Introduction to the Dynamics of Holomorphic Foliations

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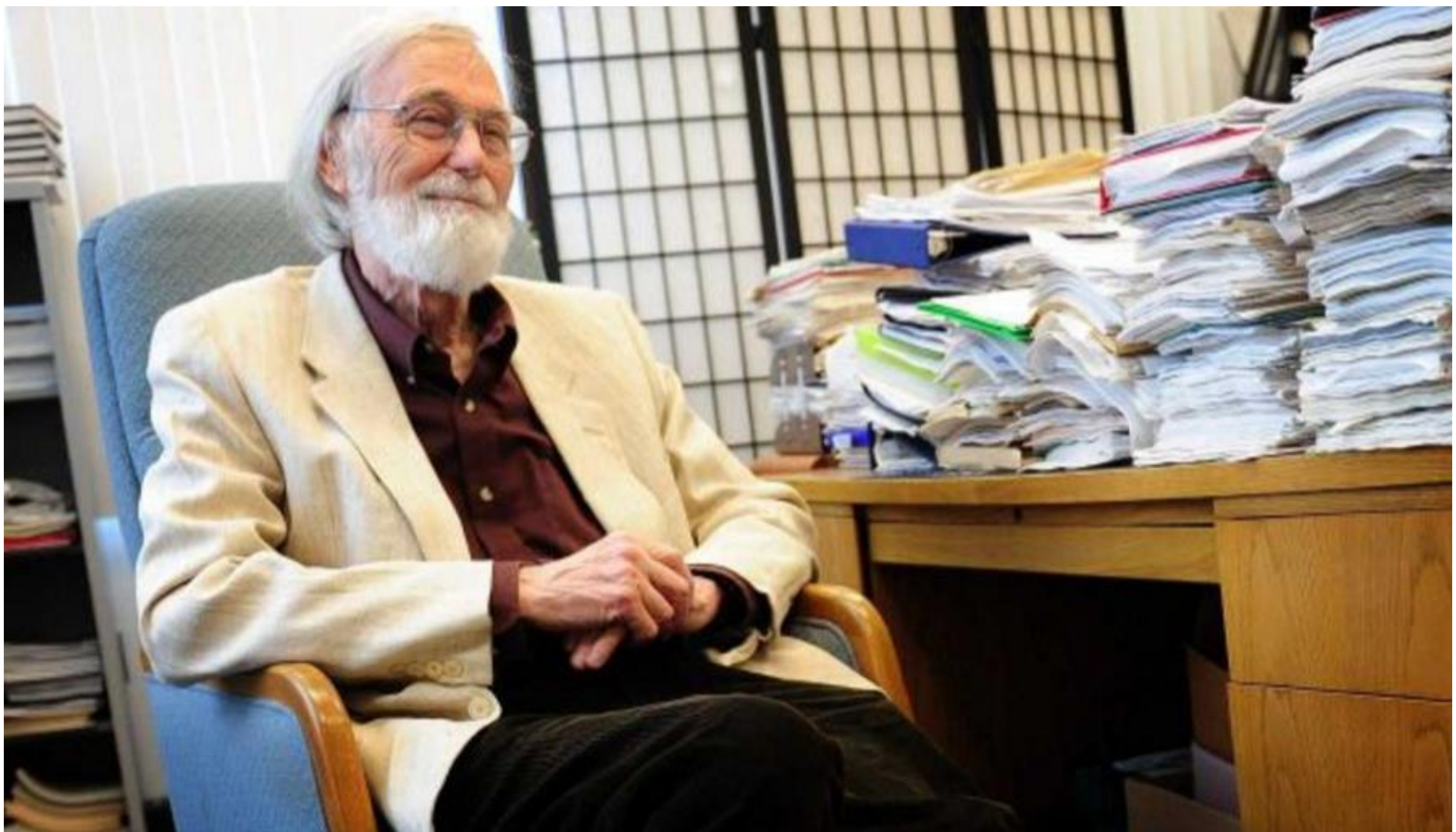
Jack “el Maravilloso”:

**You have this wonderful
ability to be everywhere in Math.
Everywhere I’ve gone, there you were,
always saying something deep and wonderful,
you make it all look so simple!!**



A Holomorphic Foliation is a Mathematical Object which

is very simple to prescribe (algebraically)

but is very elusive to describe (geometrically)

(failing to allow for a clear and complete mental grasp)

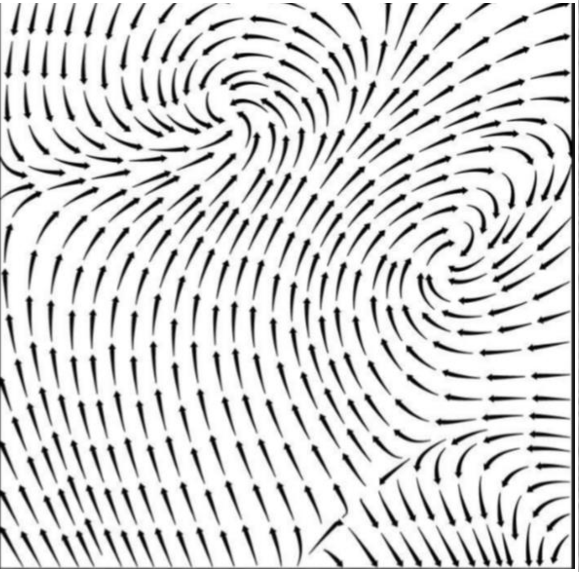
Algebraically, for $a_{j,k}, b_{j,k} \in \mathbb{C}, \quad j,k \geq 0$:

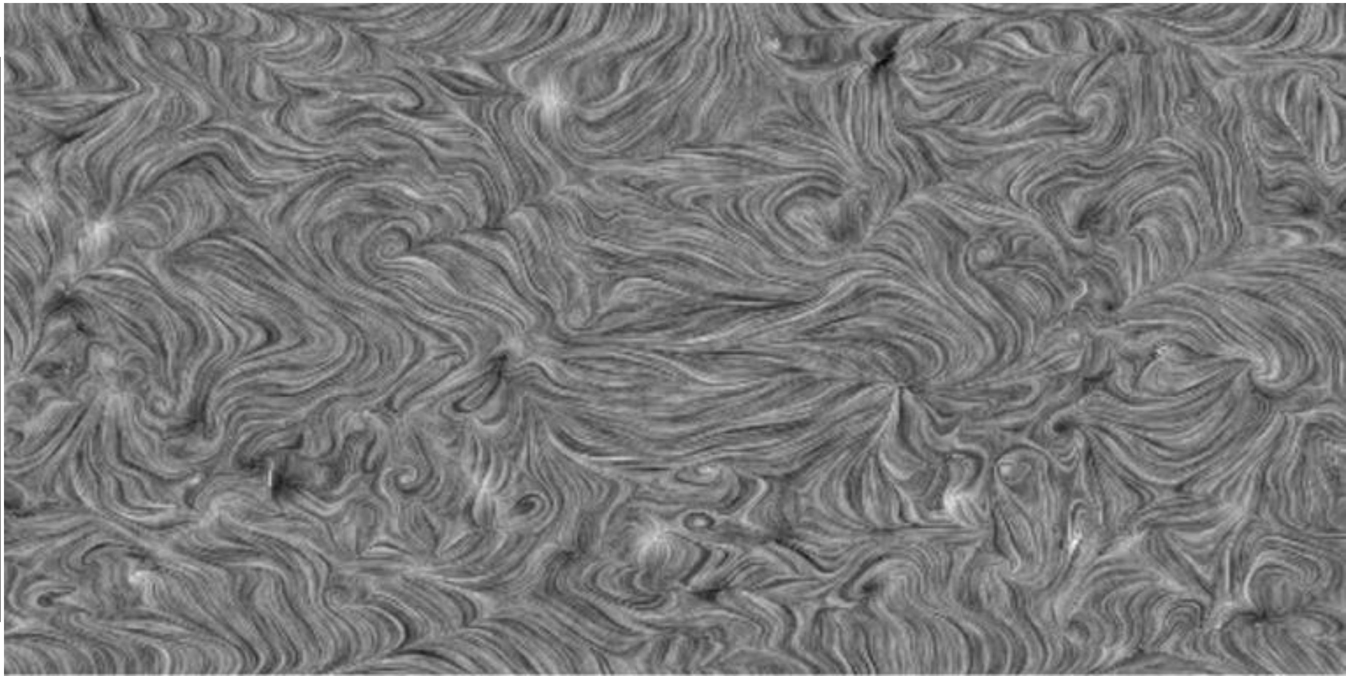
A polynomial vector field in 2 variables:

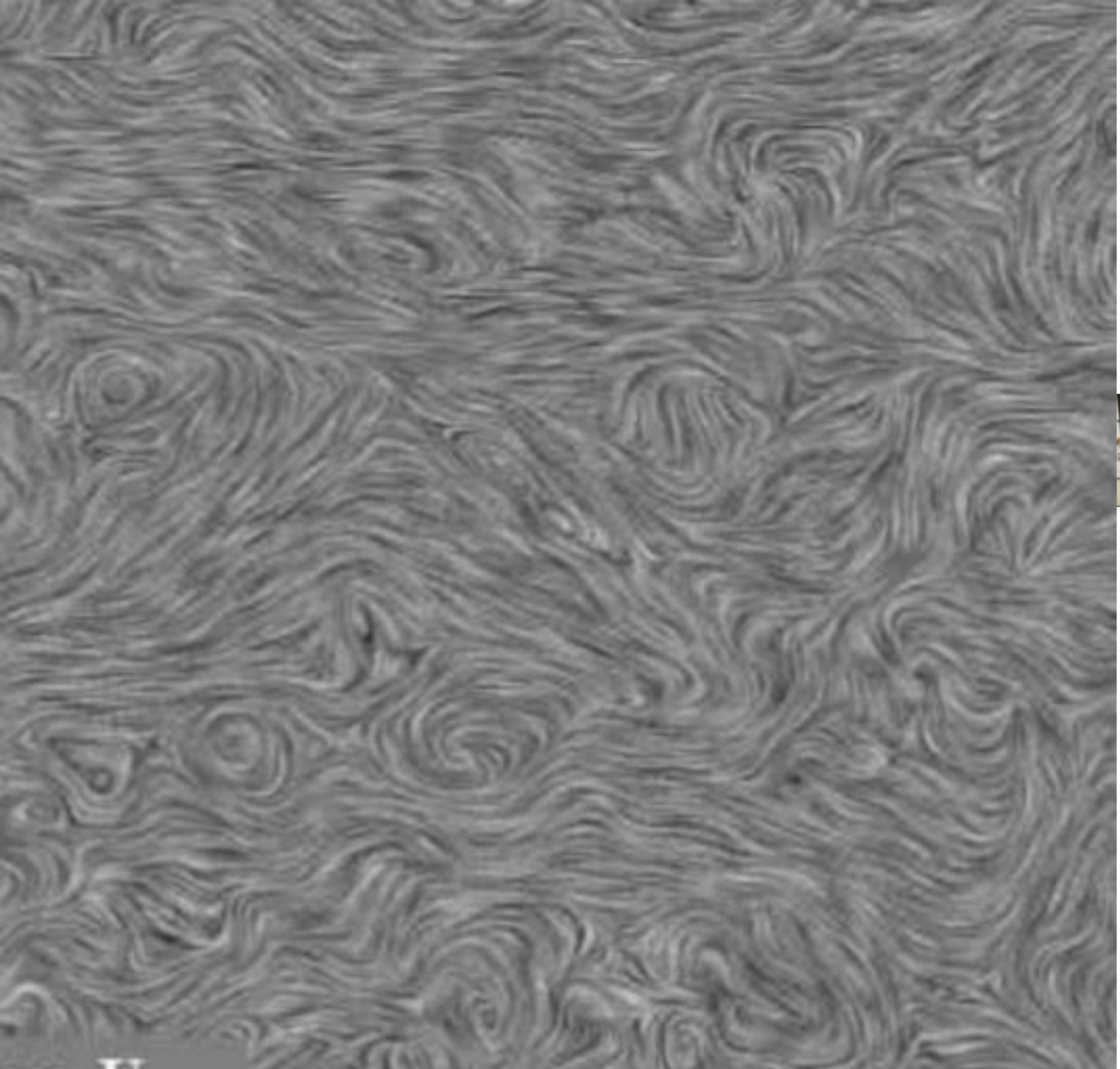
$$X := \sum_{j+k=0}^{d} a_{j,k} z^j w^k \frac{\partial}{\partial z} + \sum_{j+k=0}^{d} b_{j,k} z^j w^k \frac{\partial}{\partial w}$$

