Loop spaces, cyclic homology, and the A_∞ algebra of a Lagrangian submanifold

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Abstract of the Dissertation

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The thesis consists of three chapters. In the first chapter, we prove a variant of Jones' theorem on cyclic homology and $\mathbb{S}^1$-equivariant homology, and describe a chain level refinement of the string topology gravity algebra discovered by Chas-Sullivan. In the second chapter, we construct a new chain model of the (based and free) loop (path) space of a path-connected topological space X, defined using the fundamental groupoid of X. This may be viewed as a generalization of a classical theorem of Adams to non-simply-connected spaces. In the third chapter, we give a formulation of the Lagrangian Floer theory of a single Lagrangian submanifold in terms of its free loop space, namely lift the Fukaya A_∞ algebra of a Lagrangian submanifold L to the dg Lie algebra of cyclic invariant chains on the free loop space of L.

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Chapter 1

A cocyclic construction of $\mathbb{S}^1$-equivariant homology and application to string topology

1.1 Introduction

Let M be a closed oriented smooth manifold and $\mathcal{L}M = C^\infty(\mathbb{S}^1, M)$ be the smooth free loop space of M. In a seminal paper [5] (and a sequal [6]), Chas-Sullivan discovered rich algebraic structures on the ordinary homology and $\mathbb{S}^1$-equivariant homology of $\mathcal{L}M$, initiating the study of string topology. In particular, there is a Batalin-Vilkovisky (BV) algebra structure on (shifted) $H_*(\mathcal{L}M)$ ([5, Theorem 5.4]), which naturally induces a gravity algebra structure on (shifted) $H_*^{\mathbb{S}^1}(\mathcal{L}M)$ ([5, Section 6], [6, page 18]).

The goal of this paper is to describe a chain level refinement of the string topology gravity algebra, and compare it with an algebraic counterpart related to the de Rham dg algebra $\Omega(M)$. Along the way we also obtain results on the relation between cyclic homology and $\mathbb{S}^1$-equivariant homology, and an $\mathbb{S}^1$-equivariant version of Deligne's conjecture.

In spirit, this paper may be compared with work of Westerland [50]. Westerland gave a homotopy theoretic generalization of the gravity operations on the (negative) $\mathbb{S}^1$-equivariant homology of $\mathcal{L}M$, whereas we describe a chain level refinement.

Cyclic homology and $\mathbb{S}^1$-equivariant homology

The close connection between cyclic homology (algebra) and $\mathbb{S}^1$-equivariant homology (topology) was first systematically studied by Jones in [32]. One of the main theorems in that paper ([32, Theorem 3.3]) says that the singular chains $\{S_k(X)\}_{k\geq 0}$ of an $\mathbb{S}^1$-space X can be made into a cyclic module, such that there are natural isomorphisms between three versions of cyclic homology (positive, periodic, negative) of $\{S_k(X)\}_{k\geq 0}$ and three versions of $\mathbb{S}^1$-equivariant homology of X, in a way compatible with long exact sequences.

The first result in this paper is a theorem "cyclic dual" to Jones' theorem. As far as the author knows, such a result did not appear in the literature.

Theorem 1.1.1 (See Theorem 1.3.1)**.** *Let X be a topological space with an $\mathbb{S}^1$-action. Then $\{S_*(X \times \Delta^k)\}_{k\geq 0}$ can be made into a cocyclic chain complex, such that there are natural isomorphisms between three versions of cyclic homology of $\{S_*(X\times\Delta^k)\}_{k\geq 0}$ and three versions of $\mathbb{S}^1$-equivariant homology of X, in a way compatible with long exact sequences.*

Jones dealt with the cyclic set $\{\mathrm{Map}(\Delta^k, X)\}_k$ and the cyclic module $\{S_k(X)\}_k$, while we deal with the cocyclic space $\{X \times \Delta^k\}_k$ and the cocyclic complex $\{S_*(X \times \Delta^k)\}_k$. It is in this sense that these two theorems are "cyclic dual" to each other. In the special case that X is the free loop space of a topological space Y, Theorem 1.1.1 may also be viewed as "cyclic dual" to a result of Goodwillie ([27, Lemma V.1.4]). As does Jones' theorem, Theorem 1.1.1 has the advantage that it works for all $\mathbb{S}^1$-spaces.

The cyclic structure on singular chains plays no role in Theorem 1.1.1; what matters is the cocyclic space. Indeed, the main motivation for the author to seek for a result like Theorem 1.1.1 is to study the $\mathbb{S}^1$-equivariant homology of $\mathcal{L}M$, using a novel chain model of loop space homology defined via certain "de Rham chains", introduced by Irie [29].

Deligne's conjecture

What is called Deligne's conjecture asks whether there is an action of certain chain model of the little disks operad on the Hochschild cochain complex of an associative algebra, inducing the Gerstenhaber algebra structure on Hochschild cohomology discovered by Gerstenhaber [21]. This conjecture, as well as some variations and generalizations, has been answered affirmatively by many authors, to whom we are apologetic not to list here. What is of most interest and importance to us is work of Ward [48].

Ward ([48, Theorem C]) gave a general solution to the question when certain complex of cyclic (co)invariants admits an action of a chain model of the gravity operad, inducing the gravity algebra structure on cyclic cohomology. Recall that the gravity operad was introduced by Getzler [25] and is the $\mathbb{S}^1$-equivariant homology of the little disks operad. So Ward's result can be viewed as an $\mathbb{S}^1$-equivariant version of operadic Deligne's conjecture ([48, Corollary 5.22]).

The second result in this paper is an extension, in a special case, of Ward's theorem. To state our result, let A be a dg algebra equipped with a symmetric, cyclic, bilinear form $\langle , \rangle : A \otimes A \to \mathbb{R}$ of degree $m \in \mathbb{Z}$ satisfying Leibniz rule (see Example 1.5.9). Then $\langle , \rangle$ induces a dg A-bimodule map $\theta : A \to A^\vee[m]$, and hence a cochain map $\Theta : \mathrm{CH}(A, A) \to \mathrm{CH}(A, A^\vee[m])$ between Hochschild cochains. Let $\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])$ be the subcomplex of cyclic invariants in $\mathrm{CH}(A, A^\vee[m])$. Let $\mathsf{M}_\circlearrowleft$ be the chain model of the gravity operad that Ward constructed (see also Example 1.5.3(iii)).

Theorem 1.1.2 (See Corollary 1.6.8)**.** *Given $A, \langle , \rangle, \theta, \Theta$ as above, there is an action of $\mathsf{M}_\circlearrowleft$ on $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m]))$, giving rise to a structure of gravity algebra up to homotopy. If θ is a quasi-isomorphism and Θ restricts to a quasi-isomrphism $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])) \to \mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])$, this descends to a gravity algebra structure on the cyclic cohomology of A, which is compatible with the BV algebra structure on Hochschild cohomology.*

Here the BV algebra structure on the Hochschild cohomology of A (when θ is a quasi-isomorphism) is well-known (e.g. Menichi [42, Theorem 18]), where the BV operator is given by Connes' operator (Example 1.2.6). By compatibility with BV algebra structure we mean the content of Lemma 1.5.1. Note that Ward's original theorem only applies to the situation that θ is an isomorphism ([48, Corollary 6.2]).

Chain level structures in $\mathbb{S}^1$-equivariant string topology

Let us say more about Irie's work [29]. Using his chain model and results of Ward ([48, Theorem A, Theorem B]), Irie obtained an operadic chain level refinement of the string topology BV algebra, and compared it with a solution to the ordinary Deligne's conjecture via a chain map defined by iterated integrals of differential forms.

The third result in this paper is a similar story in the $\mathbb{S}^1$-equivariant context. Note that the string topology BV algebra induces gravity algebra structures on two versions (positive i.e. ordinary, and negative) of $\mathbb{S}^1$-equivariant homology of $\mathcal{L}M$ (Example 1.7.1).

Theorem 1.1.3 (See Theorem 1.7.6)**.** *For any closed oriented C^∞-manifold M, there exists a chain complex $\tilde{\mathcal{O}}_M^{\mathrm{cyc}}$ satisfying the following properties. Firstly, the homology of $\tilde{\mathcal{O}}_M^{\mathrm{cyc}}$ is isomorphic to the negative $\mathbb{S}^1$-equivariant homology of $\mathcal{L}M$, and $\tilde{\mathcal{O}}_M^{\mathrm{cyc}}$ admits an action of $\mathsf{M}_{\circlearrowleft}$ (hence an up-to-homotopy gravity algebra structure) which lifts the gravity algebra structure mentioned above. Secondly, there is a morphism of $\mathsf{M}_{\circlearrowleft}$-algebras*

$$\tilde{\mathcal{O}}_M^{\mathrm{cyc}} \to \Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(\Omega(M), \Omega(M)^\vee[-\dim M])) \tag{1.1.1}$$

which is induced by iterated integrals of differential forms, where the structure on RHS follows from Theorem 1.1.2 and Θ comes from Poincaré pairing. At homology level, the morphism (1.1.1) *descends to a map (part of arrow 4 below) which fits into a commutative diagram of gravity algebra homomorphisms*

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "2"] & B \arrow[d, "3"] \\ C \arrow[r, "4"] & D. \end{tikzcd} \tag{1.1.2}$$

Here A is the $\mathbb{S}^1$-equivariant homology of $\mathcal{L}M$, B is the negative cyclic cohomology of $\Omega(M)$, C is the negative $\mathbb{S}^1$-equivariant homology of $\mathcal{L}M$, D is the cyclic cohomology of $\Omega(M)$. Arrows 1, 4 are defined by iterated integrals on free loop space, and arrow 2 (resp. 3) is the connecting map in the tautological long exact sequence for $\mathbb{S}^1$-equivariant homology theories (resp. cyclic homology theories).

The crucial part of Theorem 1.1.3 is, of course, the chain level statement that fits well with structures on homology. The first part of Theorem 1.1.3 was conjectured by Ward in [48, Example 6.12], but the correct statement turns out to be more complicated, as we actually lift gravity algebra structures on negative $\mathbb{S}^1$-equivariant homology rather than $\mathbb{S}^1$-equivariant homology, whereas they are naturally related by a morphism (arrow 2).

Other than the chain level statement, part of the results at homology level is known. For example, the fact that arrow 1 is a Lie algebra homomorphism appeared in work of Abbaspour-Tradler-Zeinalian as [1, Theorem 11]; The fact that (1.1.2) commutes was of importance to Cieliebak-Volkov [10] (the arrows are only treated as linear maps there).

In a forthcoming paper, the author is going to apply results in this paper to Lagrangian Floer theory, in view of cyclic symmetry therein (Fukaya [17]).

Outline

In Section 1.2, we review cyclic homology of mixed complexes. In Section 1.3, we prove Theorem 1.1.1. In Section 1.4, we review Irie's de Rham chain complex of differentiable spaces and apply Theorem 1.1.1 to it. In Section 1.5, we review basics of operads and algebraic structures. In Section 1.6, we prove Theorem 1.1.2. In Section 1.7, we prove Theorem 1.1.3.

Conventions

Vector spaces are over $\mathbb{R}$, algebras are associative and unital, graded objects are $\mathbb{Z}$-graded. Homological and cohomological gradings are mixed by the understanding $C_* = C^{-*}, C^* = C_{-*}$.

As for sign rules, see Appendix 1.8. For the sake of convenience, we may write $(-1)^\varepsilon$ for a sign that is apparent from Koszul sign rule (Appendix 1.8.1).

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1.2 Preliminaries on cyclic homology

A convenient way to study different versions of cyclic homology is to work in the context of *mixed complexes*, which was introduced by Kassel [33]. By definition, a mixed cochain complex is a triple (C^*, b, B) consisting of a graded vector space C^* and linear maps $b : C^* \to C^{*+1}$, $B : C^* \to C^{*-1}$ such that

$$b^2 = 0, \quad B^2 = 0, \quad bB + Bb = 0.$$

Let u be a formal variable of degree 2. Define graded $\mathbb{R}[u]$-modules $C[[u]]^*$, $C[[u, u^{-1}]^*$, $C[u^{-1}]^*$ by

$$\begin{aligned} C[[u]]^n &:= \Big\{ \sum_{i \geq 0} c_i u^i \mid c_i \in C^{n-2i} \Big\}, \\ C[[u, u^{-1}]^n &:= \Big\{ \sum_{i \geq -k} c_i u^i \mid k \in \mathbb{Z}_{\geq 0},\ c_i \in C^{n-2i} \Big\}, \\ C[u^{-1}]^n &:= \Big\{ \sum_{-k \leq i \leq 0} c_i u^i \mid k \in \mathbb{Z}_{\geq 0},\ c_i \in C^{n-2i} \Big\}. \end{aligned}$$

Here the $\mathbb{R}[u]$-module structure on $C[u^{-1}]$ is induced by the identification $C[u^{-1}] = C[[u, u^{-1}]/uC[[u]]$. Then $b + uB$ is a differential on $C[[u]]^*$, $C[[u, u^{-1}]^*$, $C[u^{-1}]^*$, resulting in cohomology groups denoted by

$$\mathrm{HC}^*_{[[u]]}(C), \quad \mathrm{HC}^*_{[[u,u^{-1}]}(C), \quad \mathrm{HC}^*_{[u^{-1}]}(C).$$

These are three classical versions of *cyclic homology* of mixed complexes, called the negative, periodic and ordinary (positive) cyclic homology of (C^*, b, B), respectively. We prefer to distinguish them by suggestive symbols ($[[u]]$, $[[u,u^{-1}]$, $[u^{-1}]$) rather than names, as did in [10]. Here cohomological grading is used for cyclic homology since we deal with cochain complexes. If we move to homological grading $C_* := C^{-*}$ and replace u by v (a formal variable of degree -2), then the mixed chain complex (C_*, b, B) gives negative, periodic and ordinary (positive) cyclic homology theories

$$\mathrm{HC}_*^{[[v]]} = \mathrm{HC}^{-*}_{[[u]]}, \quad \mathrm{HC}_*^{[[v,v^{-1}]} = \mathrm{HC}^{-*}_{[[u,u^{-1}]}, \quad \mathrm{HC}_*^{[v^{-1}]} = \mathrm{HC}^{-*}_{[u^{-1}]}.$$

([10] also takes the Hom dual of C to define cyclic cohomology theories of (C, b, B), which we try to avoid in this article.)

For any mixed cochain complex (C^*, b, B), there is a *tautological exact sequence*

$$\cdots \to \mathrm{HC}^*_{[[u]]}(C) \xrightarrow{i_*} \mathrm{HC}^*_{[[u,u^{-1}]}(C) \xrightarrow{u\cdot p_*} \mathrm{HC}^{*+2}_{[u^{-1}]}(C) \xrightarrow{B_{0*}} \mathrm{HC}^{*+1}_{[[u]]}(C) \to \cdots \tag{1.2.1}$$

which is induced by the short exact sequence

$$0 \to C[[u]] \xrightarrow{i} C[[u,u^{-1}] \xrightarrow{p} C[[u,u^{-1}]/C[[u]] \to 0$$

and the $(b+uB)$-cochain isomorphism

$$(C[[u,u^{-1}]/C[[u]])^* \underset{\cong}{\xrightarrow{\cdot u}} C[u^{-1}]^{*+2}; \qquad \sum_{-k\le i\le -1} c_i u^i \mapsto \sum_{-k\le i\le -1} c_i u^{i+1}.$$

The connecting map $B_{0*} : \mathrm{HC}^{*+2}_{[u^{-1}]}(C) \to \mathrm{HC}^{*+1}_{[[u]]}(C)$ is given on cocycles by

$$B_0 : Z^{*+2}(C[u^{-1}]) \to Z^{*+1}(C[[u]]); \quad \sum_{-k\le i\le 0} c_i u^i \mapsto B(c_0).$$

Note that B_0 is not a cochain or anti-cochain map from $C[u^{-1}]^{*+2}$ to $C[[u]]^{*+1}$. Similarly, from the short exact sequences

$$0 \to C[[u]]/uC[[u]] \xrightarrow{i} C[[u,u^{-1}]/uC[[u]] \xrightarrow{p} C[[u,u^{-1}]/C[[u]] \to 0$$
$$0 \to uC[[u]] \xrightarrow{i^+} C[[u]] \xrightarrow{p_0} C[[u]]/uC[[u]] \to 0$$

one obtains the *Gysin-Connes exact sequences*

$$\cdots \to H^*(C,b) \xrightarrow{i_*} \mathrm{HC}^*_{[u^{-1}]}(C) \xrightarrow{u\cdot p_*} \mathrm{HC}^{*+2}_{[u^{-1}]}(C) \xrightarrow{B_{0*}} H^{*+1}(C,b) \to \cdots \tag{1.2.2a}$$

$$\cdots \to \mathrm{HC}^{*-2}_{[[u]]}(C) \xrightarrow{i^+_*\cdot u} \mathrm{HC}^*_{[[u]]}(C) \xrightarrow{p_{0*}} H^*(C,b) \xrightarrow{B_*} \mathrm{HC}^{*-1}_{[[u]]}(C) \to \cdots \tag{1.2.2b}$$

The connecting maps $\mathrm{HC}^{*+2}_{[u^{-1}]}(C) \xrightarrow{B_{0*}} H^{*+1}(C,b)$ and $H^*(C,b) \xrightarrow{B_*} \mathrm{HC}^{*-1}_{[[u]]}(C)$ are given on cocycles by B_0 and B, respectively.

Lemma 1.2.1. *The map* $B_{0*} : \mathrm{HC}^{*+2}_{[u^{-1}]}(C) \to \mathrm{HC}^{*+1}_{[[u]]}(C)$ *in* (1.2.1) *and the exact sequences* (1.2.2) *fit into the following commutative diagram:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(C,b) \arrow[r, "i_*"] \arrow[d, equal, "\mathrm{id}"] & \mathrm{HC}^*_{[u^{-1}]}(C) \arrow[r, "u\cdot p_*"] \arrow[d, "B_{0*}"] & \mathrm{HC}^{*+2}_{[u^{-1}]}(C) \arrow[r, "B_{0*}"] \arrow[d, "B_{0*}"] & H^{*+1}(C,b) \arrow[r] \arrow[d, equal, "\mathrm{id}"] & \cdots \\
\cdots \arrow[r] & H^*(C,b) \arrow[r, "B_*"] & \mathrm{HC}^{*-1}_{[[u]]}(C) \arrow[r, "i^+_*\cdot u"] & \mathrm{HC}^{*+1}_{[[u]]}(C) \arrow[r, "p_{0*}"] & H^{*+1}(C,b) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. The left and the right squares commute since they commute at the level of cocycles. As for the middle square, let $c = \sum_{j=-k}^{0} c_j u^j \in Z^*(C[u^{-1}])$, then $B_0(u \cdot p(c)) = B(c_{-1})$ and $i^+(u \cdot B_0(c)) = B(c_0)u$. Since c is a cocycle, $0 = (b+uB)(c) = \sum_{j=-k}^{0}(b(c_j) + B(c_{j-1}))u^j \in C[u^{-1}]$. In particular, $b(c_0) + B(c_{-1}) = 0$, so $B(c_0)u - B(c_{-1}) = (b+uB)(c_0)$ is exact. This proves $B_{0*} \circ (u \cdot p_*) = (i^+_* \cdot u) \circ B_{0*}$. $\square$

Definition 1.2.2. Let $(C^*, b, B), (C''^*, b'', B'')$ be mixed cochain complexes.

(i) A series of linear maps $\{f_i : C^* \to (C'')^{*-2i}\}_{i \in \mathbb{Z}_{\geq 0}}$ is called an ∞-*morphism* from C^* to C''^* if $\sum_{i \geq 0} u^i f_i : (C[[u,u^{-1}]^*, b+uB) \to (C''[[u,u^{-1}]^*, b''+uB'')$ is a cochain map, or equivalently, if $\{f_i\}_{i \geq 0}$ satisfies $b'' f_0 = f_0 b$ and $B'' f_{i-1} + b'' f_i = f_{i-1}B + f_i b$ $(i \geq 1)$.

(ii) An ∞-morphism $f = \{f_i\}_{i \geq 0} : C^* \to C''^*$ is called an ∞-*quasi-isomorphism* if $f_0 : (C^*, b) \to (C''^*, b'')$ is a cochain quasi-isomorphism.

(iii) Given two ∞-morphisms $\{f_i\}_{i \geq 0}, \{g_i\}_{i \geq 0} : C^* \to C''^*$, a series of linear maps $\{h_i : C^* \to (C'')^{*-2i-1}\}_{i \in \mathbb{Z}_{\geq 0}}$ is called an ∞-*homotopy* between them if $h = \sum_{i \geq 0} u^i h_i : C[[u,u^{-1}]^* \to C''[[u,u^{-1}]^*$ is a $(b+uB, b''+uB'')$-cochain homotopy between $\sum_{i \geq 0} u^i f_i$ and $\sum_{i \geq 0} u^i g_i$, or equivalently, if $\{h_i\}_{i \geq 0}$ satisfies $f_0 - g_0 = b'' h_0 + h_0 b$ and $f_i - g_i = b'' h_i + h_i b + B'' h_{i-1} + h_{i-1} B$ $(i \geq 1)$.

A morphism between mixed complexes is an ∞-morphism $\{f_i\}_{i \geq 0}$ such that $f_i = 0$ for all $i > 0$, namely a single degree 0 linear map that commutes with both b and B. A quasi-isomorphism between mixed complexes is a morphism that is also a (b,b'')-quasi-isomorphism. A homotopy between two morphisms $f,g:(C^*,b,B)\to(C''^*,b'',B'')$ is an ∞-homotopy $\{h_i\}_{i\geq 0}$ such that $h_i=0$ for all $i>0$, namely a single degree -1 linear map h satisfying $f-g=b''h+hb$ and $B''h+hB=0$.

The following important lemma goes back to [32, Lemma 2.1], and is a special case of [51, Lemma 2.3] which is stated for $\mathbb{S}^1$-complexes (an ∞-version of mixed complex). The proof is a spectral sequence argument by considering the u-adic filtration on $C[[u]]^*$ etc.

Lemma 1.2.3. *Let $\{f_i\}_{i\geq 0}:(C^*,b,B)\to(C''^*,b'',B'')$ be an ∞-quasi-isomorphism. Then $\sum_{i\geq 0}u^i f_i$ induces isomorphisms on $\mathrm{HC}^*_{[[u]]}$, $\mathrm{HC}^*_{[[u,u^{-1}]}$ and $\mathrm{HC}^*_{[u^{-1}]}$.* □

The following lemma illustrates the naturality of the tautological exact sequence and Connes-Gysin exact sequences for cyclic homology, with respect to ∞-morphisms between mixed complexes.

Lemma 1.2.4. *Let $f=\{f_i\}_{i\geq 0}:(C^*,b,B)\to(C''^*,b'',B'')$ be an ∞-morphism. Then $f=\sum_i u^i f_i$ induces a morphism between the exact sequence* (1.2.1) *for C and C'', namely there is a commutative diagram*

$$\begin{array}{ccccccc}
\cdots\mathrm{HC}^*_{[[u]]}(C) & \xrightarrow{i_*} & \mathrm{HC}^*_{[[u,u^{-1}]}(C) & \xrightarrow{u\cdot p_*} & \mathrm{HC}^{*+2}_{[u^{-1}]}(C) & \xrightarrow{B_{0*}} & \mathrm{HC}^{*+1}_{[[u]]}(C)\cdots \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\cdots\mathrm{HC}^*_{[[u]]}(C'') & \xrightarrow{i''_*} & \mathrm{HC}^*_{[[u,u^{-1}]}(C'') & \xrightarrow{u\cdot p''_*} & \mathrm{HC}^{*+2}_{[u^{-1}]}(C'') & \xrightarrow{B''_{0*}} & \mathrm{HC}^{*+1}_{[[u]]}(C'')\cdots .
\end{array}$$

Similarly, for the exact sequence (1.2.2a), *there is a commutative diagram*

$$\begin{array}{ccccccc}
\cdots H^*(C,b) & \xrightarrow{i_*} & \mathrm{HC}^*_{[u^{-1}]}(C) & \xrightarrow{u\cdot p_*} & \mathrm{HC}^{*+2}_{[u^{-1}]}(C) & \xrightarrow{B_{0*}} & H^{*+1}(C,b)\cdots \\
\downarrow{\scriptstyle f_{0*}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{0*}} \\
\cdots H^*(C'',b'') & \xrightarrow{i''_*} & \mathrm{HC}^{*-1}_{[u^{-1}]}(C'') & \xrightarrow{u\cdot p''_*} & \mathrm{HC}^{*+1}_{[u^{-1}]}(C'') & \xrightarrow{B''_{0*}} & H^{*+1}(C'',b'')\cdots .
\end{array}$$

The case of the exact sequence (1.2.2b) *is also similar.*

Proof. We only write proof for the first diagram since the others are similar. The left and the middle squares commute since they commute at the level of cocycles. Now let

$c = \sum_{j=0}^k c_{-j}u^{-j} \in Z^{*+2}(C[u^{-1}])$. $(b+uB)(c) = 0$ says $b(c_{-j}) + B(c_{-j-1}) = 0$ for all $j \in \{0,\dots,k\}$. Also recall the ∞-morphism f satisfies $B''f_{i-1} + b''f_i = f_{i-1}B + f_ib$. Using these relations, it is a straightforward calculation to see

$$\sum_{i\geq 0} f_i(B(c_0)) \cdot u^i - B''\Big(\sum_{0\leq j\leq k} f_j(c_{-j})\Big) = (b''+uB'')\Big(\sum_{i\geq 0}\sum_{0\leq j\leq k} f_{i+j+1}(c_{-j})\cdot u^i\Big).$$

The LHS is $(f\circ B_0 - B_0''\circ f)(c)$, and the RHS is exact, so commutativity of the right square is proved. □

We now discuss some important examples of mixed (co)chain complexes and their cyclic homologies. Recall that a *cosimplicial object* in some category is a sequence of objects $C(k)$ ($k \in \mathbb{Z}_{\geq 0}$) together with morphisms

$$\delta_i : C(k-1) \to C(k)\ (0 \leq i \leq k),\quad \sigma_i : C(k+1) \to C(k)\ (0 \leq i \leq k)$$

satisfying the following relations:

$$\delta_j\delta_i = \delta_i\delta_{j-1}\ (i<j); \tag{1.2.3a}$$

$$\sigma_j\sigma_i = \sigma_i\sigma_{j+1}\ (i\leq j); \tag{1.2.3b}$$

$$\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1} & (i<j) \\ \mathrm{id} & (i=j,j+1) \\ \delta_{i-1}\sigma_j & (i>j+1). \end{cases} \tag{1.2.3c}$$

A *cocyclic object* is a cosimplicial object $\{C(k)\}_k$ together with morphisms $\tau_k : C(k) \to C(k)$ satisfying the following relations:

$$\tau_k^{k+1} = \mathrm{id}; \tag{1.2.4a}$$

$$\tau_k\delta_i = \delta_{i-1}\tau_{k-1}\ (1\leq i\leq k),\qquad \tau_k\delta_0 = \delta_k; \tag{1.2.4b}$$

$$\tau_k\sigma_i = \sigma_{i-1}\tau_{k+1}\ (1\leq i\leq k),\qquad \tau_k\sigma_0 = \sigma_k\tau_{k+1}^2. \tag{1.2.4c}$$

For example, let $\Delta^0 := \mathbb{R}^0$, $\Delta^k := \{(t_1,\dots,t_k) \in \mathbb{R}^k \mid 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$ ($k>0$) be the standard simplices, then $\{\Delta^k\}_{k\in\mathbb{Z}_{\geq 0}}$ is a cocyclic set (topological space, etc.) with standard cocyclic maps $\delta_i : \Delta^{k-1} \to \Delta^k$, $\sigma_i : \Delta^{k+1} \to \Delta^k$, $\tau_k : \Delta^k \to \Delta^k$ defined by

$$\delta_i(t_1, \dots, t_{k-1}) := \begin{cases} (0, t_1, \dots, t_{k-1}) & (i = 0) \\ (t_1, \dots, t_i, t_i, \dots, t_{k-1}) & (1 \le i \le k-1) \\ (t_1, \dots, t_{k-1}, 1) & (i = k), \end{cases} \tag{1.2.5a}$$

$$\sigma_i(t_1, \dots, t_{k+1}) := (t_1, \dots, \widehat{t_{i+1}}, \dots, t_{k+1}) \text{ (miss } t_{i+1}) \ (0 \le i \le k), \tag{1.2.5b}$$

$$\tau_k(t_1, \dots, t_k) := (t_2 - t_1, \dots, t_k - t_1, 1 - t_1). \tag{1.2.5c}$$

Remark 1.2.5. Equivalently, for $\tilde{\Delta}^k := \{(s_0, s_1, \dots, s_k) \in [0,1]^{k+1} \mid s_0 + s_1 \cdots + s_k = 1\}$ $(k \ge 0)$, $\tau_k : \tilde{\Delta}^k \to \tilde{\Delta}^k$ reads $\tau_k(s_0, s_1, \dots, s_k) = (s_1, \dots, s_k, s_0)$.

Example 1.2.6 (Cocyclic complex and Connes' version of cyclic cohomology)**.** Consider the category of cochain complexes where the morphisms are degree 0 cochain maps. Let $\big((C(k)^*, d), \delta_i, \sigma_i, \tau_k\big)$ be a cocyclic cochain complex, then a mixed cochain complex (C, b, B) is obtained as follows. Let

$$\delta : C(k-1)^* \to C(k)^*; \quad c_{k-1} \mapsto (-1)^{|c_{k-1}|+k-1} \sum_{0 \le i \le k} (-1)^i \delta_i(c_{k-1}), \tag{1.2.6}$$

then $\delta^2 = 0$, $\delta d + d\delta = 0$. Let (C^*, b) be the *product* total complex of the double complex $\big(C(k)^l, d, \delta\big)^{k \in \mathbb{Z}_{\ge 0}}_{l \in \mathbb{Z}}$:

$$C^* := \prod_{l+k=*} C(k)^l = \prod_{k \ge 0} C(k)^{*-k}, \qquad b = d + \delta.$$

For later purpose we also introduce the *normalized* subcomplex (C^*_{nm}, b) of (C^*, b):

$$C^*_{\mathrm{nm}} := \prod_{k \ge 0} C_{\mathrm{nm}}(k)^{*-k}, \quad C_{\mathrm{nm}}(k) := \bigcap_{0 \le i \le k-1} \mathrm{Ker}\big(\sigma_i : C(k) \to C(k-1)\big).$$

Note that the natural inclusion $\{C_{\mathrm{nm}}(k)\} \subset \{C(k)\}$ is not cosimplicial since δ_j does not restricts to $C_{\mathrm{nm}}(k)$. The natural inclusion $(C^*_{\mathrm{nm}}, b) \hookrightarrow (C^*, b)$ is a quasi-isomorphism (see [38, Proposition 1.6.5] or [29, Lemma 2.5]). Note that so far only the cosimplicial structure on $\{C(k)^*\}_{k \ge 0}$ has come into play. Next, define the operator $B : C^* \to C^{*-1}$ by

$$B := Ns(1 - \lambda) \ \ (\textit{Connes' operator}),$$

where λ, N, s are given by (here $|c|$ is the degree of $c = (c_k)_{k\geq 0} \in \prod_{k\geq 0} C(k)$ in C^*)

$$\lambda|_{C(k)} := (-1)^k \tau_k,\ \ N|_{C(k)} := 1 + \lambda + \cdots + \lambda^k,\ \ s(c) := (-1)^{|c|-1}(\sigma_k \tau_{k+1}(c_{k+1}))_{k\geq 0}.$$

Although C_{nm} is not closed under λ, N, it is closed under s, B. For $c_{k+1} \in C_{\mathrm{nm}}(k+1)^*$, there holds $s(\lambda(c_{k+1})) = (-1)^{|c_{k+1}|}\sigma_k \tau_{k+1}^2(c_{k+1}) = (-1)^{|c_{k+1}|}\tau_k \sigma_0(c_{k+1}) = 0$, so Connes' operator B has simpler form on normalized subcomplex:

$$B|_{C_{\mathrm{nm}}} = Ns.$$

To see (C^*, b, B) is a mixed complex, define

$$b' : C^* \to C^{*+1}, \quad c \mapsto b(c) - ((-1)^{|c_{k-1}|-1}\delta_k(c_{k-1}))_{k\geq 0}.$$

It is a routine calculation to see $(b')^2 = 0$, $N(1-\lambda) = (1-\lambda)N = 0$, $(1-\lambda)b = b'(1-\lambda)$, $bN = Nb'$ and $b's + sb' = 1$. It follows that $B^2 = Ns((1-\lambda)N)s(1-\lambda) = 0$ and $bB+Bb = Nb's(1-\lambda)+Nsb'(1-\lambda) = N(1-\lambda) = 0$, as desired. The identity $(1-\lambda)b = b'(1-\lambda)$ also implies that the space of *cyclic invariants*,

$$C_{\mathrm{cyc}} := \mathrm{Ker}(1-\lambda) \subset (C, b),$$

forms a subcomplex (we denote this inclusion by i_λ). This leads to *Connes' version of cyclic cohomology* of the cocyclic cochain complex,

$$\mathrm{HC}^*_\lambda(C) = \mathrm{HC}^*_\lambda\big(C(k), d, \delta_i, \sigma_i, \tau_k\big) := H^*(C_{\mathrm{cyc}}, b).$$

Since $B = Ns(1-\lambda)$ vanishes on C_{cyc}, (C^*_{cyc}, b) is also naturally a subcomplex of $(C[[u]]^*, b + uB)$. By an argument similar to [38, Theorem 2.1.5, 2.1.8] one sees that this inclusion $I_\lambda : (C^*_{\mathrm{cyc}}, b) \hookrightarrow (C[[u]]^*, b + uB)$ induces an isomorphism

$$I_{\lambda *} : \mathrm{HC}^*_\lambda(C) \cong \mathrm{HC}^*_{[[u]]}(C). \tag{1.2.7}$$

The short exact sequence $0 \to (C_{\mathrm{cyc}}, b) \xrightarrow{i_\lambda} (C, b) \xrightarrow{p_\lambda} (C/C_{\mathrm{cyc}}, b) \to 0$ induces *Connes' long exact sequence* (we follow the presentation of [34, Section 0.14])

$$\cdots \to \mathrm{HC}^*_\lambda(C) \xrightarrow{i_{\lambda *}} H^*(C, b) \xrightarrow{B_\lambda} \mathrm{HC}^{*-1}_\lambda(C) \xrightarrow{S_\lambda} \mathrm{HC}^{*+1}_\lambda(C) \to \cdots. \tag{1.2.8}$$

Here we have made use of an isomorphism $\mathrm{HC}^{*-1}_{\lambda}(C) \cong H^*(C/C_{\mathrm{cyc}}, b)$, which is a consequence of another short exact sequence

$$0 \to (C/C_{\mathrm{cyc}}, b) \xrightarrow{1-\lambda} (C, b') \xrightarrow{N} (C_{\mathrm{cyc}}, b) \to 0 \tag{1.2.9}$$

and the fact that (C, b') is acyclic (since $b's + sb' = 1$). Lemma 1.2.9 below says (1.2.8) can be identified with (1.2.2b). Finally we mention that $\mathrm{HC}^*_{[[u]]}(C_{\mathrm{nm}}) \cong \mathrm{HC}^*_{[[u]]}(C) \cong \mathrm{HC}^*_{\lambda}(C)$, where the first isomorphism follows from Lemma 1.2.3.

A subexample of Example 1.2.6 is as follows.

Example 1.2.7 (Cyclic cohomology of dg algebras)**.** Let A^* be a dg algebra with unit 1_A. Then $\{\mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})\}_{k \geq 0}$ has the structure of a cocyclic cochain complex, where $\delta_i : \mathrm{Hom}^*(A^{\otimes k}, \mathbb{R}) \to \mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})$, $\sigma_i : \mathrm{Hom}^*(A^{\otimes k+2}, \mathbb{R}) \to \mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})$ and τ_k : $\mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R}) \to \mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})$ are

$$\begin{aligned}
\delta_i(\varphi)(a_1 \otimes \cdots \otimes a_{k+1}) &:= \begin{cases} (-1)^{\varepsilon}\varphi(a_2 \otimes \cdots \otimes a_k \otimes a_{k+1}a_1) & (i = 0) \\ \varphi(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}) & (1 \leq i \leq k), \end{cases} \\
\sigma_i(\varphi)(a_1 \otimes \cdots \otimes a_{k+1}) &:= \varphi(a_1 \otimes \cdots \otimes a_i \otimes 1_A \otimes a_{i+1} \otimes \cdots \otimes a_{k+1}) \ \ (0 \leq i \leq k), \\
\tau_k(\varphi)(a_1 \otimes \cdots \otimes a_{k+1}) &:= (-1)^{\varepsilon}\varphi(a_{k+1} \otimes a_1 \cdots \otimes a_k).
\end{aligned} \tag{1.2.10}$$

The associated mixed total complex is denoted by $\mathrm{CH}^*(A, A^\vee)$. For simplicity, denote cyclic homologies of $\mathrm{CH}^*(A, A^\vee)$ by $\mathrm{HC}^*_{[u^{-1}]}(A, A^\vee)$, $\mathrm{HC}^*_{[[u]]}(A, A^\vee) \cong \mathrm{HC}^*_{\lambda}(A, A^\vee)$, etc. Classically, $\mathrm{HC}^*_{\lambda}(A, A^\vee)$ is called *(Connes') cyclic cohomology* of A.

Let us also recall that for any dg A-bimodule M^*, there is a structure of cosimplicial complex on $\{\mathrm{Hom}^*(A^{\otimes k}, M)\}_{k \geq 0}$, where $\delta_i : \mathrm{Hom}^*(A^{\otimes k-1}, M) \to \mathrm{Hom}^*(A^{\otimes k}, M)$, σ_i :

$\mathrm{Hom}^*(A^{\otimes k+1}, M) \to \mathrm{Hom}^*(A^{\otimes k}, M)$ are

$$\delta_i(f)(a_1 \otimes \cdots \otimes a_k) := \begin{cases} (-1)^{\varepsilon} a_1 \cdot f(a_2 \otimes \cdots \otimes a_k) & (i = 0) \\ f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k) & (1 \leq i \leq k-1) \\ f(a_1 \otimes \cdots \otimes a_{k-1}) \cdot a_k & (i = k), \end{cases}$$

$$\sigma_i(f)(a_1 \otimes \cdots \otimes a_k) := f(a_1 \otimes \cdots \otimes a_i \otimes 1_A \otimes a_{i+1} \otimes \cdots \otimes \cdots \otimes a_k) \quad (0 \leq i \leq k).$$

The associated total complex, denoted by $\mathrm{CH}^*(A, M)$, is called the *Hochschild cochain complex*, whose cohomology group, denoted by $\mathrm{HH}^*(A, M)$, is called the *Hochschild cohomology.* Taking $M^* = (A^\vee)^* = \mathrm{Hom}^*(A, \mathbb{R})$ with dg A-bimodule structure satisfying

$$(d\varphi)(a) + (-1)^{|\varphi|}\varphi(da) = 0, \quad \varphi(ab) = (-1)^{(|a|+|\varphi|)|b|}(b \cdot \varphi)(a) = (\varphi \cdot a)(b), \tag{1.2.11}$$

one sees that the cosimplicial structure on $\{\mathrm{Hom}^*(A^{\otimes k}, A^\vee)\}_k$ is the same as that on $\{\mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})\}$ described previously, in view of the natural isomorphism

$$\mathrm{Hom}^*(A^{\otimes k}, A^\vee) \cong \mathrm{Hom}^*(A^{\otimes k} \otimes A, \mathbb{R}) = \mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})$$

from Hom-$\otimes$ adjunction. See Example 1.5.9 for further discussion.

Remark 1.2.8. We shall use the name "Connes' version of cyclic *co*homology" for "*co*cyclic complex", even if we work with chain complexes rather than cochain complexes. For a cocyclic chain complex $\big((C(k)_*, \partial), \delta_j, \sigma_i, \tau_k\big)$, Connes' version of cyclic cohomology is $\mathrm{HC}^\lambda_*(C) := H_*(C^{\mathrm{cyc}}, b)$ where $C^{\mathrm{cyc}}_* := \mathrm{Ker}(1 - \lambda) \subset C_*$, and $\mathrm{HC}^\lambda_*(C)$ is isomorphic to $\mathrm{HC}^{[[v]]}_*$ of the mixed chain complex $(C_*, b, B) = \big(\prod_{k \geq 0} C(k)_{*+k}, \partial + \delta, Ns(1 - \lambda)\big)$.

Lemma 1.2.9. *In the situation of Example 1.2.6, the isomorphism* (1.2.7) *and the long exact sequences* (1.2.8) *and* (1.2.2b) *fit into the following commutative diagram:*

$$\begin{tikzcd}
\cdots \mathrm{HC}^*_\lambda(C) \arrow[r, "i_{\lambda *}"] \arrow[d, "\cong"', "I_{\lambda *}"] & H^*(C, b) \arrow[r, "B_\lambda"] \arrow[d, equal, "\mathrm{id}"] & \mathrm{HC}^{*-1}_\lambda(C) \arrow[r, "-S_\lambda"] \arrow[d, "\cong"', "I_{\lambda *}"] & \mathrm{HC}^{*+1}_\lambda(C) \cdots \arrow[d, "\cong"', "I_{\lambda *}"] \\
\cdots \mathrm{HC}^*_{[[u]]}(C) \arrow[r, "p_{0*}"] & H^*(C, b) \arrow[r, "B_*"] & \mathrm{HC}^{*-1}_{[[u]]}(C) \arrow[r, "i^+_* \cdot u"] & \mathrm{HC}^{*+1}_{[[u]]}(C) \cdots
\end{tikzcd}$$

Proof. The left square commutes since it commutes at cochain level. To verify commutativity of the other two squares, we need explicit formulas of B_λ and S_λ. Since $(1-\lambda)b = b'(1-\lambda)$, there is a cochain isomorphism $(C/C_{\text{cyc}}, b) \xrightarrow[\cong]{1-\lambda} (\text{Im}(1-\lambda), b')$. Also note that $C_{\text{cyc}} = \text{Im}N$ and $N|_{C_{\text{cyc}}} = ((k+1)\text{id}_{\text{Ker}(1-(-1)^k\tau_k)})_{k\geq 0} : C_{\text{cyc}} \to C_{\text{cyc}}$ is a linear isomorphism. By definition, B_λ is the composition

$$H^*(C,b) \xrightarrow{p_{\lambda *}} H^*(C/C_{\text{cyc}}, b) \xrightarrow[\cong]{1-\lambda} H^*(\text{Im}(1-\lambda), b') \xrightarrow[\cong]{Q_{\lambda *}^{-1}} \text{HC}_\lambda^{*-1}(C),$$

where by examining (1.2.9), $Q_{\lambda *} : \text{HC}_\lambda^{*-1}(C) \xrightarrow{\cong} H^*(\text{Im}(1-\lambda), b')$ is given on cocycles by

$$Q_\lambda : Z^{*-1}(C_{\text{cyc}}, b) \to Z^*(\text{Im}(1-\lambda), b'); \quad x \mapsto (b' \circ (N|_{C_{\text{cyc}}})^{-1})(x).$$

Let us calculate that on $Z(C,b)$,

$$Q_\lambda B = Q_\lambda N s(1-\lambda) = b's(1-\lambda) = (1-sb')(1-\lambda) = 1-\lambda - s(1-\lambda)b = 1-\lambda. \quad (1.2.12)$$

Thus $(B|_{C\to C_{\text{cyc}}})_* = (Q_{\lambda *})^{-1} \circ (1-\lambda) \circ p_{\lambda *} = B_\lambda$, which says the middle square commutes. Similarly, S_λ is the composition

$$\text{HC}_\lambda^{*-1}(C) \xrightarrow[\cong]{Q_{\lambda *}} H^*(\text{Im}(1-\lambda), b') \xrightarrow[\cong]{(1-\lambda)^{-1}} H^*(C/C_{\text{cyc}}, b) \xrightarrow{R_{\lambda *}} \text{HC}_\lambda^{*+1}(C),$$

where $R_{\lambda *} : H^*(C/C_{\text{cyc}}, b) \to \text{HC}_\lambda^{*+1}(C)$ is induced by the map

$$R_\lambda : \{y \in C^* \mid b(y) \in C_{\text{cyc}}\} \to Z^{*+1}(C_{\text{cyc}}, b); \quad y \mapsto b(y).$$

(1.2.12) also holds on $Z(C/C_{\text{cyc}}, b)$, and implies $(1-\lambda)^{-1}Q_{\lambda *} = (B|_{C/C_{\text{cyc}}\to C_{\text{cyc}}})_*^{-1}$. Therefore for $x \in Z^{*-1}(C_{\text{cyc}}, b)$,

$$S_\lambda([x]) = [b(y)] \in \text{HC}_\lambda^{*+1}(C),$$

where y is any choice of elements in C^* satisfying $B(y) = x$ and $(1-\lambda)(b(y)) = 0$. For such x and y, $-b(y) - x\cdot u = (b+uB)(-y)$ is exact in $C^{*+1}[[u]]$, so $I_{\lambda *} \circ (-S_\lambda) = (i_*^+ \cdot u) \circ I_{\lambda *}$, i.e. the right square commutes. □

Example 1.2.10 ($\mathbb{S}^1$-equivariant homology theories [32])**.** Let X be a topological $\mathbb{S}^1$-space, namely a topological space with a continuous $\mathbb{S}^1$-action $F^X : \mathbb{S}^1 \times X \to X$. Let $(C_*, b) =$

$(S_*(X), \partial)$ be the singular chain complex of X, and define the rotation operator $B = J : S_*(X) \to S_{*+1}(X)$ by

$$J(a) := F_*^X([\mathbb{S}^1] \times a), \quad a \in S_*(X). \tag{1.2.13}$$

Here $[\mathbb{S}^1] \in S_1(\mathbb{S}^1)$ is the fundamental cycle of $\mathbb{S}^1$, namely $[\mathbb{S}^1] = \pi_{\mathbb{S}^1}^{\Delta^1} : \Delta^1 = [0,1] \to \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$, and $\times$ is the simplicial cross product induced by standard decomposition of $\Delta^l \times \Delta^k$ into $(k+l)$-simplices (see [28, page 278]). Then $\partial J + J\partial = 0$ since $\partial([\mathbb{S}^1] \times a) = \partial[\mathbb{S}^1] \times a + (-1)^{\deg[\mathbb{S}^1]}[\mathbb{S}^1] \times \partial a = -[\mathbb{S}^1] \times \partial a$. To see $J^2 = 0$, let us write down the cross product with $[\mathbb{S}^1]$ explicitly. For $k \in \mathbb{Z}_{\geq 0}$ and $j \in \{0, \dots, k\}$, consider the embeddings $\iota_{k,j} : \Delta^{k+1} \to \Delta^1 \times \Delta^k$ defined by

$$\iota_{k,j}(t_1, \dots, t_{k+1}) := (t_{j+1}, (t_1, \dots, t_j, t_{j+2}, \dots, t_{k+1})),$$

then for $(\sigma : \Delta^k \to X) \in S_k(X)$,

$$[\mathbb{S}^1] \times \sigma = \sum_{0 \leq j \leq k} (-1)^j (\pi_{\mathbb{S}^1}^{\Delta^1} \times \sigma) \circ \iota_{k,j} \in S_{k+1}(\mathbb{S}^1 \times X).$$

Let $F^{\mathbb{S}^1} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$, $([t], [t']) \mapsto [t + t']$ be the rotation $\mathbb{S}^1$-action on $\mathbb{S}^1$, then

$$F_*^{\mathbb{S}^1}([\mathbb{S}^1] \times [\mathbb{S}^1]) = F^{\mathbb{S}^1} \circ (\pi_{\mathbb{S}^1}^{\Delta^1} \times \pi_{\mathbb{S}^1}^{\Delta^1}) \circ \iota_{1,0} - F^{\mathbb{S}^1} \circ (\pi_{\mathbb{S}^1}^{\Delta^1} \times \pi_{\mathbb{S}^1}^{\Delta^1}) \circ \iota_{1,1} = 0.$$

From the commutative diagram

$$\begin{tikzcd}
\mathbb{S}^1 \times \mathbb{S}^1 \times X \arrow[r, "F^{\mathbb{S}^1} \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_{\mathbb{S}^1} \times F^X"] & \mathbb{S}^1 \times X \arrow[d, "F^X"] & ([t], [t'], x) \arrow[r, mapsto] \arrow[d, mapsto] & ([t + t'], x) \arrow[d, mapsto] \\
\mathbb{S}^1 \times X \arrow[r, "F^X"] & X & ([t], F^X([t'], x)) \arrow[r, mapsto] & F^X([t + t'], x)
\end{tikzcd}$$

we conclude that for any $a \in S_*(X)$,

$$J^2(a) = F_*^X([\mathbb{S}^1] \times F_*^X([\mathbb{S}^1] \times a)) = F_*^X(F_*^{\mathbb{S}^1}([\mathbb{S}^1] \times [\mathbb{S}^1]) \times a)) = 0.$$

For the mixed chain complex $(C_*, b, B) = (S_*(X), \partial, J)$, there is a natural isomorphism ([32, Lemma 5.1])

$$\mathrm{HC}_*^{[v^{-1}]}(S(X)) \cong H_*^{\mathbb{S}^1}(X) := H_*(X \times_{\mathbb{S}^1} E\mathbb{S}^1),$$

namely $\mathrm{HC}_*^{[v^{-1}]}(S(X))$ is isomorphic to the $\mathbb{S}^1$-equivariant homology of X, i.e. homology of the homotopy quotient (Borel construction). The other two cyclic homology groups of $(S_*(X), \partial, J)$ are called the *negative* and *periodic* $\mathbb{S}^1$*-equivariant homology* of X, and are denoted by

$$G_*^{\mathbb{S}^1}(X) := \mathrm{HC}_*^{[[v]]}(S(X)), \quad \widehat{H}_*^{\mathbb{S}^1}(X) := \mathrm{HC}_*^{[[v,v^{-1}]}(S(X)),$$

respectively. The tautological exact sequence (1.2.1) translates into

$$\cdots \to G_*^{\mathbb{S}^1}(X) \to \widehat{H}_*^{\mathbb{S}^1}(X) \to H_{*-2}^{\mathbb{S}^1}(X) \to G_{*-1}^{\mathbb{S}^1}(X) \to \cdots, \tag{1.2.14}$$

and the Connes-Gysin exact sequences (1.2.2) translate into

$$\cdots \to H_*(X) \to H_*^{\mathbb{S}^1}(X) \to H_{*-2}^{\mathbb{S}^1}(X) \to H_{*-1}(X) \to \cdots \tag{1.2.15a}$$

$$\cdots \to G_{*+2}^{\mathbb{S}^1}(X) \to G_*^{\mathbb{S}^1}(X) \to H_*(X) \to G_{*+1}^{\mathbb{S}^1}(X) \to \cdots. \tag{1.2.15b}$$

We end this example by mentioning that (1.2.15a) coincides with the Gysin sequence associated to the $\mathbb{S}^1$-fibration $X \times E\mathbb{S}^1 \to X \times_{\mathbb{S}^1} E\mathbb{S}^1$.

