Problem of the Month

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Theorem: Let $F(n)$ denote the number of ways 2^n can be represented as a sum of squares of four integers, where $n > 0$. Then

$$F(n) = \begin{cases} 2, & \text{if } 2|n \\ 1, & \text{if } 2 \nmid n \end{cases}$$

Proof: Constructive proof.

Let $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 2^n$ for some integers $a_1..a_4, n$ where $n > 0$.

Let us consider all possible parities of $a_1..a_4$.

Only an even number of integers among $a_1..a_4$ can be odd, otherwise the sum of their squares will be odd as well. Therefore, we only need to consider the following three cases:

1. exactly two numbers among $a_1..a_4$ are odd;
2. all of the numbers $a_1..a_4$ are odd;
3. none of the numers $a_1..a_4$ is odd.

Since different permutations do not consitute different representations, it is up to our choice to pick which a_i are odd and which are even.

Case 1: WLOG, let a_1, a_2 be odd, and let a_3, a_4 be even.

Then $a_1 = (2k_1 + 1), a_2 = (2k_2 + 1), a_3 = (2k_3), a_4 = (2k_4)$ for some $k_1..k_4 \in \mathbb{Z}^+$. Hence,

$$2^n = (2k_1 + 1)^2 + (2k_2 + 1)^2 + (2k_3)^2 + (2k_4)^2;$$
$$2^n = 4k_1^2 + 4k_2^2 + 4k_3^2 + 4k_4^2 + 2;$$
$$2^{n-1} = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2) + 1,$$

which implies that 2^{n-1} is odd. This is only possible in the case when $n = 1$ so that $2^{n-1} = 2^0 = 1$ (in this case we have $2^1 = 0^2 + 0^2 + 1^2 + 1^2$). Consequently, for $n > 1$, 2^n cannot be represented as a sum of squares of two numbers, exactly two of which are odd.

Case 2: Let $a_1..a_4$ be all odd.

Then $a_i = (2k_i + 1), 1 \leq i \leq 4$ for some $k_1..k_4 \in \mathbb{Z}^+$. Hence,

$$2^n = \sum_{i=1}^{4} (2k_i + 1)^2$$

$$2^n = \sum_{i=1}^{4} 4k_i^2 + 4k_i + 1;$$

$$2^n = \left[4 \sum_{i=1}^{4} k_i(k_i + 1) \right] + 4;$$

$$2^{n-2} = \left[\sum_{i=1}^{4} k_i(k_i + 1) \right] + 1.$$

Now, $\forall k \in \mathbb{Z}^+$: $2|k$ or $2|(k+1) \Rightarrow 2|k(k+1) \Rightarrow 2|\sum_{i=1}^{4} k_i(k_i+1)$. Hence, 2^{n-2} is odd, which is only possible in the case when $n = 2$ so that $2^{n-2} = 2^0 = 1$. Then we have $2^2 = 1^2 + 1^2 + 1^2 + 1^2$; otherwise, for $n > 2$, 2^n cannot be repesented as a sum of squares of four odd numbers. Therefore, we have proven that

For $n > 2$, 2^n can only be represented as a sum of sqaures of four *even* numbers. (I)

Case 3: Let $a_1..a_4$ be all even.
Then $a_i = 2k_i, 1 \leq i \leq 4$ for some $k_1..k_4 \in \mathbb{Z}^+$. Hence,

$$2^n = \sum_{i=1}^{4}(2k_i)^2 = 4\sum_{i=1}^{4} k_i^2 \Rightarrow 2^{n-2} = \sum_{i=1}^{4} k_i^2.$$

Thus, any representation of 2^n as a sum of squares of four even integers corresponds to some representation of 2^{n-2} as a sum of squares of four integers. Combining this with (I) and noting that the converse is also true (i.e. any representation of 2^{n-2} as a sum of four squares yields a representation of 2^n as a sum of four squares), we set a one-to-one correspondence between representations of 2^n and 2^{n-2} as a sum of four integers. Therefore, $\forall n > 2$, $F(n) = F(n-2)$. After finding empirically that $F(1) = 1$ where

$$2 = 0 + 0 + 1 + 1$$

and $F(2) = 2$, with

$$4 = 1 + 1 + 1 + 1$$
$$4 = 0 + 0 + 0 + 4$$

we establish

$$1 = F(1) = F(3) = F(5) = F(7) = ... = F(2k+1), \forall k > 0;$$
$$2 = F(2) = F(4) = F(6) = F(8) = ... = F(2k), \forall k > 0,$$

thus proving the theorem. ■

Note: we could have attained the same result by considering $a_1..a_4 \mod 8$; however, the proof presented above was chosen because it does not employ any number theory beyond divisibility by 2 (and is therefore simpler).