n\}_{n\in\mathcal{N}}$, and let $\mathbf{s}_0 \in \mathcal{A}$ be such that $g_{\mathbf{s}_0}(t_0) \in D$. By Proposition 4.11, $\ell_{\text{eucl}}(I_{t_0}(g_{\mathbf{s}_0})) < \varepsilon$, so $I_{t_0}(g_{\mathbf{s}_0}) \subset B'_n$ for each $n \in \mathcal{N}$. For $n \in \mathcal{N}$, let ψ^n be the branch of f^{-n} mapping x_n to x_0; ψ^n can be extended analytically to B'_n and hence to $I_{t_0}(g_{\mathbf{s}_0})$. Let $g_{\mathbf{s}_n}$ be the sequence of rays containing $\psi^n(g_{\mathbf{s}_0}(t_0))$. Let $t_n = F^{-n}(t_0)$. Following the inductive construction from the proof of Theorem 2.5, the arcs $g_{\mathbf{s}_n}(t_n, F(t_0))$ satisfy the following properties:

$$g_{\mathbf{s}_n}(t_n, F(t_0)) \subset B'_0 \text{ and} \tag{4.6}$$

$$I_{t_m}(g_{\mathbf{s}_n}) \subset B\left(x_0, \frac{\delta}{\eta^m}\right) \text{ for all } n > m,\ n \in \mathcal{N}. \tag{4.7}$$

Let us now show that for some sufficiently large N and some M, $\|\sigma^N \mathbf{s}_n\|_\infty \leq M$. Consider the family of addresses $\mathbf{s}_n$ constructed for x_0 and the set

$$\widetilde{\mathcal{A}}_{x_0} = \left\{\mathbf{a} \in \mathcal{S} : \mathbf{a} = \sigma^j \mathbf{s}_n \text{ for some } j \leq n \text{ and } n \in \mathcal{N}\right\}.$$

By definition, if $\mathbf{a} \in \widetilde{\mathcal{A}}_{x_0}$ then $\sigma\mathbf{a} \in \widetilde{\mathcal{A}}_{x_0}$ unless $\mathbf{a} = \mathbf{s}_0$, and for each $\mathbf{a} \in \widetilde{\mathcal{A}}_{x_0}$, $g_{\mathbf{a}}(t_0) \in \overline{U}$. It follows by Proposition 4.12 that $\|\mathbf{a}\|_\infty < M$ for all $\mathbf{a} \in \widetilde{\mathcal{A}}_{x_0}$. In particular, $\|\mathbf{s}_n\|_\infty < M$, and there is a limiting address $\mathbf{s}$ such that $g_{\mathbf{s}}$ lands at x_N by Lemma 2.2. By finiteness of $\mathcal{D}$, there exists $M' > 0$ such that the addresses of all rays coming from the construction and landing at $x \in \Lambda$ are bounded by M'. Finally, the addresses of any two rays landing together cannot differ by more than one in any entry (see Remark 3.3), so the address of any dynamic ray (not necessarily coming from this construction) landing at any $x \in \Lambda$ is bounded by $M' + 1$. □

Let us conclude by showing that also in the exponential case, there are only finitely many rays landing at each point in a hyperbolic set.

Proposition 4.13. *Let f be an exponential map with bounded postsingular set, and Λ be a hyperbolic set. Then there are only finitely many dynamic rays landing at each $x \in \Lambda$.*

Proof. For $x \in \Lambda$, let $\mathcal{A}_x$ be the set of addresses of the rays landing at x. By Theorem 4.3, each $\mathcal{A}_x$ is non-empty. Since the local dynamics near x is bijective, the set $\mathcal{A}_x$ is mapped bijectively to the set $\mathcal{A}_{f(x)}$ by the shift map σ, so there is a well defined inverse $\sigma^{-1} : \mathcal{A}_{f(x)} \to \mathcal{A}_x$. By Theorem 4.3, the norm of the addresses belonging to the set $\mathcal{A}_x$ is bounded by a constant M, so the addresses in $\mathcal{A}_x$ can be set in correspondence with sequences over $D = 2M+1$ symbols and embedded in $\mathbb{S}^1$ preserving the dynamics. In the metric of (2.2), Remark 3.3 implies that for two dynamic rays $g_{\mathbf{s}}, g_{\mathbf{s}'}$ landing together $|\mathbf{s}-\mathbf{s}'|_D < 1/D$. In the same metric, σ is locally expansive by the factor D, so uniform continuity of $\sigma^{-1} : \mathcal{A}_{f(x)} \to \mathcal{A}_x$ would give local contraction for σ^{-1} by the factor $1/D$.

Assume by contradiction that there is a sequence of points $x_n \in \Lambda$, and two sequences of angles $\mathbf{a}_n, \mathbf{a}_n{}' \in \mathcal{A}_{f(x_n)}$ such that $|\mathbf{a}_n - \mathbf{a}_n{}'|_D \to 0$, but $|\sigma^{-1}\mathbf{a}_n - \sigma^{-1}\mathbf{a}_n{}'|_D \to 1/D$.

Assume for definiteness that $\mathbf{a}'_n < \mathbf{a}_n$; by the contradiction assumption, and by $2\pi i$ periodicity of the dynamical plane, the dynamic rays of addresses $j\mathbf{a}_n$ and $(j+1)\mathbf{a}'_n$ land together for all $j \in \mathbb{Z}$. These pairs of rays divide $\mathbb{C}$ into infinitely many regions one of which contains a left half plane, and which we call V_n (see Figure 4).

