

Classification of Thurston maps with parabolic orbifolds

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Main results

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There exists an algorithm \mathcal{A} which does the following. Let f and g be marked Thurston maps and assume that every element of the canonical geometrization of f has hyperbolic orbifold. The algorithm \mathcal{A} , given the combinatorial descriptions of f and g , outputs 1 if f and g are Thurston equivalent and 0 otherwise.

Definition

A (marked) *Thurston map* is a pair (f, P_f) where $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is an orientation-preserving branched self-cover of \mathbb{S}^2 of degree $d_f \geq 2$ and P_f is a finite forward invariant set that contains all critical values of f .

In particular, the branched cover f must be postcritically finite.

Thurston equivalence

Definition

Two Thurston maps f and g are combinatorially equivalent if and only if there exist two homeomorphisms $h_1, h_2: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that the diagram

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{S}^2, P_g) \\ \downarrow f & & \downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_2} & (\mathbb{S}^2, P_g) \end{array}$$

commutes, $h_1|_{P_f} = h_2|_{P_f}$, and h_1 and h_2 are homotopic relative to P_f .

Theorem (Thurston's Theorem)

A postcritically finite branched cover $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ (except $(2, 2, 2, 2)$ -maps) is either Thurston-equivalent to a rational map g (which is then necessarily unique up to conjugation by a Möbius transformation), or f has a Thurston obstruction.

Thurston matrix and obstructions

Definition

Denote by \mathcal{C} the set of all homotopy classes of essential simple closed curves. Define the Thurston linear operator $M: \mathbb{R}^{\mathcal{C}} \rightarrow \mathbb{R}^{\mathcal{C}}$ by setting

$$M(\gamma) = \sum_{f(\gamma_i)=\gamma} \frac{1}{\deg f|_{\gamma_i}} \gamma_i.$$

Every multicurve Γ has its associated *Thurston matrix* M_{Γ} which is the restriction of M to \mathbb{R}^{Γ} .

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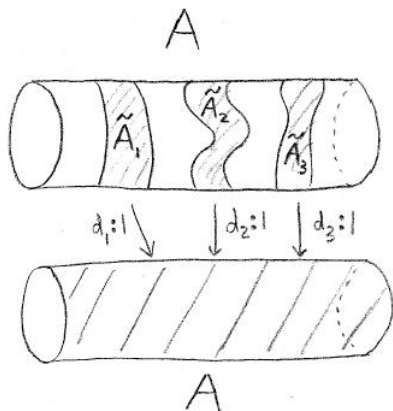
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Definition

Since all entries of M_{Γ} are non-negative real, the leading eigenvalue λ_{Γ} of M_{Γ} is also real and non-negative. A multicurve Γ is a *Thurston obstruction* if $\lambda_{\Gamma} \geq 1$.

An example of Thurston obstruction



For a rational map, we must have $\sum 1/d_i < 1$.

Definition

A *Levy cycle* is a multicurve

$$\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$$

such that each γ_i has a nontrivial preimage γ'_i , where the topological degree of f restricted to γ'_i is 1 and γ'_i is homotopic to $\gamma_{(i-1) \bmod n}$ rel Q . A Levy cycle is *degenerate* if each γ'_i bounds a disk D_i such that the restriction of f to D_i is a homeomorphism and $f(D_i)$ is homotopic to $D_{(i+1) \bmod n}$ rel Q .

Algorithm for finding Thurston obstructions

Theorem (Bonnot, Braverman, Yampolsky)

There exists an algorithm which for any Thurston map f with hyperbolic orbifold outputs either an obstruction or an equivalent rational map.

Proof.

- Enumerate all possible multicurves and start checking if any of them is an obstruction for f one-by-one.
- List all (finitely many) rational maps that *could* be equivalent to f . List all homeomorphisms classes and check whether any of them realizes equivalence one-by-one.



Conjecture

There exists an algorithm which can produce a combinatorial equivalence between two Thurston maps or say that they are not equivalent.

Main Theorem 3 is a partial resolution of this conjecture.

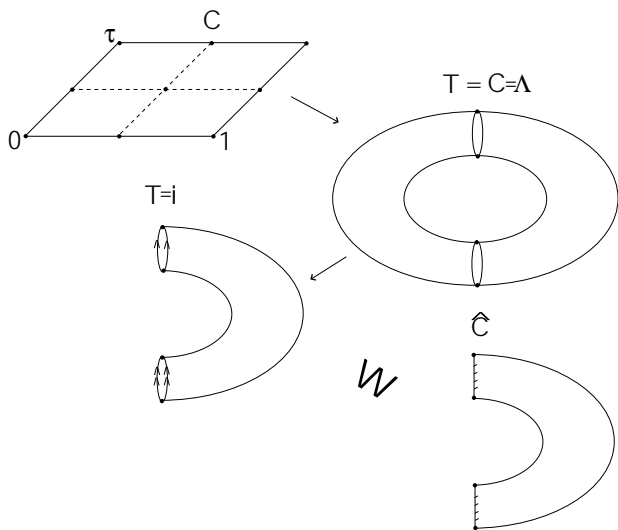
(2, 2, 2, 2)-maps

We will refer to a Thurston map that has orbifold with signature $(2, 2, 2, 2)$ simply as a $(2, 2, 2, 2)$ -map. An orbifold with signature $(2, 2, 2, 2)$ is a quotient of a torus T by an involution i ; the four fixed points of the involution i correspond to the points with ramification weight 2 on the orbifold. The corresponding branched cover $P : T \rightarrow \mathbb{S}^2$ has exactly 4 simple critical points which are the fixed points of i . It follows that a $(2, 2, 2, 2)$ -map f can be lifted to a covering self-map \hat{f} of T .