Remark 1.2.11. There seems to be no interpretation of $G_*^{\mathbb{S}^1}(X)$ and $\widehat{H}_*^{\mathbb{S}^1}(X)$ as homology groups of some spaces naturally associated to X, but there are homotopy theoretic interpretations. For example, when X is a (finite) $\mathbb{S}^1$-CW complex, [3, Lemma 4.4] says $G_*^{\mathbb{S}^1}(X)$ is naturally isomorphic to the homotopy groups of the homotopy fixed point spectrum $(\mathbf{H} \wedge X_+)^{h\mathbb{S}^1}$, where $\mathbf{H}$ is the Eilenberg-MacLane spectrum $\{K(\mathbb{Z}, n)\}$.

1.3 A cocyclic complex and an ∞-quasi-isomorphism

Let X be a topological space with $\mathbb{S}^1$-action $F^X : \mathbb{S}^1 \times X \to X$. There is a cocyclic structure on $\{X \times \Delta^k\}_{k \in \mathbb{Z}_{\geq 0}}$, where $\delta_i^{X \times \Delta^k} := \mathrm{id}_X \times \delta_i^{\Delta^k}$, $\sigma_i^{X \times \Delta^k} := \mathrm{id}_X \times \sigma_i^{\Delta^k}$ ($\delta_i^{\Delta^k}$, $\sigma_i^{\Delta^k}$ are as in (1.2.5)), and $\tau_k^{X \times \Delta^k} : X \times \Delta^k \to X \times \Delta^k$ is defined by

$$\tau_k^{X \times \Delta^k}(x, t_1, \ldots, t_k) := (F^X([t_1], x), t_2 - t_1, \ldots, t_k - t_1, 1 - t_1).$$

Taking singular chains of the cocyclic space $\{X \times \Delta^k\}_{k\geq 0}$ yields a cocyclic chain complex $\{S_*(X \times \Delta^k)\}_{k\geq 0}$. Let us denote the associated mixed complex by

$$(S_*^{X\Delta}, b, B) := \Big(\prod_{k\geq 0} S_{*+k}(X \times \Delta^k), \partial + \delta, Ns(1-\lambda)\Big).$$

The $\mathbb{S}^1$-action on X extends to $X \times \Delta^k$ where the $\mathbb{S}^1$-action on Δ^k is trivial, and then the rotation operator $J : S_*(X) \to S_{*+1}(X)$ defined in Example 1.2.10 extends component-by-component to $S_*^{X\Delta}$ by

$$J : S_*^{X\Delta} \to S_{*+1}^{X\Delta}, \quad (x_k)_{k\geq 0} \mapsto (J(x_k))_{k\geq 0}.$$

By Example 1.2.10, $J^2 = 0$ and $\partial J + J\partial = 0$. Since $\mathbb{S}^1$ acts trivially on Δ^k, J commutes with δ_i, σ_i. It follows that $\delta J + J\delta = 0$ and $J(S^{X\Delta,\mathrm{nm}}) \subset S^{X\Delta,\mathrm{nm}}$, so $(S_*^{X\Delta}, b, J)$, $(S_*^{X\Delta,\mathrm{nm}}, b, J)$ are also mixed complexes. J also commutes with $\tau_k^{X\times\Delta^k}$ because of the commutative diagram

$$\begin{tikzcd} \mathbb{S}^1 \times X \times \Delta^k \arrow[r, mapsto, "\mathrm{id}_{\mathbb{S}^1}\times\tau_k^{X\times\Delta^k}"] \arrow[d, mapsto, "F^{X\times\Delta^k}"] & \mathbb{S}^1 \times X \times \Delta^k \arrow[d, mapsto, "F^{X\times\Delta^k}"] \\ X \times \Delta^k \arrow[r, mapsto, "\tau_k^{X\times\Delta^k}"] & X \times \Delta^k \end{tikzcd} \tag{1.3.1}$$

$$\begin{tikzcd} ([t], x, t_1, \dots, t_k) \arrow[r, mapsto] \arrow[d, mapsto] & ([t], F^X([t_1], x), t_2 - t_1, \dots, 1 - t_1) \arrow[d, mapsto] \\ (F^X([t], x), t_1, \dots, t_k) \arrow[r, mapsto] & (F^X([t + t_1], x), t_2 - t_1, \dots, 1 - t_1), \end{tikzcd}$$

so $JB + BJ = 0$. We will analyze the relationship between the mixed complexes

$$(S_*^{X\Delta}, b, B), \quad (S_*^{X\Delta}, b, J), \quad (S_*(X), \partial, J).$$

If there is no risk of confusion, we shall write $\delta_i^{X\times\Delta^k}, \sigma_i^{X\times\Delta^k}, \tau_k^{X\times\Delta^k}$ and the induced maps on singular chain complexes as $\delta_i, \sigma_i, \tau_k$ for short. Note that δ_i, σ_i do not involve $\mathbb{S}^1$-action, so if we forget the $\mathbb{S}^1$-action, there is still a total complex $(S_*^{X\Delta}, b = \partial + \delta)$ from the cosimplicial chain complex $\{S_*(X \times \Delta^k)\}_{k\in\mathbb{Z}_{\geq 0}}$.

Let us state the main theorem of this section.

Theorem 1.3.1. *Let X be a topological $\mathbb{S}^1$-space. Then for both of the mixed complex structures (b,B) and (b,J) on $S_*^{X\Delta} = \prod_{k\geq 0} S_{*+k}(X\times\Delta^k)$, there are natural isomorphisms*

$$\mathrm{HC}_*^{[v^{-1}]}(S^{X\Delta}) \cong H_*^{\mathbb{S}^1}(X) \quad \text{as } \mathbb{R}[v^{-1}]\text{-modules},$$
$$\mathrm{HC}_*^{[[v,v^{-1}]}(S^{X\Delta}) \cong \widehat{H}_*^{\mathbb{S}^1}(X) \quad \text{as } \mathbb{R}[[v,v^{-1}]\text{-modules},$$
$$\mathrm{HC}_*^{[[v]]}(S^{X\Delta}) \cong G_*^{\mathbb{S}^1}(X) \quad \text{as } \mathbb{R}[[v]]\text{-modules}.$$

Moreover, these isomorphisms throw the (tautological and Connes-Gysin) exact sequences (1.2.1)(1.2.2) *for cyclic homology theories onto the (tautological and Gysin) exact sequences for $\mathbb{S}^1$-equivariant homology theories.*

Proof. The statement about isomorphisms is a consequence of Lemma 1.2.3, Corollary 1.3.5 and Proposition 1.3.7 below. The statement about long exact sequences is then a consequence of Lemma 1.2.4. □

Corollary 1.3.2. *For any topological $\mathbb{S}^1$-space X, Connes' version of cyclic cohomology of the cocyclic chain complex $\{S_*(X\times\Delta^k)\}_{k\in\mathbb{Z}_{\geq 0}}$ is naturally isomorphic to the negative $\mathbb{S}^1$-equivariant homology of X.* □

Lemma 1.3.3. *For any topological space X, the projection chain map*

$$\mathrm{pr}_0 : (S_*^{X\Delta}, b) \to (S_*(X), \partial); \quad (c_k)_{k\geq 0} \mapsto c_0$$

is a quasi-isomorphism.

Proof. Since pr_0 is surjective, it suffices to prove $\mathrm{Ker}(\mathrm{pr}_0)_* = \prod_{k\geq 1} S_{*+k}(X\times\Delta^k)$ is b-acyclic. Let us write $\tilde{S}_* := \mathrm{Ker}(\mathrm{pr}_0)_*$ and consider the decreasing filtration $\mathcal{F}_p$ ($p\in\mathbb{Z}_{\geq 1}$) on $\tilde{S}$ defined by $\mathcal{F}_p\tilde{S}_* := \prod_{k\geq p} S_{*+k}(X\times\Delta^k)$. The E_1-page of the spectral sequence of this filtration is divided into columns indexed by $q\in\mathbb{Z}_{\geq 0}$, each of which looks like

$$0 \to H_q(X\times\Delta^1) \xrightarrow{\delta_*} H_q(X\times\Delta^2) \xrightarrow{\delta_*} H_q(X\times\Delta^3) \xrightarrow{\delta_*} \cdots . \tag{1.3.2}$$

For each $k\geq 1$, the map $p_k := \sigma_0\sigma_1\cdots\sigma_{k-1} : X\times\Delta^k \to X\times\Delta^0 = X$, $(x,t_1,\dots,t_k)\mapsto (x,0) = x$ is a homotopy equivalence. Since $p_{k+1} = p_k\sigma_k$ and $\sigma_j\delta_i = \mathrm{id}$ ($i=j,j+1$), we conclude that for any $k \geq 1$ and $0 \leq i \leq k$, $H_*(X \times \Delta^{k-1}) \xrightarrow{(\delta_i)_*} H_*(X \times \Delta^k)$ is an isomorphism such that $(\delta_i)_* = (\sigma_{k-1})_*^{-1}$.Then since $\delta|_{S_q(\Delta^{k-1}\times X)} = (-1)^{q+k} \sum_{i=0}^k (-1)^i \delta_i$, (1.3.2) is nothing but

$$0 \to H_q(X \times \Delta^1) \xrightarrow[\cong]{(-1)^q(\sigma_1)_*^{-1}} H_q(X \times \Delta^2) \xrightarrow{0} H_q(X \times \Delta^3) \xrightarrow[\cong]{(-1)^q(\sigma_3)_*^{-1}} \cdots .$$

Thus all E_2-terms vanish. Finally, since the filtration $\mathcal{F}_p$ on $\tilde{S}$ is complete ($\tilde{S} = \varprojlim \tilde{S}/\mathcal{F}_p\tilde{S}$) and bounded above ($\tilde{S}_* = \mathcal{F}_1\tilde{S}_*$), standard convergence theorem [49, Theorem 5.5.10(2)] gives $H_*(\tilde{S}, b) = 0$. $\square$

Remark 1.3.4. The proof of Lemma 1.3.3 implies that more generally, for a cosimplicial complex $\{(C(k)_*, \partial)\}_{k\geq 0}$, if $\sigma_0\sigma_1\cdots\sigma_{k-1} : C(k) \to C(0)$ is a quasi-isomorphism for each $k \geq 1$, then so is $\mathrm{pr}_0 : (\prod_{k\geq 0} C(k)_{*+k}, \partial + \delta) \to (C(0)_*, \partial)$ ([29, Lemma 8.3]).

Corollary 1.3.5. *For any topological $\mathbb{S}^1$-space X, $\mathrm{pr}_0 : (S_*^{X\Delta}, b, J) \to (S_*(X), \partial, J)$ is a mixed complex quasi-isomorphism.* $\square$

Note that $S_*(X) = S_*^{X\Delta}(0) = S_*^{X\Delta,\mathrm{nm}}(0)$ by vacuum normalized condition. Since $(S_*^{X\Delta,\mathrm{nm}}, b) \hookrightarrow (S_*^{X\Delta}, b)$ is a quasi-isomorphism, Lemma 1.3.3 and Corollary 1.3.5 also hold true if $S_*^{X\Delta}$ is replaced by $S_*^{X\Delta,\mathrm{nm}}$. In the following, we may use $S_*^{X\Delta,\mathrm{nm}}$ to simplify calculation involving Connes' operator B. One could also stick with $S_*^{X\Delta}$, though.

Recall the augmentation map $\varepsilon : S_0(X) \to \mathbb{R}$, $\sum \lambda_i \cdot (\Delta^0 \xrightarrow{u_i} X) \mapsto \sum \lambda_i$.

Lemma 1.3.6. *Consider the topological $\mathbb{S}^1$-space $\mathbb{S}^1$ with rotation action on itself.*

(i) There exists a sequence of elements $\{\xi^n = (\xi_k^n)_{k\geq 0} \in S_{2n}^{\mathbb{S}^1\Delta,\mathrm{nm}}\}_{n\in\mathbb{Z}_{\geq 0}}$ such that

$$\varepsilon(\xi_0^0) = 1, \quad b(\xi^0) = 0, \quad b(\xi^n) = (J - B)(\xi^{n-1}) \ (n \geq 1).$$

(ii) Suppose $\{\xi^n\}_{n\geq 0}$, $\{\xi'^n\}_{n\geq 0}$ both satisfy conditions in (i). Then there exists a sequence of elements $\{\eta^n = (\eta_k^n)_{k\geq 0} \in S_{2n+1}^{\mathbb{S}^1\Delta,\mathrm{nm}}\}_{n\in\mathbb{Z}_{\geq 0}}$ such that

$$\xi^0 - \xi'^0 = b(\eta^0), \quad \xi^n - \xi'^n = b(\eta^n) - (J - B)(\eta^{n-1}) \ (n \geq 1).$$

Proof. (i) Consider the isomorphisms $(\mathrm{pr}_0)_* : H_0(S^{\mathbb{S}^1\Delta,\mathrm{nm}}, b) \xrightarrow{\cong} H_0(\mathbb{S}^1)$ from Lemma 1.3.3 and $\varepsilon_* : H_0(\mathbb{S}^1) \xrightarrow{\cong} \mathbb{R}$ induced by augmentation. Choose a 0-cycle ξ^0 in the homology class $(\varepsilon_* \circ (\mathrm{pr}_0)_*)^{-1}(1) \in H_0(S^{\mathbb{S}^1\Delta,\mathrm{nm}}, b)$, then $\xi^0 = (\xi^0_k)_{k\geq 0}$ is as desired. Next, since $bB = -Bb$ and $bJ = -Jb$, $B(\xi^0)$ and $J(\xi^0)$ are 1-cycles. We claim that they are homologous. Since pr_0 is a quasi-isomorphism, it suffices to look at $\mathrm{pr}_0(B(\xi^0))$ and $\mathrm{pr}_0(J(\xi^0))$. By definition,

$$\mathrm{pr}_0(B(\xi^0)) = (Ns(\xi^0))_0 = \sigma_0\tau_1(\xi^0_1), \quad \mathrm{pr}_0(J(\xi^0)) = J(\xi^0_0) = F^{\mathbb{S}^1}_*([\mathbb{S}^1] \times \xi^0_0).$$

By construction, $\xi^0_0 \in S_0(\mathbb{S}^1)$ is homologous to the map $\Delta^0 \ni 0 \mapsto [0] \in \mathbb{S}^1$, and $\xi^0_1 \in S_1(\mathbb{S}^1 \times \Delta^1)$ is homologous to the map $\Delta^1 \ni t \mapsto ([0], t) \in \mathbb{S}^1 \times \Delta^1$. So $\mathrm{pr}_0(B(\xi^0))$, $\mathrm{pr}_0(J(\xi^0))$ are homologous to

$$\Delta^1 \to \mathbb{S}^1 \times \Delta^1 \xrightarrow{\tau_1} \mathbb{S}^1 \times \Delta^1 \xrightarrow{\sigma_0} \mathbb{S}^1; \quad t \mapsto ([0], t) \mapsto ([t], 1-t) \mapsto [t],$$

$$\Delta^1 \to \Delta^1 \times \Delta^0 \to \mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow{F^{\mathbb{S}^1}} \mathbb{S}^1; \quad t \mapsto (t, 0) \mapsto ([t], [0]) \mapsto [t],$$

respectively. Namely they are both homologous to $[\mathbb{S}^1]$. This proves the existence of $\xi^1 \in S_2^{\mathbb{S}^1\Delta,\mathrm{nm}}$ satisfying $b(\xi^1) = (J - B)(\xi^0)$. Now suppose $\xi^0, \xi^1, \ldots, \xi^n$ $(n \geq 1)$ have been chosen as desired, to find ξ^{n+1}, simply notice that $(J - B)(\xi^n)$ is a $(2n + 1)$-cycle:

$$b((J - B)(\xi^n)) = -(J - B)(b(\xi^n)) = -(J - B)^2(\xi^{n-1}) = 0,$$

where $(J - B)^2 = 0$ since $J^2 = 0$, $B^2 = 0$ and $JB + BJ = 0$ (see (1.3.1)). Since $H_{2n+1}(S^{\mathbb{S}^1\Delta,\mathrm{nm}}, b) \cong H_{2n+1}(\mathbb{S}^1) = 0$ $(n \geq 1)$, $(J - B)(\xi^n)$ is exact, i.e. ξ^{n+1} exists.

(ii) By construction, ξ^0 is homologous to ξ'^0, so η^0 exists. To inductively find η^n for $n \geq 1$, simply check that $\xi^n - \xi'^n + (J - B)(\eta^{n-1})$ is a $2n$-cycle, which is then exact since $H_{2n}(\mathbb{S}^1) = 0$ $(n \geq 1)$. □

Proposition 1.3.7. *Let X be a topological $\mathbb{S}^1$-space. Denote the transposition $X \times \mathbb{S}^1 \to \mathbb{S}^1 \times X$ by ν.*

(i) Choose $\xi = \{\xi^n\}_{n\geq 0}$ as in Lemma 1.3.6(i). Define a sequence of linear maps $f^\xi = \{f^\xi_n : S_*(X) \to S^{X\Delta,\mathrm{nm}}_{*+2n}\}_{n\geq 0}$ by

$$f^\xi_n(a) := \big((F^{X\times\Delta^k} \circ (\nu \times \mathrm{id}_{\Delta^k}))_*(a \times \xi^n_k)\big)_{k\geq 0}.$$

Then f^ξ is an ∞-quasi-isomorphism from $(S_*(X), \partial, J)$ to $(S_*^{X\Delta,\mathrm{nm}}, b, B)$.

(ii) For two choices ξ, ξ', the ∞-quasi-isomorphisms $f^\xi, f^{\xi'}$ are ∞-homotopic.

Proof. (i) The verification of $f^\xi_0 \circ \partial - b \circ f^\xi_0 = 0$ is simpler than $f^\xi_n \circ \partial - b \circ f^\xi_n = B \circ f^\xi_{n-1} - f^\xi_{n-1} \circ J$ $(n \geq 1)$, so we omit it. Let us write $F_k := F^{X \times \Delta^k} \circ (\nu \times \mathrm{id}_{\Delta^k}) : X \times \mathbb{S}^1 \times \Delta^k \to X \times \Delta^k$. For $n \geq 1$,

$$\begin{aligned}
\big((f^\xi_n \circ \partial - b \circ f^\xi_n)(a)\big)_k &= (F_k)_*(\partial a \times \xi^n_k - \partial(a \times \xi^n_k) - \delta(a \times \xi^n_{k-1})) \\
&= (F_k)_*((-1)^{|a|} a \times (-\partial \xi^n_k - \delta \xi^n_{k-1})) \\
&= (-1)^{|a|} (F_k)_*(a \times (B(\xi^{n-1}_{k+1}) - J(\xi^{n-1}_k))),
\end{aligned}$$

where the last equality follows from $b(\xi^n) = (J - B)(\xi^{n-1})$. Now introduce maps

$$\begin{aligned}
&G_k : X \times \mathbb{S}^1 \times \Delta^k \to X \times \Delta^k \\
&\qquad (x, [t], t_1, \ldots, t_k) \mapsto (F^X([t + t_1], x), t_2 - t_1, \ldots, 1 - t_1), \\
&H_k : X \times \mathbb{S}^1 \times \mathbb{S}^1 \times \Delta^k \to X \times \Delta^k \\
&\qquad (x, [t], [t'], t_1, \ldots, t_k) \mapsto (F^X([t + t'], x), t_1, \ldots, t_k),
\end{aligned}$$

then

$$\begin{aligned}
G_k &= F_k \circ (\mathrm{id}_X \times \tau_k^{\mathbb{S}^1 \times \Delta^k}) = \tau_k^{X \times \Delta^k} \circ F_k, \\
H_k &= F_k \circ (\mathrm{id}_X \times F^{\mathbb{S}^1 \times \Delta^k}) = F_k \circ (F^X \times \mathrm{id}_{\mathbb{S}^1 \times \Delta^k}) \circ (\nu \times \mathrm{id}_{\mathbb{S}^1 \times \Delta^k}).
\end{aligned}$$

It follows that

$$\begin{aligned}
(-1)^{|a|} (F_k)_*(a \times B(\xi^{n-1}_{k+1})) &= B((F_k)_*(a \times \xi^{n-1}_{k+1})), \\
(-1)^{|a|} (F_k)_*(a \times J(\xi^{n-1}_k)) &= (F_k)_*(J(a) \times \xi^{n-1}_k).
\end{aligned}$$

This implies $(f^\xi_n \circ \partial - b \circ f^\xi_n)(a) = (B \circ f^\xi_{n-1} - f^\xi_{n-1} \circ J)(a)$, so f^ξ is an ∞-morphism. It remains to show f^ξ_0 is a quasi-isomorphism. Since ξ^0_0 is homologous to $\Delta^0 \ni 0 \mapsto [0] \in \mathbb{S}^1$, $\mathrm{pr}_0 \circ f^\xi_0$ is chain homotopic to $\mathrm{id}_{S_*(X)}$. Since pr_0 is a quasi-isomorphism, so is f^ξ_0.

(ii) Choose η as in Lemma 1.3.6(ii). Define a sequence of linear maps $h^\eta = \{h^\eta_n : S_*(X) \to S^{X\Delta,\mathrm{nm}}_{*+2n+1}\}_{n\geq 0}$ by

$$h^\eta_n(a) := (-1)^{|a|}\big((F^{X\times\Delta^k} \circ (\nu \times \mathrm{id}_{\Delta^k}))_*(a \times \eta^n_k)\big)_{k\geq 0}.$$

Then similar calculation as (i) shows h^η is an ∞-homotopy between f^ξ and $f^{\xi'}$. □

1.4 The story of differentiable spaces

1.4.1 Differentiable spaces and de Rham chains

Materials in this subsection are collected from Irie [29]. The notion of differentiable spaces is a modification of that utilized by Chen [8], and the notion of de Rham chains is inspired by an idea of Fukaya [16].

Let $\mathscr{U} := \coprod_{n\geq m\geq 0} \mathscr{U}_{n,m}$ where $\mathscr{U}_{n,m}$ denotes the set of oriented m-dimensional C^∞-submanifolds of $\mathbb{R}^n$. Let X be a set. A *differentialble structure* $\mathscr{P}(X)$ on X is a family of maps $\{(U,\varphi)\}$ called *plots*, such that:

- Every plot is a map φ from some $U \in \mathscr{U}$ to X;
- If $\varphi : U \to X$ is a plot, $U' \in \mathscr{U}$ and $\theta : U' \to U$ is a submersion, then $\varphi \circ \theta : U' \to X$ is a plot.

A *differentiable space* is a pair of a set and a differentiable structure on it. A map $f : X \to Y$ between differentiable spaces is called *smooth*, if $(U, f \circ \varphi) \in \mathscr{P}(Y)$ for any $(U,\varphi) \in \mathscr{P}(X)$. A subset of a differentiable space and the product of a family of differentiable spaces admit naturally induced differentiable structures ([29, Example 4.2(iii)(iv)]).

Remark 1.4.1. Differentiable structures are defined on *sets* rather than topological spaces. For later purpose, we say a differentiable structure and a topology on a set X are *compatible* if every plot is continuous.

Example 1.4.2. Here are some important examples of differentiable spaces.

(i) Let M be a C^∞-manifold. Consider two differentiable structures on it:

  a) Define $(U,\varphi) \in \mathscr{P}(M)$ if $\varphi : U \to M$ is a C^∞-map;

  b) Define $(U,\varphi) \in \mathscr{P}(M_{\mathrm{reg}})$ if $\varphi : U \to M$ is a (C^∞-)submersion.

  The set-theoretic identity map $\mathrm{id}_M : M_{\mathrm{reg}} \to M$ is smooth, but its inverse is not.

(ii) Let $\mathcal{L}M := C^\infty(\mathbb{S}^1, M)$ be the smooth free loop space of M, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. There is a differentiable structure $\mathscr{P}(\mathcal{L}^M)$ on $\mathcal{L}M$ defined by: $(U,\varphi) \in \mathscr{P}(\mathcal{L})$ iff $U \times \mathbb{S}^1 \to M$, $(u,[t]) \mapsto \varphi(u)(t)$ is a C^∞-map.

(iii) For each $k \in \mathbb{Z}_{\geq 0}$, the smooth free loop space of M with k inner marked points, denoted by $\mathcal{L}_{k+1}M$, is defined as

$$\{(\gamma, t_1, \ldots, t_k) \in \mathcal{L}M \times \Delta^k \mid \partial_t^m \gamma(0) = \partial_t^m \gamma(t_j) = 0 \ (1 \leq \forall j \leq k,\ \forall m \geq 1)\}.$$

  It has induced differentiable structure $\mathscr{P}(\mathcal{L}_{k+1}^M)$ as a subspace of $\mathcal{L}^M \times \Delta^k$, where Δ^k is viewed as a subspace of $\mathbb{R}^k$ with the differentiable structure in (i)(a).

(iv) The smooth free Moore path space of M, denoted by ΠM, is defined as

$$\{(T,\gamma) \mid T \in \mathbb{R}_{\geq 0},\ \gamma \in C^\infty([0,T], M),\ \partial_t^m \gamma(0) = \partial_t^m \gamma(T) = 0 \ (\forall m \geq 1)\}.$$

  Consider two differentiable structures $\mathscr{P}(\Pi^M), \mathscr{P}(\Pi^M_{\mathrm{reg}})$ on ΠM:

  a) Define $(U,\varphi) \in \mathscr{P}(\Pi^M)$ if $\varphi = (\varphi_T, \varphi_\gamma) : U \to \Pi M$ satisfies the following conditions: (1) $\varphi_T : U \to \mathbb{R}_{\geq 0}$ is a C^∞-map. (2) The map

$$\tilde{U} := \{(u,t) | u \in U, t \in [0, \varphi_T(u)]\} \to M; \quad (u,t) \mapsto \varphi_\gamma(u)(t)$$

  extends to a C^∞-map from an open neighborhood of $\tilde{U}$ in $U \times \mathbb{R}$ to M.

  b) Define $(U,\varphi) \in \mathscr{P}(\Pi^M_{\mathrm{reg}})$ if: $(U,\varphi) \in \mathscr{P}(\Pi^M)$ and the map $U \to M$, $u \mapsto \varphi_\gamma(u)(t_0)$ is a submersion for $t_0 = 0, T$.

(v) For each $k \in \mathbb{Z}_{\geq 0}$, the smooth free Moore loop space of M with k inner marked points, denoted by $\mathscr{L}_{k+1}M$, is defined as

$$
\begin{aligned}
\{((T_0,\gamma_0),\dots,(T_k,\gamma_k)) \in (\Pi M)^{k+1} \mid \gamma_j(T_j) = \gamma_{j+1}(0)\ (0 \leq j \leq k-1), \\
\gamma_k(T_k) = \gamma_0(0)\}, \\
= \{(T,\gamma,t_1,\dots,t_k) \in \Pi M \times \mathbb{R}^k \mid 0 \leq t_1 \leq \cdots \leq t_k \leq T,\ \gamma(0) = \gamma(T), \\
\partial_t^m \gamma(t_j) = 0\ (1 \leq \forall j \leq k,\ \forall m \geq 1)\}.
\end{aligned}
$$

Apparently there are two ways to endow the set $\mathscr{L}_{k+1}M$ with differentiable structures, namely as a subset of $(\Pi M)^{k+1}$ or of $\Pi M \times \mathbb{R}^k$. It basically follows from [29, Lemma 7.2] that these two ways are equivalent. Let us denote by $\mathscr{L}^M_{k+1}$ (resp. $\mathscr{L}^M_{k+1,\mathrm{reg}}$) the differentiable space obtained from Π^M (resp. Π^M_{reg}).

Note that the inclusion of sets $\mathcal{L}_{k+1}M = \{T = 1\} \subset \mathscr{L}_{k+1}M$, induced by the inclusion $\mathcal{L}_1 M \times \Delta^k \subset \Pi M \times \mathbb{R}^k$, is also an inclusion of differentiable spaces $\mathcal{L}^M_{k+1} \hookrightarrow \mathscr{L}^M_{k+1}$.

The *de Rham chain complex* $(C^{dR}_*(X), \partial)$ of a differentiable space X is defined as follows. For $n \in \mathbb{Z}$, let $C_n := \bigoplus_{(U,\varphi)\in\mathscr{P}(X)} \Omega_c^{\dim U - n}(U)$. For any $(U,\varphi) \in \mathscr{P}(X)$ and $\omega \in \Omega_c^{\dim U-n}(U)$, denote the image of ω under the natural inclusion $\Omega_c^{\dim U-n}(U) \hookrightarrow C_n$ by (U,φ,ω). Let $D_n \subset C_n$ be the subspace spanned by all elements of the form $(U,\varphi,\pi_!\omega') - (U',\varphi\circ\pi,\omega')$, where $(U,\varphi) \in \mathscr{P}(X)$, $U' \in \mathscr{U}$, $\omega' \in \Omega_c^{\dim U'-n}(U')$, and $\pi : U' \to U$ is a submersion. Then define $C^{dR}_n(X) := C_n/D_n$. By abuse of notation we still denote the image of (U,φ,ω) under the quotient map $C_n \to C^{dR}_n(X)$ by (U,φ,ω). Then $\partial : C^{dR}_*(X) \to C^{dR}_{*-1}(X)$ is defined by $\partial(U,\varphi,\omega) := (U,\varphi,d\omega)$. The homology of $(C^{dR}_*(X),\partial)$ is denoted by $H^{dR}_*(X)$.

Remark 1.4.3. For any oriented C^∞-manifold M, there exists $n \in \mathbb{Z}_{\geq 0}$ and an embedding $\iota : M \hookrightarrow \mathbb{R}^n$. Then $(\iota(M),\iota^{-1}) \in \mathscr{P}(M_{\mathrm{reg}}) \subset \mathscr{P}(M)$, and $(\iota(M),\iota^{-1},(\iota^{-1})^*\omega) \in C^{dR}_*(M_{\mathrm{reg}}) \subset C^{dR}_*(M)$ for any $\omega \in \Omega_c(M)$. Such a de Rham chain is independent of choices of n and ι, and by abuse of notation we write it as (M,id_M,ω). If M is closed oriented, we call $(M,\mathrm{id}_M,1)$ the *fundamental de Rham cycle* of M (or M_{reg}).

Let X, Y be differentiable spaces. The *cross product* on de Rham chains is a chain map $C_k^{dR}(X) \otimes C_l^{dR}(Y) \to C_{k+l}^{dR}(X \times Y)$, defined by

$$(U, \varphi, \omega) \times (V, \eta, \psi) := (-1)^{l \cdot \dim U}(U \times V, \varphi \times \psi, \omega \times \eta). \tag{1.4.1}$$

1.4.2 $\mathbb{S}^1$-equivariant homology of differentiable $\mathbb{S}^1$-spaces

Let X be a *differentiable* $\mathbb{S}^1$*-space*, namely X is a differentiable space with a smooth map $F^X : \mathbb{S}^1 \times X \to X$, where $\mathbb{S}^1$ is endowed with the differentiable structure in Example 1.4.2(i)(a). Let $(\mathbb{S}^1, \mathrm{id}_{\mathbb{S}^1}, 1) \in C_1^{dR}(\mathbb{S}^1)$ be the fundamental de Rham 1-cycle of $\mathbb{S}^1$. Define

$$J : C_*^{dR}(X) \to C_{*+1}^{dR}(X); \quad a \mapsto F_*^X((\mathbb{S}^1, \mathrm{id}_{\mathbb{S}^1}, 1) \times a),$$

then J is clearly an anti-chain map. We claim $J^2 = 0$. Let $g : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ be the smooth map $([t], [t']) \mapsto [t + t']$, then by the same arguments as Example 1.2.10, to see $J^2 = 0$, it suffices to prove $g_*((\mathbb{S}^1, \mathrm{id}_{\mathbb{S}^1}, 1) \times (\mathbb{S}^1, \mathrm{id}_{\mathbb{S}^1}, 1)) = 0 \in C_2^{dR}(\mathbb{S}^1)$. This is easy:

$$\begin{aligned} g_*((\mathbb{S}^1, \mathrm{id}_{\mathbb{S}^1}, 1) \times (\mathbb{S}^1, \mathrm{id}_{\mathbb{S}^1}, 1)) &\overset{(1.4.1)}{=\!=} -g_*((\mathbb{S}^1 \times \mathbb{S}^1, \mathrm{id}_{\mathbb{S}^1 \times \mathbb{S}^1}, 1)) \\ &= -(\mathbb{S}^1 \times \mathbb{S}^1, \mathrm{id}_{\mathbb{S}^1} \circ g, 1) = -(\mathbb{S}^1, \mathrm{id}_{\mathbb{S}^1}, g_!(1)) = 0. \end{aligned}$$

The middle equality on the second line holds since g is a submersion. Thus $(C_*^{dR}(X), \partial, J)$ is a mixed chain complex. One can then define positive (ordinary), periodic and negative "$\mathbb{S}^1$-equivariant de Rham homology" of X as the $\mathrm{HC}_*^{[v^{-1}]}$, $\mathrm{HC}_*^{[[v]]}$ and $\mathrm{HC}_*^{[[v, v^{-1}]}$ versions of cyclic homology of $(C_*^{dR}(X), \partial, J)$.

Consider Δ^k as a differentiable subspace of $\mathbb{R}^k$. Then the cocyclic maps $\delta_i, \sigma_i, \tau_k$ among $\{X \times \Delta^k\}_{k \in \mathbb{Z}_{\geq 0}}$, defined by the same formulae as in Section 1.3, are smooth maps between differentiable spaces. So $\{X \times \Delta^k\}_{k \in \mathbb{Z}_{\geq 0}}$ is a cocyclic differentiable space and $\{C_*^{dR}(X \times \Delta^k)\}_{k \in \mathbb{Z}_{\geq 0}}$ is a cocyclic chain complex, which gives rise to a mixed complex

$$(C_*^{dR, X\Delta}, b, B) := \Big(\prod_{k \geq 0} C_{*+k}^{dR}(X \times \Delta^k), \partial + \delta, Ns(1 - \lambda) \Big).$$

The smooth $\mathbb{S}^1$-action $\mathbb{S}^1 \times X \to X$ also extends trivially to $\mathbb{S}^1 \times X \times \Delta^k \to X \times \Delta^k$ and gives a mixed complex $(C_*^{X\Delta}, b, J)$.

There is counterpart of Theorem 1.3.1 for differentiable $\mathbb{S}^1$-spaces, whose proof is also similar. We omit the details since we will not make essential use of it.

The *smooth singular chain complex* $(C_*^{\mathrm{sm}}(X), \partial)$ of a differentiable space X, introduced in [29, Section 4.7], is defined in a similar way as the singular chain complex of topological spaces, except that only "strongly smooth" maps $\Delta^k \to X$ are considered. The homology of $(C_*^{\mathrm{sm}}(X), \partial)$ is denoted by $H_*^{\mathrm{sm}}(X)$.

Smooth singular homology is related to singular homology and de Rham homology in the following way.

- Let X be a differentiable space with a fixed compatible topology (Remark 1.4.1). Then every strongly smooth map $\Delta^k \to X$ is continuous, hence there is a natural inclusion $(C_*^{\mathrm{sm}}(X), \partial) \hookrightarrow (S_*(X), \partial)$.

- C_*^{sm}, C_*^{dR} are functors from the category of differentiable spaces to the category of chain complexes. Given a cocycle $u = (u_k)_{k\geq 0} \in C_0^{dR,\mathrm{pt}\Delta} = \prod_{k\geq 0} C_k^{dR}(\Delta^k)$ in the class $1 \in \mathbb{R} \cong H_0^{dR}(\mathrm{pt}) \cong H_0^{dR,\mathrm{pt}}$, there is a natural transformation $\iota^u : C_*^{\mathrm{sm}} \to C_*^{dR}$ defined by $\iota^u(X)_k : C_k^{\mathrm{sm}}(X) \to C_k^{dR}(X)$, $\sigma \mapsto \sigma_*(u_k)$. The homotopy class of $\iota^u(X)$ does not depend on u (since $H_n^{dR,\mathrm{pt}} = 0$ when $n > 0$).

Assumption 1.4.4. X is a differentiable space with a fixed compatible topology, such that the chain maps discussed above induce isomorphisms $H_*(X) \overset{\cong}{\leftarrow} H_*^{\mathrm{sm}}(X) \overset{\cong}{\rightarrow} H_*^{dR}(X)$.

Proposition 1.4.5. *Let X be a set which satisfies Assumption 1.4.4 and admits an $\mathbb{S}^1$-action that is both smooth (w.r.t. differentiable structure) and continuous (w.r.t. topology). Then there are natural isomorphisms $\mathrm{HC}_*^{[v^{-1}]}(C^{dR,X\Delta}) \cong H_*^{\mathbb{S}^1}(X)$, $\mathrm{HC}_*^{[[v,v^{-1}]}(C^{dR,X\Delta}) \cong \widehat{H}_*^{\mathbb{S}^1}(X)$, and $\mathrm{HC}_*^{\lambda}(C^{dR,X\Delta}) \cong \mathrm{HC}_*^{[[v]]}(C^{dR,X\Delta}) \cong G_*^{\mathbb{S}^1}(X)$, which are compatible with tautological and Connes-Gysin long exact sequences.*

Proof. Consider the mixed complex $(C_*^{\mathrm{sm},X\Delta}, b, B)$ associated to the cocyclic complex $\{C_{*+k}^{\mathrm{sm}}(X\times \Delta^k\}_k$. For any $k \geq 0$, $X \times \Delta^k$ is a differentiable and topological $\mathbb{S}^1$-space satisfying Assumption

1.4.4. By construction, the inclusions $(C^{\mathrm{sm}}_*(X \times \Delta^k), \partial) \hookrightarrow (S_*(X \times \Delta^k), \partial)$ commute with the cocyclic maps $\delta_i, \sigma_i, \tau_k$, so there is an inclusion $(C^{\mathrm{sm},X\Delta}_*, b, B) \hookrightarrow (S^{X\Delta}_*, b, B)$. On the other hand, given a choice of u, the chain maps $\iota^u(X \times \Delta^k)_* : C^{\mathrm{sm}}_*(X \times \Delta^k) \to C^{dR}_*(X \times \Delta^k)$ commute with cocyclic maps since ι^u is a natural transformation, so we obtain a morphism $(C^{\mathrm{sm},X\Delta}_*, b, B) \to (C^{dR,X\Delta}_*, b, B)$, whose homotopy class does not depend on u. Now consider the following commutative diagram of chain maps

$$\begin{array}{ccccc} (S^{X\Delta}_*, b) & \longleftarrow & (C^{\mathrm{sm},X\Delta}_*, b) & \longrightarrow & (C^{dR,X\Delta}_*, b) \\ \big\downarrow{\scriptstyle \mathrm{pr}_0} & & \big\downarrow{\scriptstyle \mathrm{pr}_0} & & \big\downarrow{\scriptstyle \mathrm{pr}_0} \\ (S_*(X), \partial) & \longleftarrow & (C^{\mathrm{sm}}_*(X), \partial) & \longrightarrow & (C^{dR}_*(X), \partial). \end{array}$$

By Lemma 1.3.3 and Remark 1.3.4, all vertical arrows are quasi-isomorphisms, and by assumption, arrows in the second row are quasi-isomorphisms. Thus arrows in the first row are quasi-isomorphisms. In this way we obtain quasi-isomorphisms of mixed complexes $(S^{X\Delta}_*, b, B) \leftarrow (C^{\mathrm{sm},X\Delta}_*, b, B) \to (C^{dR,X\Delta}_*, b, B)$, and get the desired isomorphisms by Lemma 1.2.3, Theorem 1.3.1 and Corollary 1.3.2. Compatibility with long exact sequences is a consequence of Lemma 1.2.4 and Lemma 1.2.9. $\square$

Example 1.4.6. Let M be a closed oriented C^∞-manifold. It is proved in [29, Section 5, Section 6] that Assumption 1.4.4 is satisfied for M, M_{reg} (with manifold topology) and $\mathcal{L}^M$ (with Frechét topology) in Example 1.4.2. Moreover, $\mathcal{L}^M$ is an $\mathbb{S}^1$-space that Propositon 1.4.5 applies to.

1.4.3 Application to marked Moore loop spaces

Consider the various versions of smooth loop spaces in Example 1.4.2. The following lemma is proved in [29, Section 7].

Lemma 1.4.7. *For any closed oriented C^∞-manifold M and $k \in \mathbb{Z}_{\geq 0}$, the zig-zag of smooth maps between differentiable spaces*

$$\mathscr{L}^M_{k+1,\mathrm{reg}} \xrightarrow{\mathrm{id}_{\mathscr{L}_{k+1}M}} \mathscr{L}^M_{k+1} \xleftarrow{T=1} \mathcal{L}^M_{k+1} \hookrightarrow \mathcal{L}^M \times \Delta^k$$

induces a zig-zag of isomorphisms between de Rham homology groups:

$$H_*^{dR}(\mathscr{L}_{k+1,\mathrm{reg}}^M) \xrightarrow{\cong} H_*^{dR}(\mathscr{L}_{k+1}^M) \xleftarrow{\cong} H_*^{dR}(\mathcal{L}_{k+1}^M) \xrightarrow{\cong} H_*^{dR}(\mathcal{L}^M \times \Delta^k).$$

The cocyclic structure on $\{\mathcal{L}^M \times \Delta^k\}_k$ restricts to $\{\mathcal{L}_{k+1}^M\}_k$. There is also a similar structure of cocyclic set on $\{\mathscr{L}_{k+1}M\}_k$ as follows. Regarding $\mathscr{L}_{k+1}M \subset (\Pi M)^{k+1}$, $\delta_i : \mathscr{L}_k M \to \mathscr{L}_{k+1}M$, $\sigma_i : \mathscr{L}_{k+2}M \to \mathscr{L}_{k+1}M$, $\tau_k : \mathscr{L}_{k+1}M \to \mathscr{L}_{k+1}M$ are

$$\delta_i(T,\gamma,t_1,\dots,t_{k-1}) := \begin{cases} (T,\gamma,0,t_1,\dots,t_{k-1}) & (i=0) \\ (T,\gamma,t_1,\dots,t_i,t_i,\dots,t_{k-1}) & (1 \le i \le k-1) \\ (T,\gamma,t_1,\dots,t_{k-1},T) & (i=k), \end{cases} \tag{1.4.2a}$$

$$\sigma_i(T,\gamma,t_1,\dots,t_{k+1}) := (T,\gamma,t_1,\dots,\widehat{t_{i+1}},\dots,t_{k+1}) \ (0 \le i \le k), \tag{1.4.2b}$$

$$\tau_k(T,\gamma,t_1,\dots,t_k) := (T,\gamma^{t_1},t_2-t_1,\dots,t_k-t_1,T-t_1), \tag{1.4.2c}$$

where $\gamma^{t_1}(t) := \gamma(t+t_1)$. These cocyclic maps are smooth for both $\{\mathscr{L}_{k+1}^M\}_k$ and $\{\mathscr{L}_{k+1,\mathrm{reg}}^M\}_k$. Note that if we view $\mathscr{L}_{k+1}M \subset (\Pi M)^{k+1}$, then (1.4.2c) can be written as $\tau_k((T_0,\gamma_0),\dots,(T_k,\gamma_k)) = ((T_1,\gamma_1),\dots,(T_k,\gamma_k),(T_0,\gamma_0))$.

Let us write $(C_*^{\mathscr{L}}, b, B) := \left(\prod_{k\ge 0} C_{*+k}^{dR}(\mathscr{L}_{k+1,\mathrm{reg}}^M), \partial + \delta, Ns(1-\lambda)\right)$ for the mixed total complex of the cocyclic chain complex $\{C_*^{dR}(\mathscr{L}_{k+1,\mathrm{reg}}^M)\}_k$.

Proposition 1.4.8. *For any closed oriented C^∞-manifold M, there are natural isomorphisms* $\mathrm{HC}_*^{[v^{-1}]}(C^{\mathscr{L}}) \cong H_*^{\mathbb{S}^1}(\mathcal{L}M)$, $\mathrm{HC}_*^{[[v,v^{-1}]}(C^{\mathscr{L}}) \cong \widehat{H}_*^{\mathbb{S}^1}(\mathcal{L}M)$, *and* $\mathrm{HC}_*^{\lambda}(C^{\mathscr{L}}) \cong \mathrm{HC}_*^{[[v]]}(C^{\mathscr{L}}) \cong G_*^{\mathbb{S}^1}(\mathcal{L}M)$, *which are compatible with long exact sequences.*

Proof. The smooth maps $\mathscr{L}_{k+1,\mathrm{reg}}^M \xrightarrow{\mathrm{id}} \mathscr{L}_{k+1}^M \xleftarrow{T=1} \mathcal{L}_{k+1}^M \hookrightarrow \mathcal{L}^M \times \Delta^k$ commute with cocyclic maps, inducing a zig-zag of mixed complex morphisms between the mixed total complexes associated to the cocyclic de Rham chain complexes of these cocyclic differentiable spaces. By Lemma 1.4.7, this is a zig-zag of mixed complex quasi-isomorphisms. The rest is obvious in view of Lemma 1.2.3, Proposition 1.4.5 and Example 1.4.6. □

1.5 Preliminaries on operads and algebraic structures

Let $V = \{V_i\}_{i\in\mathbb{Z}}$ be a (homologically) graded vector space.

A Lie bracket of degree $n \in \mathbb{Z}$ is a Lie bracket on $V[-n]$, namely a bilinear map $[,] : V \otimes V \to V$ of degree n satisfying shifted skew-symmetry and Jacobi identity:

$$[a, b] = -(-1)^{(|a|-n)(|b|-n)}[b, a], \quad [a, [b, c]] = [[a, b], c] + (-1)^{(|a|-n)(|b|-n)}[b, [a, c]].$$

Note that in this definition, there is no need to apply sign change (1.8.1).