Geometrically, **the phase portrait of**

$$\left(\frac{\partial z}{\partial t}, \frac{\partial w}{\partial t}\right) = X(z,w) \quad , \quad t \in \mathbb{C}$$







What’s this mess?

Structure Theorem

Equidistri-bution Properties

Menu:

Entree

Structure Theorem:

Fatou-Julia-Sullivan decomposition into a finite number of components.

(Ingredients: Quasiconformal Maps, Beltrami Equation, Teichmuller Theory.)

Main Course

Equidistribution:

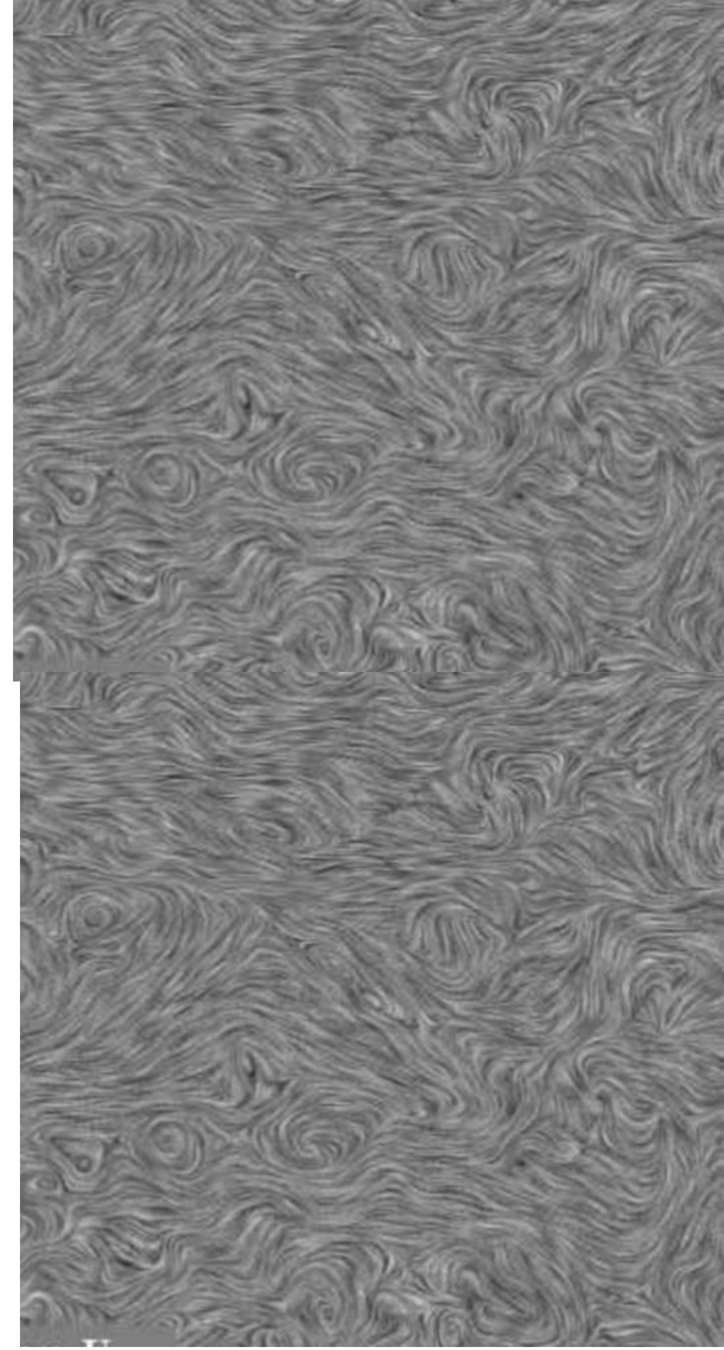
Existence of a finite number of measures capturing the assymptotic behaviour of almost every leaf.

