Manin, Mumford and Hénon

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Plan

1. The Manin-Mumford conjecture
2. The dynamical Manin-Mumford (DMM) problem
3. The case of automorphisms of $\mathbb{C}^2$

Joint work with Charles Favre (Ecole Polytechnique).

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The Manin-Mumford conjecture

Let K be a field and A be an Abelian variety over K, that is : A is a projective variety which is also an Abelian group.
If $K = \mathbb{C}$, A is a complex torus $\mathbb{C}^d/\Lambda$.

Mordell-Weil : if K is a number field, then $A(K)$ is a finitely generated abelian group.

Let $Tor(A)$ be the set of torsion points.

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Conjecture (Manin-Mumford) :

Assume $K = \overline{\mathbb{Q}}$. If $X \subset A$ is an irreducible subvariety such that $\mathrm{Tor}(A) \cap X$ is Zariski-dense in X, then X is the translate of an Abelian subvariety by a torsion point.

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Theorem (Raynaud, 1983) :

Assume $K = \overline{\mathbb{Q}}$. If $X \subset A$ is an irreducible subvariety such that $\mathrm{Tor}(A) \cap X$ is Zariski-dense in X, then X is the translate of an Abelian subvariety by a torsion point.

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This is one among many statements in number theory of the following kind : "if a subvariety contains a Zariski dense subset of special points, then it is itself special".
(other famous example : André-Oort conjecture)

Note : if $\dim X = 1$, Zariski dense just means infinite.

A particular case

Consider the multiplicative group $(\mathbb{C}^*)^2$. Torsion points are of the form (ξ_1, ξ_2), where the ξ_i are roots of unity.

(A variant of) Manin-Mumford predicts that if $X \subset (\mathbb{C}^*)^2$ is an irreducible algebraic curve defined over $\overline{\mathbb{Q}}$ containing infinitely many such points, then

$$X = (\xi_1^0, \xi_2^0) \cdot G,$$

where $G \subset (\mathbb{C}^*)^2$ is a subgroup (equivalently X admits an equation of the form $x^a y^b = u$ where a and b are coprime integers and u is a root of unity).

Example

Consider $X = \{x + y = 0\}$. Then X contains all points of the form $(\xi, -\xi)$, with ξ a root of unity. And indeed $X = (1, -1) \cdot \Delta$, where $\Delta = \{(x, x),\ x \in \mathbb{C}^*\}$ is the diagonal subgroup.

Dynamical reformulation

For every integer $d \geq 2$ there is a dynamical system $m_d : A \to A$ induced by multiplication by d.

Exercise

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Note : this point of view goes back to Northcott (1950) !

The dynamical Manin-Mumford conjecture

Conjecture (S.-W. Zhang, 1995) :

Let K be an algebraically closed field of characteristic 0. Let $f : X \to X$ be a polarized endomorphism. Let Y be an irreducible subvariety such that $\mathrm{Preper}(f) \cap Y$ is Zariski dense in Y. Then Y is preperiodic.

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By "standard" specialization arguments it may be assumed that $K \subset \mathbb{C}$, so this can be approached using holomorphic dynamics techniques.

Polarized endomorphisms include integer multiplication on Abelian varieties, as well as holomorphic self mappings of $\mathbb{P}^k(\mathbb{C})$ (we can also take products). It holds true on Abelian varieties by Raynaud's Theorem.

Dynamical Manin-Mumford (continued)

Ghioca, Tucker and Zhang (2009) found a simple counter-example to the general formulation of the conjecture : let $E = \mathbb{C}/\mathbb{Z}[i]$ and f be defined on $E \times E$ by

$$f(x,y) = (5x, (3+4i)y).$$

Then f is polarized and the torsion points of $E \times E$ are preperiodic under f. In particular the diagonal contains infinitely many preperiodic points. On the other hand the diagonal is not periodic.

Anyway we see that this map is "special".

GTZ proposed a corrected version of the conjecture with an additional technical assumption.

Dynamical Manin-Mumford (continued)

Some cases of the conjecture are known, in particular for product maps on $(\mathbb{P}^1)^n$, that is, of the form

$$(x_1, \ldots, x_n) \mapsto (f_1(x_1), \ldots, f_n(x_n)),$$

where the f_i are not Lattès maps and $Y \subset (\mathbb{P}^1)^n$ is a line (Ghioca-Tucker-Zhang, see also Medvedev-Scanlon).

