ON SINGULAR SOLUTIONS OF NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS.

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. SOME SMALL RESULTS INCLUDING FOR VISCOSITY SOLUTIONS — ALL RELATED TO MAX. PRINCIPLES.

RECALL STRONG MAX. PRINC.

FOR ELLIPTIC OPERATORS

$$L = a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i \frac{\partial}{\partial x_i} + c$$

WITH BOUNDED COEFF.

$$|b_i|, |c| \leq C$$

AND UNIFORMLY ELLIPTIC

$$\frac{|\xi|^2}{C} \leq a_{ij}\,\xi_i \xi_j \leq C|\xi|^2 \quad , \xi \in \mathbb{R}^n$$

STRONG MAX PRINC. IF $u \in C^2$

$u \geq 0$, $Lu \leq 0$ IN $\Omega \subset \mathbb{R}^n$

(Ω CONNECTED OPEN SET), $u(\bar{x}) = 0$ SOME $\bar{x} \in \Omega$

THEN $u \equiv 0$.

SIMPLE COROLLARY (NONLINEAR)

SUPPOSE $u \ge v$ IN Ω

$$F(x, u, Du, D^2u) \le F(x, v, Dv, D^2v) \text{ IN } \Omega$$

AND

$$u(\bar{x}) = v(\bar{x}) \text{ FOR SOME } \bar{x} \in \Omega$$

THEN $u \equiv v$.

HERE F IS UNIFORMLY ELLIPTIC

$$|F_u|, |F_{p_i}| \le C$$

$$\frac{|\xi|^2}{C} \le F_{u_{ij}} \xi_i \xi_j \le C|\xi|^2.$$

DEGENERATE ELLIPTICITY MEANS

$$F(x, u, p, A) \ge F(x, u, p, A+B)$$

FOR ANY NONNEGATIVE SYMMETRIC MATRIX

$$B \ge 0.$$

WE FIRST STUDY A FUNCTION

u DEFINED IN

$$\Omega \setminus \{x_0\},$$

AND A SMOOTH FUNCTION

$$v \text{ IN } \Omega,$$

WITH $u > v$ IN $\Omega \setminus \{x_0\}$

SATISFYING THERE

$$F(x, u, Du, D^2u) \le F(x, v, Dv, D^2v) \quad (2)$$

i.e. u IS A SUPERSOLUTION OF (2).

<u>QUESTION</u>: STRONG MAX. PRINC?

i.e IS

$$\liminf_{x \to x_0} (u(x) - v(x)) > 0 \ ?$$

HERE WE ASSUME ELLIPTICITY OF F BUT NOT UNIFORM ELLIPTICITY SINCE u AND Du MAY BE UNBOUNDED

MOTIVATION: TO PROVE MONOTONICITY & SYMMETRY FOR A SINGULAR SOLUTION IN, SAY, A BALL

$$\hat{B} = B_R - \{0\}.$$

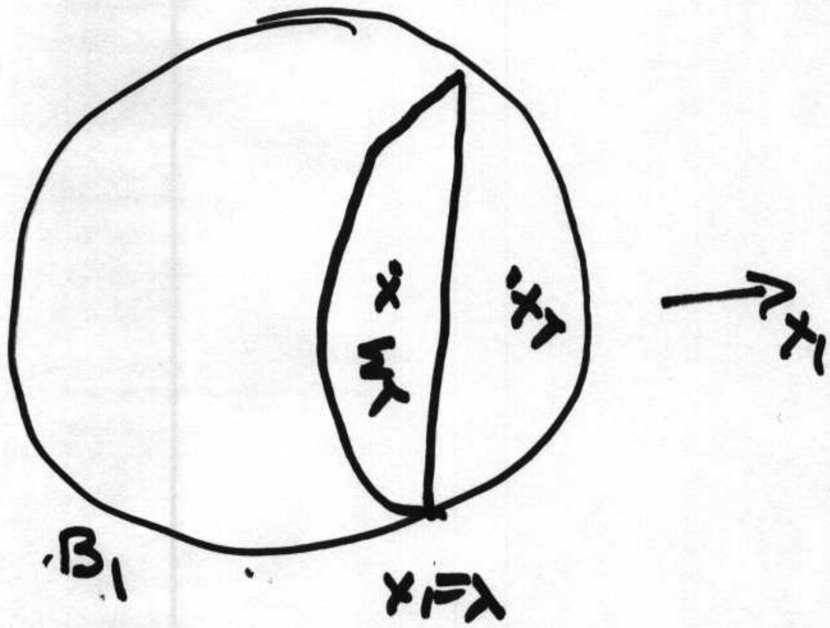
WITH $u > 0$ THERE AND $u = 0$ ON ∂B_R. HERE