(Ingredients: Hyperbolic geodesic Flow, Hopf's Argument, Partial Hyperbolic Dynamics.)

Dessert

Complex Lorenz Flow: Evidence of a Non-chaotic attractor.

Ingredients: Numerical Simulations.



Riccati Foliation:

$$X(z_1,z_2) = a(z_1)\frac{\partial}{\partial z_1} + [b_0(z_1) + b_1(z_1)z_2 + b_2(z_1)z_2^2]\frac{\partial}{\partial z_2}$$

Example:

$$1) X(z_1,z_2) = a(z_1)\frac{\partial}{\partial z_1} + b_0(z_1)\frac{\partial}{\partial z_2}$$

$$\frac{dz_2}{dz_1} = \frac{b_0(z_1)}{a(z_1)} = \sum \frac{c_j}{z_1 - d_j}$$

$$z_2(z_1) = \sum \int \frac{c_j dz_1}{z_1 - d_j} = \sum c_j Log(z_1 - d_j)$$

Additive Monodromy

$$\rho : \pi_1(\mathbb{C} - \{d_1, \ldots, d_r\}) \longrightarrow \mathbb{C}$$

$$2) Y(z_1,z_2) = a(z_1)\frac{\partial}{\partial z_1} + b_0(z_1)z_2\frac{\partial}{\partial z_2}$$

$$z_2(z_1) = e^{\sum \int \frac{c_j dz_1}{z_1 - d_j}} = \prod e^{c_j Log(z_1 - d_j)}$$

Multiplicative Monodromy

$$\rho : \pi_1(\mathbb{C} - \{d_1, \ldots, d_r\}) \longrightarrow \mathbb{C}^*$$

Ricatti Foliation:

$$X(z_1, z_2) = a(z_1)\frac{\partial}{\partial z_1} + [b_0(z_1) + b_1(z_1)z_2 + b_2(z_1)z_2^2]\frac{\partial}{\partial z_2}$$

Monodromy

$$\rho : \pi_1(\mathbb{C} - \{d_1, \ldots, d_r\}) \longrightarrow PSL(2, \mathbb{C})$$

Via the monodromy representation, Ahlfors finiteness Theorem for finitely generated discrete subgroups of $PSL(2, \mathbb{C})$ becomes a Theorem for Riccati equations

.

All you now about finitely generated subgroups of $PSL(2, \mathbb{C})$ becomes a Theorem for Riccati Equations.

Question: Does this generalize for general Holomorphic Foliations?

The space:

A compact complex 2-dimensional manifold S:

An Algebraic Surface

The Object:

A Rational Vector Field X on S

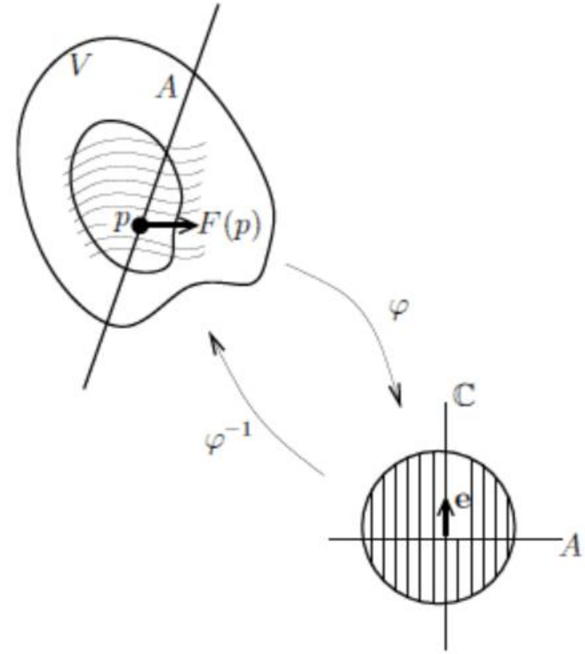
Main Algebraic or Topological Invariant:

The integer Homology class of the poles of X in $H^2(S,\mathbb{C})$

The Families of Objects:

Fixing the homology class of the pole produces finite dimensional compact families

Local Description:

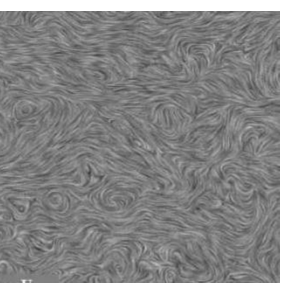


Cancel denominators:

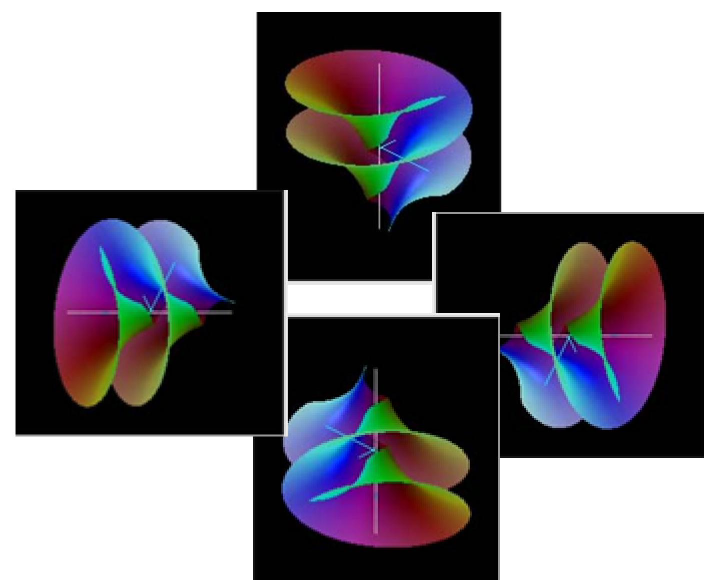
Holomorphic Vector Field X:

Non-Singular Points $X \neq 0$

Local Flow Box



$$w = z^{a+bi}$$



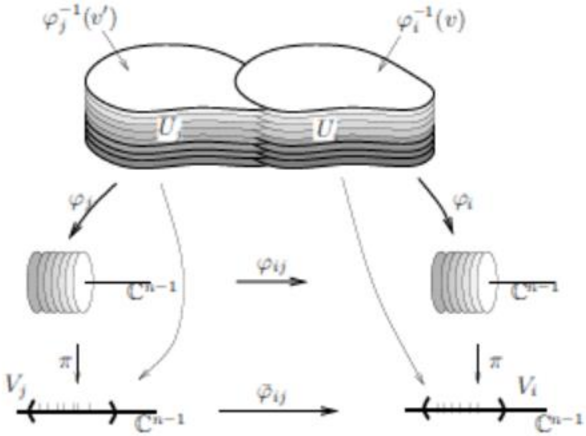
Singular Points:

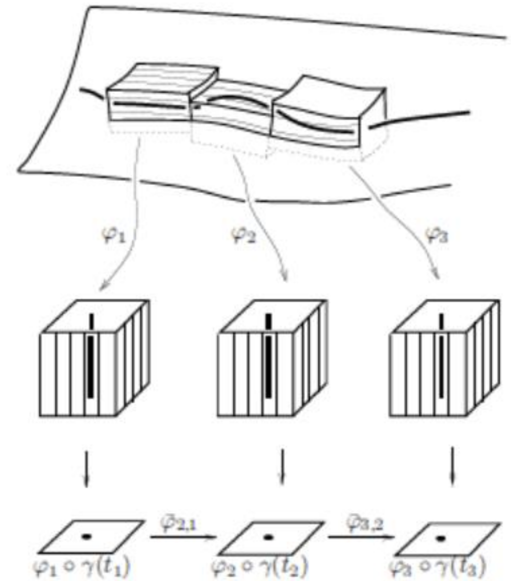
Blow Up

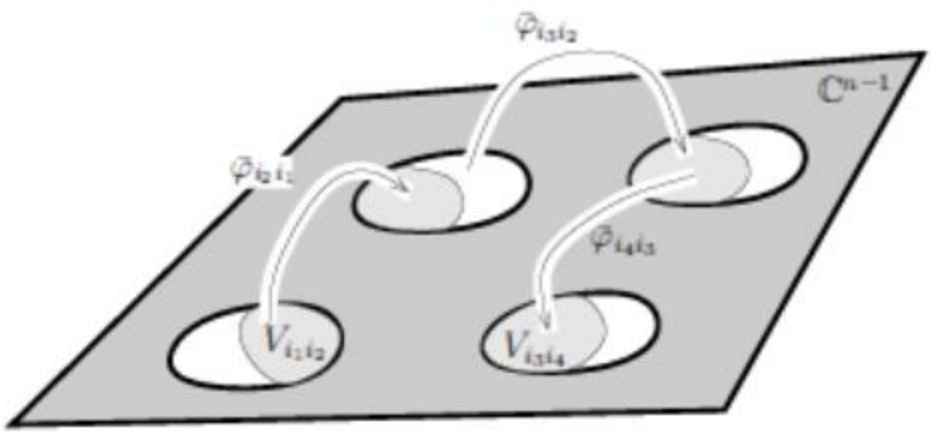
Generic Perturbation

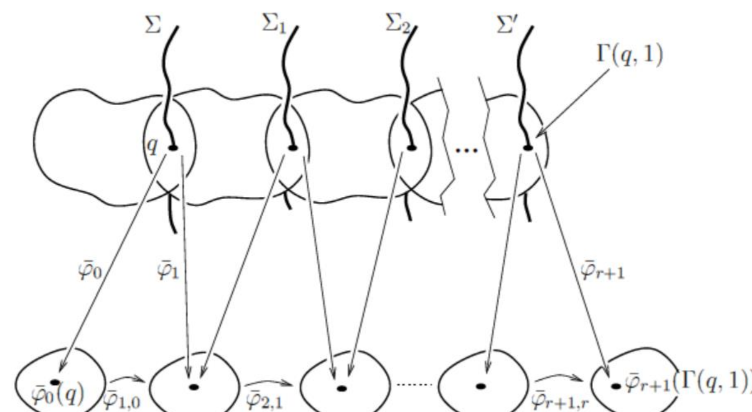
Poincaré Linealization

Transverse Dynamics
Holonomy Pseudogroup

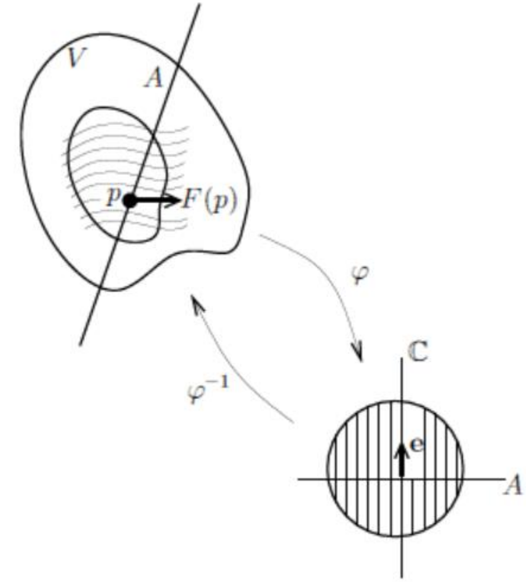








An infinitesimal automorphism of a holomorphic foliation $(S, \mathcal{F})$ is a vector field on S which in local foliated coordinates



$$(z_1, z_2) \to z_1$$

has the form

$$Y(z_1, z_2) = A(z_1)\frac{\partial}{\partial z_1} + B(z_1, z_2)\frac{\partial}{\partial z_1}$$

A , B are continuous of modulus $\varepsilon log(\varepsilon)$.

