Thurston's characterization theorem for branched covers

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Definition

A *Thurston map* is a pair (f, P_f) where $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation-preserving branched self-cover of $\mathbb{S}^2$ of degree $d_f \geq 2$ and P_f is a finite forward invariant set that contains all critical values of f.

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In particular, the branched cover f must be postcritically finite.

Thurston equivalence

Definition

Two Thurston maps f and g are combinatorially equivalent if and only if there exist two homeomorphisms $h_1, h_2 \colon \mathbb{S}^2 \to \mathbb{S}^2$ such that the diagram

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{S}^2, P_g) \\ \downarrow f & & \downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_2} & (\mathbb{S}^2, P_g) \end{array}$$

commutes, $h_1|_{P_f} = h_2|_{P_f}$, and h_1 and h_2 are homotopic relative to P_f.

Thurston's theorem

Theorem (Thurston's Theorem)

A postcritically finite branched cover $f : \mathbb{S}^2 \to \mathbb{S}^2$ *with hyperbolic orbifold is either Thurston-equivalent to a rational map* g *(which is then necessarily unique up to conjugation by a Möbius transformation), or* f *has a Thurston obstruction.*

Invariant multicurves

- a closed curve γ is *essential* if every component of $\mathbb{S}^2 \setminus \gamma$ contains at least two points of P_f (i.e. it is not homotopic to a boundary component of our surface)

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- A multicurve Γ is *invariant* if $f^{-1}(\Gamma) \subseteq \Gamma$
- A multicurve Γ is *completely invariant* if $f^{-1}(\Gamma) = \Gamma$

Curves and collars

On a hyperbolic Riemann surface:

- Every essential simple closed curve is homotopic to a hyperbolic geodesic.
- Collar Theorem. There is an explicit function $w(l)$ so that a geodesic of length l has a collar of width $w(l)$ embeded in the surface and $w(l) \to \infty$ as $l \to 0$.
- Every annulus (cylinder) is biholomorphic to a unique annulus bounded by $|z| = 1$ and $|z| = R$. The conformal modulus of the annulus is defined as $\frac{1}{2\pi} logR$.
- Schwartz Theorem. Conformal moduli of annuli are superadditive.

Thurston matrix and obstructions

Definition

Denote by $\mathcal{C}$ the set of all homotopy classes of essential simple closed curves. Define Thurston linear operator $M\colon \mathbb{R}^{\mathcal{C}} \to \mathbb{R}^{\mathcal{C}}$ by setting

$$M(\gamma) = \sum_{f(\gamma_i)=\gamma} \frac{1}{\deg f|_{\gamma_i}} \gamma_i.$$

Every multicurve Γ has its associated *Thurston matrix* M_Γ which is the restriction of M to $\mathbb{R}^{\Gamma}$.

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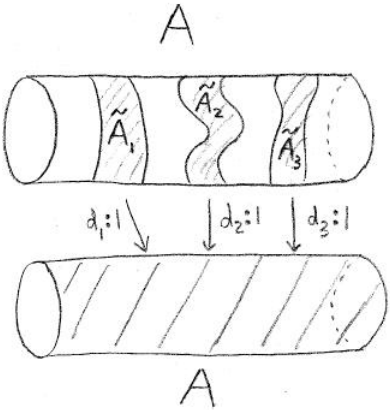
Definition

Since all entries of M_Γ are non-negative real, the leading eigenvalue λ_Γ of M_Γ is also real and non-negative. A multicurve Γ is a *Thurston obstruction* if $\lambda_\Gamma \geq 1$.

An example of Thurston obstruction



For a rational map, we must have $\sum 1/d_i < 1$.

Teichmüller space

Let $\mathcal{T}_f$ be the Teichmüller space modeled on the marked surface $(\mathbb{S}^2, P_f)$. Recall that $\mathcal{T}_f$ can be defined as the quotient of the space of all diffeomorphisms from $(\mathbb{S}^2, P_f)$ to the Riemann sphere modulo a certain equivalence relation. We write $\tau = \langle h \rangle$ if a point τ is represented by a homeomorphism h.

Moduli space

The corresponding moduli (or configuration) space $\mathcal{M}_f$ is easy to understand. It is the space of all injections h from P_f into $\mathbb{P}$ up to Moebius transformations. If we fix values of all h on selected three points of P_f to be $0, 1, \infty$, then we see that M_f is canonically isomorphic to $\mathbb{C}^{p-3} \setminus \Delta$ where Δ is a union of hyperplanes given by $z_i = z_j$ or $z_i \in \{0, 1, \infty\}$.

