November Problem of the Month

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Solution by Said Amghibech

Notation: If $\overrightarrow{A_1B_1}$ *and* $\overrightarrow{A_2B_2}$ *are two vectors, then* $\overrightarrow{A_1B_1} \wedge \overrightarrow{A_2B_2}$ *denotes their vector product.*

Let O_2 be the incenter of the triangle $A_2B_2C_2$, $\|\cdot\|$ be the Euclidean norm. Remark that the area of the polygon $A_1O_2B_1B_2$ is given by

$$Area(A_1O_2B_1) - Area(A_1B_2B_1) = \frac{1}{2}\|\overrightarrow{A_1B_1} \wedge \overrightarrow{O_2B_2}\|.$$

By using the same argument to the polygons $O_2B_1A_2C_1$ and $O_2C_1C_2A_1$, and the fact that $\|\overrightarrow{A_1B_1} \wedge \overrightarrow{O_2B_2}\| = \|\overrightarrow{B_1C_1} \wedge \overrightarrow{O_2A_2}\| = \|\overrightarrow{C_1A_1} \wedge \overrightarrow{O_2C_2}\|$ we get

$$Area(A_1B_2B_1A_2C_1C_2) = \frac{3}{2}\|\overrightarrow{A_1B_1} \wedge \overrightarrow{O_2B_2}\|.$$

Thus $Area(A_1B_2B_1A_2C_1C_2)$ is maximal if and only if (O_2B_2) is orthogonal to (A_1B_1). Then the maximum possible area of the polygon $A_1B_2B_1A_2C_1C_2$ is

$$\frac{\sqrt{3}}{2}\|\overrightarrow{A_1B_1}\|\|\overrightarrow{A_2B_2}\|.$$

If $A_1B_1 \leq 2A_2B_2$, then the maximum is the area of the big triangle.

Also $Area(A_1B_2B_1A_2C_1C_2)$ is minimal if and only if the "angle" between (O_2B_2) and (A_1B_1) is $\pi/6$. Then the minimum possible area of the polygon $A_1B_2B_1A_2C_1C_2$ is

$$\frac{\sqrt{3}}{4}\|\overrightarrow{A_1B_1}\|\|\overrightarrow{A_2B_2}\|.$$

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