The term $A(z_1)\frac{\partial}{\partial z_1}$ is a 'normal vector field which is constant along the leaves'.

$\frac{\partial A}{\partial \overline{z_1}}(z_1)$ is a 'normal Beltrami differential constant along the leaves'.

We introduce the sheaves of functions

$\mathcal{FM}$=Foliated Measurable

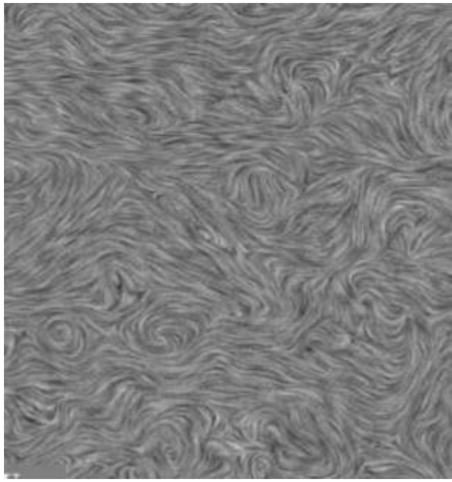
$\mathcal{FC}$=Foliated Continuous

$\mathcal{FO}$=Foliated Holomorphic

The sheaves of sections of the normal bundle $\nu^{1,0}$:

$\mathcal{FO}(\nu^{1,0})$=Foliated Holomorphic

$\mathcal{FC}(\nu^{1,0})$=Fol. Cont. sections σ with distributional derivatives in L^2 and $\bar{\partial}\sigma$ essentially bounded



The vector space of global sections

$$H^0(S, \mathcal{FC}(\nu^{1,0}))$$

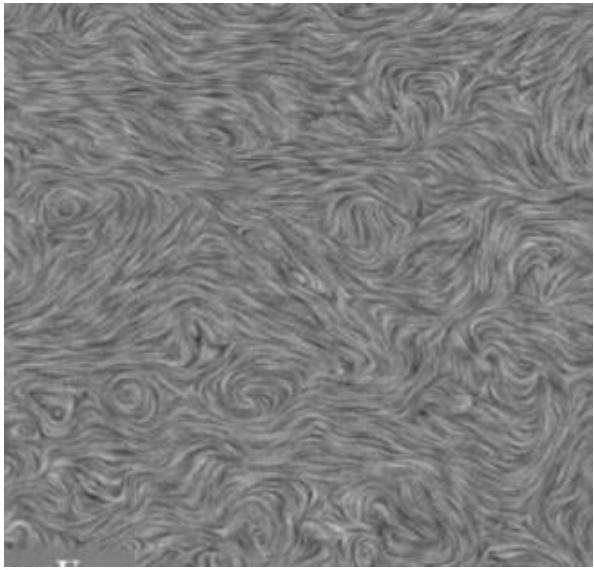
$$Fatou(\mathcal{F}) := \{x \in M \ / \exists X \in H^0(S, \mathcal{FC}(\nu^{1,0}))\ X(x) \neq 0\}$$

$$Julia(\mathcal{F}) := \{x \in M \ / X(x) = 0 \forall X \in H^0(S, \mathcal{FC}(\nu^{1,0}))\}$$

$$Fatou(\mathcal{F}) = \cup_k F_k$$

connected components (open and $\mathcal{F}$-saturated)

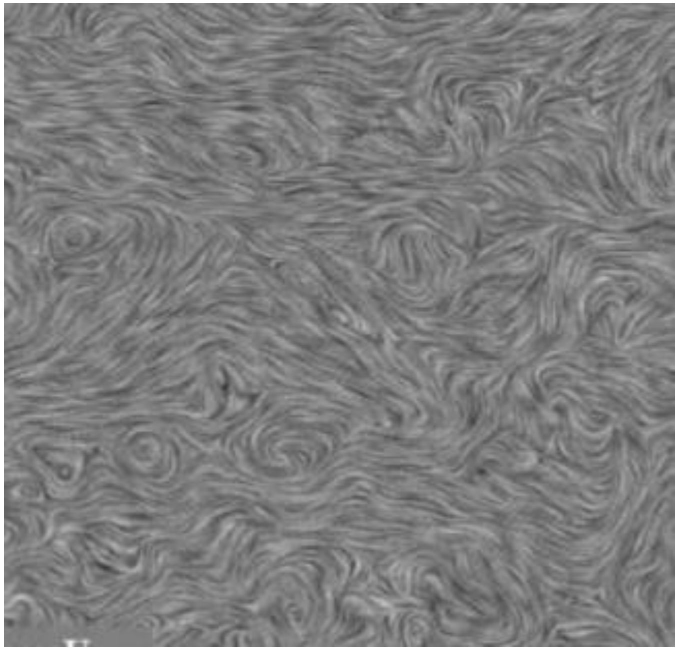
$$\mathcal{F}_k := \mathcal{F}|_{F_k}$$



The elements of $H^0(S, \mathcal{FC}(\nu^{1,0}))$ may be lifted to vector fields on M which are uniquely integrable giving rise to flows preserving the foliation.