$$F(x, u, Du, D^2u) = 0 \text{ IN } \hat{B}.$$

UNDER STANDARD CONDITIONS ON F & UNIFORMLY ELLIPTIC, IF u IS DEFINED IN ALL OF B_R THEN $u = u(r)$ AND $u_r < 0$ FOR $0 < r < R$

THE PROOF USES

METHOD OF MOVING PLANES
(A. D. ALEXANDROFF)



ONE SHOWS THAT $u(x) > u(x_\lambda)$.

BUT IF u IS NOT DEFINED AT ORIGIN, THE ARGUMENT BREAKS <u>DOWN</u> IF $\lambda < \frac{1}{2}$.

IF THE ANSWER TO QUESTION 1 WERE YES, THE METHOD COULD BE CARRIED OUT.

BUT THE ANSWER TO QUESTION 1 IS <u>NO</u> IN GENERAL

<u>EX</u> $x_0 = \{0\}$. $u = |x|$, $v = 0$

$$F(x, u, Du, D^2u) = -e^{-\Delta u} + 1 - |Du|^2$$

$$\leq 0 = F(x, v, D^2v)$$

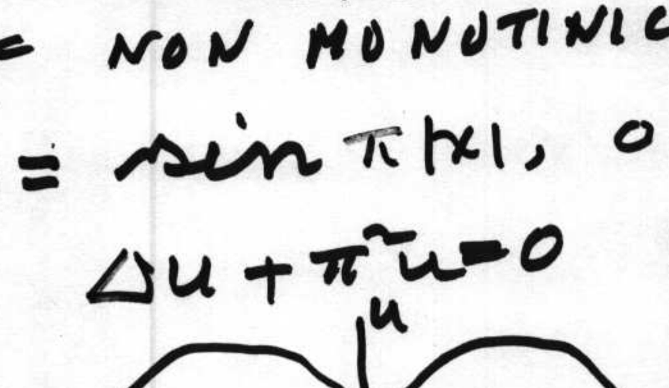
~~WE FOUND DIFFERENT SUFFICIENT CONDITIONS SO THAT ANSWER TO QUESTION 1 IS YES~~

~~WE INTRODUCED~~

EX OF NON MONOTINICITY IN 1 DIM.

$u = \sin \pi |x|$, $0 < |x| \leq 1$

$\Delta u + \pi^2 u = 0$ FOR $x \neq 0$



IN ~~HIGHER~~ HIGHER DIM. FOR

$u > 0$, $\Delta u + f(u) = 0$

IN $0 < |x| < 1$, $u = 0$ for $|x| = 1$.

IN A TALK IN ROME I POSED

THE QUESTION: IS

$u = u(r)$

$u_r < 0$ FOR $0 < r < 1$?