A structure of *Gerstenhaber algebra* is a Lie bracket of degree 1 and a graded commutative (and associative, by default) product $\cdot$ satisfying the Poisson relation:

$$[a, b \cdot c] = [a, b] \cdot c + (-1)^{(|a|+1)|b|} b \cdot [a, c].$$

A structure of *Batalin-Vilkovisky (BV) algebra* is a graded commutative product $\cdot$ and a linear map $\Delta : V_* \to V_{*+1}$ (called the BV operator) such that $\Delta^2 = 0$, and

$$\begin{aligned}\Delta(a \cdot b \cdot c) = \Delta(a \cdot b) \cdot c + (-1)^{|a|} a \cdot \Delta(b \cdot c) + (-1)^{(|a|+1)|b|} b \cdot \Delta(a \cdot c) \\ -\Delta a \cdot b \cdot c - (-1)^{|a|} a \cdot \Delta b \cdot c - (-1)^{|a|+|b|} a \cdot b \cdot \Delta c.\end{aligned} \tag{1.5.1}$$

By induction, the defining relation (1.5.1) implies that for any $k \geq 2$,

$$\begin{aligned}\Delta(a_1 \cdot a_2 \cdot \dots \cdot a_k) = \sum_{1\leq i<j\leq k} (-1)^{\varepsilon(i,j)} \Delta(a_i \cdot a_j) \cdot a_1 \cdots \widehat{a_i} \cdots \widehat{a_j} \cdots a_k \\ -(k-2) \sum_{1\leq i\leq k} (-1)^{|a_1|+\dots+|a_k|} a_1 \cdot \dots \cdot \Delta a_i \cdot \dots \cdot a_k,\end{aligned} \tag{1.5.2}$$

where $\varepsilon(i, j)$ is from Koszul sign rule. By [24, Proposition 1.2], a BV algebra is equivalently a Gerstenhaber algebra with a linear map $\Delta : V_* \to V_{*+1}$ such that $\Delta^2 = 0$ and

$$[a, b] = (-1)^{|a|}\Delta(a \cdot b) - (-1)^{|a|}\Delta a \cdot b - a \cdot \Delta b. \tag{1.5.3}$$

Following Getzler [25], a structure of *gravity algebra* is a sequence of graded symmetric linear maps $V^{\otimes k} \to V$ $(k \geq 2)$ of degree 1, $a_1 \otimes \cdots \otimes a_k \mapsto \{a_1, \dots, a_k\}$ (which we call k-th bracket), satisfying the following generalized Jacobi relations:

$$\sum_{1\le i<j\le k}(-1)^{\varepsilon(i,j)}\{\{a_i,a_j\},a_1,\ldots,\widehat{a_i},\ldots,\widehat{a_j},\ldots,a_k,b_1,\ldots,b_l\}$$
$$=\begin{cases}\{\{a_1,\ldots,a_k\},b_1,\ldots,b_l\} & (l>0)\\ 0 & (l=0).\end{cases}$$

Note that the relation for $(k,l)=(3,0)$ implies that, with sign change (1.8.1), the second bracket becomes an honest Lie bracket on $V[-1]$.

The following lemma, which goes back to [5, Theorem 6.1], is well-known to experts.

Lemma 1.5.1. *Let $(V_*,\cdot,\Delta)$ be a BV algebra, W_* be a graded vector space, with linear maps $\alpha: W_*\to V_*$, $\beta: V_*\to W_{*+1}$ such that $\Delta=\alpha\circ\beta$ and $\beta\circ\alpha=0$. Then:*

(i) W_* is a gravity algebra where the brackets $W^{\otimes k}\to W$ are

$$x_1\otimes\cdots\otimes x_k\mapsto\{x_1,\ldots,x_k\}:=\beta(\alpha(x_1)\cdot\cdots\cdot\alpha(x_k))\quad(k\ge 2).\tag{1.5.4}$$

(ii) Let $[,]$ be the Gerstenhaber bracket (1.5.3) *on V_*. Then for any $x_1,x_2\in W$,*

$$\alpha(\{x_1,x_2\})=(-1)^{|x_1|}[\alpha(x_1),\alpha(x_2)].$$

Proof. To prove (i), first note that since $\cdot$ is graded commutative, $\{x_1,\ldots,x_k\}$ is graded symmetric in its variables. Next, the generalized Jacobi relations follow from a straightforward calculation based on (1.5.2) (see the proof of [9, Theorem 8.5]), and is omitted. The proof of (ii) is trivial. □

A BV algebra homomorphism between two BV algebras is an algebra homomorphism that commutes with their BV operators. The case of gravity algebras is similar. The following lemma is obvious.

Lemma 1.5.2. *Suppose there is a commutative diagram of linear maps*

$$\begin{tikzcd}
W_* \arrow[r, "\alpha"] \arrow[d, "f"] & V_* \arrow[r, "\beta"] \arrow[d, "g"] & W_{*+1} \arrow[d, "f"] \\
W'_* \arrow[r, "\alpha'"] & V'_* \arrow[r, "\beta'"] & W'_{*+1}
\end{tikzcd}$$

such that $(V_*, W_*, \alpha, \beta)$ and $(V'_*, W'_*, \alpha', \beta')$ satisfy the assumptions in Lemma 1.5.1, and g is a BV algebra homomorphism. Then f is a gravity algebra homomorphism (for the induced structures on W,W'). □

Next we need to work in the language of operads. We collect some basics below, and refer the reader to [29, Section 2] or standard references [39][40] for more details.

Let $(\mathscr{C}, \otimes, 1_{\mathscr{C}})$ be a symmetric monoidal category. A *nonsymmetric operad* (ns operad for short) $\mathcal{O}$ in $\mathscr{C}$ consists of the following data:

- An object $\mathcal{O}(k)$ in $\mathscr{C}$ for each $k \in \mathbb{Z}_{\geq 0}$.
- Morphisms $\circ_i : \mathcal{O}(k) \otimes \mathcal{O}(l) \to \mathcal{O}(k+l-1)$ for each $1 \leq i \leq k$ and $l \geq 0$, called *partial compositions*, that are associative: for $x \in \mathcal{O}(k)$, $y \in \mathcal{O}(l)$, $z \in \mathcal{O}(m)$,

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z) \quad (1 \leq i \leq k,\ 1 \leq j \leq l,\ m \geq 0), \tag{1.5.5a}$$

$$(x \circ_i y) \circ_{l+j-1} z = (x \circ_j z) \circ_i y \quad (1 \leq i < j \leq k,\ l \geq 0,\ m \geq 0). \tag{1.5.5b}$$

- A morphism $1_{\mathcal{O}} : 1_{\mathscr{C}} \to \mathcal{O}(1)$, which a two-sided unit for $\circ_i$.

An *operad* is a ns operad such that each $\mathcal{O}(k)$ admits a right action of the symmetric group $\mathfrak{S}_k$ ($\mathfrak{S}_0$ is the trivial group), in a way compatible with partial compositions.

A (ns) operad in the symmetric monoidal category of dg (resp. graded) vector spaces is called a (ns) *dg (resp. graded) operad*. A Koszul sign $(-1)^{|y||z|}$ should appear in (1.5.5b) in graded and dg cases. Taking homology yields a functor from the category of (ns) dg operads to the category of (ns) graded operads.

Example 1.5.3. Here are some examples of dg operads and graded operads.

(i) (Endomorphism operad $\mathcal{E}nd_V$.) For any dg (resp. graded) vector space V_*, there is a dg (resp. graded) operad $\mathcal{E}nd_V$ defined as follows. For each $k \geq 0$, $\mathcal{E}nd_V(k)_* := \mathrm{Hom}_*(V^{\otimes k}, V)$, where $\mathrm{Hom}_*(V^{\otimes 0}, V) = \mathrm{Hom}_*(\mathbb{R}, V) = V_*$. For $1 \leq i \leq k$, $l \geq 0$, $f \in \mathrm{Hom}_*(V^{\otimes k}, V)$, $g \in \mathrm{Hom}_*(V^{\otimes l}, V)$, and $\sigma \in \mathfrak{S}_k$,

$$(f \circ_i g)(v_1 \otimes \cdots \otimes v_{k+l-1}) := (-1)^{\varepsilon} f(v_1 \otimes \cdots \otimes g(v_i \otimes \cdots) \otimes \cdots),$$

$$(f \cdot \sigma)(v_1 \otimes \cdots \otimes v_k) := (-1)^{\varepsilon} f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}),$$

$$1_{\mathcal{E}nd_V} := \mathrm{id}_V \in \mathrm{Hom}_0(V, V).$$

Let $\mathcal{O}$ be a (ns) graded operad or dg operad. A structure of *algebra over* $\mathcal{O}$ on V, or say an action of $\mathcal{O}$ on V, means a morphism $\mathcal{O} \to \mathcal{E}nd_V$ as (ns) operads.

(ii) (Gerstenhaber operad $\mathcal{G}er$, BV operad $\mathcal{BV}$, and gravity operad $\mathcal{G}rav$.) These are graded operads that can be defined in terms of generators subject to the relations defining Gerstenhaber/ BV/ gravity algebras. A Gerstenhaber/ BV/ gravity algebra is exactly an algebra over $\mathcal{G}er$/ $\mathcal{BV}$/ $\mathcal{G}rav$.

(iii) (Ward's construction [48].) There is a dg operad $\mathsf{M}_{\circlearrowleft}$ constructed from certain "labeled A_∞ trees", such that $H_*(\mathsf{M}_{\circlearrowleft}) \cong \mathcal{G}rav$ as graded operads, and there are explicit homotopies measuring the failure of gravity relations on $\mathsf{M}_{\circlearrowleft}$ (while Jacobi relation for the second bracket strictly holds). For this reason an algebra over $\mathsf{M}_{\circlearrowleft}$ can be viewed as a gravity algebra up to homotopy. $\mathsf{M}_{\circlearrowleft}$ is closely related to the operad of "cyclic brace operations" (Section 1.6). There are other important properties of $\mathsf{M}_{\circlearrowleft}$ that we will use later (Proposition 1.5.6(v)). Indeed the notation of Ward [48] is $\mathsf{M}_{\circlearrowright}$, but we use $\mathsf{M}_{\circlearrowleft}$ for the reason of Remark 1.5.5.

Definition 1.5.4. ([29, Definition 2.6, 2.9].) Let $\mathcal{O}$ be a ns dg operad.

(i) A *cyclic structure* $(\tau_k)_{k \geq 0}$ on $\mathcal{O}$ is a sequence of morphisms $\tau_k : \mathcal{O}(k) \to \mathcal{O}(k)$ $(k \geq 0)$ such that $\tau_k^{k+1} = \mathrm{id}_{\mathcal{O}(k)}$, $\tau_1(1_{\mathcal{O}}) = 1_{\mathcal{O}}$, and that for any $1 \leq i \leq k$, $l \geq 0$, $x \in \mathcal{O}(k)$,

$y \in \mathcal{O}(l)$,

$$\tau_{k+l-1}(x \circ_i y) = \begin{cases} \tau_k x \circ_{i-1} y & (i \geq 2) \\ (-1)^{|x||y|} \tau_l y \circ_l \tau_k x & (i = 1,\ l \geq 1) \\ \tau_k^2 x \circ_k y & (i = 1,\ l = 0). \end{cases}$$

(ii) A *multiplication* μ and a *unit* ε in $\mathcal{O}$ are elements $\mu \in \mathcal{O}(2)_0$, $\varepsilon \in \mathcal{O}(0)_0$ satisfying $\partial\mu = 0$, $\mu \circ_1 \mu = \mu \circ_2 \mu$, $\partial\varepsilon = 0$ and $\mu \circ_1 \varepsilon = \mu \circ_2 \varepsilon = 1_{\mathcal{O}}$.

Remark 1.5.5. An operad with a cyclic structure is called a *cyclic operad*. The cyclic relation in Definition 1.5.4 differs from some authors (in particular, Ward [48]) in the orientation of performing cyclic permutation, but they are equivalent. See e.g. [42, Section 3].

Let $\mathcal{O} = (\mathcal{O}(k))_{k\geq 0}$ be a ns dg operad endowed with a multiplication μ and a unit ε. Then $\{(\mathcal{O}(k)_*, \partial)\}_{k\geq 0}$ is a cosimplicial chain complex where $\delta_i : \mathcal{O}(k-1)_* \to \mathcal{O}(k)_*$, $\sigma_i : \mathcal{O}(k+1)_* \to \mathcal{O}(k)_*$ $(0 \leq i \leq k)$ are

$$\delta_i(x) := \begin{cases} \mu \circ_2 x & (i = 0) \\ x \circ_i \mu & (1 \leq i \leq k-1) \\ \mu \circ_1 x & (i = k), \end{cases} \qquad \sigma_i(x) := x \circ_{i+1} \varepsilon. \tag{1.5.6}$$

Denote the associated total complex by $(\tilde{\mathcal{O}}_*, b)$. If there is also a cyclic structure $(\tau_k)_{k\geq 0}$ on $\mathcal{O}$ such that μ is cyclically invariant, i.e. $\tau_2(\mu) = \mu$, then $\{(\mathcal{O}(k)_*, \partial), \delta_i, \sigma_i, \tau_k\}_{k\geq 0}$ is a cocyclic chain complex. Denote the associated mixed complex by $(\tilde{\mathcal{O}}_*, b, B)$.

Proposition 1.5.6. *Let $\mathcal{O} = (\mathcal{O}(k)_*, \partial)_{k\geq 0}$ be a ns dg operad. Define binary operations $\circ$ and $[,]$ on $\tilde{\mathcal{O}}_* := \prod_{k\geq 0} \mathcal{O}(k)_{*+k}$ by: for $x = (x_k)_{k\geq 0}$, $y = (y_k)_{k\geq 0}$,*

$$(x \circ y)_k := \sum_{\substack{l+m=k+1 \\ 1\leq i\leq l}} (-1)^{(i-1)(m-1)+(l-1)(|y|+m)} x_l \circ_i y_m, \tag{1.5.7a}$$

$$[x, y] := x \circ y - (-1)^{(|x|-1)(|y|-1)} y \circ x. \tag{1.5.7b}$$

Then for $(\tilde{\mathcal{O}}_*, \partial)$, statement (i)(a) below hold true.

If there is a multiplication μ and a unit ε on $\mathcal{O}$, define a binary operation $\cdot$ on $\tilde{\mathcal{O}}_*$ by

$$(x \cdot y)_k := \sum_{l+m=k} (-1)^{l|y|} (\mu \circ_1 x_l) \circ_{l+1} y_m. \tag{1.5.8}$$

Then for $(\tilde{\mathcal{O}}_*, b)$, statements (i)(b) and (ii)(a) below hold true.

If there is a cyclic structure $(\tau_k)_{k\geq 0}$ on $\mathcal{O}$, then for $\tilde{\mathcal{O}}^{\mathrm{cyc}}_* = \mathrm{Ker}(1-\lambda) \subset \tilde{\mathcal{O}}_*$, statement (iii) below holds true.

If there is a multiplication μ, a unit ε and a cyclic structure $(\tau_k)_{k\geq 0}$ on $\mathcal{O}$ such that $\tau_2(\mu) = \mu$, then for $(\tilde{\mathcal{O}}_*, b, B)$ and $\tilde{\mathcal{O}}^{\mathrm{cyc}}_*$, the other statements below hold true.

(i) *a)* *$(\tilde{\mathcal{O}}_*, \partial, \circ)$ is a dg pre-Lie algebra (with shifted grading) such that $[,]$ is a Lie bracket of degree 1.*

b) *$(\tilde{\mathcal{O}}_*, b, \circ)$ is a dg pre-Lie algebra (with shifted grading) such that $[,]$ is a Lie bracket of degree 1, and $(\tilde{\mathcal{O}}_*, b, \cdot)$ is a dg algebra.*

(ii) *a)* *$\cdot$ and $[,]$ induce a Gerstenhaber algebra structure on $H_*(\tilde{\mathcal{O}}, b)$.*

b) *$\cdot$ and Connes' operator B induce a BV algebra structure on $H_*(\tilde{\mathcal{O}}, b)$ where the BV operator is $\Delta = B_*$.*

c) *The above two structures on $H_*(\tilde{\mathcal{O}}, b)$ are related by* (1.5.3).

(iii) *$\tilde{\mathcal{O}}^{\mathrm{cyc}}_*$ is closed under the operation $[,]$. The restriction of $[,]$ to $\tilde{\mathcal{O}}^{\mathrm{cyc}}_*$ is called the cyclic bracket.*

(iv) *a)* *The BV algebra structure on $H_*(\tilde{\mathcal{O}}, b)$ obtained in (ii)(b) naturally induces gravity algebra structures on $\mathrm{HC}^{\lambda}_*(\tilde{\mathcal{O}})$, $\mathrm{HC}^{[[v]]}_*(\tilde{\mathcal{O}})$ and $\mathrm{HC}^{[v^{-1}]}_*(\tilde{\mathcal{O}})[-1]$.*

b) *The map $B_{0*} : \mathrm{HC}^{[v^{-1}]}_*(\tilde{\mathcal{O}})[-1] \to \mathrm{HC}^{[[v]]}_*(\tilde{\mathcal{O}})$ in* (1.2.1) *is a gravity algebra homomorphism. The map $I_{\lambda *} : \mathrm{HC}^{\lambda}_*(\tilde{\mathcal{O}}) \cong \mathrm{HC}^{[[v]]}_*(\tilde{\mathcal{O}})$ in* (1.2.7) *is a gravity algebra isomorphism.*

c) *The Lie bracket on $\mathrm{HC}^{\lambda}_*(\tilde{\mathcal{O}})[-1]$ induced from (iii) coincides with the second bracket of its gravity algebra structure, up to sign change* (1.8.1).

(v) $\tilde{\mathcal{O}}^{\mathrm{cyc}}_*$ *admits an action of the operad* $\mathsf{M}_{\circlearrowleft}$ *(see Example 1.5.3) which covers the cyclic bracket in (iii). Via the isomorphism* $H_*(\mathsf{M}_{\circlearrowleft}) \cong \mathcal{G}rav$*, this induces a gravity algebra structure on* $\mathrm{HC}^{\lambda}_*(\tilde{\mathcal{O}})$ *which is the same as that in (iv)(a).*

Proof. Statements (i) and (ii)(a) are exactly [29, Theorem 2.8 (i)-(iii)], which in turn follows from [48, Lemma 2.32]. Statements (ii)(b) and (ii)(c) follow from [29, Theorem 2.10], which in turn is a consequence of [48, Theorem B]. Note that [29, Theorem 2.10] uses the normalized subcomplex $\tilde{\mathcal{O}}^{\mathrm{nm}}_*$, but there is no difference on homology: as explained in [29, Section 2.5.4], the BV operator is just induced by Connes' operator $B = Ns$ on $\tilde{\mathcal{O}}^{\mathrm{nm}}$.

Statement (iii) is a straightforward consequence of [48, Corollary 3.3]. Alternatively, it is quite handy to use definition of cyclic structures to verify that if $\tau_{k_i} x_i = (-1)^{k_i} x_i$ ($x_i \in \mathcal{O}(k_i)$, $i = 1, 2$), then $\tau_{k_1+k_2-1}[x_1, x_2] = (-1)^{k_1+k_2-1}[x_1, x_2]$.

Statement (iv)(a) is an application of Lemma 1.5.1(i) to part of the exact sequences (1.2.2)(1.2.8). Note that there is a transition between (co)homological gradings.

- For $\mathrm{HC}^{[[v]]}_*$, take $V_* = H_*(\tilde{\mathcal{O}}, b)$, $W_* = \mathrm{HC}^{[[v]]}_*(\tilde{\mathcal{O}})$, $\alpha = p_{0*}$, and $\beta = B_*$.

- For $\mathrm{HC}^{[v^{-1}]}_{*-1}$, take $V_* = H_*(\tilde{\mathcal{O}}, b)$, $W_* = \mathrm{HC}^{[v^{-1}]}_*(\tilde{\mathcal{O}})[-1] = \mathrm{HC}^{[v^{-1}]}_{*-1}(\tilde{\mathcal{O}})$, $\alpha = B_{0*}$, and $\beta = i_*$.

- For HC^{λ}_*, take $V_* = H_*(\tilde{\mathcal{O}}, b)$, $W_* = \mathrm{HC}^{\lambda}_*(\tilde{\mathcal{O}})$, $\alpha = i_{\lambda*}$, and $\beta = B_{\lambda}$. Here the condition $\alpha \circ \beta = \Delta$ is satisfied because of Lemma 1.2.9.

Statement (iv)(b) follows from Lemma 1.5.2, Lemma 1.2.1 and Lemma 1.2.9. Statement (iv)(c) follows from Lemma 1.5.1(ii) and statement (ii)(c).

Statement (v) is a consequence of [48, Theorem C], where the Maurer-Cartan element $\zeta = (\zeta_k)_{k \geq 2}$ is taken as $\zeta_2 = -\mu$ and $\zeta_k = 0$ ($k \neq 2$).

To see Statement (v) covers Statement (iii), we need concrete description of the action of $\mathsf{M}_{\circlearrowleft}$ on $\tilde{\mathcal{O}}^{\mathrm{cyc}}_*$. For arity 2 it is the same as cyclic brace operations (see Example 1.6.3). □

Remark 1.5.7. The sign in (1.5.7a) comes from operadic suspension (see Appendix 1.8.3). Indeed, $\tilde{\mathcal{O}} = (\prod_{n\geq 0} \mathfrak{s}\mathcal{O}(n))[-1]$.

Remark 1.5.8. [29, Theorem 2.8 & 2.10] and [48, Theorem A & B] contain much stronger statements than Proposition 1.5.6(i)(ii) which we do not need (e.g. existence of an action of a chain model of the (framed) little 2-disks operad on $\tilde{\mathcal{O}}$ or $\tilde{\mathcal{O}}^{\mathrm{nm}}$). Proposition 1.5.6(i)(ii) themselves were known much earlier, e.g. see [42, 1.2 & Theorem 1.3].

Example 1.5.9. Let $(A^*, d, \cdot)$ be a dg algebra with unit 1_A. Then $\mathcal{E}nd_A$ admits a multiplication and a unit given by $\mu(a_1 \otimes a_2) := a_1 \cdot a_2$, $\varepsilon := 1_A$. Viewing A as a dg A-bimodule, the cosimplicial maps δ_i, σ_i in Example 1.2.7 are the same as (1.5.6) for $(\mathcal{E}nd_A, \mu, \varepsilon)$ in (i). To discuss cyclic structures, suppose there is a graded symmetric bilinear form $\langle , \rangle : A \times A \to \mathbb{R}$ of degree $m \in \mathbb{Z}$, such that

$$d\langle a, b\rangle = \langle da, b\rangle + (-1)^{|a|}\langle a, db\rangle, \quad \langle ab, c\rangle = \langle a, bc\rangle \quad (\forall a, b, c \in A). \tag{1.5.9}$$

Namely A is a dg version of Frobenius algebra, but we do not require $\dim_{\mathbb{R}} A$ to be finite or $\langle , \rangle$ to be nondegenerate. Note that since $\langle , \rangle$ is symmetric, the relation $\langle ab, c\rangle = \langle a, bc\rangle$ is equivalent to $\langle , \rangle$ being *cyclic*, i.e.

$$\langle ab, c\rangle = (-1)^{|a|(|b|+|c|)}\langle bc, a\rangle. \tag{1.5.10}$$

Now consider $A^\vee[m]$ where with a dg A-bimodule structure characterized by (1.2.11) (the degree of $\varphi \in A^\vee[m]$ is now shifted). The degree 0 map

$$\theta : A \to A^\vee[m]; \quad \theta(a)(b) := \langle a, b\rangle \ (\forall a, b \in A) \tag{1.5.11}$$

is a dg A-bimodule map, and $\mathrm{Hom}(-, \theta) : \mathrm{Hom}^*(A^{\otimes k}, A) \to \mathrm{Hom}^*(A^{\otimes k}, A^\vee[m])$ is a morphism of cosimplicial complexes. $\{\mathrm{Hom}^*(A^{\otimes k}, A^\vee[m]) = \mathrm{Hom}^{*+m}(A^{\otimes k+1}, \mathbb{R})\}_{k\geq 0}$ is moreover cocyclic with cyclic permutations $(\tau_k)_{k\geq 0}$ given in Example 1.2.7.

If θ happens to be an isomorphism, then $\{\mathrm{Hom}(-, \theta)^{-1} \circ \tau_k \circ \mathrm{Hom}(-, \theta)\}_{k\geq 0}$ endows $(\mathcal{E}nd_A, \mu, \varepsilon)$ with a cyclic structure. All statements of Proposition 1.5.6 hold for $\mathcal{O} = \mathcal{E}nd_A$.

If θ is a quasi-isomorphism, then $\widetilde{\mathcal{E}nd_A} = \mathrm{CH}^*(A,A)$ and $\mathrm{CH}^*(A,A^\vee[m])$ are quasi-isomorphic through a natural map induced by θ. In this case, let us examine the statements (i)-(v) in Proposition 1.5.6 for $\mathcal{O} = \mathcal{E}nd_A$.

(A) Statements (i) and (ii)(a) still hold honestly (they are irrelevant to θ).

(B) Statements (ii)(b), (ii)(c) "hold weakly" in the following sense: Connes' operator B on $\mathrm{CH}^*(A,A^\vee[m])$ induces a BV operator on $\mathrm{HH}^*(A,A) \cong \mathrm{HH}^*(A,A^\vee[m])$, making $\mathrm{HH}^*(A,A)$ into a BV algebra, which is compatible with its Gerstenhaber algebra structure. This is proved by Menichi [43, Theorem 18].

(C) Statement (iii) "holds weakly" in the sense that the subspace of *weakly cyclic invariants* in $\mathrm{CH}^*(A,A)$, $\Theta^{-1}(\mathrm{CH}^*_{\mathrm{cyc}}(A,A^\vee[m]))$, is closed under the bracket (1.5.7b), and hence is a dg Lie subalgebra. Here $\Theta : \mathrm{CH}^*(A,A) \to \mathrm{CH}^*(A,A^\vee[m])$ is the cochain map induced by (1.5.11), and $\mathrm{CH}^*_{\mathrm{cyc}}(A,A^\vee[m]) := \mathrm{Ker}(1-\lambda)$ is the subcomplex of cyclic invariants in $\mathrm{CH}^*(A,A^\vee[m])$, with respect to the cocyclic structure on $\{\mathrm{Hom}^{*+m}(A^{\otimes k+1},\mathbb{R})\}_{k\geq 0}$. This result is rather simple and should be well-known, e.g. it is stated without proof in [44, Lemma 4].

(D) Statement (iv) "holds weakly" in the following sense: there are gravity algebra structures on $\mathrm{HC}^*_\lambda(A,A^\vee[m]) \cong \mathrm{HC}^*_{[[u]]}(A,A^\vee[m]) \cong \mathrm{HC}^*_{[[u]]}(A,A)$ and $\mathrm{HC}^*_{[u^{-1}]}(A,A^\vee[m]) \cong \mathrm{HC}^*_{[u^{-1}]}(A,A)$, induced by the BV algebra structure on $\mathrm{HH}^*(A,A) \cong \mathrm{HH}^*(A,A^\vee[m])$ described in (C).

(E) Statement (v) "holds weakly" by Corollary 1.6.8, which largely generalizes (C).

Remark 1.5.10. Statements (C)(E) above hold true even if $\theta : A \to A^\vee[m]$ is not a quasi-isomorphism. If θ is a quasi-isomorphism, then so is $\Theta : \mathrm{CH}(A,A) \to \mathrm{CH}(A,A^\vee[m])$. If Θ also restricts to a quasi-isomorphism $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A,A^\vee[m])) \to \mathrm{CH}_{\mathrm{cyc}}(A,A^\vee[m])$, then the structures in (C)(E) are compatible with those in (B)(D).

Remark 1.5.11. Statement (C) in Example 1.5.9 is irrelevant to the algebra structure on A. It holds true when A is just a graded vector space endowed with a symmetric bilinear form $\langle,\rangle : A \times A \to \mathbb{R}$ of degree m. In this case, we shall write $\Theta^{-1}(\prod_{k\geq 0} \mathrm{Hom}^{*+m}_{\mathrm{cyc}}(A^{k+1}, \mathbb{R}))$ in place of $\Theta^{-1}(\mathrm{CH}^*_{\mathrm{cyc}}(A, A^\vee[m]))$.

1.6 Cyclic brace operations

This section is devoted to the proof of Theorem 1.1.3. Recall $\tilde{\mathcal{O}} := \prod_{k\geq 0} \mathcal{O}(k)$ if $\mathcal{O}$ is dg operad, and $\tilde{\mathcal{O}}^{\mathrm{cyc}} := \mathrm{Ker}(1-\lambda) \subset \tilde{\mathcal{O}}$ if $\mathcal{O}$ is a dg cyclic operad .

Definition 1.6.1 (Brace operations via concrete formulae)**.** Let $\mathcal{O}$ be a dg operad. For each $n \in \mathbb{Z}_{\geq 0}$, define an $(n+1)$-ary operation on $\tilde{\mathcal{O}}$ as follows. When $n = 0$, for $a \in \mathcal{O}(r)$, let $a\{\} := a$. When $n > 0$, for $a \in \mathcal{O}(r)$ and $b_j \in \mathcal{O}(t_j)$ $(1 \leq j \leq n)$, let

$$a\{b_1, \dots, b_n\} := \sum_{i_1,\dots,i_n} \pm(\cdots((a \circ_{i_1} b_1) \circ_{i_2} b_2) \cdots \circ_{i_n} b_n), \tag{1.6.1}$$

where the summation is taken over tuples $(i_1, \dots, i_n) \in \mathbb{Z}^n_{\geq 1}$ satisfying $i_{j+1} \geq i_j + t_j$ and $i_n \leq r - n + 1 + \sum_{l=1}^{n-1} t_l$. The sign $\pm$ is from iteration of (1.8.2).

Brace operations were first described by Getzler [23] in Hochschild context (generalizing the Gerstenhaber bracket [21] which corresponds to $n = 2$) and later by Gerstenhaber-Voronov [22] in operadic context. There is also an interpretation of brace operations via planar rooted trees, going back to the “minimal operad” of Kontsevich-Soibelman [37] (see also [12, Section 7-9]), which allows for a variation in the cyclic invariant setting ([48]).

Let us fix terminologies about trees before moving to more details.

- A *tree without tails* is a contractible 1-dimensional finite CW complex. A 0-cell is called a *vertex*; the closure of a 1-cell is called an *edge* (identified with $[0,1]$).

- A *tree with tails* is a tree without tails attached with copies of $[0,1)$ called *tails* by gluing each $0 \in [0,1)$ to some vertex.

The set of vertices, edges and tails in a tree T is denoted by V_T, E_T and L_T, respectively. The set of edges and tails at $v \in V_T$ is denoted by E_v and L_v, respectively. The *valence* of a vertex v is the number $|E_v \cup L_v|$. The *arity* of a vertex is its valence -1.

- An *oriented tree* is a tree with a choice of direction for each edge, from one vertex to the other. Such a choice of directions is called an orientation of the tree.

- A *rooted tree* is a tree with a choice of a distinguished tail called the *root*.

Every rooted tree is naturally oriented by directions towards the root.

- A *planar tree* is a tree with a cyclic order on $E_v \cup L_v$ for each vertex v.

Every planar tree can be embedded into the plane in a way unique up to isotopy, so that at each vertex v, the cyclic order on $E_v \cup L_v$ is counterclockwise.

Every planar rooted tree T carries a natural *total order* on $E_T \cup L_T$, which can be obtained by moving counterclockwise along the boundary of a small tubular neighborhood of T in the plane. It starts from the root and is compatible with the cyclic order on $E_v \cup L_v$ for each $v \in V_T$, and also restricts to total orders on E_T, L_T and E_v, L_v for each $v \in V_T$.

- An *n-labeled tree* is a tree T with a bijection between $\{1, 2, \ldots, n\}$ and V_T. If the number of vertices is not specified, it is just called a labeled tree.

The vertex with label i in an n-labeled tree T is denoted by $v_i(T)$, with arity $a_i(T)$.

The notion of isomorphisms of trees (with various structures) is obvious. We shall view isomorphic trees as the same.

For $n \in \mathbb{Z}_{\geq 1}$, let $\mathsf{B}^s(n)$ be the set of n-labeled planar rooted trees without non-root tails, and let $\mathsf{B}(n)$ be the vector space spanned by $\mathsf{B}^s(n)$. Let $\bar{\mathsf{B}}^s(n)$ be the set of n-labeled planar rooted trees with tails, and let $\bar{\mathsf{B}}(n)$ be the vector space spanned by $\bar{\mathsf{B}}^s(n)$.

Given $T' \in \bar{\mathsf{B}}(n)$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$, define a set

$$\mathcal{T}(T', k) := \{\ T'' \in \bar{\mathsf{B}}^s(n) \ \big|\ T'' \text{ can be obtained by attaching tails to } T' \\ \text{so that } a_i(T'') = k_i \ (1 \leq \forall i \leq n)\ \}. \tag{1.6.2}$$

Definition 1.6.2 (Brace operations via trees)**.** Let $\mathcal{O}$ be a dg operad. Define linear maps

$$\kappa_n : \mathsf{B}(n) \to \mathrm{Hom}(\tilde{\mathcal{O}}^{\otimes n}, \tilde{\mathcal{O}}), \quad \bar{\kappa}_n : \bar{\mathsf{B}}(n) \to \mathrm{Hom}(\tilde{\mathcal{O}}^{\otimes n}, \tilde{\mathcal{O}})$$

as follows. κ_n is the restriction of $\bar{\kappa}_n$. For $T' \in \bar{\mathsf{B}}^s(n)$ and $f_i \in \mathcal{O}(k_i)$ $(1 \leq i \leq n)$,

$$\bar{\kappa}_n(T')(f_1, f_2, \ldots, f_n) := \sum_{T'' \in \mathcal{T}(T', k)} \pm \bar{\kappa}_n(T'')(f_1, f_2, \ldots, f_n),$$

where by convention summation over the empty set is zero. If $a_i(T') = k_i$ $(1 \leq i \leq n)$, then $\bar{\kappa}_n(T')(f_1, \ldots, f_n)$ is the operadic composition of $f_1, \ldots, f_n$ in the obvious way described by T', where f_i is assigned to $v_i(T')$. The sign $\pm$ is from iteration of (1.8.2).

Definition 1.6.1 and Definition 1.6.2 describe the same operations on $\tilde{\mathcal{O}}$, as explained below. Consider $\beta_n \in \mathsf{B}^s(n+1)$ characterized by: $E_{\beta_n} = \{e_1, \ldots, e_n\}$, $V_{\beta_n} = \{v_1, \ldots, v_{n+1}\}$ where v_i is labeled by i, $L_{\beta_n} = \{l_1\}$, $v_1 = l_1 \cap e_1 \cap \cdots \cap e_n$, $v_{i+1} \in e_i$ $(1 \leq \forall i \leq n)$, and the cyclic order on $E_{v_1} \cup L_{v_1}$ is $(l_1, e_1, \ldots, e_n)$. Then $\kappa_{n+1}(\beta_n) \in \mathrm{Hom}(\tilde{\mathcal{O}}^{\otimes n+1}, \tilde{\mathcal{O}})$ is exactly given by (1.6.1). Moreover, putting $\mathsf{B}(n)$ in degree $n-1$, $\mathsf{B} = \{\mathsf{B}(n)\}$ carries a *reduced* (meaning $\mathsf{B}(0) = 0$ and $\mathsf{B}(1) = \mathbb{R}$) graded operad structure (see [48, Definition 2.11 & 2.13]) so that $\{\beta_n\}_{n \geq 0}$ generates B under operadic compositions and symmetric permutations, and

$$\kappa = \{\kappa_n\} : \mathsf{B} \to \mathcal{E}nd_{\tilde{\mathcal{O}}} \tag{1.6.3}$$

is a morphism of operads. B is called the *brace operad*, which tautologically controls brace operations on *brace algebras*, i.e. algebras over B. We have just seen that $\tilde{\mathcal{O}}$ is naturally a brace algebra. For the purpose of this paper we omitted details of operadic compositions on B but explained how κ is defined.

In [48, Section 3.2], Ward introduced an operad $\mathsf{B}_{\circlearrowleft}$ which he called *cyclic brace operad*. Let $\mathsf{B}^s_{\circlearrowleft}(n)$ be the set of oriented n-labeled planar trees without tails. Then $\mathsf{B}_{\circlearrowleft}(n)$ is the graded vector space spanned by $\mathsf{B}^s_{\circlearrowleft}(n)$ modulo the relation that reversing direction on an edge produces a negative sign. If there is no risk of confusion, we will by abuse of notation not distinguish $T_{\circlearrowleft} \in \mathsf{B}^s_{\circlearrowleft}(n)$ from its image in $\mathsf{B}_{\circlearrowleft}(n)$. There is a morphism of operads $\rho : \mathsf{B}_{\circlearrowleft} \to \mathsf{B}$, which is induced by maps

$$\rho^s_n : \mathsf{B}^s_{\circlearrowleft}(n) \to \mathsf{B}(n), \quad T_{\circlearrowleft} \mapsto \sum_{T \in \mathcal{R}_1(T_{\circlearrowleft})} (-1)^{\varepsilon(T_{\circlearrowleft}, T)} T, \tag{1.6.4}$$

where $\mathcal{R}_1(T_{\circlearrowleft})$ is the set of labeled planar rooted trees that can be obtained by adding a root to the (nonrooted) tree underlying $T_{\circlearrowleft}$, and $\varepsilon(T_{\circlearrowleft}, T)$ is the number of edges in $E_{T_{\circlearrowleft}} = E_T$ whose direction from $T_{\circlearrowleft}$ does not agree with the direction from the rooted structure of T. Here and hereafter, in appropriate contexts we use f^s to denote a set-theoretic map which induces a linear map f.

A natural example of cyclic brace algebras, i.e. algebras over $\mathsf{B}_{\circlearrowleft}$, is as follows.

Example 1.6.3 ([48, Corollary 3.11])**.** Let $\tilde{\mathcal{O}}$ be a dg cyclic operad. Consider $\tilde{\mathcal{O}}^{\mathrm{cyc}} \subset \tilde{\mathcal{O}}$ and $\kappa : \mathsf{B} \to \mathcal{E}nd_{\tilde{\mathcal{O}}}$ in (1.6.3). For $T \in \mathsf{B}^s_{\circlearrowleft}(n)$, $\rho(T) \in \mathsf{B}(n)$ and $(\kappa \circ \rho)(T) \in \mathrm{Hom}(\tilde{\mathcal{O}}^{\otimes n}, \tilde{\mathcal{O}})$. Restricting $(\kappa \circ \rho)(T)$ to $\tilde{\mathcal{O}}^{\mathrm{cyc}}$ gives an element in $\mathrm{Hom}((\tilde{\mathcal{O}}^{\mathrm{cyc}})^{\otimes n}, \tilde{\mathcal{O}})$. Moreover, if $f_i \in \mathcal{O}(k_i)$ $(1 \leq i \leq n)$ are cyclic invariant, then so is $(\kappa \circ \rho)(T)(f_1, \dots, f_n)$. (Such a claim appears in [48, Theorem 5.5] where it is referred to [48, Proposition 3.10], but there is no direct proof given in [48]. We will give a direct proof in a slightly different situation.) Hence $\kappa \circ \rho$ gives a morphism $\mathsf{B}_{\circlearrowleft} \to \mathcal{E}nd_{\tilde{\mathcal{O}}^{\mathrm{cyc}}}$.

Definition 1.6.4 (Cyclic brace operations)**.** Let $\mathcal{O}$ be a dg cyclic operad. The cyclic brace operations on $\tilde{\mathcal{O}}^{\mathrm{cyc}}$ are those characterized by the linear maps

$$\kappa_n \circ \rho_n : \mathsf{B}_{\circlearrowleft}(n) \to \mathrm{Hom}((\tilde{\mathcal{O}}^{\mathrm{cyc}})^{\otimes n}, \tilde{\mathcal{O}}^{\mathrm{cyc}})$$

discussed in Example 1.6.3.

Remark 1.6.5. It seems hard to write a direct formula for cyclic brace operations on $\tilde{\mathcal{O}}^{\mathrm{cyc}}$ in terms of operadic compositions, in a way as explicit as (1.6.1).

Consider $\bar{\mathsf{B}}^s_{\circlearrowleft}(n) \supset \mathsf{B}^s_{\circlearrowleft}(n)$ and $\bar{\mathsf{B}}_{\circlearrowleft}(n) \supset \mathsf{B}_{\circlearrowleft}(n)$ by extending the definitions to labeled planar trees with tails. There is a forgetful map

$$w^s_n : \bar{\mathsf{B}}^s(n) \to \bar{\mathsf{B}}^s_{\circlearrowleft}(n) \setminus \mathsf{B}^s_{\circlearrowleft}(n)$$

forgetting the choice of root but keeping the orientation from rooted structure. Note that w^s_n induces $w_n : \bar{\mathsf{B}}(n) \to \bar{\mathsf{B}}_{\circlearrowleft}(n)/\mathsf{B}_{\circlearrowleft}(n)$. There is also a map

$$r^s_n : \bar{\mathsf{B}}^s_{\circlearrowleft}(n) \setminus \mathsf{B}^s_{\circlearrowleft}(n) \to \bar{\mathsf{B}}(n), \quad T'_{\circlearrowleft} \mapsto \sum_{T' \in \mathcal{R}_0(T'_{\circlearrowleft})} (-1)^{\varepsilon(T'_{\circlearrowleft}, T')} T',$$

where $\mathcal{R}_0(T'_{\circlearrowleft})$ is the set of n-labeled planar rooted trees obtained by choosing one of the tails in $T'_{\circlearrowleft}$ as the root, and $\varepsilon(T'_{\circlearrowleft}, T')$ is defined similar to $\varepsilon(T_{\circlearrowleft}, T)$ in (1.6.4).

It is clear that $(w_n \circ r_n)(T'_{\circlearrowleft}) = |\mathcal{R}_0(T'_{\circlearrowleft})| \cdot T'_{\circlearrowleft}$. To describe $r_n \circ w_n$, consider a map

$$t^s_n : \bar{\mathsf{B}}^s(n) \to \bar{\mathsf{B}}^s(n)$$

so that T' and $t^s_n(T')$ are the same after forgetting the root, and the root of $t^s_n(T')$ is the first non-root tail of T' (if there are no non-tail roots then $t^s_n(T') = T'$). Then for any $T' \in \mathcal{R}_0(T'_{\circlearrowleft})$, we have $\mathcal{R}_0(T'_{\circlearrowleft}) = \{T', t^s_n(T'), \ldots, (t^s_n)^p(T')\}$, where $p = |\mathcal{R}_0(T'_{\circlearrowleft})| - 1$. It follows that

$$(r_n \circ w_n)(T') = \varepsilon(T'_{\circlearrowleft}, T') \cdot r_n(T'_{\circlearrowleft}) = \sum_{0 \le i \le |\mathcal{R}_0(T'_{\circlearrowleft})|-1} (-1)^{\varepsilon(T', t^i_n(T'))} \cdot t^i_n(T'). \tag{1.6.5}$$

Here $\varepsilon(T', t^i_n(T'))$ is the number of edges in $E_{T'} = E_{t^i_n(T')}$ whose direction towards the root of T' does not agree with the direction towards the root of $t^i_n(T')$.

Given $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\ge 0}$, define a map

$$\nu^s_k : \mathsf{B}^s_{\circlearrowleft}(n) \to \bar{\mathsf{B}}_{\circlearrowleft}(n), \quad T_{\circlearrowleft} \mapsto \sum_{T'_{\circlearrowleft} \in \mathcal{T}(T_{\circlearrowleft}, k)} T'_{\circlearrowleft},$$

where $\mathcal{T}(k, T_{\circlearrowleft}) \subset \bar{\mathsf{B}}^s_{\circlearrowleft}(n)$ is defined similar to $\mathcal{T}(k, T)$ in (1.6.2).

Lemma 1.6.6. *Let $\mathcal{O}$ be a dg operad. For any $T_{\circlearrowleft} \in \mathsf{B}^s_{\circlearrowleft}(n)$ and $f_i \in \mathcal{O}(k_i)$ $(1 \le i \le n)$, there holds*

$$(\kappa_n \circ \rho^s_n)(T_{\circlearrowleft})(f_1, \ldots, f_n) = (\bar{\kappa}_n \circ r_n \circ \nu^s_k)(T_{\circlearrowleft})(f_1, \ldots, f_n).$$

Proof. Consider the set of labeled planar rooted trees whose vertices have arities equal to $k = (k_1, \dots, k_n)$ in accordance with the labeling. Such a set can be represented as

$$\bigcup_{T \in \mathcal{R}_1(T_\circlearrowleft)} \mathcal{T}(T, k) = \bigcup_{T'_\circlearrowleft \in \mathcal{T}(T_\circlearrowleft, k)} \mathcal{R}_0(T'_\circlearrowleft),$$

and the result follows. $\square$

In the rest of this section, we take $\mathcal{O} = \mathcal{E}nd_A$, where A is a dg algebra endowed with a symmetric, cyclic, bilinear form $\langle , \rangle$ of degree m. Recall from Example 1.5.9 that $\langle , \rangle$ induces $\theta : A \to A^\vee[m]$ and $\Theta : \mathrm{CH}(A, A) \to \mathrm{CH}(A, A^\vee[m])$. To deal with signs, we may work with $A[1]$ instead of A. As explained in Appendix 1.8, the symmetric bilinear form $\langle , \rangle$ on A becomes anti-symmetric on $A[1]$, and the cyclic permutation τ_k on $\mathrm{Hom}(A^{\otimes k+1}, \mathbb{R})$ reads as $\tilde{\tau}_k = (-1)^k \tau_k = \lambda$ on $\mathrm{Hom}(A[1]^{\otimes k+1}, \mathbb{R})$. Since $\mathfrak{s}(\mathcal{E}nd_A) \cong \mathcal{E}nd_{A[1]}$, there is no need to take operadic suspension of $\mathsf{B}_\circlearrowleft$, and $\mathsf{B}_\circlearrowleft(n)$ stands in degree 0 when dealing with $A[1]$.

Since the pairing $\langle , \rangle$ is not necessarily nondegenerate, there is not always a cyclic structure on $\mathcal{E}nd_A$ compatible with cyclic permutations on $\{\mathrm{Hom}(A^{\otimes k+1}, \mathbb{R})\}$ via the map $\mathrm{Hom}(A^{\otimes k}, A) \to \mathrm{Hom}(A^{\otimes k}, A^\vee[m])$ induced by $\langle , \rangle$, so the discussion of Example 1.6.3 does not directly apply here. However, the following is true.

Proposition 1.6.7. $\Theta^{-1}(\prod_{k \geq 0} \mathrm{Hom}^{*+m}_{\mathrm{cyc}}(A^{\otimes k+1}, \mathbb{R}))$ *is naturally a* $\mathsf{B}_\circlearrowleft$*-algebra.*

Proof. (This proposition is irrelevant to the multiplication on A; compare Remark 1.5.11.) Similar to Example 1.6.3, it suffices to show if $T_\circlearrowleft \in \mathsf{B}^s_\circlearrowleft(n)$ and $f_i \in \mathrm{Hom}(A^{\otimes k_i}, A)$ is weakly cyclic invariant in the sense that $\lambda(\theta \circ f_i) = \theta \circ f_i$, then $(\kappa \circ \rho)(T_\circlearrowleft)(f_1, \dots, f_n)$ is weakly cyclic invariant. This is immediate from Lemma 1.6.6 and Lemma 1.6.9 below. $\square$

Corollary 1.6.8. $(\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])), d + \delta)$ *admits an action of* $\mathsf{M}_\circlearrowleft$*. Moreover, if* θ *is a quasi-isomorphism and* Θ *restricts to a quasi-isomrphism from* $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m]))$ *to* $\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])$*, this* $\mathsf{M}_\circlearrowleft$*-action lifts the gravity algebra structure on* $\mathrm{HC}^*_\lambda(A, A^\vee[m])$ *induced by the BV algebra structure on* $\mathrm{HH}^*(A, A^\vee[m])$ *(see Example 1.5.9(D)).*

Proof. As explained in the proof of [48, Theorem 5.5], to show $\mathsf{M}_{\circlearrowleft}$ acts on the space of weakly cyclic invariants, it suffices to consider cyclic brace operations, which is nothing but Proposition 1.6.7. In more details, Ward [48] defined a dg operad M using "A_∞-labeled planar rooted black/white trees" (M is isomorphic to the "minimal operad" of Kontsevich-Soibelman [37]), and $\mathsf{M}_{\circlearrowleft}$ is the nonrooted version of M. M contains B as a graded suboperad, and acts on $\mathrm{CH}(A, A)$ extending brace operations. What M does more than B to $\mathrm{CH}(A, A)$ is generated by the operation $(f, g) \mapsto \mu_A\{f, g\}$ where $\mu_A \in \mathrm{Hom}(A^{\otimes 2}, A)$ is the multiplication. The action of $\mathsf{M}_{\circlearrowleft}$ on $\Theta^{-1}(\mathrm{CH}_{\mathsf{cyc}}(A, A^\vee[m]))$ comes from an operad morphism $\mathsf{M}_{\circlearrowleft} \to \mathsf{M}$ which extends the morphism $\mathsf{B}_{\circlearrowleft} \to \mathsf{B}$ from (1.6.4). Therefore, that $\Theta^{-1}(\mathrm{CH}_{\mathsf{cyc}}(A, A^\vee[m]))$ is closed under the action of $\mathsf{M}_{\circlearrowleft}$ on $\mathrm{CH}(A, A)$ essentially follows from Proposition 1.6.7 and weakly cyclic invariance of μ_A, i.e. (1.5.10).

