SOME OPEN PROBLEMS IN SYMPLECTIC TOPOLOGY

1. Suppose that S^1 acts symplectically on the closed symplectic manifold (M, ω) with isolated fixed points. Is the action Hamiltonian? The answer is *YES* if the action is semifree (Tolman-Weitsman, Topology (2000); math/9812006), but unknown in general. (The recent preprint posted by Kim, math/0704.2639, has a sign mistake.)

2. Is it true that for every closed symplectic manifold (M, ω) the Hamiltonian group $\mathrm{Ham}(M)$ has infinite diameter with respect to the Hofer norm? This question is unknown even in the case when $M = S^2 \times S^2$ with the standard symplectic form $\omega = \lambda pr_1^*(\sigma) + pr_2^*(\sigma)$, $\lambda > 1$. If $\lambda = 1$, then $\pi_1(\mathrm{Ham}(M))$ is finite and $\mathrm{Ham}(M)$ has infinite diameter by the results of EntovPolterovich (math/0205247), but when $\lambda > 1$ their arguments show only that the universal cover $\widetilde{\mathrm{Ham}}(M)$ has infinite diameter.

3. Define (M, ω) to be `symplectically (strongly) rationally connected` (`src`, for short) if the class pt of a point in the quantum homology $QH_*(M)$ of M has nonzero square: i.e. $pt * pt \neq 0$. Does this condition have any implications for the structure of $QH_*(M)$? For example, if one chooses the coefficients Λ for $QH_*(M)$ to be a field, then $QH_*(M)$ splits as a sum of indecomposable Frobenius algebras A_i, and one could ask if one of these algebras must be a field. Here one can distinguish between summands whose unital element e_i lies in the subspace

$$\mathcal{Q}_- := \bigoplus_{i < dim\, M} H_i(M) \otimes \Lambda$$

and those where $e_i \notin \mathcal{Q}_-$. Examples were found in McDuff (math/0706.0675) of non-src manifolds with a field summand A_i with a unital element $e_i \in \mathcal{Q}-$. Hence it might be better to restrict this question to ask for field summands with $e_i \notin \mathcal{Q}_-$. Note that if such a summand did exist then, because $pt * e_i \neq 0$, we would have to have $(pt * e_i)^k \neq 0$ for all k. The answer is unknown even in the case of projective manifolds, though here the minimal model conjecture might have something to add.

4. What is the relationship between the Gromov-Witten theory of an orbifold and that of its crepant resolution? More precisely, let $\mathcal{X}$ be a Gorenstein orbifold, and suppose it admits a crepant resolution Y. (Algebraically, let X be the moduli scheme of $\mathcal{X}$, and let Y be a crepant resolution of X, if it exists.) It has been conjectured by Ruan that the GW-theories of Y and $\mathcal{X}$ are isomorphic (math/0108195). This has been formulated precisely in genus zero for manifolds satisfying a hard Lefschetz property, and proved in some cases, by Bryan-Graber (math/0610129). It has been proven in genus zero for all type A surface singularities by Coates-Corti-Iritani-Tseng (math/0704.2034). Many other special cases have been studied as well. Can this be shown to hold in genus zero for all crepant resolutions of Gorenstein orbifolds? What is the correct formulation in higher genus?