SUSANNA TERRACINI, AFTER THE

LECTURE CAME UP WITH A

POSITIVE ANSWER. BUT IT DOES

NOT WORK FOR GENERAL CASE

$F(x, u, Du, D^2 u) = 0 \quad 0 < r \leq 1$

THERE ARE COUNTEREXAMPLES

<u>POSITIVE RESULT</u>: $F(x, u, p, A)$

$= G(x, u, p, A) + f(x, u)$

IF G IS CONVEX IN (u, p, A), AND f IS

LOCALLY LIPSCHITZ IN u. (+ USUAL CONDITIONS)

§ 2 VISCOSITY SOLUTIONS AND "LOWER CANONICAL FUNCTIONS".

BEFORE CONSIDERING ~~PROBLEM~~ QUESTION 1 IN GENERAL WE CONSIDER A WIDER CLASS OF SINGULAR SOLUTIONS : VISCOSITY SOLUTIONS.

UNLIKE DISTRIBUTIONS WHICH TALKS ABOUT POSSIBLE SINGULAR SOLUTIONS IN AN OPEN SET, VISCOSITY SOLUTIONS TALK ABOUT SINGULAR SOLUTIONS AT A SINGLE POINT

UNDER SOME CONDITION WE SHOW THAT u IS A VISCOSITY SUPERSOLUTION OF (2).

RECALL VISCOSITY SUPERSOLUTION

OF $F(x, u, Du, D^2u) \le f(x)$ (3)

<u>MOTIVATION</u> SUPPOSE $u \in C^2$

AND (3) HOLDS WITH F ELLIPTIC.

IF $\varphi \in C^2$ AND $\varphi \le u$ AND

$\varphi(x_0) = u(x_0)$

THEN $\nabla \varphi = \nabla u$ THERE AND

$D^2\varphi(x_0) \le D^2 u(x_0)$ AS MATRICES

BY ELLIPTICITY

AT x_0,

$$F(x, \varphi, D\varphi, D^2\varphi) \le F(x, u, Du, D^2u) \le f.$$

THIS IS BASIS FOR

DEF'N OF VISCOSITY SUPERSOLUTION

SUPPOSE u IS CONTINUOUS (l.s.c. SUFFICES), WE SAY THAT AT A POINT $x_0 \in \Omega$ u SATISFIES

$F(x_0, u, Du, D^2u) \le f(x_0)$, VISCOSITY SENSE

PROVIDED FOR EVERY C^2 FUNCTION $\varphi \le u$ WITH

$$\varphi(x_0) = u(x_0),$$

$$F(x_0, \varphi(x_0), D\varphi, D^2\varphi) \le f(x_0).$$

SOURCE OF NOTATION

ADD $2\Delta u$

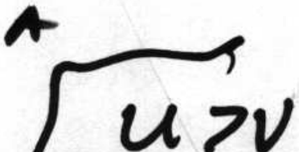
CORRESPONDING IDEA OF VISCOSITY SUBSOLUTION OF

$F(x_0, u(x_0), Du, D^2u) \geq f(x_0)$

IF u IS BOTH SUPER AND SUB VISCOSITY SOL'N IT IS CALLED A VISCOSITY SOLUTION.

Ex $u = -|x|$ IS A VISCOSITY SUPERSOLUTION AT $\{0\}$ OF <u>ANY</u> ELLIPTIC OPERATOR.

RETURN TO STRONG MAX PRINC. CONSIDER u A VISCOSITY SUPER SOL'N OF



$F(x, u, Du, D^2u) \leq F(x, v, Dv, D^2v)$

IN Ω, F ELLIPTIC (BUT NOT UNIFORMLY) VISCOSITY SENSE $v \in C^2$

WE ASSUME v is in C^2

AND u is l.s.c.