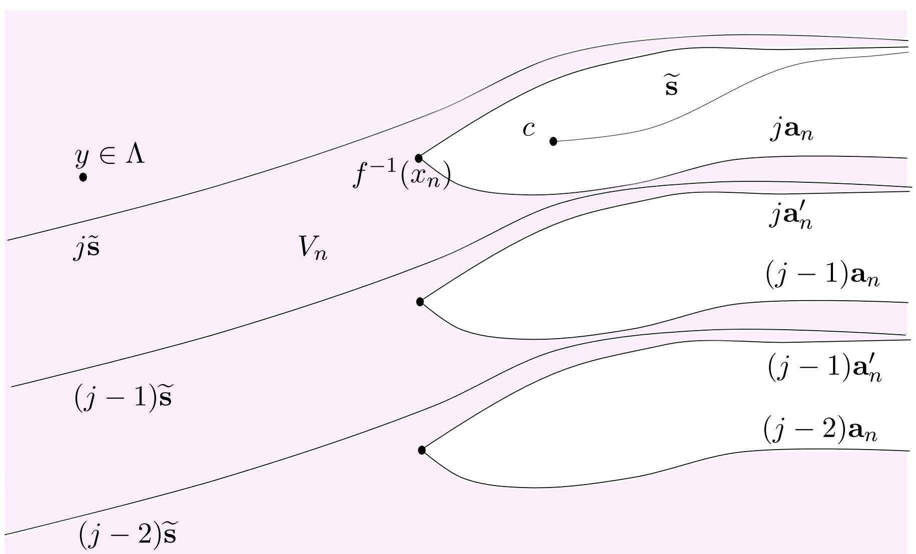

Figure 4: Illustration to the proof of Proposition 4.13. The rays are labeled by their addresses. The region V_n is shaded.

Let $Q = \bigcap \overline{V_n}$. Then $Q \cap \Lambda \neq \emptyset$ because Λ is compact and $\overline{V_n} \cap \Lambda \neq \emptyset$ for all n. Let $z \in Q \cap \Lambda$. Because $|\mathbf{a}_n - \mathbf{a}'_n|_D \to 0$, $\lim \mathbf{a}_n = \lim \mathbf{a}'_n = \mathbf{a}$ for some $\mathbf{a}$, and

$$\lim j\mathbf{a}'_n = \lim j\mathbf{a}_n = j\mathbf{a}. \tag{4.8}$$

It follows that no dynamic rays can intersect Q except for the dynamic rays of address $j\mathbf{a}$, $j \in \mathbb{N}$. However, by Corollary 4.6 the singular value is accessible by some dynamic ray $g_{\widetilde{\mathbf{s}}}$. Its countably many preimages $g_{j\widetilde{\mathbf{s}}}$ intersect any left half plane: it follows that $\widetilde{\mathbf{s}} = \mathbf{a}$ and that none of the dynamic rays of address $j\mathbf{a}$ is landing. This contradicts the fact that $\Lambda \cap Q \neq \emptyset$

and that points in Λ are accessible by Theorem 4.3. So there exists an $\varepsilon > 0$ such that if $\mathbf{a}, \mathbf{a}' \in \mathcal{A}_{f(x)}$ and $|\mathbf{a} - \mathbf{a}'| < \varepsilon$ then

$$|\sigma^{-1}\mathbf{a} - \sigma^{-1}\mathbf{a}'|_D < \frac{1}{D}\,|\mathbf{a} - \mathbf{a}'|_D. \tag{4.9}$$

Now fix a point $x_0 \in \Lambda$, and consider a finite cover of $\mathbb{S}^1$ by ε-balls, say N_0 balls. In particular, for any n, $\mathcal{A}_{x_n}$ is covered by N_0 balls of radius ε. By (4.9), their n-th preimages have diameter ε/D^n and cover $\mathcal{A}_x$. Passing to the limit, we conclude that $\mathcal{A}_x$ contains at most N_0 points. $\square$

4.4 A remark about parabolic wakes

Theorem 4.1 implies that non-recurrent parameters with bounded post-singular set always belong to parabolic wakes (see [Re1], Proposition 4 and 5, for the definition of parabolic wakes and the relation between parabolic wakes and landing of rays in the dynamical plane; see also [RS1]).

There is a combinatorial proof of this fact (see e.g. in [S1]) which can be adapted to the exponential case once Theorem 4.1 is known.

Corollary 4.14 (Corollary of Theorem 4.1)**.** *A non-recurrent parameter c with bounded postsingular set for the exponential family is contained in a parabolic wake attached to the boundary of the period one hyperbolic component W_0.*

Sketch of proof. Let $c_0 \in W_0$ be a hyperbolic parameter with an attracting fixed point $\alpha(c_0)$. Observe that for any fixed address $\mathbf{s}$, the ray $g_{\mathbf{s}}^{c_0}$ lands at a fixed point $z_{\mathbf{s}}(c_0)$. Let γ be a curve joining c to c_0 such that all $z_{\mathbf{s}}(c_0)$ can be continued analytically together with their landing rays. Call $\alpha(c)$ the analytic continuation along γ of the attracting fixed point $\alpha(c_0)$, $z_{\mathbf{s}}(c)$ the analytic continuation of $z_{\mathbf{s}}(c_0)$. Suppose that c does not belong to any parabolic wake attached to W_0. By Theorem A, $\alpha(c)$ is the landing point of at least one periodic ray, which is necessarily fixed (otherwise, there would be at least two periodic rays landing at $\alpha(c)$ determining a parabolic wake). But this gives a contradiction, because for any fixed address $\mathbf{s}$ the fixed ray $g_{\mathbf{s}}^{c}$ lands at the point $z_{\mathbf{s}}(c) \neq \alpha(c)$. $\square$

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