Now we explain why the $\mathsf{M}_{\circlearrowleft}$-action induces exactly the gravity algebra structure from BV structure on homology under quasi-isomorphism assumptions; this is just by definition. The gravity algebra structure on $\mathrm{HC}^*_\lambda(A, A^\vee[m])$ follows from Lemma 1.5.1 and (1.2.8) with $V_* = \mathrm{HH}^{-*}(A, A) = \mathrm{HH}^{-*}(A, A^\vee[m])$, $W_* = \mathrm{HC}^{-*}_\lambda(A, A^\vee[m])$, $\alpha = i_{\lambda *}$, $\beta = B_\lambda$. The product $\cdot$ (1.5.8) on $\mathrm{CH}(A, A)$ is just the cup product $(f, g) \mapsto \mu_A\{f, g\}$, so the k-th gravity bracket (1.5.4) on $\mathrm{HC}_\lambda(A, A^\vee[m])$ is induced by the operation $(f_1, f_2, \ldots, f_k) \mapsto \mu_A\{f_1, \mu_A\{f_2, \mu_A\{\cdots, \mu_A\{f_{k-1}, f_k\}\cdots\}\}\}$ at chain level, which is represented by certain black/white tree with $k-1$ adjacent black vertices labeled by μ_A. Edges with both black vertices in such a tree should be contracted, creating a new tree with a single black vertex, see [48, Appendix A, (A.10)]. This gives exactly the tree representing the generators of $H_*(\mathsf{M}_{\circlearrowleft})$, see [48, Definition 5.12, Figure 2]. □

Lemma 1.6.9. *Let $T'_{\circlearrowleft} \in \bar{\mathsf{B}}_{\circlearrowleft}(n)$, $T' \in \mathcal{R}_0(T'_{\circlearrowleft})$, $f_i \in \mathrm{Hom}(A[1]^{\otimes k_i}, A[1])$ where $k_i = a_i(T')$ $(1 \le i \le n)$. Suppose every f_i is weakly cyclic invariant. Then*

$$\theta \circ ((\bar{\kappa}_n \circ r_n)(T'_{\circlearrowleft})(f_1, \ldots, f_n)) = \varepsilon(T'_{\circlearrowleft}, T') \cdot N(\theta \circ (\bar{\kappa}_n(T')(f_1, \ldots, f_n))).$$

Proof. In view of (1.6.5), it suffices to prove the following equality:

$$\theta \circ ((\bar{\kappa}_n \circ t_n)(T')(f_1, \dots, f_n)) = \varepsilon(T', t_n(T')) \cdot \lambda(\theta \circ (\bar{\kappa}_n(T')(f_1, \dots, f_n))). \tag{1.6.6}$$

If there is only one tail or only one vertex (equivalently, no edges) in T', then t_n acts trivially on $\bar{\mathsf{B}}(n)$, and (1.6.6) is obvious. Now suppose there are at least two tails and at least one edge in T'. Then there is a unique path in T' connecting the root l_1 to the first non-root tail l_2, consisting of successive edges with successive vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ $(k \geq 1)$, where v_{i_1}, v_{i_k} are vertices of l_1, l_2, respectively. Note that these $k-1$ successive edges are the only edges in T' whose directions towards l_1 and l_2 disagree, so

$$\varepsilon(T', t_n(T')) = k - 1.$$

If $k = 1$, (1.6.6) simply follows from cyclic invariance of $\theta \circ f_{i_1}$.

If $k \geq 2$, for each $j \in \{1, \dots, k-1\}$, denote the edge joining v_{i_j} to $v_{i_{j+1}}$ by $[v_{i_j}, v_{i_{j+1}}]$, which is identified with $[0, 1]$. By removing $\frac{1}{2} \in [v_{i_j}, v_{i_{j+1}}]$ for all $1 \leq j \leq k-1$, T' is cut into k pieces, where the j-th $(1 \leq j \leq k)$ piece T_j contains v_{i_j}. By regarding $[v_{i_j}, \frac{1}{2})$ and $(\frac{1}{2}, v_{i_{j+1}}]$ as tails, these pieces become labeled planar trees, where the planar structures are induced from T', and the vertex labeling is in the same order as T': say the vertices of T_j are labeled by $i_{j,1} < i_{j,2} < \cdots < i_{j,n_j}$ in T', then put their labels as $1, 2, \dots, n_j$ in T_j. Let $T'_{j,-}$ (resp. $T'_{j,+}$) be the labeled planar rooted tree obtained by choosing the tail $(\frac{1}{2}, v_{i_j}]$ (resp. $[v_{i_j}, \frac{1}{2})$) in T_j as the root, where $(\frac{1}{2}, v_{i_1}]$ is indeed l_1 and $[v_{i_k}, \frac{1}{2})$ is indeed l_2. Suppose $(\frac{1}{2}, v_{i_j}]$ (resp. $[v_j, \frac{1}{2})$) is the p_j-th (resp. q_j-th) non-root tail in the total order on $L_{v_{i_j}}$ from planar rooted structure of $T'_{j,+}$ (resp. $T'_{j,-}$). Since l_2 is the first non-root tail in T', we have $q_1 = \cdots = q_k = 1$, and

$$p_1 + \cdots + p_k - k + 1 = |L_{T'}| - 1 = k_1 + \cdots + k_n + 1 - n =: K.$$

Denote

$$F_j^+ := \bar{\kappa}_{n_j}(T'_{j,+})(f_{i_{j,1}}, \dots, f_{i_{j,n_j}}), \quad F_j^- := \bar{\kappa}_{n_j}(T'_{j,-})(f_{i_{j,1}}, \dots, f_{i_{j,n_j}}).$$

Then for any $x_1, \dots, x_{K+1} \in A[1]$, there holds (Koszul sign $(-1)^\varepsilon$ is taken w.r.t. $A[1]$)

$$\begin{aligned}
&\langle (\bar{\kappa}_n \circ t_n)(T')(f_1, \dots, f_n)(x_1 \otimes \cdots \otimes x_K), x_{K+1} \rangle \qquad (1.6.7)\\
&= (-1)^\varepsilon \langle F_k^+ \circ_{p_k} (F_{k-1}^+ \circ_{p_{k-1}} (\cdots \circ_{p_2} F_1^+))(x_1 \otimes \cdots \otimes x_K), x_{K+1} \rangle \\
&= (-1)^\varepsilon \langle F_k^-(\cdots \otimes x_{p_k - 1}), F_{k-1}^+ \circ_{p_{k-1}} (\cdots \circ_{p_2} F_1^+)(x_{p_k} \otimes \cdots) \rangle \\
&= -(-1)^\varepsilon \langle F_{k-1}^+ \circ_{p_{k-1}} (\cdots \circ_{p_2} F_1^+)(x_{p_k} \otimes \cdots), F_k^-(\cdots \otimes x_{p_k - 1}) \rangle,
\end{aligned}$$

where the second equality follows from cyclic invariance of $\theta \circ f_{i_k}$, and the third equality follows from (anti-)symmetry of $\langle , \rangle$. Iterating the above calculation by cyclic invariance of $\theta \circ f_{i_{k-1}}, \theta \circ f_{i_{k-2}}, \dots, \theta \circ f_{i_1}$ and (anti-)symmetry of $\langle , \rangle$, we see that (1.6.7) is equal to

$$\begin{aligned}
&(-1)^k (-1)^\varepsilon \langle x_{p_1 + \cdots + p_k - k + 1}, F_1^- \circ_{q_1} (F_2^- \circ_{q_2} (\cdots \circ_{q_{k-1}} F_k^-))(\cdots \otimes x_{p_1 + \cdots + p_k - k}) \rangle \\
&= (-1)^{k-1} (-1)^\varepsilon \langle F_1^- \circ_1 (F_2^- \circ_1 (\cdots \circ_1 F_k^-))(x_{K+1} \otimes x_1 \otimes \cdots \otimes x_{K-1}), x_K \rangle \\
&= (-1)^{\varepsilon(T', t_n(T'))} (-1)^\varepsilon \langle \bar{\kappa}_n(T')(f_1, \dots, f_n)(x_{K+1} \otimes x_1 \otimes \cdots), x_K \rangle.
\end{aligned}$$

Since $\tilde{\tau}_K$ on $\mathrm{Hom}(A[1]^{\otimes K}, A[1]^\vee[m])$ corresponds to λ on $\mathrm{Hom}(A^{\otimes K}, A^\vee[m])$, this proves (1.6.6). The proof is now complete. □

Remark 1.6.10. It is easy to generalize Proposition 1.6.7 to A_∞ algebras with cyclic invariant symmetric bilinear forms (not necessarily nondegenerate), and the proof is similar.

1.7 Chain level structures in $\mathbb{S}^1$-equivariant string topology

In this section we describe chain level structures in $\mathbb{S}^1$-equivariant string topology, based on the previous results. Let us first review the initial homology level structures dicovered by Chas-Sullivan, and the chain level construction due to Irie.

Example 1.7.1 (String topology BV algebra and gravity algebra)**.** Let M be a closed oriented manifold and $\mathcal{L}M$ be its free loop space. It was discovered by Chas-Sullivan in [5][6] that:

- There is a BV algebra structure $(\Delta, \cdot)$ on $\mathbb{H}_*(\mathcal{L}M) := H_{*+\dim M}(\mathcal{L}M)$. Here Δ is induced by the $\mathbb{S}^1$-action of rotating loops (i.e. $\Delta = J_*$ where J is defined by (1.2.13)), $\cdot$ is induced by concatenation of crossing loops and is called the *loop product*. The associated Gerstenhaber bracket is called the *loop bracket*. We call this BV algebra the *string topology BV algebra*.

- There is a gravity algebra structure on $H^{\mathbb{S}^1}_{*+\dim M-1}(\mathcal{L}M)$ (as an application of Lemma 1.5.1 to part of the Gysin sequence (1.2.15a)), whose second bracket is the *string bracket* ([5, Theorem 6.1]) up to sign (1.8.1). We call this gravity algebra the *string topology gravity algebra*.

A similar application of Lemma 1.5.1 to part of the Connes-Gysin sequence (1.2.2b) for the mixed complex $(S_*(\mathcal{L}M), \partial, J)$, together with Lemma 1.2.1 and Lemma 1.5.2, yields the following lemma.

Lemma 1.7.2. *For any closed oriented manifold M, there is a gravity algebra structure on $G^{\mathbb{S}^1}_{*+\dim M}(\mathcal{L}M)$, such that the natural map $H^{\mathbb{S}^1}_{*+\dim M-1}(\mathcal{L}M) \to G^{\mathbb{S}^1}_{*+\dim M}(\mathcal{L}M)$ in* (1.2.14) *is a morphism of gravity algebras.* □

Example 1.7.3 (Irie's construction [29])**.** Given any closed oriented C^∞-manifold M, one can associate to M a ns cyclic dg operad $(\mathcal{O}_M, (\tau_k)_{k\geq 0}, \mu, \varepsilon)$ with a multiplication and a unit, defined by:

- For each $k \in \mathbb{Z}_{\geq 0}$, $(\mathcal{O}_M(k)_*, \partial) := \big(C^{dR}_{*+\dim M}(\mathscr{L}^M_{k+1,\mathrm{reg}}), \partial\big)$.

- For each $k \in \mathbb{Z}_{\geq 1}, k' \in \mathbb{Z}_{\geq 0}$ and $j \in \{1, \dots, k\}$, the partial composition $\circ_j : \mathcal{O}_M(k) \otimes \mathcal{O}_M(k') \to \mathcal{O}_M(k+k'-1)$ is defined by

$$x \circ_j x' := (\mathrm{con}_j)_*(x \,{}_{\mathrm{ev}_j}\!\times_{\mathrm{ev}_0} x'),$$

  where ${}_{\mathrm{ev}_j}\!\times_{\mathrm{ev}_0}$ is fiber product of de Rham chains with respect to evaluation maps $\mathrm{ev}_j : \mathscr{L}^M_{k+1,\mathrm{reg}} \to M_{\mathrm{reg}}$ and $\mathrm{ev}_0 : \mathscr{L}^M_{k'+1,\mathrm{reg}} \to M_{\mathrm{reg}}$ (it is well-defined because of submersive condition), and $\mathrm{con}_j : \mathscr{L}_{k+1}M\ {}_{\mathrm{ev}_j}\times_{\mathrm{ev}_0} \mathscr{L}_{k'+1}M \to \mathscr{L}_{k+k'}M$ is the concatenation map defined by inserting the second loop into the first loop at the j-th marked point.

- For each $k \in \mathbb{Z}_{\geq 0}$, $\tau_k : \mathcal{O}_M(k)_* \to \mathcal{O}_M(k)_*$ is induced by (1.4.2c).
- $1_{\mathcal{O}_M} := (M, i_1, 1) \in \mathcal{O}_M(1)_0$, $\mu := (M, i_2, 1) \in \mathcal{O}_M(2)_0$, $\varepsilon := (M, i_0, 1) \in \mathcal{O}_M(0)_0$. Here for $k \geq 0$, $i_k : M \to \mathscr{L}_{k+1}M$ is the map $p \mapsto (0, \gamma_p, 0, \ldots, 0)$, where γ_p is the constant loop of length 0 at $p \in M$.

By [29, Theorem 3.1(ii)], there is an isomorphism $H_*(\tilde{\mathcal{O}}_M, b) \cong \mathbb{H}_*(\mathcal{L}M)$ of BV algebras, where these BV algebra structures are from Proposition 1.5.6 and Example 1.7.1, respectively. (The crucial thing about $\tilde{\mathcal{O}}_M$ is the chain level structure which refines the string topology BV algebra, but we do not need to use it.)

Let $(\Omega(M)^*, d, \wedge)$ be the dg algebra of differential forms on M. For each $k \geq 0$, there is a chain map $I_k : C^{dR}_{*+\dim M}(\mathscr{L}^M_{k+1,\mathrm{reg}}) \to \mathrm{Hom}^{-*}(\Omega(M)^{\otimes k}, \Omega(M))$, called *iterated integral of differential forms*: for $\eta_1, \ldots, \eta_k \in \Omega(M)$,

$$I_k(U, \varphi, \omega)(\eta_1 \otimes \cdots \otimes \eta_k) := (-1)^{\varepsilon_0}(\varphi_0)_!(\omega \wedge \varphi_1^*\eta_1 \wedge \cdots \wedge \varphi_k^*\eta_k), \tag{1.7.1}$$

where $\varepsilon_0 := (\dim U - \dim M)(|\eta_1| + \cdots + |\eta_k|)$ and $\varphi_j := \mathrm{ev}_j \circ \varphi$ $(0 \leq j \leq k)$. Moreover, $I = (I_k)_{k\geq 0} : \mathcal{O}_M \to \mathcal{E}nd_{\Omega(M)}$ is a morphism of ns dg operads preserving multiplications and units ([29, Lemma 8.5]).

The paring $\langle \alpha, \beta \rangle := \int_M \alpha \wedge \beta$ is a graded symmetric bilinear form on $\Omega^*(M)$ of degree $m = -\dim M$, in line with Example 1.5.9. The induced dg $\Omega(M)$-bimodule map $\theta : \Omega^*(M) \to (\Omega(M)^\vee[-\dim M])^*$ is a quasi-isomorphism by Poincaré duality, hence induces a quasi-isomorphism

$$\Theta : \mathrm{CH}(\Omega(M), \Omega(M)) \xrightarrow{\simeq} \mathrm{CH}(\Omega(M), \Omega(M)^\vee[-\dim M]). \tag{1.7.2}$$

Lemma 1.7.4. *The composition*

$$\theta \circ I_k : \mathcal{O}_M(k)_* \to \mathrm{Hom}^{-*}(\Omega(M)^{\otimes k}, \Omega(M)^\vee[-\dim M]) \quad (k \geq 0) \tag{1.7.3}$$

is a morphism of cocyclic complexes.

Proof. $\{\theta\circ I_k\}_k$ is a composition of cosimplicial maps, so it suffices to check $\tau_k\circ\theta\circ I_k = \theta\circ I_k\circ\tau_k$, which is a simple computation by definition. $\square$

According to Lemma 1.7.4, $\Theta\circ I : \tilde{\mathcal{O}}_M \to \mathrm{CH}(\Omega(M), \Omega(M)^\vee[-\dim M])$ preserves cyclic invariants. Moreover, the following is true.

Lemma 1.7.5.

$$I : (\tilde{\mathcal{O}}_M^{\mathrm{cyc}})_* \to \Theta^{-1}(\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M), \Omega(M)^\vee[-\dim M])) \tag{1.7.4}$$

is a morphism of $\mathsf{M}_{\circlearrowleft}$*-algebras.*

Proof. First, (1.7.4) is a morphism of $\mathsf{B}_{\circlearrowleft}$-algebras since $I : \mathcal{O}_M \to \mathcal{E}nd_{\Omega(M)}$ is a morphism of ns operads, and the (cyclic) brace operations on the associated complexes are defined using operadic compositions. Then by the proof of Corollary 1.6.8, to show (1.7.4) is a morphism of $\mathsf{M}_{\circlearrowleft}$-algebras, it suffices to show $I_2(\mu) = \wedge$, where $\mu = (M, i_2, 1) \in \mathcal{O}_M(2)$ is the multiplication in $\mathcal{O}_M$. But this is obvious from definition. $\square$

Theorem 1.7.6. *For any closed oriented C^∞-manifold M, the ns dg operad $\mathcal{O}_M$ with $(\tau_k)_{k\geq 0}$, μ, ε in Example 1.7.3 gives rise to the following data:*

(i) A chain complex $\tilde{\mathcal{O}}_M^{\mathrm{cyc}} := \mathrm{Ker}(1-\lambda) \subset \tilde{\mathcal{O}}_M$ which is an algebra over $\mathsf{M}_{\circlearrowleft}$. In particular, $H_*(\tilde{\mathcal{O}}_M^{\mathrm{cyc}})$ carries a gravity algebra structure.

(ii) An isomorphism $H_*(\tilde{\mathcal{O}}_M^{\mathrm{cyc}}) \cong G^{\mathbb{S}^1}_{*+\dim M}(\mathcal{L}M)$ of gravity algebras, where the gravity algebra structure on $G^{\mathbb{S}^1}_{*+\dim M}(\mathcal{L}M)$ is as in Lemma 1.7.2.

(iii) A morphism $I : (\tilde{\mathcal{O}}_M^{\mathrm{cyc}})_* \to \Theta^{-1}(\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M), \Omega(M)^\vee[-\dim M]))$ of $\mathsf{M}_{\circlearrowleft}$-algebras, such that the induced map in homology fits into the following commutative diagram of gravity

algebra homomorphisms:

$$\begin{tikzcd}
H^{\mathbb{S}^1}_{*+\dim M-1}(\mathcal{L}M) \arrow[r, "1"] \arrow[dd, "2"] & \mathrm{HC}^{-*+1}_{[u^{-1}]}(\mathrm{CH}(\Omega(M),\Omega(M)^\vee[-\dim M])) \arrow[d, "3"] \\
 & \mathrm{HC}^{-*}_{[[u]]}(\mathrm{CH}(\Omega(M),\Omega(M)^\vee[-\dim M])) \arrow[d, equal, "(1.2.7)"] \\
G^{\mathbb{S}^1}_{*+\dim M}(\mathcal{L}M) \arrow[r, "4"] & \mathrm{HC}^{-*}_{\lambda}(\mathrm{CH}(\Omega(M),\Omega(M)^\vee[-\dim M])).
\end{tikzcd} \tag{1.7.5}$$

Here arrows 1, 4 are induced by (1.7.3)*, arrow 2 is as in* (1.2.14)*, arrow 3 is as in* (1.2.1)*. The gravity algebra structures are those on the (negative)* $\mathbb{S}^1$*-equivariant homology of* $\mathcal{L}M$ *(Example 1.7.1, Lemma 1.7.2) and on (negative) cyclic cohomology of* $\Omega(M)$ *(Example 1.5.9) in view of Poincaré duality.*

Proof. Statement (i) follows from Proposition 1.5.6(v). Statement (ii) follows from Proposition 1.5.6(iv) and Proposition 1.4.8. As for statement (iii), I is defined in Lemma 1.7.5; Arrows 2, 3 are gravity algebra homomorphisms by Proposition 1.5.6(iv) and Example 1.5.9; Arrows 1, 4 are gravity algebra homomorphisms by Lemma 1.7.4, Lemma 1.2.4 and Lemma 1.5.2; The diagram (1.7.5) commutes by Lemma 1.2.4. Strictly speaking, since the grading of $\mathcal{O}_M(k)_*$ has been shifted by $\dim M$ from $C^{dR}_*(\mathscr{L}^M_{k+1,\mathrm{reg}})$, there is a minor sign change for δ (1.2.6) in $\tilde{\mathcal{O}}_M$ compared to $C^{\mathscr{L}}$ (the same thing happens in [29, Lemma 8.4]), and thus we should repeat the proof of Proposition 1.4.8 under new signs and use new isomorphisms to make the diagram commute without question of signs, but this is straightforward. □

Remark 1.7.7. Statement (i) in Theorem 1.7.6 is an easy combination of work of Irie and Ward, so it is not new. But it was not known before if the chain level structures in statement (i) correctly fit with known homology level structures in string topology (it was even not known whether $H_*(\tilde{\mathcal{O}}^{\mathrm{cyc}}_M)$ is the $\mathbb{S}^1$-equivariant homology of LM), so statement (ii) is new. As for statement (iii), some (perhaps not all) of the homology level statements are known, see the discussion after Theorem 1.1.3; the chain level statement is more crucial, and is new because the fact that $\mathsf{M}_{\circlearrowleft}$ (nontrivially) acts on $\Theta^{-1}(\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M),\Omega(M)^\vee[-\dim M]))$ is new (Corollary 1.6.8).

Remark 1.7.8. It is known that if M is simply-connected, then the iterated integral map $I : (\tilde{\mathcal{O}}_M)_* \to \mathrm{CH}^{-*}(\Omega(M), \Omega(M))$ is a quasi-isomorphism (proved by K. T. Chen [7] and improved by Getzler-Jones-Petrack [26]). In this case Lemma 1.2.3 implies arrows 1, 4 in (1.7.5) are isomorphisms of gravity algebras.

Note that arrow 4 in (1.7.5) is not exactly induced by I, but is the composition $\Theta_* \circ I_*$. The author does not know an answer to the following question.

Conjecture 1.7.9. *For any closed oriented C^∞-manifold M, the quasi-isomorphism* (1.7.2) *restricts to a quasi-isomorphism on (weakly) cyclic invariants,*

$$\Theta^{-1}(\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M), \Omega(M)^\vee[-\dim M])) \xrightarrow{\simeq} \mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M), \Omega(M)^\vee[-\dim M]).$$

1.8 Appendix: Sign rules

1.8.1 Koszul sign rule

Compared to ungraded formulas, a sign $(-1)^{|a||b|}$ is produced in graded setting whenever a symbol a travels across another symbol b. For example if A, B are graded vector spaces, the graded tensor product of graded linear maps $f : A \to B$ and $g : C \to D$, $f \otimes g : A \otimes C \to B \otimes D$, is defined by $(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$.

1.8.2 Sign change rule for (de)suspension

Let $C = \{C^i\}_{i \in \mathbb{Z}}$ be a graded vector space. For any $n \in \mathbb{Z}$, define a shifted graded vector space $C[n] = \{C[n]^i\}_{i \in \mathbb{Z}}$ by $C[n]^i := C^{i+n}$. (In homological grading this turns into $C[-n]_{-i} := C_{-i-n}$.) $C[-1]$ is often denoted by ΣC and is called the suspension of C. Let $s : C \to C[-1]$; $x \mapsto sx$ be the shifted identity map which is of degree 1. By Kozsul sign rule, for $x_1, \ldots, x_k \in C$,

$$s^{\otimes k}(x_1 \otimes \cdots \otimes x_k) = (-1)^{\sum_{i=1}^k (k-i)|x_i|} sx_1 \otimes \cdots \otimes sx_k. \tag{1.8.1}$$

Here $|x_i|$ denotes the degree of x_i in C, and the sign $(-1)^{(k-i)|x_i|}$ comes from exchanging positions of $k-i$ copies of s with that of x_i. The sign change (1.8.1) identifies the graded exterior algebra of C with the graded symmetric algebra of $C[-1]$, as

$$(s \otimes s)((-1)^{|x_1||x_2|}x_2 \otimes x_1) = (-1)^{|x_1|}((-1)^{1+|sx_1||sx_2|}sx_2 \otimes sx_1).$$

The same rule applies to sign change between A and $A[1]$, the desuspension of A.

1.8.3 Operadic suspension

Let $(\mathcal{O}, \circ_i)$ be a dg operad in cohomological grading. The operadic suspension of $\mathcal{O}$ is a dg operad $(\mathfrak{s}\mathcal{O})$ with partial compositions $\tilde{\circ}_i$ satisfying

$$\begin{aligned} \mathfrak{s}\mathcal{O}(n) &= \mathcal{O}(n)[1-n], \\ a\tilde{\circ}_i b &= (-1)^{(i-1)(m-1)+(n-1)|b;\mathcal{O}(m)|}, \end{aligned} \tag{1.8.2}$$

where $a \in \mathfrak{s}\mathcal{O}(n)$, $b \in \mathfrak{s}\mathcal{O}(m)$, $|b;\mathcal{O}(m)|$ is the degree of b in $\mathcal{O}(m)$. For an explanation of signs (which comes from Koszul sign rule), see e.g. [29, Section 2.5.4].

When $\mathcal{O} = \mathcal{E}nd_A$ is the endomorphism operad of a dg algebra A, there is an isomorphism of dg operads $\mathfrak{s}\mathcal{O} = \mathfrak{s}(\mathcal{E}nd_A) \cong \mathcal{E}nd_{A[1]}$. Therefore, for signs related to $\mathfrak{s}(\mathcal{E}nd_A)$, one may alternatively use Koszul sign rule for $A[1]$ and perform (1.8.1) when necessary.

1.8.4 Cyclic permutation

If $(\mathcal{O}, \circ_i)$ is a cyclic dg operad, then $(\mathfrak{s}\mathcal{O}, \tilde{\circ}_i)$ also carries a cyclic structure where $\tilde{\tau}_k = (-1)^k\tau_k$ under the naive identification $\mathfrak{s}\mathcal{O}(k) = \mathcal{O}(k)$. On the other hand, let A be a dg algebra, consider the cocyclic complex $\{\mathrm{Hom}(A^{\otimes k+1}, \mathbb{R}), \tau_k\}$ and the operation $\tilde{\tau}_k$ on $\mathrm{Hom}(A[-1]^{\otimes k+1}, \mathbb{R})$ induced by τ_k under the linear isomorphism $s : A \to A[-1]$. Then the following equality says $\tilde{\tau}_k = (-1)^k\tau_k$ after sign change (1.8.1):

$$\tilde{\tau}_k \circ s^{\otimes k+1} = \tilde{\tau}_k \circ (s^{\otimes k} \otimes s) = (-1)^k(s \otimes s^{\otimes k}) \circ \tau_k,$$

where $s^{\otimes k}$ applies to $x_1 \otimes \cdots \otimes x_k \in A^{\otimes k}$ and s applies to $x_{k+1} \in A$. Therefore, when discussing cyclic homology theories of A under the naive identification $A = A[-1]$, cyclic invariants in $C(k)$ is $\mathrm{Ker}(1-\lambda) = \mathrm{Ker}(1-\tilde{\tau}_k)$, and $N|_{C(k)} = \sum_{i=0}^{k} \lambda^i = \sum_{i=0}^{k} \tilde{\tau}_k^i$.

Chapter 2

A chain model of path spaces and loop spaces from the fundamental groupoid

2.1 Introduction

Let X be a path-connected, locally path-connected, semilocally simply-connected topological space. This is the standing assumption throughout this chapter, leading to existence of the universal covering space of X at any base point. The purpose of this chapter is to study chain models of the following spaces of paths (loops) in X in a united way:

- (Free path space). $\mathcal{P}X := \mathrm{Map}([0,1], X)$.
- (Free loop space). $\mathcal{L}X := \mathrm{Map}(S^1, X) = \{\gamma \in \mathcal{P}X \mid \gamma(0) = \gamma(1)\}$.
- (Space of paths between two points). $\mathcal{P}_{x,x'}X := \{\gamma \in \mathcal{P}X \mid \gamma(0) = x, \gamma(1) = x'\}$, where $x, x' \in X$.
- (Based loop space). $\mathcal{L}_x X := \{\gamma \in \mathcal{L}X \mid \gamma(0) = x\} = \mathcal{P}_{x,x}X$, where $x \in X$.

The basic idea is that, instead of only considering points in X (as people usually do), we also add the information of homotopy classes of paths between points.

Since this chapter is short (no more than 20 pages in thesis format), we are lazy to write a more detailed introduction here. Let us simply mention that the main result (Theorem 2.2.1) can be viewed as a generalization of Adams' cobar theorem [2] on homology of the based loop space, and Chen's theorem [7] on iterated integrals of differential forms and homology of the free loop space. Adams' work and Chen's work both require X to be simply-connected (but see Remark 2.1.1 below), while we drop this condition by adding information of the fundamental group(oid) of X to the chain model. Of course, both the statement and proof of Theorem 2.2.1 won't appear here without the precursors.

Remark 2.1.1. Recent work of Rivera-Zeinalian [46] and Rivera [45] shows that Adams' cobar theorem is true without the simply-connectedness assumption, as long as one carefully chooses suitable versions of chains on X and L_xX. It might be interesting to compare their results with Theorem 2.2.1.

Acknowledgements

The main result in this chapter, Theorem 2.2.1, was a conjecture of Kei Irie communicated to the author in Fall 2022. (To be precise, Kei Irie proposed the free loop space version of the conjecture.) I thank Kei Irie for sharing his ideas, for feedback on my previous unsuccessful attempts of the proof, and for generously suggesting that I can write a single-authored paper based on current results. (Or this should be part of a collaboration.) I also want to thank Manuel Rivera for helpful communication, especially for teaching me Lemma 2.3.4.

2.2 Construction of the chain model

Let $\Pi_1 X$ be the fundamental groupoid of X, which consists of elements (p, q, σ) where $p, q \in X$ and σ is a (relative) homotopy class of paths in X from p to q. There is a source map

$$\mathrm{s} : \Pi_1 X \to X, \quad \mathrm{s}(p, q, \sigma) := p,$$

and a target map

$$\mathrm{t} : \Pi_1 X \to X, \quad \mathrm{t}(p, q, \sigma) = q.$$

For $p \in X$, define $[p] \in \Pi_1 X$ by $[p] := (p, p, [\text{constant path at } p])$. There is a natural surjection $\mathcal{P}X \twoheadrightarrow \Pi_1 X$ sending γ to $(\gamma(0), \gamma(1), [\gamma])$, which makes $\Pi_1 X$ into a topological space with quotient topology. There is a continuous map

$$\{(c_0, c_1) \in (\Pi_1 X)^2 \mid \mathrm{t}(c_0) = \mathrm{s}(c_1)\} \to \Pi_1 X, \quad (c_0, c_1) \mapsto c_0 * c_1, \tag{2.2.1}$$

which is induced by concatenation of two paths γ_0, γ_1 with $\gamma_0(1) = \gamma_1(0)$ in the obvious way.

For each $k \in \mathbb{Z}_{\geq 0}$, consider the following subspaces of $(\Pi_1 X)^{k+1}$.

- $\mathcal{P}^k X$ is consists of $(c_0, \ldots, c_k) \in (\Pi_1 X)^{k+1}$ such that $\mathrm{t}(c_i) = \mathrm{s}(c_{i+1})$ $(0 \leq \forall i \leq k-1)$.
- $\mathcal{L}^k X$ consists of $(c_0, \ldots, c_k) \in \mathcal{P}^k X$ such that $\mathrm{t}(c_k) = \mathrm{s}(c_0)$.
- $\mathcal{P}^k_{x,x'} X$ consists of $(c_0, \ldots, c_k) \in \mathcal{P}^k X$ such that $\mathrm{s}(c_0) = x$, $\mathrm{t}(c_k) = x'$.
- $\mathcal{L}^k_x X$ consists of $(c_0, \ldots, c_k) \in \mathcal{L}^k X$ such that $\mathrm{s}(c_0) = x$. Clearly $\mathcal{L}^k_x X = \mathcal{P}^k_{x,x} X$.

For simplicity of exposition, we introduce a symbol $\mathcal{X} \in \{\mathcal{P}, \mathcal{L}, \mathcal{P}_x, \mathcal{P}_{x,x'}, \mathcal{L}_x\}$, so that $\mathcal{X}X$ means one of $\mathcal{P}X$, $\mathcal{L}X$, $\mathcal{P}_{x,x'}X$, $\mathcal{L}_x X$, and $\mathcal{X}^k X$ means one of $\mathcal{P}^k X$, $\mathcal{L}^k X$, $\mathcal{P}^k_{x,x'} X$, $\mathcal{L}^k_x X$.

For each of $\mathcal{X} \in \{\mathcal{P}, \mathcal{L}, \mathcal{P}_{x,x'}, \mathcal{L}_x\}$, there is a cosimplicial structure (see (1.2.3)) on $\{\mathcal{X}^k X\}_{k \geq 0}$ given by operations $\delta_i : \mathcal{X}^{k-1} X \to \mathcal{X}^k X$, $\sigma_i : \mathcal{X}^{k+1} X \to \mathcal{X}^k X$, where

$$\delta_i(c_0, \ldots, c_{k-1}) := \begin{cases} (c_0, \ldots, c_{i-1}, [\mathrm{s}(c_i)], c_i, \ldots, c_{k-1}) & (0 \leq i \leq k-1) \\ (c_0, \ldots, c_{k-1}, [\mathrm{t}(c_{k-1})]) & (i = k), \end{cases}$$

$$\sigma_i(c_0, \ldots, c_{k+1}) := (c_0, \ldots, c_i * c_{i+1}, \ldots, c_{k+1}) \ \ (0 \leq i \leq k).$$

Moreover, for $\mathcal{X} = \mathcal{L}$, there is a rotation operation $\tau_k : \mathcal{L}^k X \to \mathcal{L}^k X$,

$$\tau_k(c_0, \ldots, c_k) := (c_1, \ldots, c_k, c_0) \tag{2.2.2}$$

making $\{\mathcal{L}_k X\}_{k \geq 0}$ into a cocyclic space (see (1.2.4)).

Recall the standard simplex $\Delta^k := \{(t_1, \dots, t_k) \in \mathbb{R}_{\geq 0} \mid 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$. There is a natural map (by convention $t_0 := 0$, $t_{k+1} := 1$)

$$\mathrm{e}^k : \mathcal{X}X \times \Delta^k \to \mathcal{X}^k X; \quad (\gamma, t_1, \dots, t_k) \mapsto (\gamma(t_i), \gamma(t_{i+1}), [\gamma|_{[t_i, t_{i+1}]}])_{0 \leq i \leq k}. \tag{2.2.3}$$

In fact, e^k is a quotient map. Define $(\gamma, t_1, \dots, t_k) \sim (\gamma', t'_1, \dots, t'_k)$ iff for all $0 \leq i \leq k+1$, $\gamma(t_i) = \gamma'(t'_i)$ and $[\gamma|_{[t_i, t_{i+1}]}] = [\gamma'|_{[t'_i, t'_{i+1}]}]$ (after linear reparametrization of $\gamma|_{[t_i, t_{i+1}]}$, $\gamma'|_{[t'_i, t'_{i+1}]}$), then e^k induces a homeomorphism between $(\mathcal{X}X \times \Delta^k)/\sim$ and $\mathcal{X}^k X$. There are also evaluation maps $\mathrm{ev}^k_j : \mathcal{X}^k X \to X$ $(0 \leq j \leq k+1)$,

$$\mathrm{ev}^k_j(c_0, \dots, c_k) := \begin{cases} \mathrm{s}(c_j) & (0 \leq j \leq k) \\ \mathrm{t}(c_k) & (j = k+1). \end{cases} \tag{2.2.4}$$

Putting these together gives covering maps

$$\mathcal{P}^k X \to X^{k+2}; \qquad (c_0, \dots, c_k) \mapsto (\mathrm{s}(c_0), \dots, \mathrm{s}(c_k), \mathrm{t}(c_k)) \tag{2.2.5a}$$

$$\mathcal{P}_{x,x'} X \to \{x\} \times X^k \times \{x'\}; \qquad (c_0, \dots, c_k) \mapsto (x, \mathrm{s}(c_1) \dots, \mathrm{s}(c_k), x') \tag{2.2.5b}$$

$$\mathcal{L}^k X \to X^{k+1}; \qquad (c_0, \dots, c_k) \mapsto (\mathrm{s}(c_0), \dots, \mathrm{s}(c_k)) \tag{2.2.5c}$$

$$\mathcal{L}^k_x X \to \{x\} \times X^k; \qquad (c_0, \dots, c_k) \mapsto (x, \mathrm{s}(c_1), \dots, \mathrm{s}(c_k)). \tag{2.2.5d}$$

These covering maps are compatible with cosimplicial structures, where the cosimplicial structure on $\{X^{k+2}\}_{k \geq 0}$ is defined by

$$\delta_i : X^{k+1} \to X^{k+2}, \quad (x_0, \dots, x_k) \mapsto (x_0, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_k), \quad 0 \leq i \leq k,$$

$$\sigma_i : X^{k+3} \to X^{k+2}, \quad (x_0, \dots, x_{k+2}) \mapsto (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{k+2}), \quad 0 \leq i \leq k,$$

and the cosimplicial structures on the other spaces are obtained by restriction. If X is simply-connected, these covering maps are trivial, i.e. identity maps.

Recall from Example 1.2.6 that for a cosimplicial complex $\{(C(k)_*, \partial), \delta_i, \sigma_i\}_{k \geq 0}$ there is a total complex $(\prod_{k \geq 0} C(k)_{*+k}, \partial + \delta)$, and for a cocyclic complex $\{(C(k)_*, \partial), \delta_i, \sigma_i, t_k\}_{k \geq 0}$ there is a mixed total complex $(\prod_{k \geq 0} C(k)_{*+k}, \partial + \delta, B)$, where B is Connes' operator.

Conjecture-Theorem 2.2.1. *For any path-connected, locally path-connected, semilocally simply-connected topological space X, any points $x, x' \in X$, and all of $\mathcal{X} \in \{\mathcal{P}, \mathcal{L}, \mathcal{P}_{x,x'}, \mathcal{L}_x\}$, the natural map* (2.2.3) $\mathrm{e}^k : \mathcal{X}X \times \Delta^k \to \mathcal{X}^k X$ *$(k \geq 0)$ of cosimplicial (cocyclic if $\mathcal{X} = \mathcal{L}$) spaces induces a quasi-isomorphism*

$$\begin{aligned} C_*^{\Delta}(\mathcal{X}X) &:= \left(\prod_{k \geq 0} C_{*+k}(\mathcal{X}X \times \Delta^k), \partial + \delta \right) \\ &\xrightarrow[\simeq]{\mathrm{e}_*} \left(\prod_{k \geq 0} C_{*+k}(\mathcal{X}^k X), \partial + \delta \right) =: C_*^{\mathcal{X}}(X), \end{aligned}$$

where (C_*, ∂) is any chain complex functor of an ordinary homology theory such that everything appearing in the statement is well-defined. (Unless otherwise specified, C_* stands for normalized singular chains with coefficients in a commutative ring R with unity.)

Remark 2.2.2. One can easily state a cochain (cohomology) version of Conjecture-Theorem 2.2.1, and prove it by the same method as the chain (homology) version which we will display.

Lemma 2.2.3. *For any topological space Y, the projection*

$$\mathrm{pr}_0 : C_*^{\Delta}(Y) := \left(\prod_{k \geq 0} C_{*+k}(Y \times \Delta^k), \partial + \delta \right) \to (C_*(Y), \partial)\,; \quad (a_k)_{k \geq 0} \mapsto a_0 \tag{2.2.6}$$

is a quasi-isomorphism. Moreover, if $\xi = (\xi_k)_{k \geq 0} \in \left(\prod_{k \geq 0} C_{*+k}(\Delta^k), \partial + \delta \right) = C_0^{\Delta}(\mathrm{pt})$ is a cycle such that $[\xi_0] = 1 \cdot [(\Delta^0 \to \mathrm{pt})] \in H_0(\mathrm{pt})$, then the map

$$E_\xi : C_*(Y) \to C_*^{\Delta}(Y); \quad a \mapsto (a \times \xi_k)_{k \geq 0}$$

is a chain map such that $\mathrm{pr}_0 \circ E_\xi = \mathrm{id}_{C_*(Y)}$.

Proof. pr_0 is a quasi-isomorphism by Lemma 1.3.3. The rest holds by direct computations. □

Remark 2.2.4. Throughout this chapter, we fix the choice of $\xi = (\xi_k)_{k \geq 0}$ in Lemma 2.2.3 as $\xi_k = [\Delta^k] := 1 \cdot (\Delta^k \xrightarrow{\mathrm{id}} \Delta^k) \in C_k(\Delta^k)$.

In the rest of this chapter, we will prove Conjecture 2.2.1 (and it becomes Theorem 2.2.1). In view of Lemma 2.2.3, Conjecture 2.2.1 says that for each of $\mathcal{X} \in \{\mathcal{P}, \mathcal{L}, \mathcal{P}_{x,x'}, \mathcal{L}_x\}$, $C_*^{\mathcal{X}}(X)$

is a chain model of $\mathcal{X}X$. Moreover, when $\mathcal{X} = \mathcal{L}$, the natural map $\mathrm{e}^k : \mathcal{L}X \times \Delta^k \to \mathcal{L}^k X$ preserves cocyclic structures, so by results in Chapter 1, the cyclic (co)homology groups of $C_*^{\mathcal{L}}(X)$ are isomorphic to the S^1-equivariant homology groups of $\mathcal{L}X$.

Our proof follows the strategy of Getzler-Jones-Petrack [26], which borrows ideas from Adams [2] and Chen [7].

2.3 Revisiting Adams' cobar theorem

Let us first prove Conjecture 2.2.1 for $\mathcal{X} = \mathcal{P}$.

Proposition 2.3.1. *Conjecture 2.2.1 for $\mathcal{P}X$ is true.*

Proof. Define a chain map $\beta : (C_*^{\mathcal{P}}(X), \partial + \delta) \to (C_*(X), \partial)$ by $\beta := \mathrm{t}_* \circ \mathrm{pr}_0$, where pr_0 is the projection chain map $C_*^{\mathcal{P}}(X) \to C_*(\mathcal{P}^0 X)$ as in (2.2.6), and $\mathrm{t}_* : C_*(\mathcal{P}^0 X) \to C_*(X)$ is induced by the target map $\mathrm{t} : \mathcal{P}^0 X = \Pi_1 X \to X$. Consider the following commutative diagram

$$\begin{tikzcd}
C_*^{\Delta}(\mathcal{P}X) \arrow[r, "\mathrm{e}_*"] \arrow[d, "\mathrm{pr}_0"] & C_*^{\mathcal{P}}(X) \arrow[d, "\mathrm{pr}_0"] \arrow[dr, "\beta"] & \\
C_*(\mathcal{P}X) \arrow[r, "\mathrm{e}_*^0"] & C_*(\mathcal{P}^0 X) \arrow[r, "\mathrm{t}_*"] & C_*(X),
\end{tikzcd}$$

where $\mathrm{pr}_0 : C_*^{\Delta}(\mathcal{P}X) \to C_*(\mathcal{P}X)$ is defined in the obvious way and is a quasi-isomorphism. Since the map $t \circ e^0 : \mathcal{P}X \to X$ sending γ to $\gamma(1)$ is a homotopy equivalence, we know $\mathrm{t}_* \circ \mathrm{e}_*^0 : C_*(\mathcal{P}X) \to C_*(X)$ is a quasi-isomorphism. Therefore, in order to show e_* is a quasi-isomorphism, it suffices to show β is a quasi-isomorphism. To this end, consider a chain map $\eta = \iota \circ \mathrm{c}_* : C_*(X) \to C_*^{\mathcal{P}}(X)$, where $\mathrm{c}_* : C_*(X) \to C_*(\mathcal{P}^0 X)$ is the chain map induced by the constant path map $\mathrm{c} : X \to \mathcal{P}^0 X$, $p \mapsto [p]$, and $\iota : C_*(\mathcal{P}^0 X) \to C_*^{\mathcal{P}}(X)$ is defined by $(\iota(a))_0 = a$, $(\iota(a))_k = 0$ $(\forall k > 0)$. It is clear that $\beta \circ \eta = \mathrm{id}_{C_*(X)}$. On the other hand, define a linear map $s : C_*^{\mathcal{P}}(X) \to C_*^{\mathcal{P}}(X)$,

$$(s(a))_k := (-1)^{|a|-1} f_{k*}(a_{k+1}) \ (\forall k \geq 0), \quad a = (a_k)_{k \geq 0} \in C_*^{\mathcal{P}}(X),$$

where $f_{k*} : C_*(\mathcal{P}^{k+1}X) \to C_*(\mathcal{P}^k X)$ is induced by the map $f_k : \mathcal{P}^{k+1}X \to \mathcal{P}^k X$ defined by $f_k(c_0, \dots, c_{k+1}) := (c_1, \dots, c_{k+1})$. It follows from definition that

$$\begin{aligned} f_0 \circ \delta_1 &= \mathrm{c} \circ \mathrm{t} && \text{on } \mathcal{P}^0 X, \\ f_k \circ \delta_0 &= \mathrm{id} && \text{on } \mathcal{P}^k X \ (k \geq 0), \\ f_k \circ \delta_i &= \delta_{i-1} \circ f_{k-1} && \text{on } \mathcal{P}^k X \ (k \geq 0,\, 1 \leq i \leq k+1) \ . \end{aligned}$$

Using these identities, it is easy to check $\mathrm{id}_{C_*^{\mathcal{P}}(X)} - \eta \circ \beta = \delta \circ s + s \circ \delta$. Since s also satisfies $\partial \circ s + s \circ \partial = 0$, we conclude that η is a chain homotopy inverse of β. $\square$

Theorem 2.3.2. *If X is simply-connected, then Conjecture 2.2.1 for $\mathcal{L}_x X$ is true.*

Remark 2.3.3. Theorem 2.3.2 is essentially due to Adams [2] with a slightly different statement. Let us make a short comparison between our statement and Adams':

- The chain map Adams constructed is in the opposite direction to ours. Namely, Adams' map goes from chains on X to chains on $L_x X$.
- Adams was working with a based version of normalized singular chains on X (which we will recall later) and normalized cubical chains on $\mathcal{L}_x X$, while in the statement of our theorem, the choices of chains on X and $\mathcal{L}_x X$ are flexible.

In the following, we present a detailed proof of Theorem 2.3.2 which is similar to Adams' original proof. It is worth doing so, as some ingredients in the proof will be important later.