QUESTIONS & FF

$$u(x_0) = v(x_0)$$

FOR SOME $x_0 \in \Omega$, IS

$V \equiv v$? QUESTION NO / SMOOTH / $F(...) \mid SO \ldots$

TH'M. YES

SUPPOSE u IS A VISCOSITY SOLUTION IN $\Omega - \{x_0\}$, $v \in C^2(\Omega)$. AND $u \geq v$ IN $\Omega - \{x_0\}$ AND

AND $F(x, u \ldots) \leq F(x \ldots v \ldots D^2 v)$ VISCOSITY

SENSE IN $\Omega \setminus \{x_0\}$. WHEN IS

$$\liminf_{x \to x_0} (u(x) - v(x)) > 0 \, ?$$

SUFF. CONDITION: IT INVOLVES A NEW, WE BELIEVE, CLASS OF FUNCTIONS

DEFINITION: LOWERCONICAL.

A FUNCTION u, (BOUNDED BELOW AND LSC) IS LOWERCONICAL

AT A POINT $\bar{x} \in \Omega$ IF FOR

ANY $\eta \in C^\infty$, AND FOR ANY

$\varepsilon > 0$,

$$\inf_{x \in \Omega} \left((u+\eta)(x) - (u+\eta)(\bar{x}) - \varepsilon |x - \bar{x}| \right)$$

(i.e.) SOMEWHERE < 0.

This makes sense for a function on a Riemannian manifold,

REMARK: IF ON A C^1 CURVE

$\gamma(t)$, $\gamma(0) = \bar{x}$, $u(\gamma(t))$ IS

DIFFERENTIABLE AT ~~t~~=0

THEN IT IS ~~UPPER~~ LOWER CONICAL

(AND ~~LOWER~~ UPPER CONICAL) AT $\bar{x}$. BUT

$u = |x|$ IS NOT LOWER-CONICAL AT 0?

<u>LEMMA:</u> SUPPOSE u IS (LSC), BOUNDED BELOW SATISFYING

$$F(x, u, Du, D^2u) \leq f \text{ IN } \Omega - \{\bar{x}\}$$

(f IS USC). IF u IS LOWER CONICAL AT $\bar{x}$ THEN

$$F(\bar{x}, u, Du, D^2u) \leq f(\bar{x})$$

IN VISCOSITY SENSE

<u>SO $\bar{x}$ IS A REMOVABLE SINGULTARITY</u>

<u>CONSEQUENCE</u> IF IN QUESTION

<u>u IS LOWER CONICAL AT x_0, u IS (LSC), $u > v$ IN $\Omega \sim \{x_0\}$, $v \in C^2(\Omega)$ IN VISCOSITY SENSE</u> THEN

$$\liminf_{x \to x_0} (u(x) - v(x)) > 0.$$

REMARK THE CONDITION LOWER CONICAL IS ALMOST NECESSARY FOR A VISCOSITY SUPERSOLUTION (u LSC) OF $F(x, u, Du, D^2u) \leq f(x)$, in VISCOSITY SENSE.

NAMELY [4]: SUPPOSE f IS ALSO ~~BDD~~ bounded ABOVE. SUPPOSE, ALSO, THAT

$$(*) \quad \limsup_{a \to \infty} \inf_{x \in \Omega, |(s,p)| \leq \beta} F(x, s, p, aI) = \infty \quad \forall \beta > 0$$

S−ΔU NOT

(this holds if F is uniformly elliptic.)

THEN u IS LOWER CONICAL AT EVERY POINT.

HOWEVER, ANY u SATISFIES

$$-e^{-\Delta u} \leq 0, \text{ VISCOSITY SENSE}$$

THIS OPERATOR DOES NOT SATISFY (*).

<u>REMARK</u> IF u IS SINGULAR ON A SUBMANIFOLD WE HAVE RESULTS FOR VISCOSITY SUPERSOLUTION AND STRONG MAX PRINC.