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Note : for (f, g) acting on $\mathbb{P}^1 \times \mathbb{P}^1$, the diagonal Δ contains infinitely many preperiodic points iff f and g have infinitely many preperiodic points in common ("unlikely intersection problem", cf. Baker and DeMarco)

Dynamical Manin-Mumford (continued)

Dynamical Manin-Mumford (DMM) problem :

Let X be a quasiprojective variety and $f : X \to X$ a dominant endomorphism. Describe all the positive dimensional irreducible subvarieties $Y \subset X$ such that $\mathrm{Preper}(f) \cap Y$ is Zariski dense in Y.

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In the following we study this problem for polynomial automorphisms of $\mathbb{C}^2$.

Polynomial automorphisms of $\mathbb{C}^2$

Consider the space $\mathrm{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of the affine plane : polynomial mappings with polynomial inverse.

Notice that for automorphisms, preperiodic=periodic.

An automorphism $f \in \mathrm{Aut}(\mathbb{C}^2)$ has constant Jacobian $\mathrm{Jac}(f) \in \mathbb{C}^*$.

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Basic family of examples : Hénon maps

$$h : (z, w) \mapsto (aw + p(z), az),\ \deg(p) = d,\ a \in \mathbb{C}^*$$

Polynomial automorphisms of $\mathbb{C}^2$

Friedland-Milnor : $f \in \mathrm{Aut}(\mathbb{C}^2)$ is conjugate in $\mathrm{Aut}(\mathbb{C}^2)$ to either :

- an affine map ;
- an elementary map $(x, y) \mapsto (ax + b, y + P(x))$;
- a composition $h_1 \circ \cdots \circ h_k$ where the h_i are Hénon mappings.

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The DMM problem is uninteresting in the first two cases so we assume f is a product of Hénon maps (a "Hénon-type" transformation).

Note : all this is valid over any algebraically closed field of characteristic 0.

DMM problem for polynomial automorphisms of $\mathbb{C}^2$

Theorem (Bedford-Smillie)

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Answer : yes !

Reversible mappings

A polynomial automorphism is reversible if f is conjugate to its inverse : $f^{-1} = \sigma^{-1} f \sigma$. Typically, σ is a linear involution. Note that if f is reversible then $\mathrm{Jac}(f) = \pm 1$.
Examples include all Hénon mappings of Jacobian 1 :

$$f(x, y) = (p(x) - y, x) \text{ and } \sigma(x, y) = (y, x).$$

Notice that $\mathrm{Fix}(\sigma) = \Delta = \{(x, x), x \in \mathbb{C}\}$.

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Proposition :

If $f^{-1} = \sigma^{-1} f \sigma$ and σ is an involution with a curve of fixed points C, then f admits infinitely many periodic points on C.

Note : Gomez and Meiss proved that under these assumptions, then σ is conjugate to $(x, y) \mapsto (y, x)$ so C is always the diagonal.

Reversible mappings (continued)

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If $f^{-1} = \sigma^{-1} f \sigma$ and σ is an involution with a curve of fixed points C, then f admits infinitely many periodic points on C.

Proof : indeed if $x \in f^n(\Delta) \cap \Delta$ then

$$f^{-n}(x) = \sigma f^n \sigma(x) = \sigma f^n(x) = f^n(x)$$

so $f^n(\Delta) \cap \Delta \subset \mathrm{Fix}(f^{2n})$.

On the other hand $\# f^n(\Delta) \cap \Delta \approx d^n$ so the result follows. $\square$

DMM conjecture for polynomial automorphisms

Conjecture :

These are the only examples. More precisely, if f is a Hénon-type automorphism and $C \subset \mathbb{C}^2$ an algebraic curve such that $\mathrm{Per(f)} \cap C$ is infinite then there exists $n \geq 1$ and an involution σ such that $\mathrm{Fix}(\sigma) = C$ and $\sigma f^n \sigma = f^{-n}$.

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Notice that in this case $\mathrm{Jac}(f)$ must be a root of unity.

