Problem of the Month
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1 Problem

Solve the following equation:

$$\cos^n(x) - \sin^n(x) = 1$$

where $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$.

2 Solution

Let $f(x) = \cos^n(x) - \sin^n(x)$. Note that it is enough to only consider the values of x in $[0, 2\pi)$.

2.1 Case 1: $n = 1$

When $n = 1$, we have

$$f'(x) = -\sin(x) - \cos(x).$$

So $f'(x) = 0$ when $\sin(x) = -\cos(x)$. We see $\cos(x) = 0$ isn't a case. Then we have

$$\begin{aligned} \sin(x) &= -\cos(x) \\ \tan(x) &= -1 \end{aligned}$$

So $f'(x) = 0$ when $x = \frac{3}{4}\pi, \frac{7}{4}\pi$. From this, we see $f(x)$ has local maximum at $x = \frac{7}{4}\pi$ with $f(x) > 1$ and local minimum at $x = \frac{3}{4}$ with $f(x) < 1$. Since $f(x)$ and $f'(x)$ are both continuous, there are two values for x in $[0, 2\pi)$ where $f(x) = 1$. In particular, $f(x) = 1$ only when $x = 0$ or $\frac{3}{2}\pi$.

2.2 Case 2: $n \geq 2$, n is even

We have

$$\begin{aligned} f'(x) &= -n\cos^{n-1}(x)\sin(x) - n\sin^{n-1}(x)\cos(x) \\ &= -n\cos(x)\sin(x)(\cos^{n-2}(x) + \sin^{n-2}(x)) \end{aligned}$$

Since $\cos^{n-2}(x) + \sin^{n-2}(x) > 0$, $f'(x) = 0$ only when $\cos(x) = 0$ or $\sin(x) = 0$. From this, we see $f(x)$ has local maxima at $x = 0, \pi$ and local minima at $x = \frac{1}{2}\pi, \frac{3}{2}\pi$. However, all local maxima are equal to 1 in this case. So $f(x) = 1$ only when $x = 0$ or π.

2.3 Case 3: $n \geq 2$, n is odd

From the previous part, we have

$$f'(x) = -n\cos(x)\sin(x)(\cos^{n-2}(x) + \sin^{n-2}(x))$$

In this case, $f'(x) = 0$ only when $\cos(x) = 0$ or $\sin(x) = 0$ or $\sin(x) = -\cos(x)$. From this, we see $f(x)$ has local maxima at $x = 0, \frac{3}{4}\pi, \frac{3}{2}\pi$ and local minima at $x = \frac{1}{2}\pi, \pi, \frac{7}{4}\pi$. Since $f(x) < 0$ when $x = \frac{3}{4}\pi$ and $f(x) = 1$ when $x = 0$ or $\frac{3}{2}\pi$, we see $f(x) = 1$ only when $x = 0$ or $\frac{3}{2}\pi$.

2.4 Conclusion

Generalizing the results above, we have the following:

- When n is odd, $\cos^n(x) - \sin^n(x) = 1$ for $x = 0 + 2k\pi$ and $x = \frac{3}{2}\pi + 2k\pi$ where $k \in \mathbb{Z}$.
- When n is even, $\cos^n(x) - \sin^n(x) = 1$ for $x = k\pi$ where $k \in \mathbb{Z}$.