Pullback map

Suppose we have a (unbranched) covering map $h\colon A \to B$ between finite type surfaces A and B. Then we can define $h^*\colon \mathcal{T}(B) \to \mathcal{T}(A)$ that acts by pulling back complex structures from B to A.

Maps between Teichmüller spaces

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Projection map

Suppose we have an inclusion map $i\colon A \to B$ between finite type surfaces A and B. This happens exactly when A can be obtained from B by removing finitely many points. Then we can define the forgetful projection $i_*\colon \mathcal{T}(A) \to \mathcal{T}(B)$ which "forgets" the positions of extra marked points.

Thurston's iteration

In our setting we have the unbranched cover $f\colon \mathbb{S}^2 \setminus f^{-1}(P_f) \to \mathbb{S}^2 \setminus P_f$ and the identity injection $\mathrm{id}\colon \mathbb{S}^2 \setminus f^{-1}(P_f) \to \mathbb{S}^2 \setminus P_f$ since $f^{-1}(P_f) \supset P_f$. Denote $\sigma_f = \mathrm{id}_* \circ f^*\colon \mathcal{T}_f \to \mathcal{T}_f$.

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Another definition of Thurston's iteration

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{P}, h_1(P_f)) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_\tau} \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} & (\mathbb{P}, h_\tau(P_f)) \end{array} \tag{1}$$

Fixed Points of σ_f

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Lemma

A Thurston map f is equivalent to a rational function if and only if σ_f has a fixed point.

Co-derivative of σ_f

Definition

The push-forward operator is locally defined by the formula

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Lemma

σ_f is a holomorphic self-map of $\mathcal{T}_f$ and the co-derivative of σ_f satisfies $(d\sigma_f(\tau))^* = (f_\tau)_*$ where $(f_\tau)_*$ is the push-forward operator on quadratic differentials.

Metrics on Teichmüller space

Metric definitions

For a meromorphic integrable quadratic differential on $\mathbb{P}$ we define

- its Teichmüller norm

$$\|q\|_T = 2\int_{\mathbb{P}} |q|$$

  and

- its Weil-Petersson norm

$$\|q\|_{WP} = \left(\int_{\mathbb{P}} \rho^{-2}|q|^2\right)^{1/2}$$

Estimates on the norm of $d\sigma_f^*$

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$\|(d\sigma_f)^*\|_T \leq 1.$

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Proof.

$$\int_U |g_* q| = \int_U |\sum_i g_i^* q| \leq \sum_i \int_U |g_i^* q| = \sum_i \int_{U_i} |q|$$

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Corollary (almost)

There exists at most one fixed point of σ_f, hence the uniqueness in Thurston's theorem follows.

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Corollary

σ_f is Lipschitz with respect to the WP-metric.

Boundaries of Teichmüller space

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The augmented Teichmüller space $\overline{\mathcal{T}}_f$ is homeomorphic to the WP-completion of the Teichmüller space.

Corollary

σ_f *extends continuously to* $\overline{\mathcal{T}}_f$.

No extension to the Thurston boundary

Theorem

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Recall that $\sigma_f = \mathrm{id}_* \circ f^*$ is a composition of

- $h^* : \mathcal{T}(\mathbb{S}^2, P_f) \to \mathcal{T}(\mathbb{S}^2, f^{-1}(P_f))$ that acts by pulling back complex structures
- $\mathrm{id}_* : \mathcal{T}(\mathbb{S}^2, f^{-1}(P_f)) \to \mathcal{T}(\mathbb{S}^2, P_f)$ which "forgets" the positions of extra punctures

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- $h^* : \mathcal{T}(\mathbb{S}^2, P_f) \to \mathcal{T}(\mathbb{S}^2, f^{-1}(P_f))$ that acts by pulling back complex structures — this map extends to any reasonable boundary notion
- $\mathrm{id}_* : \mathcal{T}(\mathbb{S}^2, f^{-1}(P_f)) \to \mathcal{T}(\mathbb{S}^2, P_f)$ which "forgets" the positions of extra punctures — goes in the "wrong direction", we can not push forward measured laminations

Structure of the boundary of the augmented Teichmüller space

- The augmented Teichmüller space $\overline{\mathcal{T}}_f$ is a stratified space with strata $\mathcal{S}_\Gamma$ corresponding to different multicurves Γ on $(\mathbb{S}^2, P_f)$. In particular, $\mathcal{T}_f = \mathcal{S}_\emptyset$.

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- Each point in the stratum $\mathcal{S}_\Gamma$ is a collection of complex structures on the components of the corresponding topological nodal surface with marked points.
- $\mathcal{S}_\Gamma$ is the product of Teichmüller spaces of these components.
- Within each stratum one can define its own natural Teichmüller and Weil-Petersson metrics.
- The quotient AM_f of $\overline{\mathcal{T}}_f$ by the action of the mapping class group is compact.