Main results

Theorem A

Let f be a Hénon-type automorphism, and assume that there exists an algebraic curve containing infinitely many periodic points. Then $\mathrm{Jac}(f)$ is algebraic over $\mathbb{Q}$ and $|\mathrm{Jac}(f)| = 1$, together with all its Galois conjugates.

Theorem A

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Remark :

We actually show that if $|\mathrm{Jac}(f)| \neq 1$ the number of periodic points on an algebraic curve of degree $\leq d$ is bounded by a constant $N(d, f)$ (not effective).

Main results

Theorem B

Let f be a Hénon-type automorphism, and assume that there exists an algebraic curve C containing infinitely many periodic points. Assume that the following transversality assumption (T) holds :

$$\exists p \in \mathrm{Reg}(C) \cap \mathrm{Per}(f), \text{ such that } T_pC \text{ is not periodic under } df_p.$$

Then $\mathrm{Jac}(f)$ is a root of unity.

We also obtain a result on unlikely intersections (which is related to DMM for product maps $f \otimes g$) :

Theorem C

Let f and g be two Hénon type automorphisms, defined over a number field.
Then if f and g share a set of periodic points that is Zariski dense, then there exists integers m and n such that $f^m = g^n$.

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This is a generalization of results by Baker-DeMarco and Yuan-Zhang for one-dimensional rational maps.

Strategy of proof

The proofs of these results rely on the approach of Szpiro, Ullmo and Zhang to the Manin-Mumford conjecture (equistribution of points of small height).

Interesting phenomenon : while the statements are purely algebraic, the proofs use arithmetic tools (in particular the notion of height). So in a first stage we assume that all mappings are defined over a number field.

For Theorems A and B, "standard" specialization arguments then show that the result holds in the complex case as well, and actually also on every algebraically closed field of characteristic zero. (for Theorem C this is still in progress)

Applying the equidistribution theorem

Recall that f is assumed to be a product of Hénon mappings $f = h_1 \circ \cdots \circ h_k$ with $h_i(x, y) = (a_i y + p_i(x), x)$ and $\deg(f) = d$, defined over $\overline{\mathbb{Q}}$. Then one can define the (forward) Green function

$$z \mapsto G^+(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^n(z)\|,$$

and similarly G^- in backward time. G^+, G^- and $G = \max(G^+, G^-)$ are plurisubharmonic and $G(z) = \log\|z\| + O(1)$ at infinity.

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To be able to apply arithmetic equidistribution, it is important that the same theory makes sense also in the non-Archimedean fields $\mathbb{C}_p$, $p \geq 2$ prime (this is non-trivial and due to Kawaguchi).

Applying the equidistribution theorem

Then equidistribution theorems due to Autissier and Thuillier imply :

Proposition :

Let f be an automorphism defined over a number field. Assume that (p_n) is an infinite sequence of periodic points inside a curve C. Then C is defined over $\overline{\mathbb{Q}}$ and the sequence of probability measures μ_n equidistributed over the Galois conjugates of p_n converges to $\alpha\Delta(G^+|_C)$ where α is a positive rational number depending only on C.

Applying the equidistribution theorem

But we can also do the same in negative time, and μ_n also converges to $\beta\Delta(G^-|_C)$. Finally :

Proposition :

Let f be an automorphism defined over a number field, possessing infinitely many periodic points on a curve C. There exists positive rational numbers $\alpha(C)$ and $\beta(C)$ and a harmonic function H such that along C, $\alpha G^+ = \beta G^- + H$.

In the following we assume $H = 0$ and $\alpha = \beta = 1$, which does not affect the argument.

Proof of Theorem B

Recall the statement

Theorem B

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Then $\mathrm{Jac}(f)$ is a root of unity.

Recall that here f is supposed to be defined on a number field. Iterating, we may assume that assume that p is fixed.

Proof of Theorem B

We will prove the theorem under the simplifying assumption that p is a saddle point (by Bedford, Lyubich and Smillie, most periodic points are saddles, so this is not unreasonable).