IN OUR PROOF THAT A VISCOSITY SUPERSOLUTION OF

$$F(y, u, Du, D^2 u) \leq f$$

(UNIFORM ELLIPTICITY IS NOT ASSUMED) OUTSIDE A POINT $\bar{x}$ IS ONE ALSO AT $\bar{x}$ IF u IS LOWER CONICAL AT $\bar{x}$ WE USE:

SHARPENING OF THE HOPF LEMMA

HERE L IS UNIFORMLY ELLIPTIC OPERATOR

~~THM LET $B \subset \Omega$ BE A BALL. THERE EXIST CONSTANTS $\varepsilon, \mu > 0$ DEPENDING ONLY ON C (ELLIPTICITY CONSTANT), B AND Ω S.T. IF U IS (LSC)~~

THM LET $B \subset \Omega$ BE A BALL. $\exists$ CONSTANTS $\bar{\varepsilon}, \mu > 0$, DEPENDING ONLY ON C (ELLIPTICITY CONSTANT), B AND Ω, SUCH THAT IF u (LSC) AND

$$Lu \leq \bar{\varepsilon} \text{ IN } \Omega, \text{ VISCOSITY SENSE}$$

AND

$$u \geq 0 \text{ IN } \Omega$$
$$u \geq 1 \text{ IN } B,$$

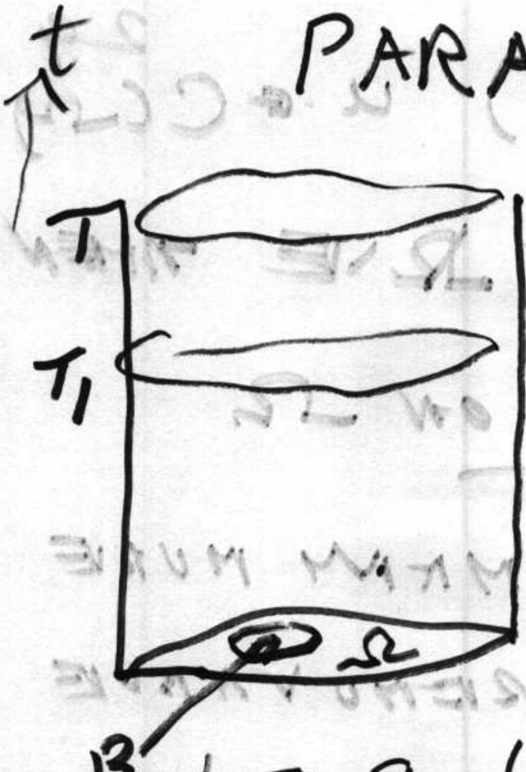
THEN

$$u(x) \geq \mu \operatorname{dist.}(x, \partial\Omega).$$

IN USUAL HOPF LEMMA, $\bar{\varepsilon} = 0$.

WE HAVE 2 PROOFS OF THIS. ONE FOLLOWS FROM PARABOLIC VERSION:

PARABOLIC VERSION



Ω HAS C^2 BOUNDARY

B IS A BALL IN Ω

$\Omega \subset \mathbb{R}^n$

$$L = a_{ij}(x,t)\,\partial_{ij} + b_i(x,t)\,\partial_i + c(x,t)$$

UNIFORMLY ELLIPTIC, COEFF. DEPEND ON t.

<u>THEOREM</u> [4]. $\exists\ \varepsilon, \mu > 0$ DEPENDING ONLY ON ELLIPTICITY CONSTANTS, n, Ω, B AND $0 < T_1 < T$. SUCH THAT IF u IS (LSC) IN

$$\bar{\Omega} \times (0, T]$$

AND

(i) $u(x,0) \geq 1$ IN B

(ii) $(L - \partial_t) u \leq \varepsilon$ IN $\Omega \times (0,T]$ VISCOSITY SENSE