We need some preparations. Recall the R-module of normalized singular n-chains of X is

$$C_n(X) = \langle \text{maps } \sigma : \Delta^n \to X \rangle / \langle \text{degenerate maps} \rangle = \langle \text{nondegenerate } \sigma : \Delta^n \to X \rangle.$$

We shall use $\mathrm{Map}(\Delta^n, X)$ to denote the set of nondegenerate maps $\Delta^n \to X$, or shall not explicitly mention that degenerate chains have been modded out when such a thing is obvious. The normalized singular chains of X form a dg coassociative coalgebra $(C_*(X), \partial, \Delta)$, where $\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$ is the Alexander-Whitney coproduct,

$$\Delta(\sigma) := \sum_{0 \leq s \leq n} \sigma_{0,\dots,s} \otimes \sigma_{s,\dots,n}, \quad \sigma \in \mathrm{Map}(\Delta^n, X),$$

where for $0 \le k \le l \le n$, $\sigma_{k,\dots,l} : \Delta^{l-k} \to X$ is the restriction of σ to the subsimplex of Δ^n with vertices $k, k+1, \dots, l$. For $0 \le i \le k$, define chain maps $\delta_i : C_*(X)^{\otimes k+1} \to C_*(X)^{\otimes k+2}$, $\sigma_i : C_*(X)^{\otimes k+3} \to C_*(X)^{\otimes k+2}$ by

$$\delta_i(a_0 \otimes \cdots \otimes a_k) := a_0 \otimes \cdots \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_k, \tag{2.3.1a}$$

$$\sigma_i(a_0 \otimes \cdots \otimes a_{k+2}) := \begin{cases} a_0 \otimes \cdots \otimes a_i \otimes a_{i+2} \otimes \cdots \otimes a_{k+2} & (\deg a_i = 0) \\ 0 & (\deg a_i > 0). \end{cases} \tag{2.3.1b}$$

Then $\{(C_*(X)^{\otimes k+2}, \partial), \delta_i, \sigma_i\}_{k\ge 0}$ is a cosimplicial chain complex. If $x \in X$ is a base point, then via the natural inclusion $C_*(\{x\}) \hookrightarrow C_*(X)$, $\{C_*(\{x\}) \otimes C(X)^{\otimes k} \otimes C_*(\{x\})\}_{k\ge 0}$ becomes a cosimplicial subcomplex of $\{C_*(X)^{\otimes k+2}\}_{k\ge 0}$. Note that

$$C_n(\{x\}) = \begin{cases} R \cdot (\Delta^0 \to \{x\}) \cong R & (n = 0) \\ 0 & (n \ne 0), \end{cases}$$

so $C_*(\{x\})$ can be identified with the trivial coalgebra R with $\Delta(1) = 1 \otimes 1$, and the inclusion $C_*(\{x\}) \hookrightarrow C_*(X)$ can be identified with a coaugmentation $R \to C_*(X)$.

For $k \in \mathbb{Z}_{\ge 0}$, there is a quasi-isomorphism $\mathcal{AW}_k : C_*(X^{k+2}) \to (C(X)^{\otimes k+2})_*$ defined by

$$\mathcal{AW}_k(\sigma) := \sum_{0 \le s_1 \le \cdots \le s_{k+1} \le n} (\pi^1 \circ \sigma_{0,\dots,s_1}) \otimes (\pi^2 \circ \sigma_{s_1,\dots,s_2}) \otimes \cdots \otimes (\pi^{k+2} \circ \sigma_{s_{k+1},\dots,n}), \tag{2.3.2}$$

where $\pi^j : X^{k+2} \to X$ is projection onto the j-th factor. (This is just a generalization of the usual Alexander-Whitney map $\mathcal{AW}_0 : C_*(X \times X) \to (C(X) \otimes C(X))_*$.) It is not hard to see $\{\mathcal{AW}_k\}_{k\ge 0}$ is a map of cosimplicial chain complexes, generalizing the fact that $\Delta = \mathcal{AW}_0 \circ \mathrm{Diag}_*$, where $\mathrm{Diag}_* : C_*(X) \to C_*(X \times X)$ is induced by the diagonal map. Then by a comparison argument, $\{\mathcal{AW}_k\}_{k\ge 0}$ induces a quasi-isomorphism

$$\left(\prod_{k\ge 0} C_{*+k}(X^{k+2}), \partial + \delta \right) \xrightarrow{\simeq} \left(\prod_{k \ge 0} \left(C(X)^{\otimes k+2}\right)_{*+k}, \partial + \delta \right). \tag{2.3.3}$$

Similarly, restricting to $C_*(\{x\} \times X^k \times \{x\})$, $\mathcal{AW}_k$ gives a quasi-isomorphism

$$C_*(\{x\} \times X^k \times \{x\}) \xrightarrow{\simeq} (C(\{x\}) \otimes C(X)^{\otimes k} \otimes C(\{x\}))_* \tag{2.3.4}$$

which is compatible with cosimplicial structures. Thus

$$\left(\prod_{k\geq 0} C_{*+k}(\{x\}\times X^k\times\{x\}), \partial+\delta\right) \simeq \left(\prod_{k\geq 0}(C(\{x\})\otimes C(X)^{\otimes k}\otimes C(\{x\}))_{*+k}, \partial+\delta\right).$$

When X is simply-connected, $\mathcal{L}_x^k X = \{x\}\times X^k\times\{x\}$. In order to prove Theorem 2.3.2, it suffices to show the map $C_*(\mathcal{L}_x X)\to \big(\prod_{k\geq 0}(C(\{x\})\otimes C(X)^{\otimes k}\otimes C(\{x\}))_{*+k}, \partial+\delta\big)$ defined as

$$C_*(\mathcal{L}_x X) \underset{\simeq}{\xrightarrow{E_\xi}} C_*^{\Delta}(\mathcal{L}_x X) \xrightarrow{\mathrm{e}_*} C_*^{\mathcal{L}_x}(X) \underset{\simeq}{\xrightarrow{(\mathcal{AW}_k)_{k\geq 0}}} \left(\prod_{k\geq 0}(C(\{x\})\otimes C(X)^{\otimes k}\otimes C(\{x\}))_{*+k}, \partial+\delta\right)$$

is a quasi-isomorphism, where E_ξ is described in Lemma 2.2.3.

For technical reasons, we will need a based version of normalized singular chains. For any space X, $x\in X$, and $m\in\mathbb{Z}_{\geq 0}$, define a subcomplex $C_*^{[m]}(X,x)$ of $C_*(X)$ such that $C_n^{[m]}(X,x)$ is generated by maps $\sigma:\Delta^n\to X$ which send the m-skeleton of Δ^n to x, modulo degenerate singular simplices. It is clear that $\Delta: C_*(X)\to C_*(X)\otimes C_*(X)$ satisfies $\Delta(C_*^{[m]}(X,x))\subset C_*^{[m]}(X,x)\otimes C_*^{[m]}(X,x)$, so $C_*^{[m]}(X,x)$ is a dg subcoalgebra of $C_*(X)$, and there are cosimplicial complexes $\{C_*^{[m]}(X,x)^{\otimes k+2}\}_{k\geq 0}$ and $\{C_*^{[m]}(\{x\},x)\otimes C_*^{[m]}(X,x)^{\otimes k}\otimes C_*^{[m]}(\{x\},x)\}_{k\geq 0}$ similar to the discussion above.

The following lemma is pointed out to the author by Manuel Rivera.

Lemma 2.3.4. *If $\pi_i(X,x)=0$ $(0\leq \forall i\leq m)$, then the inclusion $(C_*^{[m]}(X,x),\partial)\subset (C_*(X),\partial)$ is a quasi-isomorphism.* □

We can now prove Theorem 2.3.2.

Proof of Theorem 2.3.2. Consider the fibration $\pi_P:\mathcal{P}X\to X\times X$, $\pi(\gamma)=(\gamma(0),\gamma(1))$. We follow [41, Section 5.3] to define an increasing filtration $\{\mathcal{F}_p\}_{p\geq 0}$ on $C_*(\mathcal{P}X)$. For $i,j\in\mathbb{Z}_{\geq 0}$, let $S(i,j)$ be the set of nondecreasing maps $\{0,\dots,i\}\to\{0,\dots,j\}$. There is an obvious identification between $S(i,j)$ and the set of simplicial maps $\Delta^i\to\Delta^j$ which are order-preserving on vertices. Now we are ready to define

$$\mathcal{F}_p(C_n(\mathcal{P}X)) := \big\langle \sigma:\Delta^n\to\mathcal{P}X \mid \exists i\leq p,\ \tau:\Delta^i\to X\times X,\ \varphi\in S(n,i) \\ \text{s.t. } \pi_P\circ\sigma=\tau\circ\varphi\big\rangle.$$

Then $\{\mathcal{F}_p\}$ is bounded, leading to a convergent spectral sequence $\{E^r_{p,q}\}$ which is exactly the Serre spectral sequence of the fibration $\pi_P : \mathcal{P}X \to X \times X$. Since X is simply-connected,

$$E^2_{p,q} = H_p(X \times X; H_q(\mathcal{L}_x X)).$$

Consider the chain map (which is a quasi-isomorphism by Proposition 2.3.1)

$$\Phi := \mathcal{AW} \circ \mathrm{e}_* \circ E_\xi : (C_*(\mathcal{P}X), \partial) \to \left(\prod_{k\geq 0}(C(X)^{\otimes k+2})_{*+k}, \partial + \delta\right).$$

Define an increasing filtration $\{\tilde{\mathcal{F}}_p\}_{p\geq 0}$ on $\prod_{k\geq 0}(C(X)^{\otimes k+2})_{*+k}$ by

$$\tilde{\mathcal{F}}_p\left(\prod_{k\geq 0}(C(X)^{\otimes k+2})_{n+k}\right) := \prod_{k\geq 0} \bigoplus_{\substack{i_0+\cdots+i_{k+1}=n+k \\ i_0+i_{k+1}\leq p}} C_{i_0}(X) \otimes \cdots \otimes C_{i_{k+1}}(X),$$

which induces a spectral sequence $\{\tilde{E}^r_{p,q}\}$. One verifies by definition that $\Phi(\mathcal{F}_p) \subset \tilde{\mathcal{F}}_p$, so Φ induces a morphism of spectral sequences $\Phi_* : \{E^r_{p,q}\} \to \{\tilde{E}^r_{p,q}\}$. However, the filtration $\{\tilde{\mathcal{F}}_p\}$ on $\prod_{k\geq 0}(C(X)^{\otimes k+2})_{*+k}$ is only bounded below, but not exhaustive. In order to prove convergence of $\{\tilde{E}^r_{p,q}\}$ and calculate it, we will need the help of $C^{[1]}_*(X, x)$.

Let us first figure out what $(\tilde{E}^0_{p,q}, d^0)$ is. By definition,

$$\begin{aligned}
\tilde{E}^0_{p,q} &= \tilde{\mathcal{F}}_p\left(\prod_{k\geq 0}(C(X)^{\otimes k+2})_{p+q+k}\right) / \tilde{\mathcal{F}}_{p-1}\left(\prod_{k\geq 0}(C(X)^{\otimes k+2})_{p+q+k}\right) \\
&= \prod_{k\geq 0} \bigoplus_{\substack{j_1+j_2=p \\ i_1+\cdots+i_k=q+k}} C_{j_1}(X) \otimes C_{i_1}(X) \otimes \cdots \otimes C_{i_k}(X) \otimes C_{j_2}(X) \ \cong \ \prod_{k\geq 0} A_p \otimes B(k)_{q+k},
\end{aligned}$$

where

$$A_p := \bigoplus_{j_1+j_2=p} C_{j_1}(X) \otimes C_{j_2}(X), \quad B(k)_{q+k} := \bigoplus_{i_1+\cdots+i_k=q+k} R \otimes C_{i_1}(X) \otimes \cdots \otimes C_{i_k}(X) \otimes R.$$

The two factors R in $B(k)$ can be both identified with $C_*(\{x\})$, and $B_* := \prod_{k\geq 0} B(k)_{*+k}$ is just the subcomplex $\prod_{k\geq 0}(C(\{x\}) \otimes C(X)^{\otimes k+2} \otimes C(\{x\}))_{*+k}$ of $\prod_{k\geq 0}(C(X)^{\otimes k+2})_{*+k}$, namely $B_* = C^{\mathcal{L}_x}_*(X)$. We claim that $d^0 : \tilde{E}^0_{p,q} \to \tilde{E}^0_{p,q-1}$ is equal to $\left(\mathrm{id}_{A_p} \otimes (\partial_{B(k)} + \delta|_{B(k)})\right)_{k\geq 0}$. Since d^0 is induced by the boundary map $\partial + \delta$ on $\prod_{k\geq 0}(C(X)^{\otimes k+2})_{*+k}$, the claim means that on $(\tilde{\mathcal{F}}_p/\tilde{\mathcal{F}}_{p-1})(C(X)^{\otimes k+2})_{p+q+k} = A_p \otimes B(k)_{q+k}$,

$$\begin{aligned}
&\partial_{C(X)^{\otimes k+2}} + \delta|_{C(X)^{\otimes k+2}} - \mathrm{id}_{A_p} \otimes (\partial_{B(k)} + \delta|_{B(k)}) \\
&= \partial_{A_p} \otimes \mathrm{id}_{B(k)} \pm (\delta_0|_{C(X)^{\otimes k+2}} - \mathrm{id}_{A_p} \otimes \delta_0|_{B(k)}) \pm (\delta_{k+1}|_{C(X)^{\otimes k+2}} - \mathrm{id}_{A_p} \otimes \delta_{k+1}|_{B(k)}) \\
&= 0 \mod \tilde{\mathcal{F}}_{p-1}.
\end{aligned}$$

Clearly $\partial_{A_p} \otimes \mathrm{id}_{B(k)} = 0 \mod \tilde{\mathcal{F}}_{p-1}$. As for $\delta_0|_{C(X)^{\otimes k+2}} - \mathrm{id}_{A_p} \otimes \delta_0|_{B(k)}$, by definition (2.3.1a), for any $j_1 + j_2 = p$, $i_1 + \cdots + i_k = q+k$, and $a_s \in C_{j_s}(X)$ $(s = 1, 2)$, $b_t \in C_{i_t}(X)$ $(t = 1, \dots, k)$,

$$\begin{aligned}
&(\delta_0|_{C(k)'} - \mathrm{id}_{A_p} \otimes \delta_0|_{B(k)})(a_1 \otimes b_1 \otimes \cdots \otimes b_k \otimes a_2) \\
&= \begin{cases} 0 & (j_1 = 0) \\ (\Delta(a_1) - a_1 \otimes 1) \otimes b_1 \otimes \cdots \otimes b_k \otimes a_2 & (j_1 > 0). \end{cases}
\end{aligned}$$

Since $\Delta(a_1) - a_1 \otimes 1 \in \bigoplus_{j<j_1} C_j(X) \otimes C_{j_1-j}(X)$, we have

$$(\Delta(a_1) - a_1 \otimes 1) \otimes b_1 \otimes \cdots \otimes b_k \otimes a_2 \in \tilde{\mathcal{F}}_{p-1}(C(X)^{\otimes k+2})_{p+q+k}.$$

This proves $\delta_0|_{C(X)^{\otimes k+2}} - \mathrm{id}_{A_p} \otimes \delta_0|_{B(k)} = 0 \mod \tilde{\mathcal{F}}_{p-1}$, and in the same way we can prove $\delta_{k+1}|_{C(X)^{\otimes k+2}} - \mathrm{id}_{A_p} \otimes \delta_{k+1}|_{B(k)} = 0 \mod \tilde{\mathcal{F}}_{p-1}$.Thus we have proved

$$\left(\tilde{E}^0_{p,*}, d^0\right) \cong \left(\prod_{k \geq 0} A_p \otimes B(k)_{*+k}, \left(\mathrm{id}_{A_p} \otimes (\partial_{B(k)} + \delta|_{B(k)})\right)_{k \geq 0}\right). \tag{2.3.5}$$

Now we need $C^{[1]}_*(X, x)$. Since X is simply-connected, the inclusion $C^{[1]}_*(X, x) \hookrightarrow C_*(X)$ is a quasi-isomorphism, and so is $\prod_{k\geq 0}(C^{[1]}(X, x)^{\otimes k+2})_{*+k} \hookrightarrow \prod_{k \geq 0}(C(X)^{\otimes k+2})_{*+k}$. Let $\prod_{k \geq 0} C(k)'_{*+k}$ be the normalized subcomplex of $\prod_{k \geq 0}(C^{[1]}(X, x)^{\otimes k+2})_{*+k}$, i.e.

$$C(k)'_{*+k} := (C^{[1]}(X, x)^{\otimes k+2})_{*+k} \cap \textstyle\bigcap_{0 \leq i \leq k-1} \ker \sigma_i.$$

The inclusion $\prod_{k \geq 0} C(k)'_{*+k} \hookrightarrow \prod_{k \geq 0}(C^{[1]}(X, x)^{\otimes k+2})_{*+k}$ is a quasi-isomorphism by [29, Lemma 2.5]. Since $C^{[1]}_0(X, x) = R \cdot (\Delta^0 \to \{x\})$ has rank 1, from the definition of σ_i (2.3.1b), we have

$$\begin{aligned}&(C^{[1]}_{i_0}(X,x)\otimes\cdots\otimes C^{[1]}_{i_{k+1}}(X,x))\cap\textstyle\bigcap_{0\le i\le k-1}\ker\sigma_i\\&=\begin{cases}C^{[1]}_{i_0}(X,x)\otimes\cdots\otimes C^{[1]}_{i_{k+1}}(X,x) & \text{if } i_j>0\ (\forall j\in\{1,\dots,k\}),\\ 0 & \text{if } i_j=0\ (\exists j\in\{1,\dots,k\}).\end{cases}\end{aligned}\tag{2.3.6}$$

Moreover, $C^{[1]}_1(X,x)=0$ since the constant singular 1-chain $\Delta^1\to\{x\}$ is degenerate. Thus if $0\neq a_0\otimes a_1\otimes\cdots\otimes a_k\otimes a_{k+1}\in C(k)'$, then $\deg a_i\ge 2$ $(1\le\forall i\le k)$. This fact is crucial and has the following consequences.

(i) Consider the restriction of $\{\tilde{\mathcal{F}}_p\}$ to $\prod_{k\ge0}C(k)'_{*+k}$. Then $\tilde{\mathcal{F}}_{q-k}C(k)'_{q+k}=C(k)'_{q+k}$. Thus $\tilde{\mathcal{F}}_q\prod_{k\ge0}C(k)'_{q+k}=\prod_{k\ge0}C(k)'_{q+k}$, so the filtration on $\prod_{k\ge0}C(k)'_{*+k}$ is bounded. Denote the associated convergent spectral sequence by $\{'\tilde{E}^r_{p,q}\}$.

(ii) $C(k)'_{q+k}=0$ whenever $q+k<2k$, i.e. $k>q$. Thus $\prod_{k\ge0}C(k)'_{*+k}=\bigoplus_{k\ge0}C(k)'_{*+k}$.

Let $B'_*=\prod_{k\ge0}B(k)'_{*+k}$ be the normalized subcomplex of $\prod_{k\ge0}(R\otimes C^{[1]}(X,x)^{\otimes k}\otimes R)_{*+k}$, where R is identified with $C^{[1]}_*(\{x\},x)$. Explicitly,

$$B(k)'_{q+k}=\bigoplus_{\substack{i_1+\cdots+i_k=q+k\\ i_1,\dots,i_k>0}}R\otimes C^{[1]}_{i_1}(X,x)\otimes\cdots\otimes C^{[1]}_{i_k}(X,x)\otimes R.$$

Then $(B'_*,\partial+\delta)\hookrightarrow(B_*,\partial+\delta)$ is a quasi-isomorphism. Similarly, there are quasi-isomorphisms $\prod_{k\ge0}A_p\otimes B(k)'_{*+k}\overset{\simeq}{\hookrightarrow}\prod_{k\ge0}A_p\otimes(R\otimes C^{[1]}(X,x)^{\otimes k}\otimes R)_{*+k}\overset{\simeq}{\hookrightarrow}\prod_{k\ge0}A_p\otimes B(k)_{*+k}$. By (2.3.5),

$$\tilde{E}^1_{p,q}=H_q(E^0_{p,*},d^0)=H_q\left(\prod_{k\ge0}A_p\otimes B(k)_{*+k},\partial+\delta\right)\cong H_q\left(\prod_{k\ge0}A_p\otimes B(k)'_{*+k},\partial+\delta\right).$$

For the same reason as (ii), we have $B(k)'_{q+k}=0$ whenever $k>q$, thus $\prod_{k\ge0}A_p\otimes B(k)'_{*+k}=\bigoplus_{k\ge0}A_p\otimes B(k)'_{*+k}$. Since A_p is a free R-module, we then have

$$\begin{aligned}\tilde{E}^1_{p,q}&\cong H_q\left(\bigoplus_{k\ge0}A_p\otimes B(k)'_{*+k},\partial+\delta\right)\\&\cong A_p\otimes\bigoplus_{k\ge0}H_q\left(B(k)'_{*+k},\partial+\delta\right)=A_p\otimes H_q(B'_*)\cong A_p\otimes H_q(B_*),\\ \tilde{E}^2_{p,q}&\cong H_p(A_*\otimes H_q(B))\cong H_p\left((C(X)^{\otimes2})_*\otimes H_q(C^{\mathcal{L}_x}X)\right).\end{aligned}$$

Thus $\{\tilde{E}^r_{p,q}\}$ is first-quadrant for $r \geq 1$. Clearly $\tilde{E}^2_{0,q} = H_q(C^{\mathcal{L}_x}X)$. Since $B(k)'_k = 0$ when $k > 0$, we have $H_0(B_*, \partial + \delta) \cong H_0(B'_*, \partial + \delta) = H_0(B(0)_*, \partial) = R$, so $\tilde{E}^2_{p,0} = H_p(C(X)^{\otimes 2})$.

Now we prove convergence of $\{\tilde{E}^r_{p,q}\}$. Recall (i) $\{'\tilde{E}^r_{p,q}\} \Rightarrow H_{p+q}(\prod_{k\geq 0} C(k)'_{*+k})$. Let $A'_p := \bigoplus_{j_1+j_2=p} C^{[1]}_{j_1}(X,x) \otimes C^{[1]}_{j_2}(X,x)$. In the same way as calculating $E^r_{p,q}$ $(r = 0,1,2)$, we get $('\tilde{E}^0_{p,*}, d^0) = \left(\bigoplus_{k\geq 0} A'_p \otimes B(k)'_{*+k}, \partial_B + \delta_B\right)$, $'\tilde{E}^1_{p,q} = A'_p \otimes H_q(B'_*)$, $'\tilde{E}^2_{p,q} = H_p(A'_* \otimes H_q(B'))$. Since $H_q(B') = H_q(B)$ and $A'_* \simeq A_*$, by universal coefficient theorem, $'\tilde{E}^2_{p,q} \cong \tilde{E}^2_{p,q}$. This isomorphism of E^2-pages is induced by the inclusion $\prod_{k\geq 0} C(k)'_{*+k} \hookrightarrow \prod_{k\geq 0} C(k)_{*+k}$, so it also induces $'\tilde{E}^r_{p,q} \cong \tilde{E}^r_{p,q}$ $(\forall r \geq 2)$. Thus $\{\tilde{E}^r_{p,q}\} \Rightarrow H_{p+q}(\prod_{k\geq 0} C(k)'_{*+k}) \cong H_{p+q}(\prod_{k\geq 0} C(k)_{*+k})$.

By Proposition 2.3.1, $\Phi_* : H_*(\mathcal{P}X) \to H_*(C^P(X))$ is an isomorphism, so the morphism $\Phi_* : \{E^r_{p,q}\} \to \{\tilde{E}^r_{p,q}\}$ induces an isomorphism $E^\infty_{p,q} \cong \tilde{E}^\infty_{p,q}$ for all $p, q \geq 0$. By previous calculation, Φ_* also induces an isomorphism $E^2_{p,0} = H_p(X \times X) \cong H_p(C(X)^{\otimes 2}) = \tilde{E}^2_{p,0}$ for all $p \geq 0$. Then by Zeeman's comparison theorem ([41, Theorem 3.26]), $H_q(\mathcal{L}_x X) \cong E^2_{0,q} \cong \tilde{E}^2_{0,q} \cong H_q(C^{\mathcal{L}_x}X)$ for all $q \geq 0$. Here the conditions for applying [41, Theorem 3.26] are satisfied because of the Universal Coefficient Theorem. The proof of Theorem 2.3.2 is now complete. □

2.4 Proof of the conjecture in general

Lemma 2.4.1. *Let $f : X \to Y$ be a weak homotopy equivalence between topological spaces. Then for each of $\mathcal{X} \in \{\mathcal{P}, \mathcal{L}, \mathcal{P}_{x,x'}, \mathcal{L}_x\}$, Conjecture 2.2.1 is true for $\mathcal{X}X$ iff it is true for $\mathcal{X}Y$. (Here $\mathcal{P}_{x,x'}X$ corresponds to $\mathcal{P}_{f(x),f(x')}Y$.)*

Proof. The fundamental groupoid $\Pi_1 X$ is a fibration over X whose fiber at $x \in X$ is the universal cover $\tilde{X}$ of X based at x. If $f : X \to Y$ is a weak homotopy equivalence, so is the map $\tilde{X} \to \tilde{Y}$ induced by f. Using the long exact sequence of homotopy groups induced by a fibration we know $\Pi_1 X$ is weakly homotopy equivalent to $\Pi_1 Y$. The space $\mathcal{L}^k X$ is the pull back of $\Pi_1(X)^{k+1}$ along certain diagonals in X^{2k+2}, so $\mathcal{L}^k X$ is weakly homotopy equivalent to $\mathcal{L}^k Y$. Recall that a weak homotopy equivalence induces isomorphisms of homology groups with arbitrary coefficients ([28, Proposition 4.21]). By a comparison argument ([49, Theorem 5.5.11]), f induces a quasi-isomorphism between $C_*^{\mathcal{L}}(X)$ and $C_*^{\mathcal{L}}(Y)$. Similar arguments show that f induces quasi-isomorphisms $C_*^{\mathcal{X}}(X) \simeq C_*^{\mathcal{X}}(Y)$ for other choices of $\mathcal{X}$. On the other hand, $f : X \to Y$ clearly induces a weak homotopy equivalence between $\mathcal{P}X \simeq X$ and $\mathcal{P}Y \simeq Y$. Since there is a fibration $\mathcal{P}_{x,x'}X \hookrightarrow \mathcal{P}X \twoheadrightarrow X \times X$ (and similar for Y), by comparing the long exact sequences of homotopy groups induced by the fibrations, we see f induces a weak homotopy equivalence between $\mathcal{P}_{x,x'}X$ and $\mathcal{P}_{f(x),f(x')}Y$, and in particular, a weak homotopy equivalence between $\mathcal{L}_xX$ and $\mathcal{L}_{f(x)}Y$. Since there is a fibration $\mathcal{L}_xX \hookrightarrow \mathcal{L}X \twoheadrightarrow X$, the same argument shows f induces a weak homotopy equivalence between $\mathcal{L}X$ and $\mathcal{L}Y$. It remains to verify commutativity of diagrams, which is straightforward. □

Lemma 2.4.2. *If Conjecture 2.2.1 for $\mathcal{L}_{x_0}X$ is true ($\exists x_0 \in X$), then for $\mathcal{P}_{x,x'}X$ ($\forall x, x' \in X$) it is also true.*

Proof. Suppose Conjecture 2.2.1 is true for $\mathcal{L}_{x_0}X$. Since X is path-connected, for any $x, x' \in X$, there exists $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(\frac{1}{2}) = x$, $\gamma(1) = x'$. Consider the space $X \vee [0,1]$ where $0 \in [0,1]$ is glued to $x_0 \in Z$. Then $f = \mathrm{id}_X \vee \gamma : X \vee [0,1] \to X$ is a homotopy equivalence such that $f(\frac{1}{2}) = x$, $f(1) = x'$. On the other hand, the quotient map $g : X \vee [0,1] \to X$ contracting $[0,1]$ is a homotopy equivalence such that $g(\frac{1}{2}) = g(1) = x_0$. Applying Lemma 2.4.1 to $X \xleftarrow{f} X \vee [0,1] \xrightarrow{g} X$ finishes the proof. □

Corollary 2.4.3. *If X is simply-connected, then Conjecture 2.2.1 for $\mathcal{P}_{x,x'}X$ is true.* □

Theorem 2.4.4. *Conjecture 2.2.1 for $\mathcal{L}_xX$ is true in general.*

Proof. Let $\pi : \tilde{X} \to X$ be the universal covering space of X based at x, and fix $\tilde{x} \in \pi^{-1}(x)$. There is a homeomorphism $\mathcal{L}_xX \cong \coprod_{\alpha\in\pi_1(X,x)} \mathcal{P}_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X}$, so $C_*(\mathcal{L}_xX) = \bigoplus_{\alpha\in\pi_1(X,x)} C_*(\mathcal{P}_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X})$. Similarly, for each $k \in \mathbb{Z}_{\geq 0}$, there is a homeomorphism $\mathcal{L}_x^kX \cong \coprod_{\alpha\in\pi_1(X,x)} \mathcal{P}^k_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X}$, and

$C_*(\mathcal{L}^k_x X) = \bigoplus_{\alpha\in\pi_1(X,x)} C_*(\mathcal{P}^k_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X})$. Consider the following commutative diagram:

$$\begin{tikzcd}
\bigoplus\limits_{\alpha\in\pi_1(X,x)} C_*(\mathcal{P}_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X}) \arrow[r, "\bigoplus_\alpha \mathrm{e}_*\circ E_\xi"] \arrow[d, equal] & \bigoplus\limits_{\alpha\in\pi_1(X,x)} C_*^{\mathcal{P}_{\tilde{x},\alpha\cdot\tilde{x}}}(\tilde{X}) = \bigoplus\limits_{\alpha\in\pi_1(X,x)} \prod\limits_{k\geq 0} C_{*+k}(\mathcal{P}^k_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X}) \arrow[d, hook] \\
C_*(\mathcal{L}_x X) \arrow[r, "\mathrm{e}_*\circ E_\xi"] & C_*^{\mathcal{L}_x}(X) = \prod\limits_{k\geq 0} \bigoplus\limits_{\alpha\in\pi_1(X,x)} C_{*+k}(\mathcal{P}^k_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X}).
\end{tikzcd} \tag{2.4.1}$$

By Corollary 2.4.3, the upper horizontal map in (2.4.1) is a quasi-isomorphism. Therefore, in order to prove the Conjecture for $\mathcal{L}_x X$, i.e. the lower horizontal map is a quasi-isomorphism, it suffices to prove the right vertical map in (2.4.1) is a quasi-isomorphism.

By the proof of Lemma 2.4.2, for any $\alpha \in \pi_1(X,x)$, there is a zig-zag of homotopy equivalences

$$\tilde{X} \xleftarrow{\simeq} \tilde{X}\vee[0,1] \xrightarrow{\simeq} \tilde{X} \quad \text{s.t.} \quad \tilde{x} \leftarrow\!\shortmid \tfrac{1}{2} \mapsto \tilde{x},\ \alpha\cdot\tilde{x} \leftarrow\!\shortmid 1 \mapsto \tilde{x}.$$

Then by the proof of Lemma 2.4.1, there are zig-zags of quasi-isomorphisms

$$\begin{aligned}
&C_*(\mathcal{P}_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X}) \xleftarrow{\simeq} C_*(\mathcal{P}_{\frac{1}{2},1}(\tilde{X}\vee[0,1])) \xrightarrow{\simeq} C_*(\mathcal{L}_{\tilde{x}}\tilde{X}),\\
&C_*(\mathcal{P}^k_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X}) \xleftarrow{\simeq} C_*(\mathcal{P}^k_{\frac{1}{2},1}(\tilde{X}\vee[0,1])) \xrightarrow{\simeq} C_*(\mathcal{L}^k_{\tilde{x}}\tilde{X}), \quad k\in\mathbb{Z}_{\geq 0}.
\end{aligned}$$

Thus there is a commutative diagram

$$\begin{tikzcd}
\bigoplus\limits_{\alpha\in\pi_1(X,x)} \prod\limits_{k\geq 0} C_{*+k}(\mathcal{P}^k_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X}) \arrow[r, hook] & \prod\limits_{k\geq 0} \bigoplus\limits_{\alpha\in\pi_1(X,x)} C_{*+k}(\mathcal{P}^k_{\tilde{x},\alpha\cdot\tilde{x}}\tilde{X}) \\
\bigoplus\limits_{\alpha\in\pi_1(X,x)} \prod\limits_{k\geq 0} C_{*+k}(\mathcal{P}^k_{\frac{1}{2},1}(\tilde{X}\vee[0,1])) \arrow[r, hook] \arrow[u, "\simeq"] \arrow[d, "\simeq"'] & \prod\limits_{k\geq 0} \bigoplus\limits_{\alpha\in\pi_1(X,x)} C_{*+k}(\mathcal{P}^k_{\frac{1}{2},1}(\tilde{X}\vee[0,1])) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
\bigoplus\limits_{\alpha\in\pi_1(X,x)} \prod\limits_{k\geq 0} C_{*+k}(\mathcal{L}^k_{\tilde{x}}\tilde{X}) \arrow[r, hook] & \prod\limits_{k\geq 0} \bigoplus\limits_{\alpha\in\pi_1(X,x)} C_{*+k}(\mathcal{L}^k_{\tilde{x}}\tilde{X})
\end{tikzcd} \tag{2.4.2}$$

Since $\tilde{X}$ is simply-connected, $\mathcal{L}^k_{\tilde{x}}\tilde{X} = \{\tilde{x}\}\times\tilde{X}^k\times\{\tilde{x}\}$. So there are quasi-isomorphisms

$$\begin{aligned}
C_*(\mathcal{L}^k_{\tilde{x}}\tilde{X}) &\xrightarrow[\simeq]{(2.3.4)} (C(\{\tilde{x}\})\otimes C(\tilde{X})^{\otimes k}\otimes C(\{\tilde{x}\}))_*\\
&\xhookleftarrow{\simeq} (C^{[1]}(\{\tilde{x}\},\tilde{x})\otimes C^{[1]}(\tilde{X},\tilde{x})^{\otimes k}\otimes C^{[1]}(\{\tilde{x}\},\tilde{x}))_*.
\end{aligned} \tag{2.4.3}$$

Let $C^{[1]}_+(\tilde{X},\tilde{x}) \subset C^{[1]}(\tilde{X},\tilde{x})$ be the part in positive grading. Since $C^{[1]}_1(\tilde{X},\tilde{x}) = 0$ and $C^{[1]}_0(\tilde{X},\tilde{x}) = R$, by a fact similar to (ii) in Theorem 2.3.2, the normalized subcomplexes of

$$\bigoplus_{\alpha\in\pi_1(X,x)} \prod_{k\geq 0} \Big(C^{[1]}(\{\tilde{x}\},\tilde{x}) \otimes C^{[1]}(\tilde{X},\tilde{x})^{\otimes k} \otimes C^{[1]}(\{\tilde{x}\},\tilde{x}) \Big)_{*+k}$$

and

$$\prod_{k\geq 0} \bigoplus_{\alpha\in\pi_1(X,x)} \Big(C^{[1]}(\{\tilde{x}\},\tilde{x}) \otimes C^{[1]}(\tilde{X},\tilde{x})^{\otimes k} \otimes C^{[1]}(\{\tilde{x}\},\tilde{x}) \Big)_{*+k}$$

are both equal to

$$\bigoplus_{\alpha\in\pi_1(X,x)} \bigoplus_{k\geq 0} \Big(C^{[1]}(\{\tilde{x}\},\tilde{x}) \otimes C^{[1]}_+(\tilde{X},\tilde{x})^{\otimes k} \otimes C^{[1]}(\{\tilde{x}\},\tilde{x}) \Big)_{*+k}.$$

Combining this with (2.4.3)(2.4.2)(2.4.1) finishes the proof. □

Theorem 2.4.5. *Conjecture 2.2.1 for $\mathcal{L}X$ is true in general.*

Proof. Consider the fibration $\pi_L : \mathcal{L}X \to X$, $\gamma \mapsto \gamma(0)$. Define an increasing filtration $\{\mathcal{F}_p\}_{p\geq 0}$ on $C_*(\mathcal{L}X)$ by

$$\mathcal{F}_p(C_n(\mathcal{L}X)) := \big\langle \sigma : \Delta^n \to \mathcal{L}X \mid \exists i \leq p,\ \tau : \Delta^i \to X,\ \varphi \in S(n,i) \\ \text{s.t. } \pi_L \circ \sigma = \tau \circ \varphi \big\rangle.$$

Then $\{\mathcal{F}_p\}$ is bounded, and the associated convergent spectral sequence $\{E^r_{p,q}\}$ is the Serre spectral sequence of the fibration $\pi_L : \mathcal{L}X \to X$. In particular,

$$E^1_{p,q} = \bigoplus_{\sigma\in\mathrm{Map}(\Delta^p,X)} H_q(\mathcal{L}_{\sigma(0)}X), \quad E^2_{p,q} = H_p(X;\mathcal{H}_q(\mathcal{L}_xX)),$$

where $\mathcal{H}_q(\mathcal{L}_xX)$ is the local system of groups $H_q(\mathcal{L}_xX)$ on X induced by the fibration π_L.

Recall the evaluation map (2.2.4):

$$\mathrm{ev}^k_0 : \mathcal{L}^kX \to X^{k+1}, \quad (c_0,\dots,c_k) \mapsto \mathrm{s}(c_0).$$

Define a filtration $\{\tilde{\mathcal{F}}_p\}$ on $C^{\mathcal{L}}_*(X)$ by $\tilde{\mathcal{F}}_p(C^{\mathcal{L}}_n(X)) := \prod_{k\geq 0} \tilde{\mathcal{F}}_p C_{n+k}(\mathcal{L}^kX)$, where for each k,

$$\tilde{\mathcal{F}}_p C_{n+k}(\mathcal{L}^kX) := \big\langle \sigma : \Delta^{n+k} \to \mathcal{L}^kX \mid \exists i \leq p,\ \tau : \Delta^i \to X,\ \varphi \in S(n+k,i) \\ \text{s.t. } \mathrm{ev}^k_0 \circ \sigma = \tau \circ \varphi \big\rangle.$$

Let $\{\tilde{E}^r_{p,q}\}$ be the associated spectral sequence. We claim that there is an isomorphism

$$\tilde{E}^1_{p,q} = H_q\left(\tilde{E}^0_{p,*}, d^0\right) \cong H_q\left(\prod_{k\geq 0}\bigoplus_{\sigma\in\mathrm{Map}(\Delta^p,X)} C_{*+k}(\mathcal{L}^k_{\sigma(0)}X), \partial+\delta\right), \quad \forall p,q, \tag{2.4.4}$$

where $\sigma(0)$ is (image of) the 0-th vertex of σ. Let us first construct maps that will be proved to induce (2.4.4). For $k\in\mathbb{Z}_{\geq 0}$, $\sigma\in\mathrm{Map}(\Delta^p,X)$, $f\in\mathrm{Map}(\Delta^{q+k},\mathcal{L}^k_{\sigma(0)}X)$, define $\sigma\#f\in\mathrm{Map}(\Delta^p\times\Delta^{q+k},\mathcal{L}^kX)$ as follows. For $u\in\Delta^p$, let $\gamma_u:[0,1]\to\Delta^p$, $\gamma_u(t):=tu$. Then $(\sigma\circ\gamma_u)(0)=(\sigma\circ\gamma_u^{-1})(1)=\sigma(0)$. For $v\in\Delta^{q+k}$, write $f(v)=(c_0(v),\dots,c_k(v))$. Then define

$$(\sigma\#f)(u,v) := \left((\sigma\circ\gamma_u^{-1})*c_0(v), c_1(v),\dots,c_{k-1}(v), c_k(v)*(\sigma\circ\gamma_u)\right).$$

Let $\mathrm{Sh}(q+k,p)$ be the set of $(q+k,p)$ shuffles (Definition 3.6.1). For any $\tau\in\mathrm{Sh}(q+k,p)$, there is an embedding

$$\iota_\tau:\Delta^{p+q+k}\to\Delta^p\times\Delta^{q+k}, \quad (t_1,\dots,t_{p+q+k})\mapsto((t_{\tau(q+k+1)},\dots,t_{\tau(p+q+k)}),(t_{\tau(1)},\dots,t_{\tau(q+k)})).$$

Now define an R-linear map (defined on generators)

$$\begin{aligned}\psi_k : \bigoplus_{\sigma\in\mathrm{Map}(\Delta^p,X)} C_{q+k}(\mathcal{L}^k_{\sigma(0)}X) &\to \tilde{\mathcal{F}}_pC_{p+q+k}(\mathcal{L}^kX),\\ (\sigma,f) &\mapsto \sum_{\tau\in\mathrm{Sh}(q+k,p)}\mathrm{sgn}(\varepsilon_\tau)\cdot(\sigma\#f)\circ\iota_\tau.\end{aligned}$$

On the other hand, for each $k\in\mathbb{Z}_{\geq 0}$, there is an R-linear map

$$\begin{aligned}\phi_k:\tilde{\mathcal{F}}_pC_{p+q+k}(\mathcal{L}^kX) &\to \bigoplus_{\sigma\in\mathrm{Map}(\Delta^p,X)} C_{q+k}(\mathcal{L}^k_{\sigma(0)}X)\\ (g:\Delta^{p+q+k}\to\mathcal{L}^kX) &\mapsto (\mathrm{ev}_0\circ g_{q+k,\dots,p+q+k}, g_{0,\dots,q+k}).\end{aligned}$$

Here $g_{0,\dots,q+k}$ is restriction of g to the simplex $\Delta^{q+k}_{0,\dots,q+k}\subset\Delta^{p+q+k}$ spanned by vertices $0,\dots,q+k$, and $g_{q+k,\dots,p+q+k}$ is obtained similarly. We need to show ϕ_k is well-defined, i.e. $g_{0,\dots,q+k}(\Delta^{q+k})\subset\mathcal{L}^k_{\sigma(0)}X$ where $\sigma=\mathrm{ev}_0\circ g_{q+k,\dots,p+q+k}$. Since $g\in\tilde{\mathcal{F}}_p$, there exists $i\leq p$, $\tau:\Delta^i\to X$ and $\varphi\in S(p+q+k,i)$ such that $\mathrm{ev}_0\circ g=\tau\circ\varphi$. By abuse of notation, let j be the j-th vetex in a simplex Δ^n. If $\phi_k(g)\neq 0$, then $\mathrm{ev}_0\circ g_{q+k,\dots,p+q+k}=\tau\circ\varphi|_{\Delta^p_{q+k,\dots,p+q+k}}:\Delta^p\to X$

is nondegenerate, so $i = p$ and $\varphi(j+q+k) = j$ $(0 \le \forall j \le p)$. Then since φ is order-preserving on vertices, $\varphi(j) = 0$ for all $0 \le j \le q+k$, and so $\varphi|_{\Delta^{q+k}_{0,\ldots,q+k}}$ is constant. Thus $\mathrm{ev}_0 \circ g_{0,\ldots,q+k} = \tau \circ \varphi|_{\Delta^{q+k}_{0,\ldots,q+k}} \equiv (\mathrm{ev}_0 \circ g)(q+k) = \sigma(0)$, as desired. The previous argument also proves $\phi_k|_{\tilde{\mathcal{F}}_{p-1}} = 0$. Thus we have defined R-linear maps

$$\psi_k : \bigoplus_{\sigma \in \mathrm{Map}(\Delta^p, X)} C_{q+k}(\mathcal{L}^k_{\sigma(0)} X) \to \tilde{\mathcal{F}}_p C_{q+k}(\mathcal{L}^k X)/\tilde{\mathcal{F}}_{p-1} C_{q+k}(\mathcal{L}^k X),$$

$$\phi_k : \tilde{\mathcal{F}}_p C_{q+k}(\mathcal{L}^k X)/\tilde{\mathcal{F}}_{p-1} C_{q+k}(\mathcal{L}^k X) \to \bigoplus_{\sigma \in \mathrm{Map}(\Delta^p, X)} C_{q+k}(\mathcal{L}^k_{\sigma(0)} X).$$

By [41, Lemma 5.23], ψ_k is a ∂-chain map. By [41, Lemma 5.25], ϕ_k is a ∂-chain map. By [41, Lemma 5.24], ψ_k and ϕ_k are ∂-chain homotopy inverse to each other. Thus ϕ_k induces an isomorphism

$$\bigoplus_{\sigma \in \mathrm{Map}(\Delta^p, X)} H_* \left(C(\mathcal{L}^k_{\sigma(0)} X), \partial \right) \cong H_* \left(\tilde{\mathcal{F}}_p C_{q+k}(\mathcal{L}^k X)/\tilde{\mathcal{F}}_{p-1} C_{q+k}(\mathcal{L}^k X), \partial \right).$$

Moreover, it is clear from definition that $(\phi_k)_{k \ge 0}$ is a map of cosimplicial chain complexes. Since $\tilde{E}^0_{p,q} = \prod_{k \ge 0} \tilde{\mathcal{F}}_p C_{q+k}(\mathcal{L}^k X)/\tilde{\mathcal{F}}_{p-1} C_{q+k}(\mathcal{L}^k X)$ and d^0 is induced by $\partial + \delta$, we conclude that $(\phi_k)_{k \ge 0}$ induces the isomorphism (2.4.4).

To calculate the RHS of (2.4.4), we proceed as follows. By the proof of Theorem 2.4.4,

$$\prod_{k \ge 0} \bigoplus_{\sigma \in \mathrm{Map}(\Delta^p, X)} C_{*+k}(\mathcal{L}^k_{\sigma(0)} X) = \prod_{k \ge 0} \bigoplus_{\sigma \in \mathrm{Map}(\Delta^p, X)} \bigoplus_{\alpha \in \pi_1(X, \sigma(0))} C_{*+k}(\mathcal{P}^k_{\tilde{x}_\sigma, \alpha \cdot \tilde{x}_\sigma} \tilde{X}_\sigma),$$

where for $\sigma \in \mathrm{Map}(\Delta^p, X)$, $\pi_\sigma : \tilde{X}_\sigma \to X$ is the universal covering space of X based at $\sigma(0)$, and $\tilde{x}_\sigma \in \pi_\sigma^{-1}(\sigma(0))$. Again, by similar arguments as the proof of Theorem 2.4.4 (use simply-connectedness of $\tilde{X}_\sigma$ to pass from $C_*(\tilde{X}_\sigma)$ to $C^{[1]}_*(\tilde{X}_\sigma, \tilde{x}_\sigma)$, use $\tilde{X}_\sigma \vee [0,1]$ to pass from $\mathcal{P}^k_{\tilde{x}_\sigma, \alpha \cdot \tilde{x}_\sigma} \tilde{X}_\sigma$ to $\mathcal{L}^k_{\tilde{x}_\sigma} \tilde{X}_\sigma$, use the normalized subcomplex of the total complex to pass from $\prod_k$ to $\bigoplus_k$), we can show the inclusion

$$\bigoplus_{\sigma \in \mathrm{Map}(\Delta^p, X)} \prod_{k \ge 0} \bigoplus_{\alpha \in \pi_1(X, \sigma(0))} C_{*+k}(\mathcal{P}^k_{\tilde{x}_\sigma, \alpha \cdot \tilde{x}_\sigma} \tilde{X}_\sigma) \hookrightarrow \prod_{k \ge 0} \bigoplus_{\sigma \in \mathrm{Map}(\Delta^p, X)} \bigoplus_{\alpha \in \pi_1(X, \sigma(0))} C_{*+k}(\mathcal{P}^k_{\tilde{x}_\sigma, \alpha \cdot \tilde{x}_\sigma} \tilde{X}_\sigma)$$

is a quasi-isomorphism, and $\{\tilde{E}^r_{p,q}\}$ converges. Thus

$$\tilde{E}^1_{p,q} = H_q \left(\tilde{E}^0_{p,*}, d^0 \right) \cong \bigoplus_{\sigma \in \mathrm{Map}(\Delta^p, X)} H_q \left(C^{\mathcal{L}_{\sigma(0)}}_*(X), \partial + \delta \right).$$

By Theorem 2.4.4, the filtration-preserving chain map $\mathrm{e}_* \circ E_\xi : C_*(\mathcal{L}X) \to C_*^{\mathcal{L}}(X)$ induces isomorphisms $E^1_{p,q} \cong \tilde{E}^1_{p,q}$ for all p, q. Then by classical comparison theorem ([49, Theorem 5.2.12]), $\mathrm{e}_* \circ E_\xi$ induces an isomorphism $H_*(\mathcal{L}X) \cong H_*(C^{\mathcal{L}}(X))$. $\square$

Remark 2.4.6. In the proof of Theorem 2.4.5, although $(\phi_k)_{k\geq 0}$ is a map of cosimplicial complexes, $(\psi_k)_{k\geq 0}$ does not seem to be a map of cosimplicial complexes. It seems not true that $\psi_k \circ \delta_0 - \delta_0 \circ \psi_k = 0 \mod \tilde{\mathcal{F}}_{p-1}$; the same problem occurs for δ_k.