Definition of σ_f on the boundary

We represent points in $\overline{\mathcal{T}}_f$ not only by homeomorphisms but also by continuous maps from $(\mathbb{S}^2, P_f)$ to a nodal Riemann surface that are allowed to send a whole simple closed curve to a node. Consider such an h representing some point in $\overline{\mathcal{T}}_f$.

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We complete this diagram as before

$$
\begin{array}{ccc}
(\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (R_1, P_f) \\
\Big\downarrow f & & \Big\downarrow \{f^{C_i}\} \\
(\mathbb{S}^2, P_f) & \xrightarrow{h} & (R, P_f)
\end{array}
$$

Action of σ_f on $\overline{\mathcal{T}}_f$

Theorem

The map σ_f as defined above is continuous on $\overline{\mathcal{T}}_f$.

Remark.

Note that by definition σ_f maps any stratum $\mathcal{S}_\Gamma$ into the stratum $\mathcal{S}_{f^{-1}(\Gamma)}$, therefore invariant boundary strata are in one-to-one correspondence with completely invariant multicurves.

Classification of invariant boundary strata

Lemma

If $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ *is a completely invariant positive multicurve and* $\lambda_\Gamma \geq 1$*, then* $\mathcal{S}_\Gamma$ *is weakly attracting.*

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Lemma

If $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ *is a completely invariant multicurve and* $\lambda_\Gamma < 1$*, then* $\mathcal{S}_\Gamma$ *is weakly repelling.*

Sketch of the proof of Thurston's theorem

Pick any starting point $\tau \in \mathcal{T}_f$ and consider $\tau_n = \sigma_f^n(\tau)$. Take an accumulation point in $\overline{\mathcal{M}}_f$ of the projection of τ_n to the moduli space on the stratum of smallest possible dimension.
For simplicity we assume that τ_n accumulates on some $\tau_0 \in \mathcal{S}_\Gamma$.

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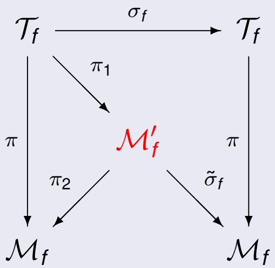
- If $\Gamma = \emptyset$ then τ_0 is a fixed point of σ_f.
- If $\Gamma \neq \emptyset$ then Γ must be a Thurston obstruction. Otherwise, $\mathcal{S}_\Gamma$ is weakly repelling and therefore τ_n can not have an accumulation point there.

Lemma

There exists an intermediate cover $\mathcal{M}'_f$ of $\mathcal{M}_f$ (so that $\mathcal{T}_f \xrightarrow{\pi_1} \mathcal{M}'_f \xrightarrow{\pi_2} \mathcal{M}_f$ are covers and $\pi_2 \circ \pi_1 = \pi$) such that

i. *π_2 is finite,*

ii.



commutes for some map $\tilde{\sigma}_f \colon \mathcal{M}'_f \to \mathcal{M}_f$,

iii. *If $\pi_1(\tau_1) = \pi_1(\tau_2)$ then $f_{\tau_1} = f_{\tau_2}$ up to pre- and post-composition by Moebius transformations.*

Pilgrim's theorems

Definition

The *canonical* obstruction Γ_f is the set of all homotopy classes of curves γ that satisfy $l(\gamma, \sigma_f^n(\tau)) \to 0$ for all $\tau \in \mathcal{T}_f$

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If for a Thurston map with hyperbolic orbifold its canonical obstruction is empty then it is Thurston equivalent to a rational function. If the canonical obstruction is not empty then it is a Thurston obstruction.

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Pilgrim's conjecture

The action on any invariant stratum on the boundary is given by pullbacks of complex structures by a collection of maps σ_{fC} for all components C of any surface in the stratum. Combinatorics of the process is very simple: we have a map from a finite set into itself, every component is pre-periodic. The whole action, therefore, can be characterized by studying cycles of components. For each cycle Y there are three cases, the composition f^Y of all coverings in the cycle is either of the following:

- a homeomorphism,
- a Thurston map with a parabolic orbifold,
- a Thurston map with a hyperbolic orbifold.

Definition

The *canonical* obstruction Γ_f is the set of all homotopy classes of curves γ that satisfy $I(\gamma, \sigma_f^n(\tau)) \to 0$ for all $\tau \in \mathcal{T}_f$

Theorem

If a cycle Y of components a topological surface corresponding to the stratum $\mathcal{S}_{\Gamma_f}$ has hyperbolic orbifold then f^Y is not obstructed and, hence, equivalent to a rational map.