AND

(iii) $u \geq 0$ ON PARABOLIC BOUNDARY

<u>THEN</u>

$u(x,t) \geq \mu \, dist(x, \partial\Omega)$ ON $\Omega \times [T_1, T]$

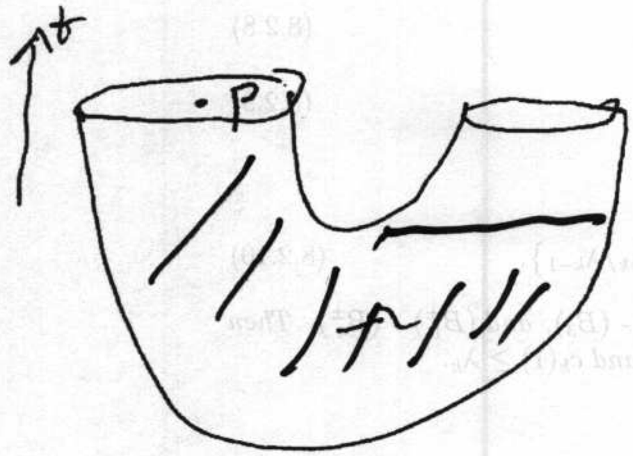
<u>NOTE</u>: u NEED NOT BE > 0 EVERYWHERE

(20)"

HARVEY & LAWSON HAVE WRITTEN A VERY INTERESTING PAPER ON REMOVABLE SINGULARITIES FOR FULLY NONLINEAR ELLIPTIC EQUATIONS. THEY INCLUDE A SIMPLER PROOF THAN OURS FOR REMOVABLE SINGULARITY UNDER CONDITION LOWER CONICAL.

§3

STRONG MAX PRINC FOR NONLINEAR PARABOLIC OPERATOR



u is lsc in Ω

v is $C^2(\Omega)$

$u \geq v$ in $\Omega \cup$ TOP.

$$F(x,t,u,Du,D^2u) - u_t \leq F(x,t,v,Dv,D^2v) - v_t$$

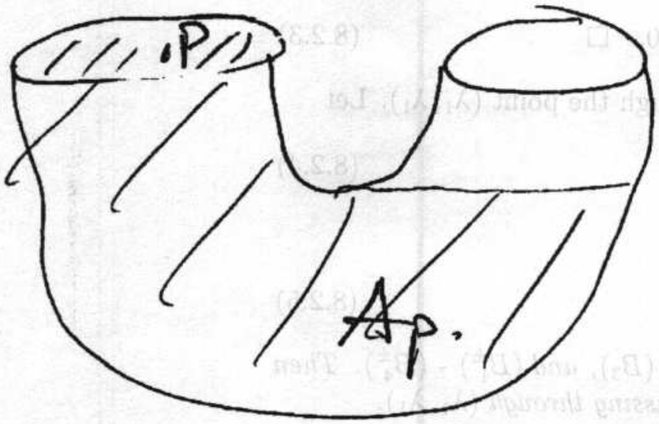
IN VISCOSITY SENSE IN Ω AND ON TOP BOUNDARY

F IS ELLIPTIC BUT NOT UNIFORMLY

THEOREM [?]: SUPPOSE $u \geq v$ in $\Omega \cup$ TOP AND $u = v$ AT A POINT P ON TOP

THEN $u \equiv v$ in Ω_P

WHERE S_p IS THE SET OF POINTS IN $\Omega \cup TOP$ WHICH MAY BE CONNECTED TO P BY A CONTINUOUS CURVE ON WHICH f IS NON DECREASING



WE THANK H. MATANO FOR SUGGESTING WE CONSIDER THIS PROBLEM

IN THE PROOFS WE USE RATHER STANDARD COMPARISON FUNCTIONS. BUT WHEN WORKING WITH VISCOSITY (SUPER)SOLUTIONS WE CANNOT ~~SUBTRACT~~ COMPARE ~~SOLUTIONS~~ FUNCTIONS BY SUBTRACTING ONE FROM ANOTHER.

WE COMPARE FUNCTIONS BY MOVING ONE DOWN UNTIL IT LIES BELOW THE OTHER, AND THEN MOVING IT UP UNTIL IT JUST TOUCHES THE OTHER FROM BELOW.

References

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