2.5 Some remarks

First remark. For $\mathcal{X} = \mathcal{L}_x$, there is a concatenation map

$$\mathrm{con}_x : \mathcal{L}^k_x X \times \mathcal{L}^{k'}_x X \to \mathcal{L}^{k+k'}_x X$$
$$((c_0, \dots, c_k), (c'_0, \dots, c'_{k'})) \mapsto (c_0, \dots, c_{k-1}, c_k * c'_0, c'_1, \dots, c'_{k'}),$$

which induces a product operation on $C^{\mathcal{L}_x}_*(X) = \prod_{k\geq 0} C_{*+k}(\mathcal{L}^k_x X)$:

$$(a_k)_{k\geq 0} \cdot (b_k)_{k\geq 0} := \left(\sum_{k_1+k_2=k} (\mathrm{con}_x)_*(a_{k_1} \times b_{k_2}) \right)_{k\geq 0}. \tag{2.5.1}$$

It is clear that $(C^{\mathcal{L}_x}_*(X), \partial + \delta, \cdot)$ is a dg associative algebra. (Possibly there are some signs in (2.5.1), but let us forget about it at the moment).

Let $\tilde{\mathcal{L}}_x X$ be the based Moore loop space of X, then $C_*(\tilde{\mathcal{L}}_x X)$ is a dg associative algebra whose product is induced by concatenation of based Moore loops. Let $\{(\tilde{\mathcal{L}}_x)_k X\}_{k\geq 0}$ be the cosimplicial space defined similar to $\{\mathscr{L}_{k+1} M\}_{k\geq 0}$ in Example 1.4.2(v). There are natural maps $(\tilde{\mathcal{L}}_x)_k X \to \mathcal{L}^k_x X$, and Theorem 2.2.1 holds when we use $(\tilde{\mathcal{L}}_x)_k X$ in place of $\mathcal{L}_x X \times \Delta^k$.

Proposition 2.5.1. *The quasi-isomorphism* $\mathrm{e}_* \circ E_\xi : C_*(\tilde{\mathcal{L}}_x X) \to C^{\mathcal{L}_x}_*(X)$ *obtained by Theorem 2.2.1 and Lemma 2.2.3 induces an isomorphism of homology groups as R-algebras.*

Proof. If we choose $\xi_k = 1 \cdot (\Delta^k \xrightarrow{\mathrm{id}} \Delta^k)$ and handle signs in (2.5.1), then we may check $\mathrm{e}_* \circ E_\xi : C_*(\tilde{\mathcal{L}}_x X) \to C^{\mathcal{L}_x}_*(X)$ is a dg algebra map. Alternatively, for any choice of ξ (all choices of ξ are homologous in $C^{\Delta}_*(\mathrm{pt})$), we can argue as follows. There is a dg algebra structure on $\prod_{k\ge 0} C_{*+k}((\tilde{\mathcal{L}}_x)_k X)$ defined in the same way as $C^{\mathcal{L}_x}_*(X)$, and $\mathrm{e}_* : \prod_{k\ge 0} C_{*+k}((\tilde{\mathcal{L}}_x)_k X) \to C^{\mathcal{L}_x}_*(X)$ is a dg algebra map. The quasi-isomorphism $\mathrm{pr}_0 : \prod_{k\ge 0} C_{*+k}((\tilde{\mathcal{L}}_x)_k X) \to C_*(\tilde{\mathcal{L}}_x X)$ is clearly a dg algebra map. Since pr_0 and E_ξ are chain homotopy inverse to each other, we get the conclusion. $\square$

Second remark. For $\mathcal{X} = \mathcal{L}$, one can define string topology operations on $C^{\mathcal{L}}_*(X)$ as long as the chain complex C_* is good for transversality purposes. For example, C_* may be the (regular) de Rham chain complex defined by Irie [29] (see Section 1.4). See also Chapter 3.

Third remark. Let (M, g) be a Riemannian manifold, and consider piecewise smooth paths and loops on M. For each of $\mathcal{X} \in \{\mathcal{P}, \mathcal{L}, \mathcal{L}_x, \mathcal{P}_{x,x'}\}$, there is a length function $l : \mathcal{X}M \to \mathbb{R}_{\ge 0}$, $l(\gamma) := \int_0^1 |\dot\gamma(t)|_g dt$. For $a \in \mathbb{R}_{\ge 0} \cup \{\infty\}$, define $(\mathcal{X}M)^a := l^{-1}([0,a]) \subset \mathcal{X}M$. Similarly, Define a function $l_0 : \Pi_1 M \to \mathbb{R}_{\ge 0}$ by

$$l_0(p,q,\sigma) := \inf \left\{ l(\gamma) \;\middle|\; \gamma : [0,1] \to M,\ \gamma(0) = p,\ \gamma(1) = q,\ [\gamma] = \sigma \right\},$$

and for each $k \in \mathbb{Z}_{\ge 0}$, define a function $l_k : \mathcal{X}^k M \to \mathbb{R}_{\ge 0}$ by $l_k(c_0, \ldots, c_k) := l_0(c_0 * \cdots * c_k)$. For $a \in \mathbb{R}_{\ge 0} \cup \{\infty\}$, define $(\mathcal{X}^k M)^a := l_k^{-1}([0,a]) \subset \mathcal{X}^k M$. Then $(\mathcal{X}^k M)^a$ is a cosimplicial subspace of $\mathcal{X}^k M$, and we have a cosimplicial chain complex $C_*((\mathcal{X}^k M)^a)$. Denote its total complex by $C^{\mathcal{X}}_*(M)^a$.

We have the following refined version of Conjecture 2.2.1.

Conjecture 2.5.2. *For any $a \in \mathbb{R}_{\ge 0} \cup \{\infty\}$, and for all of $\mathcal{X} \in \{\mathcal{P}, \mathcal{L}, \mathcal{L}_x, \mathcal{P}_{x,x'}\}$, the natural map $\mathrm{e}_* : C^{\Delta}_*(\mathcal{X}M)^a \to C^{\mathcal{X}}_*(M)^a$ is a quasi-isomorphism.*

Conjecture 2.5.2 is open to the author. It seems that the proof of Theorem 2.2.1 presented in this chapter does not work for Conjecture 2.5.2, even in the case M is simply-connected.

If Conjecture 2.5.2 is true, it might be useful in studying *quantitative* aspects of symplectic geometry (e.g. symplectic capacities). Some work of Irie [31] is in this flavour.

Chapter 3

Cyclic loop bracket and Fukaya A_∞ algebra

3.1 Introduction

Let (M, ω_M) be a symplectic manifold which is closed or convex at infinity, and let $L \subset M$ be an embedded Lagrangian submanifold which is connected, closed, oriented. Assume $L \subset M$ is relatively spin ([18, Definition 1.6]) and fix a relative spin structure, which is used to orient moduli spaces of pseudoholomorphic disks $(D, \partial D) \to (M, L)$ ([18, Theorem 8.1.1]). These are the standing assumptions of Fukaya-Oh-Ohta-Ono [18], where they rigorously constructed the Lagrangian Floer theory of such (M, L):

- (Fukaya-Oh-Ohta-Ono [18]) There is a gapped filtered (homotopy-)unital A_∞ algebra structure $\mathfrak{m} = \{\mathfrak{m}_k\}_{k \geq 0}$ on a version of simplicial chain complex $\mathbb{Q}\mathcal{X}_L$ of L with $\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}$ coefficients, obtained by counting pseudoholomorphic disks $(D, \partial D) \to (M, L)$ with mark points on ∂D. ($\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}$ is the Novikov ring over $\mathbb{Q}$, see (3.4.2).) The homotopy equivalence class of this filtered A_∞ algebra is independent of choices of the almost complex structure on (M, ω_M) and virtual perturbations made in the construction.

- (Fukaya [17]) Using differential forms $\Omega^*(L)$ in place of $\mathbb{Q}\mathcal{X}_L$ in the construction, this filtered A_∞ algebra is cyclic (over $\mathbb{R}$), and is unique up to pseudo-isotopy.

Fukaya-Oh-Ohta-Ono also studied the Lagrangian intersection Floer theory for a pair of Lagrangian submanifolds, but the discussion in this chapter is restricted to a single L (and so its Hamiltonian perturbations).

In [16], over $\mathbb{R}$, Fukaya outlined another construction of the A_∞ algebra associated to $L \subset M$, using the free loop space of L. Fukaya's observation is that, every map $u : (D, \partial D) \to (M, L)$ restricts to a loop $u|_{\partial D} : \mathbb{S}^1 = \partial D \to L$, and if one copies this for moduli spaces of pseudoholomorphic disks bounded by L, then one gets a chain $\mathcal{M}$ on $\mathcal{L}L$, which is Maurer-Cartan with respect to chain level loop bracket by disk bubbling ($\mathcal{M}$ being Maurer-Cartan corresponds to the A_∞ relations). Moreover, $\mathcal{M}$ should be $\mathbb{S}^1$-equivariant since $\mathbb{S}^1 \subset \mathrm{Aut}(D)$, and is Maurer-Cartan with respect to chain level string bracket.

In order for Fukaya's idea to work, one needs a chain complex $\mathbf{C}_*$ such that $H_*(\mathbf{C}) \cong H_*(\mathcal{L}L; \mathbb{R})$, or better a chain complex $\mathbf{C}_*^{\mathbb{S}^1}$ such that $H_*(\mathbf{C}^{\mathbb{S}^1}) \cong H_*^{\mathbb{S}^1}(\mathcal{L}L; \mathbb{R})$, such that the following conditions are satisfied:

(i) Chain level loop bracket on $\mathbf{C}_*$ / string bracket on $\mathbf{C}_*^{\mathbb{S}^1}$ can be defined.

(ii) Definition of chains fit with geometry of holomorphic disks. Namely, one can really define a chain $\mathcal{M}$ in $\mathbf{C}_* \widehat{\otimes} \Lambda_{0,\mathrm{nov}}$ or $\mathbf{C}_*^{\mathbb{S}^1} \widehat{\otimes} \Lambda_{0,\mathrm{nov}}$ from moduli spaces of pseudoholomoprhic disks $(D, \partial D) \to (M, L)$, and $\mathcal{M}$ satisfies the Maurer-Cartan equation.

(iii) There is a naturally defined chain map $\mathbf{C}_*$ or $\mathbf{C}_*^{\mathbb{S}^1} \to \prod_{k \geq 0} \mathrm{Hom}(C_*(L; \mathbb{R})^{\otimes k}, C_*(L; \mathbb{R}))$ which send Maurer-Cartan elements in $\mathbf{C}_*$ or $\mathbf{C}_*^{\mathbb{S}^1}$ to A_∞ operations on $C_*(L)$, where $C_*(L; \mathbb{R})$ is a suitable version of (co)chains on L over $\mathbb{R}$.

In [29], Irie worked out details of (i)(iii) for $\mathbf{C}_*$. In [30], Irie worked out details of (ii) and then realized some applications outlined in [16], including a proof of Audin's conjecture and the classification of prime oriented (embedded) Lagrangian submanifolds in $(\mathbb{C}^3, \omega_{\mathrm{std}})$ up to diffeomorphism. Irie did not work in the $\mathbb{S}^1$-equivariant situation, and his motivation was not to study the (cyclic) A_∞ algebra of L.

The purpose of this chapter was initially to work out details of Fukaya's proposal mentioned above in the cyclic invariant setting, based on results of the previous two chapters. In order to keep unity of all three chapters, we omit details corresponding to Step (ii) as well as applications to the Lagrangian Floer theory of L in this thesis, but present details corresponding to Steps (i)(iii). Details of Step (ii) will appear in the author's future writings.

Let us state the outcome of Steps (i)(ii)(iii) altogether, though. Note that we obtain a chain complex $\mathbf{C}^{\mathrm{cyc}}_*$ such that $H_*(\mathbf{C}^{\mathrm{cyc}}) \cong G^{\mathbb{S}^1}_*(\mathcal{L}L)$ instead of $H^{\mathbb{S}^1}_*(\mathcal{L}L)$.

- Moduli spaces of marked holomorphic disks $(D, \partial D) \to (M, L)$ push forward to a gapped filtered element $\mathcal{M} \in \mathbf{C}^{\mathrm{cyc}} \widehat{\otimes} \Lambda_{0,\mathrm{nov}}$, which is Maurer-Cartan with respect to the cyclic loop bracket: $\partial \mathcal{M} = \{\mathcal{M}, \mathcal{M}\}$.

- There is a dg Lie algebra homomorphism $\mathbf{C}^{\mathrm{cyc}} \to \prod_{k \geq 0} \mathrm{Hom}^{\mathrm{cyc}}(\Omega(L)^{\otimes k}, \Omega(L))$ sending $\mathcal{M}$ to the gapped filtered cyclic A_∞ deformation of $(\Omega^*(L), d, \wedge)$ defined by Fukaya-Oh-Ohta-Ono [18] and Fukaya [17].

- The gauge equivalence class of $\mathcal{M}$ is independent of choices of the almost complex structure on (M, ω_M) and artificial choices (Kuranishi structures, CF perturbations, etc.) made in the construction, which implies that the cyclic A_∞ deformation of $(\Omega^*(L), d, \wedge)$ induced by $\mathcal{M}$ is unique up to pseudo-isotopy.

Let us point out what (in addition to working in the cyclic invariant setting) is improved in Step (ii) compared to what Irie did in [30].

- In [30], in order to handle families of disks and loops in a consistent way, starting from moduli spaces of holomorphic disks, Irie first obtained maps to the continuous loop space, then used a technical C^0-approximation to obtain maps to the smooth loop space. If we use the chain model in Chapter 2 (combine it with de Rham chains), then there is no need to care about smoothness of loops, and one only needs to care about evaluation maps at mark points. Then there is no essential technical difficulty on the loop space side. Thus the technical details to handle Step (ii) are greatly simplified.

- It becomes much easier to check the (cyclic) A_∞ operations induced by $\mathcal{M}$ agrees with the definition of Fukaya-Oh-Ohta-Ono, when there is not need to apply extra C^0-approximation to get smooth loops. (Irie did not do this.)

- We generalize the notion of de Rham chains to include plots that are maps from manifolds with corners. This is not an essential improvement but is just a matter of convenience for Step (ii), but it is also interesting and might be useful elsewhere.

Remark 3.1.1. We have worked out a cyclic invariant story, which at homology level corresponds to the negative $\mathbb{S}^1$-equivariant homology of $\mathcal{L}L$ rather than the $\mathbb{S}^1$-equivariant homology. This fits very well with cyclic A_∞ algebras, and is better than the non-cyclic invariant story in applications. However, a truly $\mathbb{S}^1$-equivariant story of the Lagrangian Floer theory of $L \subset M$ with potential applications, as proposed by Fukaya [16], is still a mystery.

Outline

In Section 3.2, we introduce a version of de Rham chain complexes defined via manifolds with corners. In Section 3.3, we combine results in Chapter 1 and Chapter 2 to present a chain model for chain level string bracket (cyclic bracket) and iterated integrals, as well as the $[0, 1]$-parameterized version. In Section 3.4, we review basics about (cyclic) A_∞ algebras. In Section 3.5, we establish the equivalence between pseudo-isotopy of (cyclic) A_∞ algebras and gauge equivalence of corresponding Maurer-Cartan elements in the (cyclic) Hochschild cochain complex. In Section 3.6, we briefly discuss homological algebra of L_∞ algebras that is relavant to gauge equivalence of Maurer-Cartan elements.

Conventions

Unless otherwise specified:

- (Algebraic structures, sign rule) The same as Chapter 1.
- (Orientations) We follows [20].

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3.2 de Rham chain complex via manifolds with corners

In Section 1.4, we reviewed Irie's construction of the de Rham chain complex of differentiable spaces, where smooth manifolds without boundary are utilized to define the de Rham chain complex. To establish results in this chapter, it is most natural to include manifolds with corners into the definition of plots, and we choose to do so. (This is also the original proposal of Fukaya in [16], and it is tricky to make things rigorous.) For simplicity, we will not discuss the general theory of differentiable spaces in this setting (which is straightforward but requires more writing), but only present what we need. In order not to cause confusions, we will use a different notation for de Rham chains (involving manifolds with corners) compared to the notation in Section 1.4.

3.2.1 Some facts about manifolds with corners

Terminologies about (smooth) manifolds with corners in this chapter are adapted from [18].

Definition 3.2.1. Let P be a manifold with corners, and $n = \dim P$.

(i) (Corner structure stratification). For $k \in \{0, \ldots, n\}$, let $S_k(P)$ be the closure of the set of points in P whose neighborhoods are diffeomorphic to open neighborhoods of 0 in $\mathbb{R}^{n-k} \times [0,1)^k$. Clearly $\overset{\circ}{S}_k(P) := S_k(P) \setminus S_{k+1}(P)$ carries a structure of a smooth manifold without boundary of dimension $n-k$, which is uniquely determined by the structure of P.

(ii) (Charts of product type). For $p \in \overset{\circ}{S}_k(P)$, we can choose a chart (V_p, ϕ_p) of P at p, such that $\phi_p : V_p \xrightarrow{\cong} [V_p] \times [0,1)^k$, $\phi_p(p) = (\bar{p}, 0)$, where $[V_p]$ is an open subset of $\mathbb{R}^{n-k}$. For simplicity, we may just write $V_p = [V_p] \times [0,1)^k$ when there is no risk of confusion.

Definition 3.2.2. Let P, Q be manifolds with corners, and $f : Q \to P$ be a continuous map.

(i) (Smooth map). f is called a smooth map if for any $q \in P$, $p = f(q) \in Q$, there exists $\varepsilon > 0$, charts of product type $V_q = [V_q] \times [0,1)^{k_q}$ at $q = (\bar{q}, 0)$ and $V_p = [V_p] \times [0,1)^{k_p}$ at $p = (\bar{p}, 0)$, such that $f(V_q) \subset V_p$, and the restriction of f to V_q extends to a smooth map (in the usual sense) from $[V_q] \times (-\varepsilon, 1)^{k_q}$ to $[V_p] \times (-\varepsilon, 1)^{k_p}$.

(ii) (Corner stratified smooth map). f is called a corner stratified smooth map if for any $q \in Q$, $p = f(q) \in P$, there exist charts of product type $V_q = [V_q] \times [0,1)^{l+k}$ at $q = (\bar{q}, 0)$ and $V_p = [V_p] \times [0,1)^k$ at $p = (\bar{p}, 0)$, and a smooth map $\bar{f}_q : V_q \to [V_p]$, such that f is of the form

$$f(x, (s_1, \ldots, s_l, t_1, \ldots, t_k)) = \big(\bar{f}_q(x, (s_1, \ldots, s_l, t_1, \ldots, t_k)), t_1, \ldots, t_k\big)$$

in coordinates $x \in [V_q]$, $(s_1, \ldots, s_l) \in [0,1)^l$, $(t_1, \ldots, t_k) \in [0,1)^k$.

(iii) (Corner stratified submersion). f is called a corner stratified submersion if for any $q \in Q$, $p = f(q) \in P$, there exist $V_q = [V_q] \times [0,1)^{l+k}$, $V_p = [V_p] \times [0,1)^k$, $\bar{f}_q : V_q \to [V_p]$ which satisfy the conditions in (ii) and the following condition:

For any $(t_1, \ldots, t_k) \in [0,1)^k$ and $i \in \{0, \ldots, l\}$, the map

$$[V_q] \times \overset{\circ}{S}_i([0,1)^l) \to [V_p]; \quad (x, (s_1, \ldots, s_l)) \mapsto \bar{f}_q(x, (s_1, \ldots, s_l, t_1, \ldots, t_k))$$

is a submersion between smooth manifolds without boundary.

Note that if P has no boundaries or corners, then a corner stratified smooth map $Q \to P$ is the same as a smooth map $Q \to P$.

Lemma 3.2.3. *Let $f : Q \to P$ be a corner stratified submersion between manifolds with corners. Then for any $q \in Q$, $p = f(q) \in P$, there exist charts of product type $V_q = [V_q] \times [0,1)^{l+k}$ at $q = (\bar{q}, 0)$ and $V_p = [V_p] \times [0,1)^k$ at $p = (\bar{p}, 0)$, and an open subset $[V_{q,p}]$ of $\mathbb{R}^{\dim[V_q]-\dim[V_p]}$, such that $[V_q] = [V_p] \times [V_{q,p}]$, and that f is of the form*

$$f(y, z, s, t) = (y, t)$$

in coordinates $y \in [V_p] \subset \mathbb{R}^{\dim[V_p]}$, $z \in [V_{q,p}] \subset \mathbb{R}^{\dim[V_q]-\dim[V_p]}$, $s \in [0,1)^l$, $t \in [0,1)^k$.

Proof. Choose charts as in Definition 3.2.2(iii), and assume $[V_q]$ is precompact. Write $\dim[V_p] = m$, $\dim[V_q] = m + n$. By definition, the map $[V_q] \to [V_p]$, $x \mapsto \bar{f}_q(x, 0, 0)$ is a submersion. So there is $\varepsilon \in (0, 1)$ such that for any $(s, t) = (s_1, \dots, s_l, t_1, \dots, t_k) \in [0, \varepsilon)^{l+k}$, the map $[V_q] \to [V_p]$, $x \mapsto \bar{f}_q(x, s, t)$ is a submersion. By inverse function theorem, there is a neighborhood B_q of q in $\mathbb{R}^{m+n}$, a neighborhood B_p of p in $\mathbb{R}^m$, and an open set $[U_{q,p}] \subset \mathbb{R}^n$, which depend on ε but are independent of $(s, t) \in [0, \varepsilon)^l \times [0, \varepsilon)^k$, and diffeomorphisms

$$\Phi_{s,t} : [V_q] \cap B_q \xrightarrow{\cong} ([V_p] \cap B_p) \times [U_{q,p}] \subset \mathbb{R}^m \times \mathbb{R}^n$$

which depend smoothly on $(s, t) \in [0, \varepsilon)^{l+k}$, such that

$$\bar{f}_q(\cdot, s, t) \circ \Phi_{s,t}^{-1} : ([V_p] \cap B_p) \times [U_{q,p}] \to \mathbb{R}^m$$

is the map $(y, z) \mapsto y$. Then, the diffeomorphism

$$\begin{aligned} g_q : ([V_p] \cap B_p) \times [U_{q,p}] \times [0, \varepsilon)^{l+k} &\xrightarrow{\cong} ([V_q] \cap B_q) \times [0, \varepsilon)^{l+k} \\ (y, z, s, t) &\mapsto (\Phi_{s,t}^{-1}(y, z), s, t) \end{aligned}$$

is a coordinate change on $([V_q] \cap B_q) \times [0, \varepsilon)^{l+k}$, such that

$$(f \circ g_q)(y, z, s, t) = f(\Phi_{s,t}^{-1}(y, z), s, t) = (\bar{f}_q(\Phi_{s,t}^{-1}(y, z), s, t), t) = (y, t).$$

Finally we rescale the coordinates on $[0, \varepsilon)^{l+k}$, and the proof is complete. $\square$

Definition 3.2.4 (Integration along the fiber)**.** Let $f : Q \to P$ be a corner stratified submersion between oriented manifolds with corners. Define a linear map $f_! : \Omega_c^*(Q) \to \Omega_c^{*+\dim P - \dim Q}(P)$ in the following way.

- For any $q \in Q$, $p = f(q)$, choose charts $V_q = [V_p] \times [V_{q,p}] \times [0,1)^{l+k}$, $V_p = [V_p] \times [0,1)^k$ as in Lemma 3.2.3. Let $d = \dim[V_p]$, $y_1, \dots, y_d$ be coornidates on $[V_p] \subset \mathbb{R}^n$, and $t_1, \dots, t_k$ be coordinates on $[0,1)^k$. If $\omega \in \Omega_c^*(Q)$ is supported in V_q, then in the chosen charts, ω can be uniquely written as

$$\omega = \sum_{|I| \leq k, |J| \leq d} dt_I \wedge dy_J \wedge \omega_{IJ},$$

  where I, J are multi-indices, and each ω_{IJ} satisfies $\omega_{IJ}(\frac{\partial}{\partial t_i}, \cdots) = \omega_{IJ}(\frac{\partial}{\partial y_j}, \cdots) = 0$ $(1 \leq \forall i \leq k,\ 1 \leq \forall j \leq d)$. Define

$$f_!\omega := \sum_{|I| \leq k, |J| \leq d} dt_I \wedge dy_J \wedge \int_{[V_{q,p}] \times [0,1)^l} \omega_{IJ}.$$

  Then $f_!\omega \in \Omega_c^{*+\dim P - \dim Q}(P)$ and $f_!\omega$ is supported in V_p.

- In general, for $\omega \in \Omega_c^*(Q)$, define $f_!\omega$ by patching together local definitions with the aid of a partition of unity on Q. The definition of $f_!\omega$ does not depend on the choice of the partition of unity.

Definition 3.2.5 (Normalized boundary)**.** Let P be a manifold with corners. Define a manifold with corners ∂P and a map $\pi : \partial P \to S_1(P)$ as follows.

For any $p \in \overset{\circ}{S}_k(P)$, choose a chart of product type $V_p = [V_p] \times [0,1)^k$ at $p = (\bar{p}, 0)$, and assume $[V_p]$ is connected. Then

$$S_1(V_p) = \bigcup_{1 \leq i \leq k} [V_p] \times [0,1)^{i-1} \times \{0\} \times [0,1)^{k-i}.$$

Define

$$\partial V_p := \coprod_{1 \leq i \leq k} [V_p] \times [0,1)^{i-1} \times \{0\} \times [0,1)^{k-i},$$

and define $\pi|_{\partial V_p} : \partial V_p \to S_1(V_p)$ in the obvious way, so that the restriction of $\pi|_{\partial V_p}$ to each component of ∂V_p is the natural inclusion. The coordinate changes among charts $\{V_p\}_{p\in P}$ restricts componentwise to coordinate changes among $\{\partial V_p\}_{p\in P}$, and the charts ∂V_p ($p \in P$) glue to be a manifold with corners, which is denoted by ∂P. The map $\pi : \partial P \to S_1(P)$ has been defined locally.

We call ∂P the normalized boundary of P and $\pi : \partial P \to S_1(P)$ the normalization map.

The normalization map $\pi : \partial P \to S_1(P)$ induces a map $\overset{\circ}{S}_k(\partial P) \to \overset{\circ}{S}_{k+1}(P)$ for each $k \geq 0$, which is a $(k+1)$-fold covering map. The composition $\pi : \partial P \to S_1(P) \hookrightarrow P$ is a smooth map. If $\omega \in \Omega^*(P)$, denote

$$\omega|_{\partial P} := \pi^*\omega \in \Omega^*(\partial P). \tag{3.2.1}$$

If ω is compactly supported, so is $\omega|_{\partial P}$.

Remark 3.2.6. If P is a manifold with boundary (without corners of codimension ≥ 2), then ∂P agrees with the boundary of P in the usual sense. If $S_2(P) \neq \varnothing$, then ∂P is not a subset of P, and (3.2.1) is written by abuse of notation.

Lemma-Definition 3.2.7 (Decompositon of the normalized boundary with respect to a corner stratified smooth map)**.** *Let $f : Q \to P$ be a corner stratified smooth map between manifolds with corners. Then $\partial Q = \partial^v Q \coprod \partial^h Q$, where $\partial^v Q$, $\partial^h Q$ are manifolds with corners which can be characterized locally in the following way.*

For any $q \in Q$, $p = f(q)$, choose charts $V_q = [V_q] \times [0,1)^l \times [0,1)^k$ and $V_p = [V_p] \times [0,1)^k$ as in Definition 3.2.2(ii). Then

$$V_q \cap \partial^v Q = \coprod_{1\leq i\leq k} [V_q] \times [0,1)^l \times [0,1)^{i-1} \times \{0\} \times [0,1)^{k-i},$$
$$V_q \cap \partial^h Q = \coprod_{1\leq i\leq l} [V_q] \times [0,1)^{i-1} \times \{0\} \times [0,1)^{l-i} \times [0,1)^k.$$

$\partial^v Q$, $\partial^h Q$ are called the horizonal and horizontal boundaries of Q (with respect to f), respectively. The corner stratified smooth map $f : Q \to P$ induces corner stratified smooth maps

$$f|_{\partial^v Q} : \partial^v Q \to \partial P, \quad f|_{\partial^h Q} : \partial^h Q \to P$$

in an obvious way. (The symbols $f|_{\partial^v Q}, f|_{\partial^h Q}$ are abuse of notation.) In particular,

- *If f is a corner stratified submersion, then so are $f|_{\partial^v Q}, f|_{\partial^h Q}$.*
- *If P is a manifold without boundary or corner, then $\partial^v Q = \varnothing$ and $\partial^h Q = \partial Q$.*

Lemma 3.2.8. *Let $f : Q \to P$ be a corner stratified submersion between manifolds with corners. Then for any $\omega \in \Omega_c^*(Q)$,*

$$(f_!\omega)|_{\partial P} = (f|_{\partial^v Q})_!(\omega|_{\partial^v Q}).$$

Proof. It suffices to prove it locally, which is obvious from definitions. □

Lemma 3.2.9 (Stokes' formula)**.** *Let $f : Q \to P$ be a corner stratified submersion between oriented manifolds with corners. Then*

$$df_!\omega - f_! d\omega = (-1)^{|\omega| + \dim Q}(f|_{\partial^h Q})_!(\omega|_{\partial^h Q}), \qquad \forall \omega \in \Omega_c^*(Q).$$

Proof. The proof is the same as Stokes' theorem on oriented manifolds with boundary. □

3.2.2 de Rham chain complex of manifolds without boundary via manifolds with corners

Let

$$\mathscr{V} := \coprod_{n \geq m \geq 0} \mathscr{V}_{n,m}, \tag{3.2.2}$$

where $\mathscr{V}_{n,m}$ is the set of oriented m-dimensional submanifolds (possibly with corners) of $\mathbb{R}^n$. Let N be an oriented manifold without boundary or corner. Define

$$\bar{\mathscr{P}}(N) := \{(V, \varphi) \mid V \in \mathscr{V},\ \varphi : V \to N \text{ is a smooth map}\}.$$

For every $n \in \mathbb{Z}_{\geq 0}$, consider the vector space

$$\bigoplus_{(V,\varphi)\in\bar{\mathscr{P}}(N)} \Omega_c^{\dim V-n}(V). \tag{3.2.3}$$

For any $(V,\varphi) \in \bar{\mathscr{P}}(N)$ and $\omega \in \Omega_c^{\dim V-n}(V)$, denote by (V,φ,ω) the image of ω under the natural inclusion $\Omega_c^{\dim V-n}(V) \hookrightarrow$ (3.2.3). Consider the subspace Z_n of (3.2.3) generated by

$$\big\{(V,\varphi,\pi_!\omega) - (V',\varphi\circ\pi,\omega) \mid (V,\varphi) \in \bar{\mathscr{P}}(N),\ V' \in \mathscr{V},\ \omega \in \Omega_c^{\dim V'-n}(V'),$$
$$\pi : V' \to V \text{ is a corner stratified submersion}\big\}.$$

Then define

$$\bar{C}_n^{dR}(N) := \left(\bigoplus_{(V,\varphi)\in\bar{\mathscr{P}}(N)} \Omega_c^{\dim V-n}(V)\right)/Z_n.$$

We shall write the image of (V,φ,ω) in $\bar{C}_n^{dR}(N)$ as $[(V,\varphi,\omega)]$.

Define a linear map $\partial : \bar{C}_*^{dR}(N) \to \bar{C}_{*-1}^{dR}(N)$ on generators by

$$\partial[(V,\varphi,\omega)] := [(V,\varphi,d\omega)] + (-1)^{|\omega|+\dim V}[(\partial V,\varphi|_{\partial V},\omega|_{\partial V})]. \tag{3.2.4}$$

Lemma 3.2.10. *$\partial : \bar{C}_*^{dR}(N) \to \bar{C}_{*-1}^{dR}(N)$ in (3.2.4) is well-defined.*

Proof. Consider a generator $(V,\varphi,\pi_!\omega) - (V',\varphi\circ\pi,\omega) \in Z_n$. Then we check

$$\begin{aligned}
&[(V,\varphi,d\pi_!\omega)] + (-1)^{|\pi_!\omega|+\dim V}[(\partial V,\varphi|_{\partial V},(\pi_!\omega)|_{\partial V})]\\
&= [(V,\varphi,\pi_!d\omega + (-1)^{|\omega|+\dim V'}(\pi|_{\partial^h V'})_!(\omega|_{\partial^h V'}))] + (-1)^{|\pi_!\omega|+\dim V}[(\partial V,\varphi|_{\partial V},(\pi_!\omega)|_{\partial V})]\\
&= [(V',\varphi\circ\pi,d\omega)] + (-1)^{|\omega|+\dim V'}[(\partial^h V',\varphi\circ\pi|_{\partial^h V'},\omega|_{\partial^h V'}) + (\partial^v V',\varphi\circ\pi|_{\partial^v V'},\omega|_{\partial^v V'})]\\
&= [(V',\varphi\circ\pi,d\omega)] + (-1)^{|\omega|+\dim V'}[(\partial V',\varphi\circ\pi|_{\partial V'},\omega|_{\partial V'})],
\end{aligned}$$

where $\partial^h V'$, $\partial^v V'$ are said with respect to $\pi : V' \to V$. The first equality follows from Lemma 3.2.9 (Stokes' formula), and the second equality follows from Lemma 3.2.8. $\square$

Lemma 3.2.11. *$\partial : \bar{C}_*^{dR}(N) \to \bar{C}_{*-1}^{dR}(N)$ in (3.2.4) satisfies $\partial^2 = 0$.*

Proof. By definition,

$$\begin{aligned}
&\partial^2[(V,\varphi,\omega)]\\
&= \partial[(V,\varphi,d\omega)] + (-1)^{|\omega|+\dim V}\partial[(\partial V,\varphi|_{\partial V},\omega|_{\partial V})]\\
&= [(\partial V,\varphi|_{\partial V},(-1)^{|d\omega|+\dim V}(d\omega)|_{\partial V} + (-1)^{|\omega|+\dim V}d(\omega|_{\partial V}))] + [(\partial^2 V,\varphi|_{\partial^2 V},\omega|_{\partial^2 V})]\\
&= [(\partial^2 V,\varphi|_{\partial^2 V},\omega|_{\partial^2 V})].
\end{aligned}$$

We claim there is an orientation-reversing diffeomorphism $r : \partial^2 V \to \partial^2 V$ such that

$$\varphi|_{\partial^2 V}\circ r = \varphi|_{\partial^2 V}, \quad r^*(\omega|_{\partial^2 V}) = \omega|_{\partial^2 V}, \quad r^2 = \mathrm{id}. \tag{3.2.5}$$

The normalization map $\partial(\partial V) \to S_1(\partial V)$ induces a diffeomorphism $\pi_1 : \mathring{S}_0(\partial^2 V) \xrightarrow{\cong} \mathring{S}_1(\partial V)$, and the normalization map $\partial V \to S_1(V)$ induces a double covering map $\pi_2 : \mathring{S}_1(\partial V) \to \mathring{S}_2(V)$. Let $r' : \mathring{S}_0(\partial^2 V) \xrightarrow{\cong} \mathring{S}_0(\partial^2 V)$ be the only nontrivial deck transformation over the covering map $\pi_2\circ\pi_1 : \mathring{S}_0(\partial^2 V) \to \mathring{S}_2(V)$. It is easy to see r' extends to a diffeomorphism $r : \partial^2 V \xrightarrow{\cong} \partial^2 V$ which is orientation-reversing and satisfies (3.2.5). Then we have

$$\begin{aligned}
[(\partial^2 V,\varphi|_{\partial^2 V},\omega|_{\partial^2 V})] &= [(\partial^2 V,\varphi|_{\partial^2 V}\circ r,\omega|_{\partial^2 V})]\\
&= [(\partial^2 V,\varphi|_{\partial^2 V},r_!(\omega|_{\partial^2 V}))] = -[(\partial^2 V,\varphi|_{\partial^2 V},\omega|_{\partial^2 V})],
\end{aligned}$$

which implies $[(\partial^2 V,\varphi|_{\partial^2 V},\omega|_{\partial^2 V})] = 0$. □

We have proved $(\bar{C}^{dR}_*(N),\partial)$ is a chain complex. Denote its homology by $\bar{H}^{dR}_*(N)$. We also need a transverse version of de Rham chain complex of M. Define

$$\bar{\mathscr{P}}(N_{\mathrm{reg}}) := \{(V,\varphi) \mid V\in\mathscr{V},\ \varphi : V\to N \text{ is a corner stratified submersion}\}.$$

Using $\bar{\mathscr{P}}(N_{\mathrm{reg}})$ in place of $\bar{\mathscr{P}}(N)$, we can define a chain complex $(\bar{C}^{dR}_*(N_{\mathrm{reg}}),\partial)$ in the same way as defininig $(\bar{C}^{dR}_*(N),\partial)$. There is a natural chain map $\bar{C}^{dR}_*(N_{\mathrm{reg}}) \to \bar{C}^{dR}_*(N)$ induced by the natural inclusion $\bar{\mathscr{P}}(N_{\mathrm{reg}}) \subset \bar{\mathscr{P}}(N)$.

Lemma 3.2.12. *The natural map $\bar{C}^{dR}_*(N_{\mathrm{reg}}) \to \bar{C}^{dR}_*(N)$ is a quasi-isomorphism.*

Proof. The proof is the same as [29, Proposition 5.2]. We remark that in [29], de Rham chain complexes $C_*^{dR}(N)$, $C_*^{dR}(N_{\mathrm{reg}})$ are defined using plots consisting of maps from manifolds without boundary, while in this section, de Rham chain complexes $\bar{C}_*^{dR}(N)$, $\bar{C}_*^{dR}(N_{\mathrm{reg}})$ are defined using plots consisting of maps from manifolds with corners. In order to apply the proof of $C_*^{dR}(N_{\mathrm{reg}}) \simeq C_*^{dR}(N)$ ([29, Proposition 5.2]) to this lemma, it suffices to observe the following obvious fact: If V is a manifold with corners, $\varphi : V \to N$ is smooth, $\varphi(V) \subset W \subset N$, and $F : N \times \mathbb{R}^D \to N$ is a smooth map such that for any $x \in W$, $F|_{\{x\}\times\mathbb{R}^D} : \mathbb{R}^D \to N$ is a submersion, then $F \circ (\varphi \times \mathrm{id}_{\mathbb{R}^D}) : V \times \mathbb{R}^D \to N$ is a corner stratified submersion. $\square$

Recall the definitions of $\mathscr{P}(N)$ and $\mathscr{P}(N_{\mathrm{reg}})$ from Example 1.4.2. There are natural inclusions $\mathscr{P}(N) \subset \bar{\mathscr{P}}(N)$ and $\mathscr{P}(N_{\mathrm{reg}}) \subset \bar{\mathscr{P}}(N_{\mathrm{reg}})$, which induce natural chain maps $C_*^{dR}(N) \to \bar{C}_*^{dR}(N)$ and $C_*^{dR}(N_{\mathrm{reg}}) \to \bar{C}_*^{dR}(N_{\mathrm{reg}})$.

Lemma 3.2.13. *The natural map $i : C_*^{dR}(N_{\mathrm{reg}}) \to \bar{C}_*^{dR}(N_{\mathrm{reg}})$ is an isomorphism of chain complexes. Moreover, $(C_*^{dR}(N_{\mathrm{reg}}), \partial) \xrightarrow{\cong} (\bar{C}_*^{dR}(N_{\mathrm{reg}}), \partial) \xrightarrow{\cong} (\Omega_c^{\dim N - *}(N), d)$.*

Proof. There is a chain map

$$j : (\bar{C}_*^{dR}(N_{\mathrm{reg}}), \partial) \to (\Omega_c^{\dim N-*}(N), d); \quad [(V, \varphi, \omega)] = [(N, \mathrm{id}_N, \varphi_! \omega)] \mapsto \varphi_! \omega,$$

and a chain map

$$k : (\Omega_c^{\dim N-*}(N), d) \to (C_*^{dR}(N_{\mathrm{reg}}), \partial); \quad \omega \mapsto [(N, \mathrm{id}_N, \omega)].$$

It is clear that $k \circ j \circ i = \mathrm{id}_{C_*^{dR}(N_{\mathrm{reg}})}$, $i \circ k \circ j = \mathrm{id}_{\bar{C}_*^{dR}(N_{\mathrm{reg}})}$, $j \circ i \circ k = \mathrm{id}_{\Omega_c^{\dim N-*}(N)}$. $\square$

Corollary 3.2.14. $\bar{H}_*^{dR}(N) \cong \bar{H}_*^{dR}(N_{\mathrm{reg}}) \cong H_*^{dR}(N_{\mathrm{reg}}) \cong H_{c,dR}^{\dim N-*}(N) \cong H_*(N; \mathbb{R})$. $\square$

3.3 Chain level string bracket and iterated integral of differential forms

In this section, N is a closed oriented smooth manifold.

3.3.1 Chain model of $\mathcal{L}N$ and cyclic loop bracket

Consider the cocyclic space $\{\mathcal{L}^kN\}_{k\geq 0}$ (see Section 2.2). For each k, $\mathcal{L}^kN$ is a smooth oriented manifold of dimension $(k+1)\cdot\dim N$. There are smooth evaluation maps

$$\mathrm{ev}^k_i : \mathcal{L}^kN \to N, \quad (c_0,\dots,c_k)\mapsto \mathrm{s}(c_i) \ \ (0\leq i\leq k).$$

From Section 3.2.2, we have de Rham chain complexes $(\bar{C}^{dR}_*(\mathcal{L}^kN),\partial)$ and $(\bar{C}^{dR}_*((\mathcal{L}^kN)_{\mathrm{reg}}),\partial)$. Define

$$\tilde{\mathscr{P}}(\mathcal{L}^kN) := \left\{(V,\varphi)\in\bar{\mathscr{P}}(\mathcal{L}^kN) \mid \mathrm{ev}^k_i\circ\varphi : V\to N \text{ is a submersion } (0\leq\forall i\leq k)\right\}.$$

Using $\tilde{\mathscr{P}}(\mathcal{L}^kN)$ in place of $\bar{\mathscr{P}}(\mathcal{L}^kN)$ or $\bar{\mathscr{P}}((\mathcal{L}^kN)_{\mathrm{reg}})$, we can define a chain complex $(\tilde{C}^{dR}_*(\mathcal{L}^kN),\partial)$ in the same way as defining $\bar{C}^{dR}_*(\mathcal{L}^kN)$ or $\bar{C}^{dR}_*((\mathcal{L}^kN)_{\mathrm{reg}})$. Similar to Lemma 3.2.12, we see the natural inclusions $\bar{\mathscr{P}}((\mathcal{L}^kN)_{\mathrm{reg}})\subset\tilde{\mathscr{P}}(\mathcal{L}^kN)\subset\bar{\mathscr{P}}(\mathcal{L}^kN)$ induce natural quasi-isomorphisms $\bar{C}^{dR}_*((\mathcal{L}^kN)_{\mathrm{reg}})\simeq\tilde{C}^{dR}_*(\mathcal{L}^kN)\simeq\bar{C}^{dR}_*(\mathcal{L}^kN)$.

Recall from Example 1.7.3 (Irie's construction) that $((C^{dR}_{*+\dim N}(\mathscr{L}^N_{k+1,\mathrm{reg}}),\partial))_{k\geq 0}$ is a ns cyclic dg operad with a multiplication and a unit. In the same way, we can define a structure of a ns cyclic dg operad on $((\tilde{C}^{dR}_{*+\dim N}(\mathcal{L}^kN),\partial))_{k\geq 0}$, with a multiplication μ and a unit ε. Let us spell out the structures below for clarity.

- For $k\in\mathbb{Z}_{\geq 1}, k'\in\mathbb{Z}_{\geq 0}$ and $j\in\{1,\dots,k\}$, there is a chain map

  $$\circ_j : \tilde{C}^{dR}_{l+\dim N}(\mathcal{L}^kN)\otimes\tilde{C}^{dR}_{l'+\dim N}(\mathcal{L}^{k'}N)\to\tilde{C}^{dR}_{l+l'+\dim N}(\mathcal{L}^{k+k'-1}N)$$

  defined as follows. If $x=[(V,\varphi,\omega)]$ and $x'=[(V',\varphi',\omega')]$, set $\varphi_j:=\mathrm{ev}^k_j\circ\varphi$ and $\varphi'_0:=\mathrm{ev}^{k'}_0\circ\varphi'$. Then

  $$x\circ_j x' := (-1)^{l'(\dim V-\dim N)}[(V\times_{(\varphi_j,\varphi'_0)}V',\mathrm{con}_j\circ(\varphi_j\times\varphi'_0),\omega\times\omega')].$$

  Here the fiber product $V\times_{(\varphi_j,\varphi'_0)}V'$ is a smooth manifold with corners since φ_j,φ'_0 are corner stratified submersions to N, and con_j is the concatenation map

$$\mathrm{con}_j : \mathcal{L}^k N \times_{(\mathrm{ev}^k_j, \mathrm{ev}^{k'}_0)} \mathcal{L}^{k'} N \to \mathcal{L}^{k+k'-1} N$$
$$((c_0, \dots, c_k), (c'_0, \dots, c'_{k'}))$$
$$\mapsto \begin{cases} (c_0, \dots, c_{j-2}, c_{j-1} * c'_0, c'_1, \dots, c'_{k'-1}, c'_{k'} * c_j, \dots, c_k) & (k' \geq 1), \\ (c_0, \dots, c_{j-2}, c_{j-1} * c'_0 * c_j, c_{j+1}, \dots, c_k) & (k' = 0), \end{cases}$$

where $c_{j-1} * c'_0$ etc. are as in (2.2.1). The operations $\circ_j$ are associative (1.5.5).

- For each $k \in \mathbb{Z}_{\geq 0}$, $\tau_k : \tilde{C}^{dR}_{*+\dim N}(\mathcal{L}^k N) \to \tilde{C}^{dR}_{*+\dim N}(\mathcal{L}^k N)$ is induced by (2.2.2).
- $\mu := [(N, i_2, 1)] \in \tilde{C}^{dR}_{\dim N}(L^2 N)$, $\varepsilon := [(N, i_0, 1)] \in \tilde{C}^{dR}_{\dim N}(L^0 N)$. Here

$$i_k : N \to \mathcal{L}^k N; \quad p \mapsto ([p], \dots, [p]), \tag{3.3.1}$$

where $[p] = (p, p, [\text{constant path at } p]) \in \Pi_1 N$.