An orbifold with signature $(2, 2, 2, 2)$ has a unique affine structure of the quotient \mathbb{R}^2/G where

$$G = \langle z \mapsto z + 1, z \mapsto z + i, z \mapsto -z \rangle .$$

(2, 2, 2, 2)-maps



Classification of $(2, 2, 2, 2)$ -maps

Theorem (Main Theorem 2)

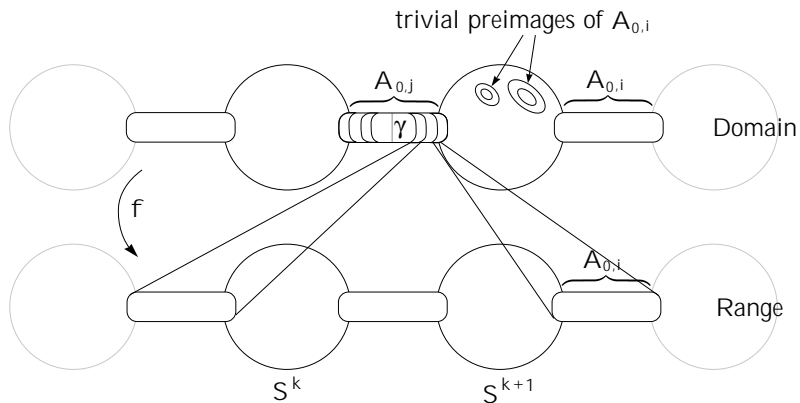
Let f be a $(2, 2, 2, 2)$ -map (with extra marked points) such that the associated matrix is hyperbolic. Then either f is equivalent to a quotient of an affine map or f admits a degenerate Levy cycle.

Furthermore, in the former case the affine map is defined uniquely up to conjugacy.

Corollary

There exists an algorithm which for any $(2, 2, 2, 2)$ -map f with hyperbolic matrix outputs either a degenerate Levy cycle or an equivalent quotient of an affine map.

Pilgrim's decomposition of a Thurston map



Theorem

The canonical obstruction Γ is a unique minimal Thurston obstruction with the following properties.

- If the first-return map F of a cycle of components in S_Γ is a $(2, 2, 2, 2)$ -map, then every curve of every simple Thurston obstruction for F has two postcritical points of f in each complementary component and the two eigenvalues of \hat{F}_* are equal or non-integer.*
- If the first-return map F of a cycle of components in S_Γ is not a $(2, 2, 2, 2)$ -map or a homeomorphism, then there exists no Thurston obstruction of F .*

Computing Canonical Obstructions

Theorem

There exists an algorithm which for any Thurston map f finds its canonical obstruction Γ_f .

Proof.

- 1 Run the BBY algorithm to get an obstruction Γ .
- 2 Decompose f along Γ .
- 3 Check conditions of the previous theorem. Either they are satisfied or we can construct an obstruction within one of the decomposition pieces.
- 4 Once we have found a maximal obstruction we check the conditions of the characterization theorem for all of its subsets.



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Definition

Let f be a $(2, 2, 2, 2)$ -map and let z be an f -periodic point with period n . Fix a universal cover F of f and take a point \tilde{z} in the fiber of z . If $z \notin P$, we define the *Nielsen index* $\text{ind}_{F,n}(\tilde{z})$ to be the unique element g of the orbifold group G such that $F^n(\tilde{z}) = g \cdot \tilde{z}$. If $z \in P$ then the Nielsen index of z is defined up to pre-composition with the symmetry around z .

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Definition

Let f be a $(2, 2, 2, 2)$ -map and let z_1, z_2 be f -periodic points with period n . We say that z_1 and z_2 are in the same *Nielsen class of period n* if there exists a universal cover F_n of f^n and points \tilde{z}_1, \tilde{z}_2 in the fibers of z_1, z_2 respectively, such that both \tilde{z}_1 and \tilde{z}_2 are fixed by F_n .

Strategy of the proof

- A map f admits a degenerate Levy cycle if and only if there exist two distinct periodic points in P_f in the same Nielsen class.
- If there are points in the same Nielsen class, one can find a curve that separates them from other marked points which will generate a degenerate Levy cycle.
- If all points have distinct Nielsen indexes, they define a conjugacy between f and the appropriate quotient of an affine map on Q . It can be shown that in the absence of Levy cycles such a conjugacy can be promoted to a conjugacy on the whole sphere.