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Choose local coordinates (x, y) so that $W^u_{\mathrm{loc}} = \{y = 0\}$ and $W^s_{\mathrm{loc}} = \{x = 0\}$.
Denote by u and s the respective unstable and stable eigenvalues at p (which is 0 in our coordinates). Then

$$f(x, y) = (ux + h.o.t., sy + h.o.t.)$$

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Rescaling lemma

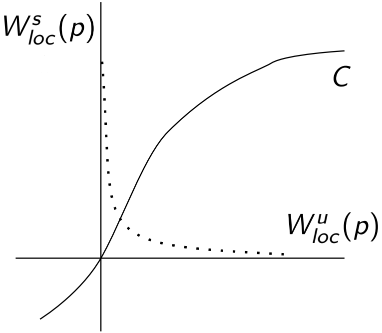
There exist local coordinates (x, y) as above so that if ξ and η are small enough, then

$$f^n\left(\frac{\xi}{u^n}, \eta\right) \to (\xi, 0) \text{ uniformly in } (\xi, \eta) \text{ as } n \to \infty.$$

Proof of Theorem B (continued)

Recall that by equidistribution we have that $G^+|_C = G^-|_C$. We now use this symmetry and the rescaling lemma to show that $|us| = |\mathrm{Jac}(f)| = 1$.

Due to (T) the local picture is like this :



Let $y = \psi(x)$ be a local equation of C.

Proof of Theorem B (continued)

Assume by contradiction that $|us| > 1$. Then

$$d^n G^+\left(\frac{\xi}{u^n}, \psi\left(\frac{\xi}{u^n}\right)\right) = G^+ \circ f^n\left(\frac{\xi}{u^n}, \psi\left(\frac{\xi}{u^n}\right)\right) \to G^+(\xi, 0)$$

by the rescaling Lemma. But by symmetry $G^+(x, \psi(x)) = G^-(x, \psi(x))$, so the above quantity equals

$$d^n G^-\left(\frac{\xi}{u^n}, \psi\left(\frac{\xi}{u^n}\right)\right) = G^- \circ f^{-n}\left(\frac{\xi}{u^n}, \psi\left(\frac{\xi}{u^n}\right)\right)$$
$$\approx G^-\left(0, s^{-n}\psi\left(\frac{\xi}{u^n}\right)\right) \to G^-(0,0) = 0$$

We conclude that $G^+|_{W^u_{\mathrm{loc}}(p)} \equiv 0$, a contradiction since $W^u_{\mathrm{loc}}(p) \not\subset K^+$. Hence $|us| \leq 1$
Reversing the roles of u and s we conclude that $|us| = |\mathrm{Jac}(f)| = 1$.

Proof of Theorem B (continued)

Notice that all Galois conjugates of $\mathrm{Jac}(f)$ have modulus 1 since our assumption on periodic points is purely algebraic.

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How to proceed to prove that $\mathrm{Jac}(f)$ is a root of unity ? We use the classical

Lemma

An algebraic number α is a root of unity iff $|\alpha|_v = 1$ for every place v of $\mathbb{Q}[\alpha]$. Explicitly, this means that for $p \in \mathcal{P} \cup \{\infty\}$, if τ is any embedding of $\mathbb{Q}[\alpha]$ into $\mathbb{C}_p$, then $|\tau(\alpha)|_p = 1$.

For Archimedean places v, we just proved that $|\mathrm{Jac}(f)|_v = 1$ (assuming that p is a saddle at such places).

Proof of Theorem B (continued)

Now fix a non-Archimedean place v. All that was said before makes sense in $\mathbb{C}_v$, including the existence of Green functions, the equidistribution theorem, etc. (this relies on the technology of Berkovich spaces, non-Archimedean potential theory, etc).

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Consider our periodic point $p \in C$ satisfying the transversality assumption (T).

Non-archimedean Lemma

If the place v is non-Archimedean, then either $|u|_v = |s|_v = 1$ or p is a saddle.

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In the first case we directly get that $|\mathrm{Jac}(f)|_v = 1$. In the second we repeat the above proof (rescaling and $G^+ = G^-$ on C), which again works over $\mathbb{C}_v$, to conclude that $|us|_v = 1$.

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So at all places we have that $|\mathrm{Jac}(f)|_v = 1$, therefore $\mathrm{Jac}(f)$ is a root of unity. □

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But since $G^+|_C = G^-|_C$, $|\chi^u| = |\chi^s|$, i.e. $\chi^s = -\chi^u$, therefore $|\mathrm{Jac}(f)| = 1$ and we are done. □

Thanks !