Problem of the Month

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Solution by Roman Kogan

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Problem:
There are 100 seats on a plain. All seats are assigned to passengers. Passengers enter the plane one at a time. First, enters a crazy old lady, who takes any seat with uniform probability (by chance, this may happen to be her own seat). Each subsequent passenger acts as follows: if his/her seat is vacant, then he/she takes this seat; but if his/her seat is already taken, he/she takes any of the remaining seats with uniform probability.
What is the probability that the last passenger takes his/her own seat?

Solution: Let's consider the problem for n passengers on the plane, and let's number them 1 through n according to the order at which they enter the plane. Consider the old lady. If she takes her seat (which happens with probability $\frac{1}{n}$), every other passenger (including the last one) gets his seat with probability 1, since they will not be forced to take seats other than their own. If the old lady takes the seat of last passenger, then, obviously, the last passenger has no chance to get his seat. Now let's look at the case where the old lady occupies the place of the k'th passenger, where $1 < k < n$. Passengers $2 \ldots (k-1)$ take their own seats, since they are vacant.
Now passenger k starts to act exactly like the old lady:

1. he takes a seat randomly;
2. if he takes the seat of old lady, which we can think of as being *his* seat now, then passengers $k+1, k+2, \ldots, n$ get to their seats with probability 1;
3. if he takes the last passenger's seat, last passenger gets his seat with probability 0.

Therefore, if we let $f(x)$ be the probability that the last passenger takes his seat on a plane with x seats (according to the conditions stated at the problem), then the probability that the last person takes his seat *given* that old lady took the seat of k'th passenger is $f(k)$. Since the seat of k'th passenger is taken with probability $\frac{1}{n}$ (both old lady and a person whose seat was occupied select new seat with with uniform randomness), we have

$$f(n) = \tfrac{1}{n} + \tfrac{1}{n}\left(\textstyle\sum_{i=2}^{n-1} f(i)\right).$$

We observe that in case $n = 2$, the old lady takes her seat with probability $\frac{1}{2}$, in which case the last passenger always gets his sit, and that she gets the seat of last passenger with probability $\frac{1}{2}$ as well; hence $f(2) = \frac{1}{2}$. Our claim is that $f(n) = \frac{1}{2}$ for all n, and we prove it using full induction on n. We have shown the base case above. Now let $f(n) = \frac{1}{2}$ for $n \leq k$. Then $f(k+1) = \frac{1}{k+1} + \frac{1}{k+1}\sum_{i=2}^{k} f(i) = \frac{1}{k+1} + \frac{k-1}{2(k+1)} = \frac{k+1}{2(k+1)} = \frac{1}{2}$, as desired. The claim follows by the principle of full mathematical induction.
□

Answer: The last passenger gets his seat with probability $\frac{1}{2}$