Lemma 3.3.1. *There is a morphism of cyclic dg operads*

$$\Phi = (\Phi_k)_{k \geq 0} : ((C^{dR}_{*+\dim N}(\mathscr{L}^N_{k+1,\mathrm{reg}}), \partial))_{k \geq 0} \to ((\tilde{C}^{dR}_{*+\dim N}(\mathcal{L}^k N), \partial))_{k \geq 0} \tag{3.3.2}$$

which preserves multiplicatioins and units, and induces a quasi-isomorphism

$$\Phi_* : \left(\prod_{k \geq 0} C^{dR}_{*+k+\dim N}(\mathscr{L}^N_{k+1,\mathrm{reg}}), \partial + \delta \right) \xrightarrow{\simeq} \left(\prod_{k \geq 0} \tilde{C}^{dR}_{*+k+\dim N}(\mathcal{L}^k N), \partial + \delta \right).$$

Proof. The morphism $(\Phi_k)_{k \geq 0}$ is defined by

$$C^{dR}_{*+\dim N}(\mathscr{L}^N_{k+1,\mathrm{reg}}) \to \tilde{C}^{dR}_{*+\dim N}(\mathcal{L}^k N),$$
$$[(U, \varphi, \omega)] \in \mapsto [(U, \phi_k \circ \varphi, \omega)],$$

where $\phi_k : \mathscr{L}^N_{k+1,\mathrm{reg}} \to \mathcal{L}^k N$ is the set-theoretic map $(\gamma_i, T_i)_{0 \leq i \leq k} \mapsto (\gamma_i(0), \gamma_i(T_i), [\gamma_i])_{0 \leq i \leq k}$. Clearly this morphism preserves μ and ε. The fact that Φ_* is a quasi-isomorhism follows from Theorem 2.2.1 and some commutative diagrams which we omit here. $\square$

Proposition-Definition 3.3.2 (Chain level loop bracket and cyclic loop bracket)**.** *There are dg Lie algebras* $(\mathbf{C}^{\mathcal{L},\mathrm{cyc}}_*(N), b, [,]) \subset (\mathbf{C}^{\mathcal{L}}_*(N), b, [,])$ *whose bracket* $[,]$ *is of degree 1, where*

$$\mathbf{C}^{\mathcal{L}}_*(N) := \prod_{k\geq 0} \tilde{C}^{dR}_*(\mathcal{L}^k N),$$

$$\mathbf{C}^{\mathcal{L},\mathrm{cyc}}_*(N) := \prod_{k\geq 0} \tilde{C}^{dR,\mathrm{cyc}}_*(\mathcal{L}^k N),$$

$$\tilde{C}^{dR,\mathrm{cyc}}_*(\mathcal{L}^k N) := \tilde{C}^{dR}_{*+\dim N+k}(\mathcal{L}^k N) \cap \ker(1-(-1)^k\tau_k),$$

and for $x = (x_k)_{k\geq 0}$, $y = (y_k)_{k\geq 0}$ *in* $\mathbf{C}^{\mathcal{L}}_*(N)$,

$$bx := \partial x + \delta x = (\partial x_k + \delta x_{k-1})_{k\geq 0}, \quad (x_{-1} \textit{ is vaccum})$$

$$\begin{aligned}(\delta x)_k &:= \sum_{0\leq i\leq k} (-1)^{|x|+i}\delta_i x_{k-1} \\ &= (-1)^{|x|}\mu \circ_2 x_{k-1} + \sum_{0<i<k}(-1)^{|x|+i} x_{k-1}\circ_i \mu + (-1)^{|x|+k}\mu\circ_1 x_{k-1}, \\ [x,y] &:= x\circ y - (-1)^{(|x|-1)(|y|-1)} y\circ x, \\ (x\circ y)_k &:= \sum_{\substack{k_1+k_2=k+1\\ 1\leq i\leq k_1}} (-1)^{(i-1)(k_2-1)+(k_1-1)(|y|+k_2)} x_{k_1}\circ_i y_{k_2}.\end{aligned}$$

There are isomorphisms

$$\mathbf{H}^{\mathcal{L}}_*(N) := H_*(\mathbf{C}^{\mathcal{L}}_*(N), b) \cong H_{*+\dim N}(\mathcal{L}N;\mathbb{R}),$$

$$\mathbf{H}^{\mathcal{L},\mathrm{cyc}}_*(N) := H_*(\mathbf{C}^{\mathcal{L},\mathrm{cyc}}_*(N), b) \cong G^{\mathbb{S}^1}_{*+\dim N}(\mathcal{L}N;\mathbb{R}),$$

where $G^{\mathbb{S}^1}_*$ *is negative* $\mathbb{S}^1$*-equivariant homology (Example 1.2.10). Under these isomorphisms,* $[,]$ *corresponds to the loop bracket on* $H_{*+\dim N}(\mathcal{L}N;\mathbb{R})$ *and the string bracket on* $G^{\mathbb{S}^1}_{*+\dim N}(\mathcal{L}N;\mathbb{R})$ *(Example 1.7.1, Lemma 1.7.2).*

Proof. Direct consequences of Proposition 1.5.6, Example 1.7.3 and Lemma 3.3.1. □

3.3.2 Model of $[0,1]\times\mathbf{C}^{\mathcal{L}}$ and $[0,1]\times\mathbf{C}^{\mathcal{L},\mathrm{cyc}}$

For each $k\in\mathbb{Z}_{\geq 0}$, $[0,1]\times\mathcal{L}^k N$ is an oriented manifold with boundary. For $V\in\mathscr{V}$ (3.2.2), a map $\varphi : V\to[0,1]\times\mathcal{L}^k N$ and an interval $I\subset\mathbb{R}$, we denote $\varphi =: (\varphi_{[0,1]}, \varphi_{\mathcal{L}})$, and

$V_I := \varphi_{[0,1]}^{-1}(I)$.

Fix $\epsilon_0 \in (0, \frac{1}{4})$. Let $\mathscr{P}_{\epsilon_0}^{[0,1]}(\mathcal{L}^k N)$ be the set consisting of tuples $(V, \varphi, \sigma_+, \sigma_-)$ satisfying the following conditions:

- $V \in \mathscr{V}$, $\varphi : V \to [0,1] \times \mathcal{L}^k N$ is a corner stratified smooth map.
- $(\varphi_{[0,1]}, \mathrm{ev}_i^k \circ \varphi_{\mathcal{L}}) : V \to [0,1] \times N$ is a corner stratified submersion $(0 \leq \forall i \leq k)$.
- $\sigma_+ : V_{[1-\epsilon_0,1]} \xrightarrow{\cong} [1-\epsilon_0, 1] \times V_1$ is a diffeomorphism such that
$$\varphi|_{V_{[1-\epsilon_0,1]}} = (i_{[1-\epsilon_0,1]} \times \varphi_{\mathcal{L}}|_{V_1}) \circ \sigma_+,$$
where $i_{[1-\epsilon_0,1]} : [1-\epsilon_0, 1] \to [0,1]$ is the inclusion map.
- $\sigma_- : V_{[0,\epsilon_0]} \xrightarrow{\cong} [0, \epsilon_0] \times V_0$ is a diffeomorphism such that
$$\varphi|_{V_{[0,\epsilon_0]}} = (i_{[0,\epsilon_0]} \times \varphi_{\mathcal{L}}|_{V_0}) \circ \sigma_-,$$
where $i_{[0,\epsilon_0]} : [0, \epsilon_0] \to [0,1]$ is the inclusion map.

For $(V, \varphi, \sigma_+, \sigma_-) \in \mathscr{P}_{\epsilon_0}^{[0,1]}(\mathcal{L}^k N)$, define

$$\mathscr{A}_{\epsilon_0}^*(V, \varphi, \sigma_+, \sigma_-) := \left\{ \omega \in \Omega_c^*(V) \;\middle|\; \omega|_{V_{[1-\epsilon_0,1]}} = \sigma_+^*(1 \times \omega|_{V_1}),\ \omega|_{V_{[0,\epsilon_0]}} = \sigma_-^*(1 \times \omega|_{V_0}) \right\}.$$

For $n \in \mathbb{Z}_{\geq 0}$, define (we omit ϵ_0 in the notation for $C_n^{dR,[0,1]}(\mathcal{L}^k N)$ etc.)

$$C_n^{dR,[0,1]}(\mathcal{L}^k N) := \left(\bigoplus_{(V,\varphi) \in \mathscr{P}_{\epsilon_0}^{[0,1]}(\mathcal{L}^k N)} \mathscr{A}_{\epsilon_0}^{\dim V - n - 1}(V, \varphi, \sigma_+, \sigma_-) \right) / Z_n^{[0,1]},$$

where $Z_n^{[0,1]}$ is a subspace generated by vectors

$$(V, \varphi, \sigma_+, \sigma_-, \omega) - (V', \varphi', \sigma'_+, \sigma'_-, \omega')$$

such that there exists a corner stratified submersion $\pi : U' \to U$ satisfying

$$\varphi' = \varphi \circ \pi,$$

$$\omega = \pi_! \omega',$$

$$\sigma_+ \circ \pi|_{V'_{[1-\epsilon_0,1]}} = (\mathrm{id}_{[1-\epsilon_0,1]} \times \pi|_{V'_1}) \circ \sigma'_+,$$

$$\sigma_- \circ \pi|_{V'_{[0,\epsilon_0]}} = (\mathrm{id}_{[0,\epsilon_0]} \times \pi|_{V'_0}) \circ \sigma'_-.$$

For $(V,\varphi)\in\mathscr{P}^{[0,1]}(\mathcal{L}^kN)$, let ∂^hV be the horizontal boundary of V with respect to φ (Lemma-Definition 3.2.7). Define a linear map $\partial: C_*^{dR,[0,1]}(\mathcal{L}^kN)\to C_{*-1}^{dR,[0,1]}(\mathcal{L}^kN)$ by

$$\begin{aligned}\partial[(V,\varphi,\sigma_+,\sigma_-,\omega)] &:= [(V,\varphi,\sigma_+,\sigma_-,d\omega)]\\ &\qquad + (-1)^{|\omega|+\dim V}[(\partial^hV,\varphi|_{\partial^hV},\sigma_+|_{\partial^hV},\sigma_-|_{\partial^hV},\omega|_{\partial^hV})].\end{aligned}$$

Note that here we take ∂^hV instead of ∂V. Similar to Lemma 3.2.10 and Lemma 3.2.11, we can prove $\partial: C_*^{dR,[0,1]}(\mathcal{L}^kN)\to C_{*-1}^{dR,[0,1]}(\mathcal{L}^kN)$ is well-defined and $\partial^2=0$, so $(C_*^{dR,[0,1]}(\mathcal{L}^kN),\partial)$ is a chain complex. We omit details of the proof, but point out the following lemma which is needed in the proof.

Lemma 3.3.3. *Suppose X,Y,Z are oriented manifolds with corners, and $f:X\to Y$, $g:Y\to Z$ are corner-stratified submersions. Let $\partial^{h,f}X$ be the horizontal boundary of X with respect to f. Similarly, there are $\partial^{v,f}X$, $\partial^{h,g}Y$, $\partial^{v,g}Y$, $\partial^{h,g\circ f}X$, $\partial^{v,g\circ f}X$. Then*

$$\partial^{h,g\circ f}X=\partial^{h,f}X\coprod(\partial^{v,f}X\cap\partial^{h,g\circ f}X).$$

There is a corner stratified submersion

$$f|_{\partial^{v,f}X\cap\partial^{h,g\circ f}X}:\partial^{v,f}X\cap\partial^{h,g\circ f}X\to\partial^{h,g}Y.$$

For any $\omega\in\Omega_c^*(X)$, there holds

$$(f_!\omega)|_{\partial^{h,g}Y}=(f|_{\partial^{v,f}X\cap\partial^{h,g\circ f}X})_!(\omega|_{\partial^{v,f}X\cap\partial^{h,g\circ f}X}).$$

There is an orientation-reversing diffeomorphism $r:(\partial^{h,g})^2Y\to(\partial^{h,g})^2Y$ such that

$$r^2=\mathrm{id},\quad g|_{(\partial^{h,g})^2Y}\circ r=g|_{(\partial^{h,g})^2Y},\quad r^*(\eta|_{(\partial^{h,g})^2Y})=\eta|_{(\partial^{h,g})^2Y}\ (\forall\eta\in\Omega^*(Y)).$$

Remark 3.3.4. Concerning psedoholomorphic disks $(D,\partial D)\to(M,L)$, the appearance of ϵ_0 in the definition of $C_*^{dR,[0,1]}(\mathcal{L}^kL)$ corresponds to choosing almost complex structures J_t ($t\in[0,1]$) on M such that $J_t\equiv J_0$ for $t\in[0,\epsilon_0]$ and $J_t\equiv J_1$ for $t\in[1-\epsilon_0,1]$. The same applies when choosing a $[0,1]$-family of Kuranishi structures, CF perturbations, etc.

Similar to $((\tilde{C}^{dR}_{*+\dim N}(\mathcal{L}^k N), \partial))_{k\geq 0}$, there is a structure of a ns cyclic dg operad on $((C^{dR,[0,1]}_{*+\dim N}(\mathcal{L}^k N), \partial))_{k\geq 0}$, with a multiplication $\mu^{[0,1]}$ and a unit $\varepsilon^{[0,1]}$:

- For $k \in \mathbb{Z}_{\geq 1}, k' \in \mathbb{Z}_{\geq 0}$ and $j \in \{1, \dots, k\}$, the partial composition chain map

$$\circ_j : C^{dR,[0,1]}_{l+\dim N}(\mathcal{L}^k N) \otimes C^{dR,[0,1]}_{l'+\dim N}(\mathcal{L}^{k'} N) \to C^{dR,[0,1]}_{l+l'+\dim N}(\mathcal{L}^{k+k'-1} N)$$

  is defined as follows. For $x = [(V, \varphi, \sigma_+, \sigma_-, \omega)]$, $x' = [(V', \varphi', \sigma'_+, \sigma'_-, \omega')]$, set

$$\varphi_j := (\varphi_{[0,1]}, \mathrm{ev}^k_j \circ \varphi_{\mathcal{L}}), \quad \varphi'_j := (\varphi'_{[0,1]}, \mathrm{ev}^{k'}_0 \circ \varphi'_{\mathcal{L}}).$$

  Then

$$x \circ_j x' := (-1)^{l'(\dim V - \dim N - 1)} [(V \times_{(\varphi_j, \varphi'_0)} V', \tilde{\varphi}, \tilde{\sigma}_+, \tilde{\sigma}_-, \omega \times \omega')],$$

  where $\tilde{\varphi} : V \times_{(\varphi_j, \varphi'_0)} V' \to [0,1] \times \mathcal{L}^{k+k'-1} N$ is defined by

$$\tilde{\varphi}(v, v') := (\varphi_{[0,1]}(v), \mathrm{con}_j(\varphi_{\mathcal{L}}(v), \varphi'_{\mathcal{L}}(v'))),$$

  and $\tilde{\sigma}_+, \tilde{\sigma}_-$ are defined as follows:

$$\begin{aligned}
\rho_+(v, v') &:= \mathrm{pr}_{[1-\epsilon_0, 1]} \circ \sigma_+(v) = \mathrm{pr}_{[1-\epsilon_0, 1]} \circ \sigma'_+(v'), \\
\tilde{\sigma}_+(v, v') &:= (\rho_+(v, v'), (\mathrm{pr}_{V_1} \circ \sigma_+(v), \mathrm{pr}_{V_1} \circ \sigma'_+(v'))), \\
\rho_-(v, v') &:= \mathrm{pr}_{[0, \epsilon_0]} \circ \sigma_-(v) = \mathrm{pr}_{[0, \epsilon_0]} \circ \sigma'_-(v'), \\
\tilde{\sigma}_-(v, v') &:= (\rho_-(v, v'), (\mathrm{pr}_{V_0} \circ \sigma_-(v), \mathrm{pr}_{V_0} \circ \sigma'_-(v'))).
\end{aligned}$$

- For $k \in \mathbb{Z}_{\geq 0}$, $\tau_k : C^{dR,[0,1]}_{*+\dim N}(\mathcal{L}^k N) \to C^{dR,[0,1]}_{*+\dim N}(\mathcal{L}^k N)$ is induced by $\mathrm{id}_{[0,1]} \times (\tau_k)_{\mathcal{L}^k N}$.

- $\mu^{[0,1]} \in C^{dR,[0,1]}_{\dim N}(\mathcal{L}^2 N)$, $\varepsilon^{[0,1]} \in C^{dR,[0,1]}_{\dim N}(\mathcal{L}^0 N)$ are defined by

$$\begin{aligned}
\mu^{[0,1]} &:= [([0,1] \times N, \mathrm{id}_{[0,1]} \times i_2, \mathrm{id}_{[1-\epsilon_0,1] \times N}, \mathrm{id}_{[0,\epsilon_0] \times N}, 1)], \\
\varepsilon^{[0,1]} &:= [([0,1] \times N, \mathrm{id}_{[0,1]} \times i_0, \mathrm{id}_{[1-\epsilon_0,1] \times N}, \mathrm{id}_{[0,\epsilon_0] \times N}, 1)],
\end{aligned}$$

  where $i_k : N \to \mathcal{L}^k N$ $(k = 2, 0)$ is the embedding (3.3.1).

As a consequence, similar to Proposition-Definition 3.3.2, there are natural dg Lie algebra structures $(b,[,])$ on $\mathbf{C}_*^{\mathcal{L},[0,1]}(N)$ and $\mathbf{C}_*^{\mathcal{L},[0,1],\mathrm{cyc}}(N)$, where

$$\begin{aligned}\mathbf{C}_*^{\mathcal{L},[0,1]}(N) &:= \prod_{k\geq 0} C^{dR,[0,1]}_{*+\dim N+k}(\mathcal{L}^kN),\\ \mathbf{C}_*^{\mathcal{L},[0,1],\mathrm{cyc}}(N) &:= \prod_{k\geq 0} C^{dR,[0,1],\mathrm{cyc}}_{*+\dim N+k}(\mathcal{L}^kN),\\ C_*^{dR,[0,1],\mathrm{cyc}}(\mathcal{L}^kN) &:= C_*^{dR,[0,1]}(\mathcal{L}^kN)\cap \ker(1-(-1)^k\tau_k).\end{aligned}$$

For any $k\in\mathbb{Z}_{\geq 0}$, there is a chain map

$$\begin{aligned} i^{(k)} : \tilde{C}_*^{dR}(\mathcal{L}^kN) &\to C_*^{dR,[0,1]}(\mathcal{L}^kN)\\ [(V,\varphi,\omega)] &\mapsto [([0,1]\times V, \mathrm{id}_{[0,1]}\times\varphi, \mathrm{id}_{[1-\epsilon_0,1]\times V}, \mathrm{id}_{[0,\epsilon_0]\times V}, 1\times\omega)],\end{aligned}$$

and there are chain maps

$$\begin{aligned} e_+^{(k)}, e_-^{(k)} : C_*^{dR,[0,1]}(\mathcal{L}^kN) &\to \tilde{C}_*^{dR}(\mathcal{L}^kN)\\ e_+^{(k)} : [(V,\varphi,\sigma_+,\sigma_-,\omega)] &\mapsto [(V_1, \varphi|_{V_1}, \omega|_{V_1})],\\ e_-^{(k)} : [(V,\varphi,\sigma_+,\sigma_-,\omega)] &\mapsto [(V_0, \varphi|_{V_0}, \omega|_{V_0})].\end{aligned}$$

Clearly, $(i^{(k)})_{k\geq 0}$ and $(e_\pm^{(k)})_{k\geq 0}$ are morphisms of ns cyclic dg operads, and

$$i^{(2)}(\mu) = \mu^{[0,1]},\quad i^{(0)}(\varepsilon)=\varepsilon^{[0,1]},\quad e_\pm^{(2)}(\mu^{[0,1]})=\mu,\quad e_\pm^{(0)}(\varepsilon^{[0,1]})=\varepsilon.$$

It follows that $(i^{(k)})_{k\geq 0}, (e_\pm^{(k)})_{k\geq 0}$ induce dg Lie algebra homomorphisms

$$\begin{aligned} i : &\quad \mathbf{C}_*^{\mathcal{L}}(N)\to \mathbf{C}_*^{\mathcal{L},[0,1]}(N),\quad \mathbf{C}_*^{\mathcal{L},\mathrm{cyc}}(N)\to \mathbf{C}_*^{\mathcal{L},\mathrm{cyc},[0,1]}(N),\\ e_\pm : &\quad \mathbf{C}_*^{\mathcal{L},[0,1]}(N)\to \mathbf{C}_*^{\mathcal{L}}(N),\quad \mathbf{C}_*^{\mathcal{L},\mathrm{cyc},[0,1]}(N)\to \mathbf{C}_*^{\mathcal{L},\mathrm{cyc}}(N).\end{aligned}$$

Clearly $e_\pm\circ i = \mathrm{id}_{\mathbf{C}_*^{\mathcal{L}}(N)}$ and $e_\pm\circ i|_{\mathbf{C}_*^{\mathcal{L},\mathrm{cyc}}(N)} = \mathrm{id}_{\mathbf{C}_*^{\mathcal{L},\mathrm{cyc}}(N)}$.

Lemma 3.3.5. $(e_+,e_-) : \mathbf{C}_*^{\mathcal{L},[0,1]}(N)\to \mathbf{C}_*^{\mathcal{L}}(N)\oplus\mathbf{C}_*^{\mathcal{L}}(N)$ *is surjective. The same is true when restricting to* $\mathbf{C}_*^{\mathcal{L},\mathrm{cyc},[0,1]}(N)$.

Proof. It suffices to show $(e_+^{(k)}, e_-^{(k)}) : C_*^{dR,[0,1]}(\mathcal{L}^k N) \to \tilde{C}_*^{dR}(\mathcal{L}^k N) \oplus \tilde{C}_*^{dR}(\mathcal{L}^k N)$ is surjective for all k. Let t be the coordinate function on $[0,1]$, and let $\xi \in C^\infty([0,1],[0,1])$ be a smooth function such that $\xi(t) = 1$ for all $t \in [0, \epsilon_0]$ and $\xi(t) = 0$ for all $t \in [1-\epsilon_0, 1]$. If $x = [(V, \varphi, \omega)]$ is a generator of $\tilde{C}_*^{dR}(\mathcal{L}^k N)$, let

$$\tilde{x} := [([0,1] \times V, \mathrm{id}_{[0,1]} \times \varphi, \mathrm{id}_{[1-\epsilon_0,1]\times V}, \mathrm{id}_{[0,\epsilon_0]\times V}, \xi \times \omega)].$$

Then $e_+(\tilde{x}) = 0$, $e_-(\tilde{x}) = x$. Thus the image of (e_+, e_-) contains $0 \oplus \tilde{C}_*^{dR,\mathrm{cyc}}(\mathcal{L}^k N)$. Similarly, the image of (e_+, e_-) contains $\tilde{C}_*^{dR}(\mathcal{L}^k N) \oplus 0$. This completes the proof for $\mathbf{C}_*^{\mathcal{L},[0,1]}(N)$. The same proof applies to $\mathbf{C}_*^{\mathcal{L},\mathrm{cyc},[0,1]}(N)$. □

Lemma 3.3.6. *$i \circ e_\pm$ is chain homotopic to $\mathrm{id}_{\mathbf{C}_*^{\mathcal{L},[0,1]}(N)}$. The same is true for $\mathbf{C}_*^{\mathcal{L},\mathrm{cyc},[0,1]}(N)$.*

Proof. Since the chain complexes are over $\mathbb{R}$, it suffices to prove $(i \circ e_\pm)_* = \mathrm{id}_{\mathbf{H}_*^{\mathcal{L},[0,1]}(N)}$. Since $e_\pm \circ i = \mathrm{id}_{\mathbf{C}_*^{\mathcal{L}}(N)}$, it suffices to prove $i, e_\pm$ are quasi-isomorphisms, which is true if $i^{(k)}, e_\pm^{(k)}$ are quasi-isomorphisms for each k. By Remark 3.3.7, we can modify the proof of [30, Lemma 4.8] to show that for each k, $i^{(k)} \circ e_\pm^{(k)}$ is chain homotopic to $\mathrm{id}_{C_*^{dR,[0,1]}(\mathcal{L}^k N)}$. This completes the proof for $\mathrm{id}_{\mathbf{C}_*^{\mathcal{L},[0,1]}(N)}$. As for $\mathrm{id}_{\mathbf{C}_*^{\mathcal{L},\mathrm{cyc},[0,1]}(N)}$, it suffices to prove $i|_{\mathbf{C}_*^{\mathcal{L},\mathrm{cyc}}(N)}, e_\pm|_{\mathbf{C}_*^{\mathcal{L},\mathrm{cyc},[0,1]}(N)}$ are quasi-isomorphisms, which follows from the result for $i, e_\pm$ and lemmas in Section 1.2. □

Remark 3.3.7. The way we define $C_*^{dR,[0,1]}(\mathcal{L}^k N)$ is equivalent to Irie's ([30, Section 4.4]), modulo that we have included manifolds with corners into the definition of de Rham chains. Irie's definition is to consider $(U, \psi, \tau_+, \tau_-)$ where $\psi : U \to \mathbb{R} \times \mathcal{L}^k N$, $\tau_+ : U_{[1,\infty)} \cong [1,\infty) \times U_1$, $\tau_- : U_{(-\infty,0]} \cong (-\infty, 0] \times U_0$, instead of $(V, \varphi, \sigma_+, \sigma_-)$. Let us denote $C_*^{dR,[0,1]}(\mathcal{L}^k N)$ by C_* and denote the chain complex defined by Irie (but in our manifolds-with-corners setting) by C'_*. The advantage of Irie's C'_* is that one does not need to distinguish between horizontal boundaries and vertical boundaries when taking the boundary operator, and it is easy to construct corner stratified submersion to $\mathbb{R} \times N$ (compared to $[0,1] \times N$). The advantage of our C_* is that it fits with geometric constructions in a straightforward way, as we will see in later sections. In the following, we show C_*, C'_* are chain homotopy equivalent.

(i) Define a chain map $f : C_* \to C'_*$ as follows. Let $(V, \varphi, \sigma_+, \sigma_-) \in \mathscr{P}^{[0,1]}_{\epsilon_0}(\mathcal{L}^k N)$ and $\omega \in \mathscr{A}_{\epsilon_0}(V, \varphi, \sigma_+, \sigma_-)$. We can glue $[1,\infty) \times V_1$ to $[1-\epsilon_0, 1] \times V_1 \subset V$ in the obvious way, and extend ω to $[1,\infty) \times V_1$ as $1 \times \omega|_{V_1}$. Similarly we glue $(-\infty, 0] \times V_1$ to V and extend ω. Let $U := ((-\infty, 0] \times V_1) \cup V \cup ([1,\infty) \times V_1)$, and define $\psi = (\psi_{\mathbb{R}}, \psi_{\mathcal{L}}) : U \to \mathbb{R} \times \mathcal{L}^k N$ by

$$\psi|_{(-\infty,0]\times V_1}(t,x) = (t, \varphi_{\mathcal{L}}(x)), \quad \psi|_{[1,\infty)\times V_1}(t,x) = (t, \varphi_{\mathcal{L}}(x)), \quad \psi|_V = \varphi.$$

Then $(\psi_{\mathbb{R}}, \mathrm{ev}^k_i \circ \psi_{\mathcal{L}})$ is a corner stratified submersion. There are natural diffeomorphisms

$$\tau_+ : U_{[1,\infty)} := \varphi_{\mathbb{R}}^{-1}([1,\infty)) \xrightarrow{\cong} [1,\infty) \times U_1, \quad \tau_- : U_{(-\infty,0]} \xrightarrow{\cong} (-\infty, 0] \times U_0$$

such that ψ and $\tilde{\omega}$ (the extension of ω) are of product forms in the obvious way on $U_{[1,\infty)}$, $U_{(-\infty,0]}$ (via τ_+, τ_-). Then $f([(V, \varphi, \sigma_+, \sigma_-, \omega)]) := [(U, \psi, \tau_+, \tau_-, \tilde{\omega})]$.

(ii) Define a chain map $g : C'_* \to C_*$ as follows. Let U be a manifold with corners, $\psi : U \to \mathbb{R} \times \mathcal{L}^k N$ be a smooth map such that $(\psi_{\mathbb{R}}, \mathrm{ev}^k_i \circ \psi_{\mathcal{L}}) : U \to \mathbb{R} \times N$ $(0 \le i \le k)$ is a corner stratified submersion, and $\tau_+ : U_{[1,\infty)} \xrightarrow{\cong} [1,\infty) \times U_1$, $\tau_- : U_{(-\infty,0]} \xrightarrow{\cong} (-\infty, 0] \times U_0$, such that $\psi|_{U_{[1,\infty)}}$, $\psi|_{U_{(-\infty,0]}}$ are of product form in the above sense. Suppose $\omega \in \Omega^*(U)$ satisfies that $\omega|_{U_{[0,1]}}$ is compactly supported, and $\omega|_{U_{[1,\infty)}}, \omega|_{U_{(-\infty,0]}}$ are of product form $1 \times \omega|_{U_1}, 1 \times \omega|_{U_0}$ via τ_+, τ_-, respectively. Fix $\xi \in C^\infty(\mathbb{R}, \mathbb{R})$ which satisfies

$$\xi(t) = 1 \ (t \le \epsilon_0), \quad \xi(t) = -1 \ (t \ge 1 - \epsilon_0), \quad |\xi'(t)| < \frac{1}{\epsilon_0} \ (0 \le t \le 1).$$

Note that ξ exists because $\epsilon_0 \in (0, \frac{1}{4})$. Then the map $\lambda : \mathbb{R} \to \mathbb{R}$, $t \mapsto t + \xi(t)\epsilon_0$ is a diffeomorphism which satisfies

$$\lambda(t) = t + \epsilon_0 \ (t \le \epsilon_0), \quad \lambda(t) = t - \epsilon_0 \ (t \ge 1 - \epsilon_0).$$

We can define $(V, \varphi, \sigma_+, \sigma_-) \in \mathscr{P}^{[0,1]}_{\epsilon_0}(\mathcal{L}^k N)$ and $\bar{\omega} \in \mathscr{A}_{\epsilon_0}(V, \varphi, \sigma_+, \sigma_-)$ by

$$
\begin{aligned}
V &:= U_{[-\epsilon_0, 1+\epsilon_0]} \\
\varphi &:= (\lambda \circ \psi_{\mathbb{R}}|_{U_{[-\epsilon_0,1+\epsilon_0]}}, \psi_{\mathcal{L}}|_{U_{[-\epsilon_0,1+\epsilon_0]}}) \\
\sigma_+ &:= (\varphi_{[0,1]}|_{U_{[1,1+\epsilon_0]}}, \mathrm{pr}_{U_1} \circ \tau_+|_{U_{[1,1+\epsilon_0]}}) \\
\sigma_- &:= (\varphi_{[0,1]}|_{U_{[-\epsilon_0,0]}}, \mathrm{pr}_{U_0} \circ \tau_-|_{U_{[-\epsilon_0,0]}}) \\
\bar{\omega} &:= \omega|_{U_{[-\epsilon_0,1+\epsilon_0]}}.
\end{aligned}
$$

Then $g([(U, \psi, \tau_+, \tau_-, \omega)]) := [(V, \phi, \sigma_+, \sigma_-, \bar{\omega})]$.

Let us show f, g are chain homotopy inverse to each other. Fix $\chi \in C^\infty([0,1],[0,1])$ which satisfies $\chi(s) = 0$ for s near 0 and $\chi(s) = 1$ for s near 1. Define a smooth map

$$\mu : [0,1] \times \mathbb{R} \to \mathbb{R}, \quad (s,t) \mapsto t + \chi(s)\xi(t)\epsilon_0.$$

Then $\mu_0 = \mathrm{id}_{\mathbb{R}}$, $\mu_1 = \lambda$, and for any $s \in [0,1]$, $\mu_s := \mu(s, \cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism which satisfies

$$\mu_s(t) = t + \chi(s)\epsilon_0 \ (t \le \epsilon_0), \quad \mu_s(t) = t - \chi(s)\epsilon_0 \ (t \ge 1 - \epsilon_0).$$

If $(V, \varphi, \sigma_+, \sigma_-) \in \mathscr{P}^{[0,1]}_{\epsilon_0}(\mathcal{L}^k N)$ and $\omega \in \mathscr{A}_{\epsilon_0}(V, \varphi, \sigma_+, \sigma_-)$, denote $[(V^1, \varphi^1, \sigma^1_+, \sigma^1_-, \omega^1)] = (g \circ f)([(V, \varphi, \sigma_+, \sigma_-, \omega))]) \in C_*$, where $V^1 = ([-\epsilon_0, 0] \times V_0) \cup V \cup ([0, \epsilon_0] \times V_1)$. For any $s \in [0,1]$, we can define $(V^s, \varphi^s, \sigma^s_+, \sigma^s_-) \in \mathscr{P}^{[0,1]}_{\epsilon_0}(\mathcal{L}^k N)$ and $\omega^s \in \mathscr{A}_{\epsilon_0}(V^s, \varphi^s, \sigma^s_+, \sigma^s_-)$ in a similar way, where $V^s = ([-\chi(s)\epsilon_0, 0] \times V_0) \cup V \cup ([0, \chi(s)\epsilon_0] \times V_1)$ and φ^s is defined using μ_s. Let $\overline{V} := \cup_{s \in [0,1]} V^s$, $\overline{\varphi} : \overline{V} \to [0,1] \times \mathcal{L}^k N$, $\overline{\varphi}(s,x) := \varphi^s(x)$, and define $\overline{\sigma_+}$, $\overline{\sigma_-}$, $\overline{\omega}$ in the obvious way, so that $[(\overline{V}, \overline{\varphi}, \overline{\sigma_+}, \overline{\sigma_-}, \overline{\omega})] \in C_{*+1}$. Define a linear map $h : C_* \to C_{*+1}$ by $h([(V, \varphi, \sigma_+, \sigma_-, \omega)]) := [(\overline{V}, \overline{\varphi}, \overline{\sigma_+}, \overline{\sigma_-}, \overline{\omega})]$. Then we can check $\partial \circ h + h \circ \partial = g \circ f - \mathrm{id}_{C_*}$. Similarly we can show $f \circ g \simeq \mathrm{id}_{C'_*}$. □

By Lemma 3.3.5 and Lemma 3.3.6, $\mathbf{C}^{\mathcal{L},[0,1]}_*(N)$ is a dg Lie algebra model of $[0,1] \times \mathbf{C}^{\mathcal{L}}_*(N)$, and $\mathbf{C}^{\mathcal{L},\mathrm{cyc},[0,1]}_*(N)$ is a dg Lie algebra model of $[0,1] \times \mathbf{C}^{\mathcal{L},\mathrm{cyc}}_*(N)$, in the following sense.

Definition 3.3.8. Let $\left(\overline{B} = \{\overline{B}^i\}_{i\in\mathbb{Z}}, d, \{,\}\right)$ be a dg Lie algebra over $\mathbb{R}$. A *dg Lie algebra model of* $[0,1]\times\overline{B}$ is a dg Lie algebra $\widetilde{\overline{B}}$ together with dg Lie algebra homomorphisms

$$\overline{\mathtt{Incl}} : \overline{B} \to \widetilde{\overline{B}}, \quad \overline{\mathtt{Eval}}_{t=0} : \widetilde{\overline{B}} \to \overline{B}, \quad \overline{\mathtt{Eval}}_{t=1} : \widetilde{\overline{B}} \to \overline{B}$$

satisfying the following conditions:

- $\overline{\mathtt{Eval}}_{t=0} \circ \overline{\mathtt{Incl}} = \overline{\mathtt{Eval}}_{t=1} \circ \overline{\mathtt{Incl}} = \mathrm{id}_{\overline{B}}$;
- $\overline{\mathtt{Incl}}$, $\overline{\mathtt{Eval}}_{t=0}$, $\overline{\mathtt{Eval}}_{t=1}$ are (co)chain homotopy equivalences;
- $\overline{\mathtt{Eval}}_{t=0} \oplus \overline{\mathtt{Eval}}_{t=1} : \widetilde{\overline{B}} \to \overline{B} \oplus \overline{B}$ is surjective.

3.3.3 Iterated integrals of differential forms parametrized by [0,1]

There is an iterated integral map

$$J_k : \tilde{C}^{dR}_{*+\dim N}(\mathcal{L}^k N) \to \mathrm{Hom}^{-*}(\Omega(N)^{\otimes k}, \Omega(N)) \tag{3.3.3}$$

$$J_k([(V,\varphi,\omega)])(\eta_1\otimes\cdots\otimes\eta_k) := (-1)^{(\dim V - \dim N)(|\eta_1|+\cdots+|\eta_k|)}(\varphi_0)_!(\omega\wedge\varphi_1^*\eta_1\wedge\cdots\wedge\varphi_k^*\eta_k),$$

which is defined in the same way as the map $I_k : C^{dR}_{*+\dim N}(\mathscr{L}^N_{k+1,\mathrm{reg}}) \to \mathrm{Hom}^{-*}(\Omega(N)^{\otimes k}, \Omega(N))$ in (1.7.1). Clearly $J_k \circ \Phi_k = I_k$ where Φ_k is the map (3.3.2).

For later purposes, we need to discuss smooth dependence of a linear map $\Omega(N)^{\otimes k} \to \Omega(N)$. Let us make the following definition, which is motivated from [20, Definition 21.25].

Definition 3.3.9 (Smoothly extendable multi-linear maps)**.**

(i) We say $\psi_k \in \mathrm{Hom}(\Omega(N)^{\otimes k}, \Omega(N))$ is *smoothly extendable*, if for any smooth manifold S and smooth sections $\eta_1,\dots,\eta_k \in \Gamma(S\times N, \mathrm{pr}_N^*\Omega_N)$,

$$(s,x) \mapsto \psi_k(i_s^*\eta_1\otimes\cdots\otimes i_s^*\eta_k)(x), \qquad (s,x)\in S\times N$$

is also an element in $\Gamma(S\times N, \mathrm{pr}_N^*\Omega_N)$. Here $\mathrm{pr}_N : S\times N\to N$ is the projection onto N, and $i_s : N \to S\times N$ is the embedding $N \cong \{s\}\times N \hookrightarrow S\times N$.

(ii) We say $\varphi_k \in \mathrm{Hom}(\Omega(N)^{\otimes k}, \Omega([0,1]\times N))$ is *smoothly extendable*, if for any smooth manifold S and smooth sections $\eta_1, \dots, \eta_k \in \Gamma(S\times N, \mathrm{pr}_N^*\Omega_N)$,

$$(s,t,x) \mapsto \varphi_k(i_s^*\eta_1 \otimes \cdots \otimes i_s^*\eta_k)(t,x), \qquad (s,t,x) \in S\times[0,1]\times N$$

is an element in $\Gamma(S\times[0,1]\times N, \mathrm{pr}_{[0,1]\times N}^*\Omega_{[0,1]\times N})$. Here $\mathrm{pr}_{[0,1]\times N} : S\times[0,1]\times N \to [0,1]\times N$ and $i_s : [0,1]\times N \hookrightarrow S\times[0,1]\times N$ are defined similar to (i).

The space of smoothly extendable linear maps $\Omega(N)^{\otimes k} \to \Omega(N)$ (resp. $\to \Omega([0,1]\times N)$) is denoted by $\mathrm{Hom}_\diamond(\Omega(N)^{\otimes k}, \Omega(N))$ (resp. $\mathrm{Hom}_\diamond(\Omega(N)^{\otimes k}, \Omega([0,1]\times N))$).

Remark 3.3.10. It is clear that whether S has boundaries (corners) does not influence the meaning of smooth extendability in Definition 3.3.9. Let us list some simple facts about smooth extendability that will be useful later:

(i) For $\psi_k \in \mathrm{Hom}(\Omega(N)^{\otimes k}, \Omega(N))$, ψ_k is smoothly extendable if and only if $\mathrm{pr}_N^* \circ \psi_k : \Omega(N)^{\otimes k} \to \Omega([0,1]\times N)$ is smoothly extendable.

(ii) If $\varphi_k \in \mathrm{Hom}_\diamond(\Omega(N)^{\otimes k}, \Omega([0,1]\times N))$, then $i_t^* \circ \varphi_k \in \mathrm{Hom}_\diamond(\Omega(N)^{\otimes k}, \Omega(N))$ ($\forall t \in [0,1]$).

(iii) If $\varphi_k \in \mathrm{Hom}_\diamond(\Omega(N)^{\otimes k}, \Omega([0,1]\times N))$, then for any $\eta_1, \dots, \eta_k \in \Omega([0,1]\times N)$,

$$\Big(\ (t,x) \mapsto \varphi_k(i_t^*\eta_1 \otimes \cdots \otimes i_t^*\eta_k)(t,x)\ \Big) \in \Omega([0,1]\times N).$$

Indeed, setting $S=[0,1]$, $\varphi_k(i_t^*\eta_1 \otimes \cdots \otimes i_t^*\eta_k)(t,x)$ is C^∞ in (t,x) because it is the pull-back of $\varphi_k(i_s^*\eta_1 \otimes \cdots \otimes i_s^*\eta_k)(t,x)$ via the smooth diagonal map

$$[0,1]\times N \to [0,1]^2 \times N, \quad (t,x) \mapsto (t,t,x).$$

This fact is the main reason why we introduce such a notion of smooth extendability.

(iv) The exterior derivative d and the wedge product $\wedge$ are smoothly extendable, and $\mathcal{E}nd^\diamond_{\Omega(N)} := (\mathrm{Hom}_\diamond(\Omega(N)^{\otimes k}, \Omega(N)))_{k\geq 0}$ is a dg sub-operad of $\mathcal{E}nd_{\Omega(N)}$ (Example 1.5.3(i), Example 1.5.9). Thus $\mathrm{CH}^*_\diamond(\Omega(N), \Omega(N))$ is a dg Lie subalgebra of $\mathrm{CH}^*(\Omega(N), \Omega(N))$.

Example 3.3.11. Suppose V is a smooth manifold with corners, $\omega \in \Omega_c(V)$, $f_1, \ldots, f_k : V \to N$ are smooth maps, and $(g, f_0) : V \to [0,1] \times N$ is a corner stratified submersion. Define $\psi_k : \Omega(N)^{\otimes k} \to \Omega(N)$, $\varphi_k : \Omega(N)^{\otimes k} \to \Omega([0,1] \times N)$ by

$$\varphi_k(\eta_1 \otimes \cdots \otimes \eta_k) := (g, f_0)_!(\omega \wedge f_1^*\eta_1 \wedge \cdots \wedge f_k^*\eta_k),$$
$$\psi_k(\eta_1 \otimes \cdots \otimes \eta_k) := (f_0)_!(\omega \wedge f_1^*\eta_1 \wedge \cdots \wedge f_k^*\eta_k).$$

By looking at local expressions, it is easy to see φ_k, ψ_k are smoothly extendable.

By Lemma 1.7.5 and Lemma 3.3.1, the iterated integral maps $(J_k)_{k \geq 0}$ (3.3.3) induce dg Lie algebra homomorphisms

$$\begin{aligned} J = (J_k)_{k\geq 0} : \quad & \mathbf{C}^{\mathcal{L}}_*(N) \to \mathrm{CH}^{-*}(\Omega(N), \Omega(N)), \\ & \mathbf{C}^{\mathcal{L},\mathrm{cyc}}_*(N) \to \mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(N), \Omega(N)), \end{aligned}$$

where $\mathrm{CH}^*_{\mathrm{cyc}}(\Omega(N), \Omega(N))$ is the weakly cyclic subcomplex of $\mathrm{CH}^*(\Omega(N), \Omega(N))$ with respect to Poincaré pairing (Section 1.7). (Here and hereafter, we write $\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(N), \Omega(N))$ in place of $\Theta^{-1}(\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(N), \Omega(N)^\vee[-\dim N]))$ in Section 1.7.) By Example 3.3.11, we have

$$J(\mathbf{C}^{\mathcal{L}}_*(N)) \subset \mathrm{CH}^{-*}_\diamond(\Omega(N), \Omega(N)), \quad J(\mathbf{C}^{\mathcal{L},\mathrm{cyc}}_*(N)) \subset \mathrm{CH}^{-*}_{\mathrm{cyc},\diamond}(\Omega(N), \Omega(N)).$$

Similarly, we can define a linear map (using the same formula as J 3.3.3 except for signs)

$$\tilde{J} : \mathbf{C}^{\mathcal{L},[0,1]}_*(N) \to \prod_{k \geq 0} \mathrm{Hom}^{-*-k}_\diamond(\Omega(N)^{\otimes k}, \Omega([0,1] \times N))$$

which restricts to a linear map $\mathbf{C}^{\mathcal{L},\mathrm{cyc},[0,1]}_*(N) \to \prod_{k\geq 0} \mathrm{Hom}^{-*-k}_{\mathrm{cyc},\diamond}(\Omega(N)^{\otimes k}, \Omega([0,1] \times N))$.

In Section 3.5, we will see that $\prod_{k \geq 0} \mathrm{Hom}^{*-k}_\diamond(\Omega(N)^{\otimes k}, \Omega([0,1] \times N))$ carries a natural dg Lie algebra structure, and it is a dg Lie algebra model of $[0,1] \times \mathrm{CH}^*_\diamond(\Omega(N), \Omega(N))$. The same holds for the (weakly) cyclic invariant subcomplexes.

3.4 (Cyclic) A_∞ algebras and (cyclic) A_∞ deformations

Deformation problems (in characteristic zero) are in principle governed by dg Lie algebras (or L_∞ algebras) via solutions of Maurer-Cartan equation (modulo gauge equivalence), see for example [15]. Following this philosophy, in this subsection we discuss (cyclic) filtered A_∞ deformations of the de Rham dg algebra $\Omega(X)$ over the Novikov ring, and relate it to iterated integrals of differential forms on chains on loop spaces.

We use the sign convention of Fukaya for A_∞ algebras, namely transited from (de)suspension (Appendix 1.8). ([18] does not use (1.8.1) to view dg algebras as A_∞ algebras, but [20] does.)

3.4.1 Coderivations on the tensor coalgebra

We discuss $\mathbb{R}$-vector spaces below, but everything extends to modules over a commutative ring with unity. First recall that a graded vector space $D = \{D^i\}_{i\in\mathbb{Z}}$ is called a graded coalgebra if there is a degree 0 linear map (called comultiplication) $\Delta : D \to D \otimes D$ which is coassociative, namely $(\mathrm{id}_D \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_D) \circ \Delta$. Let D be such a graded coalgebra. Then one can define $\Delta^n : D \to D^{\otimes(n+1)}$ $(n \in \mathbb{Z}_{\geq 0})$ recursively by

$$\Delta^0 := \mathrm{id}_D,\ \Delta^n := (\Delta \otimes \mathrm{id}_D) \circ \Delta^{n-1}.$$

A linear map $\epsilon : D^0 \to \mathbb{R}$ is called a counit if $(\epsilon \otimes \mathrm{id}_D) \circ \Delta = \mathrm{id}_D = (\mathrm{id}_D \otimes \epsilon) \circ \Delta$, and a counit is unique if it exists. A coderivation of degree d is a linear map $\Phi : D^* \to D^{*+d}$ such that $\Delta \circ \Phi = (\Phi \otimes \mathrm{id}_D + \mathrm{id}_D \otimes \Phi) \circ \Delta$, where Koszul sign rule is applied in $\Phi \otimes \mathrm{id}_D$ and $\mathrm{id}_D \otimes \Phi$. The space of coderivations, $\mathrm{Coder}(D)$, is a graded Lie algebra under the Lie bracket $[,]$ defined by

$$[\Phi, \Phi'] := \Phi \circ \Phi' - (-1)^{|\Phi||\Phi'|}\Phi' \circ \Phi, \qquad \Phi, \Phi' \in \mathrm{Coder}(D). \tag{3.4.1}$$

Let $V = \{V^i\}_{i\in\mathbb{Z}}$ be a graded vector space. Consider the tensor algebra of V:

$$T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}, \quad T_+(V) := \bigoplus_{k=1}^{\infty} V^{\otimes k},$$

where the grading is given on each component by $|v_1 \otimes \cdots \otimes v_k| = |v_1| + \cdots + |v_k|$. There is a graded coalgebra structure on $T(V)$ defined by

$$\Delta(v_1 \otimes \cdots \otimes v_n) := \sum_{k=0}^{n} (v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_n), \quad \epsilon : V^0 \twoheadrightarrow V^{\otimes 0} \xrightarrow{\cong} \mathbb{R},$$

which also restricts to a graded coalgebra structure (without counit) on $T_+(V)$.

There is a natural correspondence between $\mathrm{Hom}(T(V), V)$ and $\mathrm{Coder}(T(V))$ as follows.

- For $k \geq 0$, a graded linear map $\varphi_k : V^{\otimes k} \to V$ can be extended to a coderivation $\hat{\varphi}_k$:
  - If $n \geq k$, $\hat{\varphi}_k(v_1 \otimes \cdots \otimes v_n) :=$
$$\sum_{i=1}^{n-k+1} (-1)^{|\varphi_k|(|v_1|+\cdots+|v_{i-1}|)} v_1 \otimes \cdots \otimes v_{i-1} \otimes \varphi_k(v_i \otimes \cdots \otimes v_{i+k-1}) \otimes v_{i+k} \otimes \cdots \otimes v_n.$$
    Here when $k = 0$, $\varphi_k(v_i \otimes \cdots \otimes v_{i+k-1})$ simply means $\varphi_0(1)$.
  - If $n < k$, $\hat{\varphi}_k(v_1 \otimes \cdots \otimes v_n) := 0$.
- For any coderivation $\hat{\varphi}$ on $T(V)$, denote its $\mathrm{Hom}(V^{\otimes k}, V)$ component by φ_k, then $\hat{\varphi}$ can be recovered as $\hat{\varphi} = \sum_{k \geq 0} \hat{\varphi}_k$. This is a finite sum when evaluating on $T(V)$.

By restricting to $k \geq 1$, one also identifies $\mathrm{Hom}(T_+(V), V)$ with $\mathrm{Coder}(T_+(V))$.