Since the $\mathbb{C}$-codimension of the foliation is 1 we may multiply X by $e^{2i\pi\theta}$, and obtain that the foliation $\mathcal{F}_k$ is transitive, *i.e.* there are ambient leaf preserving isotopies sending any leaf in F_k to any other leaf of F_k.

Theorem 1 (Ghys,*,Saludes, 2001) Let $\mathcal{F}$ be a holomorphic foliation with Poincaré type singularities in the compact complex surface S and let $\mathcal{F}_k$ be the restriction of $\mathcal{F}$ to some connected component F_k of the Fatou set. Then there are three exclusive cases:



1) Wandering component: the leaves of $\mathcal{F}_k$ are closed in F_k.

2) Semi-wandering component: the closures of the leaves of $\mathcal{F}_k$ form a real codimension 1 foliation of F_k which has the structure of a fiber bundle over a 1-dimensional manifold.

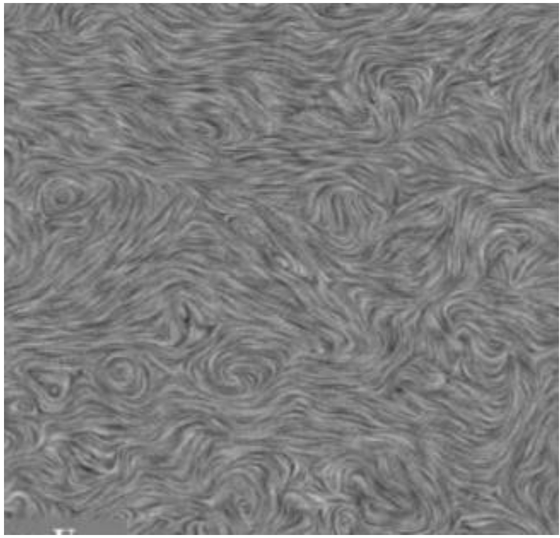
3) Dense component: the leaves of $\mathcal{F}_k$ are dense in F_k.

Theorem 2 1) Let F_k be a "wandering component" of the Fatou set. Then the leaf space of $\mathcal{F}_k$ is a finite Riemann surface Σ_k, *i.e.* it is Hausdorff and compact minus a finite number of points. The natural projection $F_k \to \Sigma_k$ has the structure of a locally trivial fiber bundle.

2) There is a finite number of "wandering components" in the Fatou set.

Theorem 3 (semi-wandering components): Let F_k be a "semi-wandering component" of the Fatou set. Then the closures of the leaves of $\mathcal{F}_k$ define a real analytic foliation $\bar{\mathcal{F}}_k$ given by a locally trivial fibration of F_k on the circle or an interval. The foliation $\mathcal{F}_k$ is a G-Lie foliation, where $G = \mathbb{C}$ or $Aff(\mathbb{R})$. The lift of $\mathcal{F}_k$ to the universal cover $\widetilde{F_k}$ is given by a locally trivial fibration of $\widetilde{F_k}$ onto some strip $\{z \in \mathbb{C} \mid \alpha < \Im(z) < \beta\}$ (with $-\infty \leq \alpha < \beta \leq +\infty$).

Theorem 4 (dense components): Let F_k be a "dense component" of the Fatou set. Then $\mathcal{F}_k$ is an ergodic foliation in F_k (with respect to the Lebesgue measure class of M). There are two possibilities:



1) $\mathcal{F}_k$ is an $\mathbb{R}^2$-Lie foliation. The Julia set consists of a finite number of compact leaves and the Fatou set is connected. The foliation is defined by a meromorphic closed basic 1-form having poles on the Julia set.

2) $\mathcal{F}_k$ is an $Aff(\mathbb{R})$-Lie foliation.

The lift of $\mathcal{F}_k$ to the universal cover $\widetilde{F_k}$ of F_k is given by the fibers of a locally trivial fibration of $\widetilde{F_k}$ onto $\mathbb{C}$ (in case 1) or onto the upper half space (in case 2).

We may then further decompose the Julia set of $\mathcal{F}$ in the measurable category. An $\mathcal{F}$-invariant measurable set $J \subset M$ is said to be *recurrent in the measurable sense* if there is no transversal disc D containing a Borel set $B \subset J \cap D$ with positive (2-dimensional) Lebesgue measure and such that distinct points in B are in distinct leaves of $\mathcal{F}$.

Theorem 5 Let $(M,\mathcal{F})$ be a transversely holomorphic foliated compact manifold such that the Lebesgue measure of the Julia set is positive. Then there is a (Lebesgue) measurable foliated partition of the Julia set $Julia(\mathcal{F}) = J_0 \cup \ldots \cup J_r$, $r \geq 0$ such that:

1) For $k \geq 1$ the sets J_k have positive Lebesgue measure and $\mathcal{F}|_{J_k}$ is ergodic with respect to the Lebesgue measure class. The space of essentially bounded measurable basic Beltrami differentials on J_k is 1-dimensional.

2) J_0 is empty or it is a recurrent set in the measurable sense. There are no non-zero essentially bounded measurable basic Beltrami differentials on J_0.

Main Point:
Infinitesimal Teichmuller Theory

Teichmuller Theory
Foliated Beltrami Equation:

A measurable Beltrami coefficient is $\mu \in \mathcal{FM}(\nu^{1,0} \otimes \nu^{*0,1})$, with $\|\mu\|_\infty < 1$

Foliated Beltrami Equation

$$\frac{\partial \phi}{\partial \overline{z}} = \mu \frac{\partial \phi}{\partial z}$$

The solutions construct for us a new transversely holomorphic foliation $\mathcal{F}_\mu$ on S which is topologically $\mathcal{F}$ but has new transversal structure.

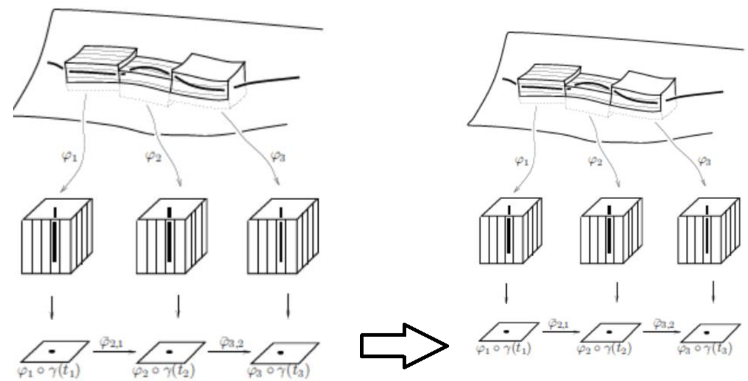
We have a universal family of these parametrized by a ball

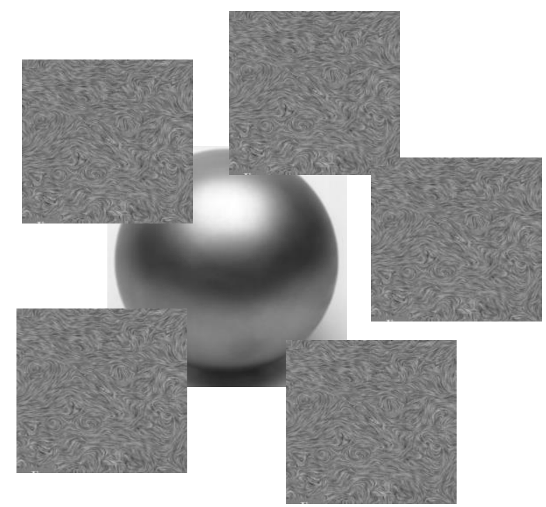
$$H^0(S, \mathcal{FM}(\nu^{1,0} \otimes \nu^{*0,1}))_1$$

$$Teich(\mathcal{F}) := H^0(S, \mathcal{FM}(\nu^{1,0} \otimes \nu^{*0,1}))_1 / \sim$$

New Category:
Transversely Holomorphic Foliations

$\mathcal{F}$ $\qquad\qquad$ $\mathcal{F}_\mu$





Infinitesimal Teichmuller Theory

Understand the map

$$\overline{\partial} : H^0(S, \mathcal{FC}(\nu^{1,0})) \longrightarrow H^0(S, \mathcal{FM}(\nu^{1,0} \otimes \nu^{*0,1}))$$

Fundamental Fact:

The Kernel is finite dimensional
$H^0(S, \mathcal{FO}(\nu^{1,0}))$
The image is closed of finite codimension
and the cokernel embeds in
$H^1(S, \mathcal{FC}(\nu^{1,0}))$.

Remark:
Both are finite dimensional
or both are infinite dimensional

Right hand side is with measurable
coefficients
So no problem on gluing on open sets

New Ingredient for the main course: Geodesic Flow on Comp. Hyperbolic Surfaces:

Let C be a compact Riemann surface of genus $g \geq 2$ provided with its hyperbolic metric, obtained from the Poincaré metric on the unit disc and the Uniformization Theorem. Let T^1C be the unit tangent bundles to C

$$\varphi : T^1S \times \mathbb{R} \longrightarrow T^1C$$

the geodesic flow. T^1C has the Liouville measure $dLiouv$ (hyperbolic metric on C, Haar measure on T^1_pS) which is φ-invariant.

Theorem(E. Hopf): The geodesic flow φ is ergodic, i.e. For almost any initial $v_p \in T^1S$ the geodesic starting at v_p equidistributes on T^1S according to $dLiouv$ i.e.

$$lim_{T\to\infty} \frac{\varphi(v_p, [0,T]_*(dLebes_{[0,t]})}{T} = dLiouv$$

Another new ingredient:

Oseledec's Theorem: Let μ be an ergodic invariant measure on the dynamical system $\phi : M \times \mathbb{R} \to \mathcal{M}$ and C a multiplicative cocycle of the dynamical system such that for each $t \in T$, the maps $x \to \log \|C(x,t)\|$ and $x \to \log \|C(x,t)^{-1}\|$ are L^1-integrable with respect to μ. Then for μ-almost all x and each non-zero vector $u \in \mathbb{R}^n$ the limit

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log \frac{\|C(x,t)u\|}{\|u\|}$$

exists and assumes, depending on u but not on x, up to n different values. These are the Lyapunov exponents. Further, if $\lambda_1 > ... > \lambda_m$ are the different limits then there are subspaces

$$R^n = R_1 \supset ... R_m \supset R_{m+1} = \{0\}$$

such that the limit is λ_j for $u \in \mathbb{R}_j - \mathbb{R}_{j+1}$ and $j = 1, ..., m$.