If there is a fixed graded skew-symmetric bilinear form $\langle , \rangle$ on V, let $\mathrm{Hom}_{\mathrm{cyc}}(V^{\otimes k}, V)$ be the subspace of $\mathrm{Hom}(V^{\otimes k}, V)$ consisting of $\varphi_k : V^{\otimes k} \to V$ such that

$$\langle \varphi_k(v_1 \otimes \cdots \otimes \cdots \otimes v_0), v_{k+1} \rangle = (-1)^{|v_0|(|v_1|+\cdots+|v_k|)} \langle \varphi_k(v_0 \otimes v_1 \otimes \cdots \otimes \cdots \otimes v_{k-1}), v_k \rangle.$$

We say $\varphi_k \in \mathrm{Hom}(V^{\otimes k}, V)$ is *cyclic* if it is in $\mathrm{Hom}_{\mathrm{cyc}}(V^{\otimes k}, V)$, and denote the subspace of cyclic linear maps in $\mathrm{Hom}(T(V), V) = \mathrm{Coder}(T(V))$ by $\mathrm{Hom}_{\mathrm{cyc}}(T(V), V) = \mathrm{Coder}_{\mathrm{cyc}}(T(V))$. It is clear that $\mathrm{Coder}_{\mathrm{cyc}}(T(V))$ is closed under the bracket $[,]$ (3.4.1) on $\mathrm{Coder}(T(V))$.

Let $C = \{C^i\}_{i \in \mathbb{Z}}$ be a graded vector space, recall that $C[1] = \{C[1]^i\}_{i \in \mathbb{Z}}$, $C[1]^i := C^{i+1}$ (see Appendix 1.8). For simplicity we still denote $sx \in C[1]$ by x, and to avoid ambiguity, we shall write $|x|$ for the original degree of $x \in C$, and write $|x|'$ for its degree considered in $C[1]$, so that $|x|' = |x| - 1$. The degree of $\varphi_k \in \mathrm{Hom}(C[1]^{\otimes k}, C[1])$ is then $|\varphi_k|' = |\varphi_k| + k - 1$. If $\langle , \rangle$ is a graded symmetric bilinear form on C, define

$$\langle x, y \rangle' := (-1)^{|y|} \langle x, y \rangle.$$

Then $\langle x, y \rangle'$ is graded skew-symmetric if degrees of x, y are considered in $C[1]$.

Definition 3.4.1 (A_∞ algebras and cyclic A_∞ algebras)**.**

(i) A structure of A_∞ algebra on C is a sequence of linear maps $\{\mathfrak{m}_k : C[1]^{\otimes k} \to C[1]\}_{k\geq 0}$ of degree 1 such that the coderivation $\hat{\mathfrak{m}} = \sum_{k\geq 0} \hat{\mathfrak{m}}_k$ on $(T(C[1]), \Delta)$ is a codifferential, i.e. $\hat{\mathfrak{m}}\hat{\mathfrak{m}} = 0$.

(ii) If a graded symmetric bilinear form $\langle , \rangle$ on C is given, a structure of cyclic A_∞ algebra on $(C, \langle , \rangle)$ is a sequence of linear maps $\{\mathfrak{m}_k : C[1]^{\otimes k} \to C[1]\}_{k\geq 0}$ of degree 1 such that $(C, \{\mathfrak{m}_k\}_{k\geq 0})$ is an A_∞ algebra and $\mathfrak{m}_k$ is cyclic with respect to $\langle , \rangle'$.

Remark 3.4.2. In Definition 3.4.1(ii), one usually requires $\langle , \rangle$ to be nondegenerate. For example, nondegeneracy of $\langle , \rangle$ is needed in the construction of the canonical model of cyclic filtered A_∞ algebras in [17].

The condition $\hat{\mathfrak{m}}\hat{\mathfrak{m}} = 0$ (A_∞ relation) is equivalent to $\mathfrak{m} \circ \hat{\mathfrak{m}} = (\hat{\mathfrak{m}} \circ \hat{\mathfrak{m}})|_{T(C[1]) \to C[1]} = 0$, which explicitly says that for each $n \geq 0$ and $x_1, \ldots, x_n \in C[1]$,

$$\sum_{\substack{k_1+k_2=n+1 \\ 1\leq i\leq k_1}} (-1)^{|x_1|'+\cdots+|x_{i-1}|'} \mathfrak{m}_{k_1}(x_1 \otimes \cdots \otimes x_{i-1} \otimes \mathfrak{m}_{k_2}(x_i \otimes \cdots \otimes x_{i+k_2-1}) \otimes \cdots \otimes x_n) = 0.$$

We say an A_∞ algebra $(C, (\mathfrak{m}_k)_{k\geq 0})$ is *curved* if $\mathfrak{m}_0 \neq 0$, and is *strict* (or *uncurved*) if $\mathfrak{m}_0 = 0$. In case C is strict, A_∞ relation implies $\mathfrak{m}_1\mathfrak{m}_1 = 0$, so $H(C, \mathfrak{m}_1)$ is defined.

Example 3.4.3. Let $(A, d, \cdot)$ be a dg algebra. In view of (1.8.1), let us define $(\mathfrak{m}_k)_{k\geq 0}$ by

$$\mathfrak{m}_1(a) = da, \quad \mathfrak{m}_2(a \otimes b) = (-1)^{|a|} a \cdot b,$$

and $\mathfrak{m}_k = 0$ for other k. Then it is easy to see $(A, \mathfrak{m}_1, \mathfrak{m}_2)$ becomes an A_∞ algebra. If $\langle , \rangle$ is a symmetric bilinear form on A, then $(A, \mathfrak{m}_1, \mathfrak{m}_2, \langle , \rangle)$ is a cyclic A_∞ algebra if and only if $(A, d, \cdot, \langle , \rangle)$ is a dg Frobenius algebra (1.5.9). In particular, if N is a closed oriented manifold, then $(\Omega^*(N), d, \wedge, \langle , \rangle_N)$ is a cyclic A_∞ algebra, where $\langle , \rangle_N$ is the Poincaré pairing:

$$\langle \omega, \eta \rangle_N := \int_N \omega \wedge \eta, \quad \omega, \eta \in \Omega^*(N).$$

Recall that for every graded Lie algebra $(B, [,])$, each element $x \in B$ gives rise to a derivation $\mathrm{ad}_x := [x, \cdot]$ on B. If x has degree 1 and $[x, x] = 0$, then $\mathrm{ad}_x \mathrm{ad}_x = \frac{1}{2}[\mathrm{ad}_x, \mathrm{ad}_x] = \frac{1}{2}\mathrm{ad}_{[x,x]} = 0$, and $(B, \mathrm{ad}_x, [,])$ becomes a dg Lie algebra. If $(C, (\mathfrak{m}_k)_{k\geq 0})$ is an A_∞ algebra, then $\mathrm{Coder}(T(C[1]))$ is a graded Lie algebra, $\hat{\mathfrak{m}}$ has degree 1 and $[\hat{\mathfrak{m}}, \hat{\mathfrak{m}}] = 0$, so $(\mathrm{Coder}(T(C[1])), \mathrm{ad}_{\hat{\mathfrak{m}}}, [,])$ is a dg Lie algebra. Similarly, if $(C, (\mathfrak{m}_k)_{k\geq 0}, \langle,\rangle)$ is a cyclic A_∞ algebra, then $(\mathrm{Coder}_{\mathrm{cyc}}(T(C[1])), \mathrm{ad}_{\hat{\mathfrak{m}}}, [,])$ is a dg Lie algebra.

Definition 3.4.4 (Hochschild cochain complex of (cyclic) A_∞ algebras)**.**

(i) If $(C, \mathfrak{m})$ is an A_∞ algebra, we call $(\mathrm{Coder}(T(C[1])), \mathrm{ad}_{\hat{\mathfrak{m}}}, [,])$ the Hochschild cochain complex of $(C, \mathfrak{m})$, which is a dg Lie algebra.

(ii) If $(C, \mathfrak{m}, \langle,\rangle)$ is a cyclic A_∞ algebra, we call $(\mathrm{Coder}(T(C[1])), \mathrm{ad}_{\hat{\mathfrak{m}}}, [,])$ the cyclic Hochschild cochain complex of $(C, \mathfrak{m}, \langle,\rangle)$, which is a dg Lie algebra.

Example 3.4.5. Let $(A, \mathfrak{m}_1, \mathfrak{m}_2)$ be a dg algebra as in Example 3.4.3. There are identifications

$$\mathrm{Coder}(T(A[1])) \cong \mathrm{Hom}(T(A[1]), A[1]) = \prod_{k\geq 0} \mathrm{Hom}(A[1]^{\otimes k}, A[1]) \cong \prod_{k\geq 0} \mathrm{Hom}(A^{\otimes k}[k], A[1])$$

$$\hat{\varphi} = \sum_{k\geq 0} \hat{\varphi}_k \quad \longleftrightarrow \quad \varphi = (\varphi_k)_{k\geq 0} \quad \longleftrightarrow \quad \text{sign change (1.8.1).}$$

For homogeneous $\hat{\varphi}, \hat{\psi} \in \mathrm{Coder}(T(A[1]))$,

$$\begin{aligned}&(\varphi \circ \hat{\psi})(a_1 \otimes \cdots \otimes a_k)\\ = &\sum_{\substack{l+m=k+1\\ 1\leq i\leq l}} (-1)^{|\psi_m|'(|a_1|'+\cdots+|a_{i-1}|')} \varphi_l(a_1 \otimes \cdots \otimes \psi_m(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes \cdots \otimes a_k),\end{aligned}$$

which up to sign is the same as operad composition in $\mathcal{E}nd_A$ (Example 1.5.3). Under sign change (1.8.1), the commutator bracket $[,]$ on $\mathrm{Coder}(T(A[1]))$ coincides with Gerstenhaber bracket (1.5.7b) on $\mathrm{CH}(A, A)$. Moreover, $\mathrm{ad}_{\hat{\mathfrak{m}}} := [\hat{\mathfrak{m}}, \cdot] = [\hat{\mathfrak{m}}_1 + \hat{\mathfrak{m}}_2, \cdot]$ coincides with $d - \delta$. (Here it is $d - \delta$ instead of $d + \delta$ because we use the sign convention of Irie [29] for Hochschild differntial. See [29, Section 2.5.4] and the proof of Proposition 1.5.6: the operadic Maurer-Cartan element $\zeta = (\zeta_k)_{k\geq 0}$ is taken as $\zeta_2 = -\mu$ and $\zeta_k = 0$ $(k \neq 2)$. If ζ is taken as $\zeta_2 = \mu$ instead, then $\mathrm{ad}_{\hat{\mathfrak{m}}}$ corresponds to $d + \delta$).

3.4.2 Deformation over the Novikov ring

Let R be a commutative ring with unity, the *universal Novikov ring* over R defined in [18] is

$$\Lambda^R_{0,\mathrm{nov}} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i} \;\middle|\; a_i \in R,\ \lambda_i \in \mathbb{R}_{\geq 0},\ n_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty \right\}, \tag{3.4.2}$$

where T and e are formal generators such that $\deg T = 0$, $\deg e = 2$. Degree assumption means that for any graded $\Lambda^R_{0,\mathrm{nov}}$-module $C = \{C^m\}_{m\in\mathbb{Z}}$, it holds that $T^\lambda e^n C^m \subset C^{m+2n}$. Since e has even degree, it does not affect signs appearing in various formulae.

Since we always work with $R = \mathbb{R}$, for simplicity let us write $\Lambda_{0,\mathrm{nov}}$ in place of $\Lambda^{\mathbb{R}}_{0,\mathrm{nov}}$. Similarly define

$$\begin{aligned}
\Lambda_{\mathrm{nov}} &:= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i} \;\middle|\; a_i \in \mathbb{R},\ \lambda_i \in \mathbb{R},\ n_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty \right\}, \\
\Lambda^+_{0,\mathrm{nov}} &:= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i} \;\middle|\; a_i \in \mathbb{R},\ \lambda_i \in \mathbb{R}_{>0},\ n_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty \right\}.
\end{aligned}$$

Then Λ_{nov} is the fraction field of $\Lambda_{0,\mathrm{nov}}$, and $\Lambda^+_{0,\mathrm{nov}} \subset \Lambda_{0,\mathrm{nov}}$ is an ideal. There is a ring isomorphism

$$\Lambda_{0,\mathrm{nov}}/\Lambda^+_{0,\mathrm{nov}} \cong \mathbb{R}[e, e^{-1}] =: R_e,$$

and a splitting of $\mathbb{R}$-algebras $\Lambda_{0,\mathrm{nov}} = \Lambda^+_{0,\mathrm{nov}} \oplus R_e$.

There is a natural decreasing filtration on Λ_{nov} defined by

$$\mathcal{F}^\lambda \Lambda_{\mathrm{nov}} := \left\{ \sum_i a_i T^{\lambda_i} e^{n_i} \in \Lambda_{\mathrm{nov}} \;\middle|\; \lambda_i \geq \lambda \right\}, \quad \lambda \in \mathbb{R},$$

which restricts to $\Lambda_{0,\mathrm{nov}}, \Lambda^+_{0,\mathrm{nov}}$. A *filtered* $\Lambda_{0,\mathrm{nov}}$*-module* is a $\Lambda_{0,\mathrm{nov}}$-module C endowed with a decreasing filtration $\mathcal{F}^\lambda$, called *energy filtration*, such that $\mathcal{F}^\lambda \Lambda_{0,\mathrm{nov}} \cdot \mathcal{F}^{\lambda'} C \subset \mathcal{F}^{\lambda+\lambda'} C$ and that $\bigcap_\lambda \mathcal{F}^\lambda C = \{0\}$ (i.e. we assume the filtration is Hausdorff). Such filtration equips C with a non-Archimedean norm and a metric

$$\|x\| := \exp\left(-\sup\{\lambda \mid x \in \mathcal{F}^\lambda C\}\right), \quad d(x,y) := \|x - y\|. \tag{3.4.3}$$

$\Lambda_{\mathrm{nov}}, \Lambda_{0,\mathrm{nov}}, \Lambda^+_{0,\mathrm{nov}}$ are complete with respect to this metric.

We will take completion of filtered $\Lambda_{0,\mathrm{nov}}$-modules when necessary, and the completion of C is denoted by $\widehat{C}$. The completed tensor product of C, C' is written as $C\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}C' := (C\otimes_{\Lambda_{0,\mathrm{nov}}} C')\widehat{}$. There is a natural isomorphism $C\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}C' \cong \widehat{C}\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}\widehat{C}'$, so a finite completed tensor product is defined without ambiguity.

We say a $\Lambda_{0,\mathrm{nov}}$-module homomorphism $\varphi : C \to C'$ is filtered if $\varphi(\mathcal{F}^\lambda C) \subset \mathcal{F}^\lambda C'$ $(\forall\lambda)$, or equivalently, its operator norm $\|\varphi\| \le 1$. Let $\mathrm{Hom}(C, C')$ be the $\Lambda_{0,\mathrm{nov}}$-module of filtered homomorphisms $C \to C'$. It is easy to see $\mathrm{Hom}(C, C')$ is naturally filtered, and is complete as long as C' is complete. For any C, C', $\mathrm{Hom}(C, \widehat{C}') = \mathrm{Hom}(\widehat{C}, \widehat{C}')$.

Now let $\overline{C} = \{\overline{C}^m\}_{m\in\mathbb{Z}}$ be a graded vector space, viewed as trivially filtered, i.e. $\mathcal{F}^0\overline{C} = \overline{C}$ and $\mathcal{F}^\lambda\overline{C} = 0$ $(\forall\lambda > 0)$. Then $\overline{C} \otimes \Lambda_{0,\mathrm{nov}}$ inherits energy filtration from $\Lambda_{0,\mathrm{nov}}$. Write $C = \{C^m\}_{m\in\mathbb{Z}} = \overline{C}\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$, so that

$$C^m = \left\{ \sum_{i=0}^{\infty} c_i T^{\lambda_i} e^{n_i} \;\middle|\; c_i \in \overline{C}^{m-2n_i},\ \lambda_i \in \mathbb{R}_{\ge 0},\ n_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

Similar to $T(V)$ where V is a vector space, there is a structure of a graded filtered coalgebra over $\Lambda_{0,\mathrm{nov}}$ on the uncompleted tensor algebra $T(C) = \bigoplus_{k\ge 0} C^{\otimes_{\Lambda_{0,\mathrm{nov}}} k}$, and there is an identification $\mathrm{Hom}(T(C), C) = \mathrm{Coder}(T(C))$.

Let us recall the completed tensor algebra $\widehat{T}(C)$ defined in [18, Definition 3.2.16]. Let $C^{\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}} k}$ denote the k-fold completed tensor product of C, then $\widehat{T}(C)$ is defined as a completion of $\bigoplus_{k\ge 0} C^{\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}} k}$ with respect to *both* energy filtration $\bigoplus_{k\ge 0} \mathcal{F}^\lambda(C^{\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}} k})$, $\lambda \in \mathbb{R}_{\ge 0}$ and length filtration $\bigoplus_{k\ge n} C^{\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}} k}$, $n \in \mathbb{Z}_{\ge 0}$. Concretely,

$$\widehat{T}(C) := \Big\{ \sum_{k=0}^{\infty} \mathbf{x}_k \text{ formal sum } \mid \mathbf{x}_k \in \mathcal{F}^{\lambda_k}(C^{\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}} k}),\ \lim_{k\to\infty} \lambda_k = \infty \Big\}.$$

The coalgebra structure on $T(C)$ uniquely extends to a (formal) coalgebra structure on $\widehat{T}(C)$, and every coderivation on $T(C)$ uniquely extends to a formal coderivation on $\widehat{T}(C)$.

Definition 3.4.6 (Filtered A_∞ algebras and cyclic filtered A_∞ algebras).

(i) A structure of *filtered A_∞ algebra* on $C = \overline{C}\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$ is a sequence of degree 1 filtered homomorphisms $\mathfrak{m} = \left\{\mathfrak{m}_k : C[1]^{\otimes_{\Lambda_{0,\mathrm{nov}}} k} \to C[1]\right\}_{k\geq 0}$ such that the following conditions are satisfied:

- $\hat{\mathfrak{m}} = \sum_{k\geq 0} \hat{\mathfrak{m}}_k \in \mathrm{Coder}(T(C[1]))^1$ is a codifferential, i.e. $\hat{\mathfrak{m}}\hat{\mathfrak{m}} = 0$. (Equivalently, $\hat{\mathfrak{m}} \in \mathrm{Coder}(\widehat{T}(C[1]))^1$ is a formal codifferential.)
- There is a strict A_∞ algebra structure $\overline{\mathfrak{m}}$ on $\overline{C}$ such that $\mathfrak{m}_k \equiv \overline{\mathfrak{m}}_k \otimes \mathrm{id}_{R_e} \mod \Lambda^+_{0,\mathrm{nov}}$ on $\overline{C}[1]^{\otimes k} \otimes R_e$ for each $k \geq 0$. In case $k = 0$ this says $\mathfrak{m}_0 : \Lambda_{0,\mathrm{nov}} \to C[1]$ satisfies $\mathfrak{m}_0(1) \in \mathcal{F}^{\lambda_0}C[1]$ for some $\lambda_0 > 0$.

(ii) Fix a symmetric bilinear form $\langle,\rangle$ on $\overline{C}$, which extends in the obvious way to a $\Lambda_{0,\mathrm{nov}}$-valued symmetric bilinear form on $C = \overline{C}\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$. A structure of *cyclic filtered A_∞ algebra* on C is a sequence of degree 1 filtered homomorphisms $\mathfrak{m} = \left\{\mathfrak{m}_k : C[1]^{\otimes_{\Lambda_{0,\mathrm{nov}}} k} \to C[1]\right\}_{k\geq 0}$ such that $(C, \mathfrak{m})$ is a filtered A_∞ algebra, and $\mathfrak{m}_k$ is cyclic with respect to $\langle,\rangle'$. (Equivalently, $\hat{\mathfrak{m}} \in \mathrm{Coder}_{\mathrm{cyc}}(\widehat{T}(C[1]))^1$ is a formal codifferential.)

From now on, we call a strict A_∞ algebra over $\mathbb{R}$ an *unfiltered A_∞ algebra.*

Definition 3.4.7 (Filtered A_∞ deformations and cyclic filtered A_∞ deformations)**.**

(i) Let $(\overline{C}, \overline{\mathfrak{m}})$ be an unfiltered A_∞ algebra. A *filtered A_∞ deformation* of $(\overline{C}, \overline{\mathfrak{m}})$ over $\Lambda_{0,\mathrm{nov}}$ is a filtered A_∞ algebra structure $\mathfrak{m}$ on $C = \overline{C}\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$ satisfying Definition 3.4.6(i).

(ii) Let $(\overline{C}, \overline{\mathfrak{m}}, \langle,\rangle)$ be cyclic unfiltered A_∞ algebra. A *cyclic filtered A_∞ deformation* of $(\overline{C}, \overline{\mathfrak{m}}, \langle,\rangle)$ over $\Lambda_{0,\mathrm{nov}}$ is a cyclic filtered A_∞ algebra structure $\mathfrak{m}$ on $(C = \overline{C}\widehat{\otimes}\Lambda_{0,\mathrm{nov}}, \langle,\rangle)$ satisfying Definition 3.4.6(ii).

Remark 3.4.8. (i) Our definition of filtered A_∞ deformations is narrower than [18, Definition 3.2.34]. Namely, we insist on the same underlying unfiltered A_∞ algebra $(\overline{C}, \overline{\mathfrak{m}})$ rather than a weak homotopy equivalent one as [18, Definition 3.2.34] did.

(ii) Filtered A_∞ algebras can be described by $T(C[1])$ without using $\widehat{T}(C[1])$. But the introduction of $\widehat{T}(C[1])$ is natural, which makes it possible to interpret filtered A_∞ homomorphisms (Definition 3.5.1) as morphisms between formal coalgebras.

We want to use *dg Lie algebra formalism* to describe filtered A_∞ deformations (c.f. [13]). Let $(\overline{C}, \overline{\mathfrak{m}})$ be an unfiltered A_∞ algebra and $C = \overline{C}\widehat{\otimes}\Lambda_{0,\text{nov}}$. For each $k \geq 0$, there is an obvious map $\text{Coder}(T(\overline{C}[1])) \to \text{Hom}(\overline{C}[1]^{\otimes k}, \overline{C}[1]) \to \text{Hom}(C[1]^{\otimes_{\Lambda_{0,\text{nov}}} k}, C[1])$, inducing $\text{Coder}(T(\overline{C}[1]))\widehat{\otimes}\Lambda_{0,\text{nov}} \to \text{Hom}(C[1]^{\otimes_{\Lambda_{0,\text{nov}}} k}, C[1])$ since the latter is complete. Thus we obtain a natural map $\text{Coder}(T(\overline{C}[1]))\widehat{\otimes}\Lambda_{0,\text{nov}} \to \text{Coder}(T(C[1]))$, which is easily seen to be an embedding of dg Lie algebras: $(\text{Coder}(T(\overline{C}[1]))\widehat{\otimes}\Lambda_{0,\text{nov}}, \text{ad}_{\hat{\overline{\mathfrak{m}}}}, [,]) \subset (\text{Coder}(T(C[1])), \text{ad}_{\hat{\overline{\mathfrak{m}}}}, [,])$.

We say a filtered A_∞ deformation $(C, \mathfrak{m})$ of $(\overline{C}, \overline{\mathfrak{m}})$ is *uniform* if $\hat{\mathfrak{m}} \in \text{Coder}(T(\overline{C}[1]))\widehat{\otimes}\Lambda_{0,\text{nov}}$. For example, gapped filtered A_∞ deformations, defined below, are uniform.

Consider the additive monoid $\mathbb{R}_{\geq 0} \times 2\mathbb{Z}$. Let E, μ be projection onto its two factors.

Definition 3.4.9. A discrete submonoid $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ is a submonoid satisfying

- $E(G) \subset \mathbb{R}_{\geq 0}$ is discrete.
- $G \cap (\{0\} \times 2\mathbb{Z}) = \{(0,0)\}$, and $G \cap (\{\lambda\} \times 2\mathbb{Z})$ is a finite set for any $\lambda > 0$.

Definition 3.4.10. A (cyclic) filtered A_∞ deformation $(C, \mathfrak{m})$ of $(\overline{C}, \overline{\mathfrak{m}})$ is called *gapped* (and *G-gapped* if we want to specify G) if there is a discrete submonoid $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$, and a sequence of linear maps $\left\{\mathfrak{m}_{k,\beta} : \overline{C}[1]^{\otimes k} \to \overline{C}[1]\right\}_{k \geq 0}$ for each $\beta \in G$, such that

$$\mathfrak{m}_k = \sum_{\beta \in G} T^{E(\beta)} e^{\frac{\mu(\beta)}{2}} \mathfrak{m}_{k,\beta}.$$

Notice that by definition $\mathfrak{m}_{k,(0,0)} = \overline{\mathfrak{m}}_k$. An unfiltered A_∞ algebra is trivially gapped.

Let $\left(\overline{B} = \{\overline{B}^i\}_{i \in \mathbb{Z}}, d, \{,\}\right)$ be a dg Lie algebra over $\mathbb{R}$, and let $B = \overline{B}\widehat{\otimes}\Lambda_{0,\text{nov}}$ be its trivial extension over $\Lambda_{0,\text{nov}}$. To be consistent with L_∞ language (Definition 3.6.15), we only consider Maurer-Cartan elements with norm (3.4.3) less than 1. Thus we set

$$\text{MC}(B) := \{x \in (\overline{B}\widehat{\otimes}\Lambda^+_{0,\text{nov}})^1 \mid dx - \frac{1}{2}\{x, x\} = 0\}.$$

Definition 3.4.11. $x \in \mathrm{MC}(B)$ is called *gapped* (and *G-gapped* to specify G) if there is a discrete submonoid $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$, and $x_\beta \in \overline{B}^{1-\mu(\beta)}$ for each $\beta \in G$, such that

$$x = \sum_{\beta \in G} T^{E(\beta)} e^{\frac{\mu(\beta)}{2}} x_\beta.$$

Denote the set of G-gapped Maurer-Cartan elements in B by $\mathrm{MC}_G(B)$. Notice that by definition, $x_{(0,0)} = 0$ for any $x \in \mathrm{MC}_G(B)$.

Remark 3.4.12. If $G \subset G'$, then G-gappedness implies G'-gappedness. Thus if there are finitely many elements that are gapped for different G_i, we can take a large common G so that everything is G-gapped. This applies not only to gapped filtered A_∞ algebras and gapped Maurer-Cartan elements, but also to gapped filtered A_∞ homomorphisms (Definition 3.5.1).

Lemma 3.4.13. *There are bijections*

$$\{\textit{uniform filtered } A_\infty \textit{ deformations of } (\overline{C}, \overline{\mathfrak{m}})\} \leftrightarrow \mathrm{MC}\big(\mathrm{Coder}(T(\overline{C}[1])) \widehat{\otimes} \Lambda_{0,\mathrm{nov}}, \mathrm{ad}_{\hat{\overline{\mathfrak{m}}}}, [,]\big),$$
$$\{G\textit{-gapped filtered } A_\infty \textit{ deformations of } (\overline{C}, \overline{\mathfrak{m}})\} \leftrightarrow \mathrm{MC}_G\big(\mathrm{Coder}(T(\overline{C}[1])) \widehat{\otimes} \Lambda_{0,\mathrm{nov}}, \mathrm{ad}_{\hat{\overline{\mathfrak{m}}}}, [,]\big).$$

Similar results hold for (uniform / G-gapped) cyclic filtered A_∞ deformations.

Proof. Since $\Lambda_{0,\mathrm{nov}} = R_e \oplus \Lambda_{0,\mathrm{nov}}^+$, $\hat{\mathfrak{m}} \in (\mathrm{Coder}(T(\overline{C}[1])) \widehat{\otimes} \Lambda_{0,\mathrm{nov}})^1$ can be uniquely written as

$$\hat{\mathfrak{m}} = \hat{\overline{\mathfrak{m}}} - \hat{\mathfrak{n}}, \qquad \hat{\mathfrak{n}} \in (\mathrm{Coder}(T(\overline{C}[1])) \widehat{\otimes} \Lambda_{0,\mathrm{nov}}^+)^1.$$

One readily checks that A_∞ relation $\hat{\mathfrak{m}}\hat{\mathfrak{m}} = 0$ is equivalent to $\hat{\overline{\mathfrak{m}}}\hat{\mathfrak{n}} + \hat{\mathfrak{n}}\hat{\overline{\mathfrak{m}}} - \hat{\mathfrak{n}}\hat{\mathfrak{n}} = 0$, or say, $\mathrm{ad}_{\hat{\overline{\mathfrak{m}}}}\hat{\mathfrak{n}} - \frac{1}{2}[\hat{\mathfrak{n}}, \hat{\mathfrak{n}}] = 0$. Then the desired bijection is simply $\hat{\mathfrak{m}} \leftrightarrow \hat{\mathfrak{n}}$. Clearly $\hat{\mathfrak{m}}$ is G-gapped iff $\hat{\mathfrak{n}}$ is G-gapped. If $\overline{\mathfrak{m}}$ is cyclic, then $\hat{\mathfrak{m}}$ is cyclic iff $\hat{\mathfrak{n}}$ is cyclic. $\square$

Remark 3.4.14. In the above we are using the usual (unshifted, cohomological) grading of a dg Lie algebra, so Maurer-Cartan elements are of degree 1. If grading is shifted (or opposite), then Maurer-Cartan elements are of a shifted (or opposite) degree.

3.5 "Pseudo-isotopy = gauge equivalence"

The purpose of this section is to establish equivalence between the following two notions (whose definitions will be given), which enhances the 1-1 correspondence in Lemma 3.4.13:

- Pseudo-isotopy of (cyclic) filtered A_∞ deformations of an A_∞ algebra.
- Gauge equivalence of Maurer-Carten elements in the (cyclic) Hochschild cochain complex of the A_∞ algebra.

In Sections 3.5.1,3.5.2, we collect some materials from [17, 18], and there is nothing new. In Section 3.5.3, we collect some materials from [17, 18, 20], and prove the main result.

Before discussing (cyclic) A_∞ algebras, let us introduce some notations.

Let $(D, \Delta), (D', \Delta')$ be graded coalgebras, a linear map $F : D \to D'$ is called a coalgebra homomorphism if $\Delta' \circ F = (F \otimes F) \circ \Delta$. Fix such an $F \in \mathrm{Hom}((D, \Delta), (D', \Delta'))$, then the space of *coderivations with respect to* F, denoted by $\mathrm{Coder}(D, D'; F)$, consists of linear maps $\Phi : D \to D'$ such that $\Delta' \circ \Phi = (\Phi \otimes F + F \otimes \Phi) \circ \Delta$.

Let $\overline{V}, \overline{W}$ be graded vector spaces, then every $\overline{f} = (\overline{f}_k)_{k\geq 1} \in \prod_{k\geq 1} \mathrm{Hom}(\overline{V}^{\otimes k}, \overline{W})$ can be uniquely extended to a coalgebra homomorphism

$$\hat{\overline{f}} := \sum_{m\geq 1} (\otimes^m \overline{f}) \circ \Delta^{m-1} \in \mathrm{Hom}((T_+(\overline{V}), \Delta), (T_+(\overline{W}), \Delta)).$$

This assignment identifies $\mathrm{Hom}(T_+(\overline{V}), \overline{W})$ with $\mathrm{Hom}((T_+(\overline{V}), \Delta), (T_+(\overline{W}), \Delta))$. Fix such an $\hat{\overline{f}}$, then every $\varphi = (\varphi_k)_{k\geq 0} \in \prod_{k\geq 0} \mathrm{Hom}(\overline{V}^{\otimes k}, \overline{W})$ can be uniquely extended to

$$\hat{\varphi} := (\hat{\overline{f}} \otimes \varphi \otimes \hat{\overline{f}}) \circ \Delta^2 \in \mathrm{Coder}(T(\overline{V}), T(\overline{W}); \hat{\overline{f}}). \tag{3.5.1}$$

This assignment identifies $\mathrm{Hom}(T(\overline{V}), \overline{W})$ with $\mathrm{Coder}(T(\overline{V}), T(\overline{W}); \hat{\overline{f}})$ ([18, Lemma 4.4.43]).

Let $V = \overline{V}\widehat{\otimes}\Lambda_{0,\mathrm{nov}}, W = \overline{W}\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$, then each $f = (f_k)_{k\geq 0} \in \prod_{k\geq 0} \mathrm{Hom}(V^{\otimes_{\Lambda_{0,\mathrm{nov}}} k}, W)$

with $\|f_0\| < 1$ can be uniquely extended to a (formal) filtered coalgebra homomorphism

$$\hat{f} := \sum_{m \geq 0} (\otimes^m f) \circ \Delta^{m-1} : \widehat{T}(V) \to \widehat{T}(W) \tag{3.5.2}$$

$$\hat{f}(v_1 \otimes \cdots \otimes v_n) := \sum_{\substack{k_1+\cdots+k_m=n \\ m\geq 0,\ k_j \geq 0}} (-1)^{\varepsilon} f_{k_1}(v_1 \otimes \cdots \otimes v_{k_1}) \otimes \cdots \otimes f_{k_m}(v_{n-k_m+1} \otimes \cdots \otimes v_n),$$

where $(-1)^{\varepsilon}$ is Koszul sign. Notice that (3.5.2) converges in $\widehat{T}(W)$ since $\|f_0\| < 1$, $\|f_k\| \leq 1$. Here $f_{k_j}(\dots)$ means $f_0(1)$ if $k_j = 0$, so in particular $\hat{f}(1) = 1 + f_0(1) + f_0(1) \otimes f_0(1) + \cdots$ (this is the only place that $m = 0$ in (3.5.2) makes sense).

3.5.1 (Cyclic) A_∞ homomorphisms

Definition 3.5.1. Let $(\overline{C}, \overline{\mathfrak{m}}), (\overline{C}', \overline{\mathfrak{m}}')$ be unfiltered A_∞ algebras, and let $(C, \mathfrak{m}), (C', \mathfrak{m}')$ be filtered A_∞ algebras that are filtered A_∞ deformations of $\overline{C}$, $\overline{C}'$, respectively.

(i) An *(unfiltered) A_∞ homomorphism* from $(\overline{C}, \overline{\mathfrak{m}})$ to $(\overline{C}', \overline{\mathfrak{m}}')$ is a sequence of degree 0 linear maps $\bar{\mathfrak{f}} = \{\bar{\mathfrak{f}}_k : \overline{C}[1]^{\otimes k} \to \overline{C}'[1]\}_{k \geq 1}$ such that $\hat{\overline{\mathfrak{m}}}' \circ \hat{\bar{\mathfrak{f}}} = \hat{\bar{\mathfrak{f}}} \circ \hat{\overline{\mathfrak{m}}}$.

(ii) A *(filtered) A_∞ homomorphism* from $(C, \mathfrak{m})$ to $(C', \mathfrak{m}')$ is a sequence of degree 0 filtered homomorphisms $\mathfrak{f} = \{\mathfrak{f}_k : C[1]^{\otimes_{\Lambda_{0,\mathrm{nov}}} k} \to C'[1]\}_{k \geq 0}$ such that $\|\mathfrak{f}_0\| < 1$, $\hat{\mathfrak{m}}' \circ \hat{\mathfrak{f}} = \hat{\mathfrak{f}} \circ \hat{\mathfrak{m}}$, and $\mathfrak{f} \equiv \bar{\mathfrak{f}} \otimes \mathrm{id}_{R_e} \mod \Lambda_{0,\mathrm{nov}}^+$ for some unfiltered A_∞ homomorphism $\bar{\mathfrak{f}} : (\overline{C}, \overline{\mathfrak{m}}) \to (\overline{C}', \overline{\mathfrak{m}}')$.

(iii) An A_∞ homomorphism $\mathfrak{f}$ is called *strict* if $\mathfrak{f}_0 = 0$, and is called *linear* if $\mathfrak{f}_k = 0$ $(\forall k \neq 1)$.

(iv) A filtered A_∞ homomorphism $\mathfrak{f} : (C, \mathfrak{m}) \to (C', \mathfrak{m}')$ is called *gapped* (or *G-gapped* to specify G), if there is a discrete submonoid G (Definition 3.4.9) and $f_{k,\beta} \in \mathrm{Hom}\big(\overline{C}[1]^{\otimes k}, \overline{C}'[1]\big)$ $(k \geq 0, \beta \in G)$, such that C, C' are G-gapped and $\mathfrak{f}_k = \sum_{\beta \in G} T^{E(\beta)} e^{\frac{\mu(\beta)}{2}} \mathfrak{f}_{k,\beta}$.

The condition $\hat{\mathfrak{m}}' \circ \hat{\mathfrak{f}} = \hat{\mathfrak{f}} \circ \hat{\mathfrak{m}}$ is equivalent to $\mathfrak{m}' \circ \hat{\mathfrak{f}} = \mathfrak{f} \circ \hat{\mathfrak{m}}$, namely for each $n \geq 0$ and

$x_1, \ldots, x_n \in C[1]$,

$$\sum_{\substack{r \geq 0 \\ k_1+\cdots+k_r=n}} \mathfrak{m}'_r(\mathfrak{f}_{k_1}(x_1 \otimes \cdots \otimes x_{k_1}) \otimes \cdots \otimes \mathfrak{f}_{k_r}(x_{n-k_r+1} \otimes \cdots \otimes x_n))$$
$$= \sum_{\substack{l_1+l_2=n+1 \\ 1 \leq i \leq l_1}} (-1)^{|x_1|'+\cdots+|x_{i-1}|'} \mathfrak{f}_{l_1}(x_1 \otimes \cdots \otimes x_{i-1} \otimes \mathfrak{m}_{l_2}(x_i \otimes \cdots \otimes x_{i+l_2-1}) \otimes \cdots \otimes x_n).$$

Definition 3.5.2. Let $(C, \mathfrak{m}, \langle, \rangle_{\overline{C}}), (C', \mathfrak{m}', \langle, \rangle_{\overline{C}'})$ be G-gapped cyclic filtered A_∞ algebras. A G-gapped filtered A_∞ homomorphism $\mathfrak{f} : C \to C'$ is called *cyclic* if

$$\langle \mathfrak{f}_{1,(0,0)}(x_1), \mathfrak{f}_{1,(0,0)}(x_2) \rangle_{\overline{C}'} = \langle x_1, x_2 \rangle_{\overline{C}}$$

for all $x_1, x_2 \in C$, and

$$\sum_{\beta_1+\beta_2=\beta} \sum_{k_1+k_2=k} \langle \mathfrak{f}_{k_1,\beta_1}(x_1 \otimes \cdots \otimes x_{k_1}), \mathfrak{f}_{k_2,\beta_2}(x_{k_1+1} \otimes \cdots \otimes x_k) \rangle_{\overline{C}} = 0$$

for all $(k, \beta) \neq (2, (0,0))$ and $x_1, \ldots, x_k \in C$.

Definition 3.5.3. (i) The A_∞ *composition* $\mathfrak{f}^2 \circ \mathfrak{f}^1 = \big((\mathfrak{f}^2 \circ \mathfrak{f}^1)_k\big)$ of A_∞ homomorphisms $\mathfrak{f}^1, \mathfrak{f}^2$ is given by composition of corresponding coalgebra homomorphisms. It is easy to see if $\mathfrak{f}^1, \mathfrak{f}^2$ are cyclic, then $\mathfrak{f}^2 \circ \mathfrak{f}^1$ is cyclic.

(ii) An A_∞ homomorphism is an A_∞ *isomorphism* if it is invertible with respect to A_∞ composition. It is easy to see if $\mathfrak{f}$ is cyclic and invertible, then $\mathfrak{f}^{-1}$ is also cyclic.

The following lemma is used in [18, Proposition 5.4.5].

Lemma 3.5.4. *Let $\mathfrak{f} : C \to C'$ be a G-gapped filtered A_∞ homomorphism. The following are equivalent:*

(i) $\mathfrak{f} : C \to C'$ is a filtered A_∞ isomorphism;

(ii) $\bar{\mathfrak{f}} : \overline{C} \to \overline{C}'$ is an unfiltered A_∞ isomorphism;

(iii) $\mathfrak{f}_1 : C[1] \to C'[1]$ is a filtered $\Lambda_{0,\mathrm{nov}}$-module isomorphism;

(iv) $\bar{\mathfrak{f}}_1 = \mathfrak{f}_{1,(0,0)} : \overline{C}[1] \to \overline{C}'[1]$ *is a vector space isomorphism.*

Proof. "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iv)" and "(iii) $\Rightarrow$ (iv)" are obvious, so it suffices to prove "(iv) $\Rightarrow$ (i),(iii)". We shall make induction on energy filtration, so let us write $E(G) = \{E_0 = 0, E_1, E_2, \dots\}$ where each $E_i < E_{i+1}$, and for any G-gapped element $x = \{x_{k,\beta}\}$ write

$$x_k = \sum_{\beta \in G} T^{E(\beta)} e^{\frac{\mu(\beta)}{2}} x_{k,\beta} = \sum_{i \geq 0} T^{E_i} x_{k,i}, \qquad x_{k,i} := \sum_{\beta \in G, E(\beta) = E_i} e^{\frac{\mu(\beta)}{2}} x_{k,\beta}.$$

Notice that in particular $x_{k,0} = x_{k,(0,0)}$ since $G \cap (\{0\} \times 2\mathbb{Z}) = \{(0,0)\}$.

To prove "(iv) $\Rightarrow$ (iii)", define $\mathfrak{g}'_{1,i}$ inductively on i by $\mathfrak{g}'_{1,0} = \mathfrak{f}_{1,0}^{-1}$ and $\mathfrak{g}'_{1,i} = -\sum_{p<i, E_p+E_q=E_i} \mathfrak{g}'_{1,p} \circ \mathfrak{f}_{1,q} \circ \mathfrak{f}_{1,0}^{-1}$ for $i > 0$. Then $\mathfrak{g}'_1 = \sum_{i \geq 0} T^{E_i} \mathfrak{g}'_{1,i}$ satisfies

$$\mathfrak{g}'_1 \circ \mathfrak{f}_1 = \sum_{i \geq 0} T^{E_i} \sum_{E_p+E_q=E_i} \mathfrak{g}'_{1,p} \circ \mathfrak{f}_{1,q} = \mathrm{id}_{C[1]}.$$

In the same way there exists $\mathfrak{f}'_1$ such that $\mathfrak{f}'_1 \circ \mathfrak{g}'_1 = \mathrm{id}_{C'[1]}$, so $\mathfrak{f}_1 = \mathfrak{f}'_1 = (\mathfrak{g}'_1)^{-1}$ and $\mathfrak{f}_1$ is an isomorphism.

To prove "(iv) $\Rightarrow$ (i)", first define a strict total order on $\mathbb{Z}^2_{\geq 0}$ by $(k,i) < (k',i')$ iff $i < i'$ or $i = i', k < k'$, then define $\mathfrak{g}_{k,i}$ inductively on (k,i) by $\mathfrak{g}_{0,0} = 0$, $\mathfrak{g}_{1,0} = \mathfrak{f}_{1,0}^{-1}$ and

$$\mathfrak{g}_{k,i} = -\sum_{(l,i_0)<(k,i)} \sum_{\substack{k_1+\cdots+k_l=k \\ E_{i_0}+\cdots+E_{i_l}=E_i}} \mathfrak{g}_{l,i_0} \circ (\mathfrak{f}_{k_1,i_1} \otimes \cdots \otimes \mathfrak{f}_{k_l,i_l}) \circ (\mathfrak{f}_{1,0}^{-1})^{\otimes k}$$

for $(k,i) > (1,0)$. Let $\mathfrak{g} = (\mathfrak{g}_k)_{k \geq 0} = \sum_{i \geq 0} T^{E_i} (\mathfrak{g}_{k,i})_{k \geq 0}$, then

$$(\mathfrak{g} \circ \hat{\mathfrak{f}})|_{C[1]^{\hat{\otimes}_{\Lambda_{0,\mathrm{nov}}} k}} = \sum_{i \geq 0} T^{E_i} \sum_{l \geq 0} \sum_{\substack{k_1+\cdots+k_l=k \\ E_{i_0}+\cdots+E_{i_l}=E_i}} \mathfrak{g}_{l,i_0} \circ (\mathfrak{f}_{k_1,i_1} \otimes \cdots \otimes \mathfrak{f}_{k_l,i_l})$$

satisfies $(\mathfrak{g} \circ \hat{\mathfrak{f}})|_{C[1]} = \mathrm{id}_{C[1]}$ and $(\mathfrak{g} \circ \hat{\mathfrak{f}})|_{C[1]^{\hat{\otimes}_{\Lambda_{0,\mathrm{nov}}} k}} = 0$ for $1 \neq k \geq 0$, so $\hat{\mathfrak{g}} \circ \hat{\mathfrak{f}} = \mathrm{id}_{\widehat{T}(C[1])}$. In the same way there exists $\mathfrak{f}'$ such that $\hat{\mathfrak{f}}' \circ \hat{\mathfrak{g}} = \mathrm{id}_{\widehat{T}(C'[1])}$, so $\mathfrak{f} = \mathfrak{f}' = \mathfrak{g}^{-1}$ as coalgebra homomorphisms. Since $\mathfrak{f}$ is an A_∞ homomorphism, so is $\mathfrak{g}$, and thus $\mathfrak{f}$ is an A_∞ isomorphism. $\square$

Remark 3.5.5. In general $\mathfrak{g}'_1 = \mathfrak{f}_1^{-1} \neq (\mathfrak{f}^{-1})_1 = \mathfrak{g}_1$, since $\mathfrak{g}_1$ contains terms from $\mathfrak{f}_0$.

3.5.2 Homotopy equivalence of A_∞ algebras

Let $(\overline{C}, \overline{\mathfrak{m}})$ be an unfiltered A_∞ algebra, and let $(C, \mathfrak{m})$ be a gapped filtered A_∞ deformation of $(\overline{C}, \overline{\mathfrak{m}})$.

Definition 3.5.6. A gapped filtered A_∞ algebra $(\widetilde{C}, \tilde{\mathfrak{m}})$ together with gapped filtered A_∞ homomorphisms

$$\mathrm{Incl} : (C, \mathfrak{m}) \to (\widetilde{C}, \tilde{\mathfrak{m}}), \quad \mathrm{Eval}_{t=0} : (\widetilde{C}, \tilde{\mathfrak{m}}) \to (C, \mathfrak{m}), \quad \mathrm{Eval}_{t=1} : (\widetilde{C}, \tilde{\mathfrak{m}}) \to (C, \mathfrak{m})$$

is said to be an *(A_∞ algebra) model of* $[0,1] \times C$ if the following holds:

- Incl, $\mathrm{Eval}_{t=0}$, $\mathrm{Eval}_{t=1}$ are linear A_∞ homomorphisms;
- $\mathrm{Eval}_{t=0} \circ \mathrm{Incl} = \mathrm{Eval}_{t=1} \circ \mathrm{Incl} = \mathfrak{id}_C$;
- The underlying unfiltered maps $\overline{\mathrm{Incl}}_1$, $(\overline{\mathrm{Eval}}_{t=0})_1$, $(\overline{\mathrm{Eval}}_{t=1})_1$ are cochain homotopy equivalences between the underlying unfiltered cochain complexes $(\overline{C}, \overline{\mathfrak{m}}_1)$, $(\overline{\widetilde{C}}, \overline{\tilde{\mathfrak{m}}}_1)$;
- $(\mathrm{Eval}_{t=0})_1 \oplus (\mathrm{Eval}_{t=1})_1 : \widetilde{C} \to C \oplus C$ is surjective.

The notion of a model of $[0,1] \times \overline{C}$ is similar.

Remark 3.5.7. (i) Over a field, for example over $\mathbb{R}$ in our setting, a quasi-isomorphism between cochain complexes must be a cochain homotopy equivalence.

(ii) The gapping condition is crucial in the proof of Lemma 3.5.12 and Proposition 3.5.14.

Example 3.5.8. Let $\mathcal{A}^0([0,1])$ be a linear subspace of the space of absolutely continuous functions on $[0,1]$, which contains all constant functions and is closed under differentiation, integration and multiplication. We further assume that there exists $u(t) \in \mathcal{A}^0([0,1])$ such that $u(0) \neq u(1)$. For example, $\mathcal{A}^0([0,1])$ can be taken as the space of (piecewise) polynomial or (piecewise) smooth functions on $[0,1]$. Then

$$\mathcal{A}([0,1]) = \mathcal{A}^0([0,1]) \oplus \mathcal{A}^1([0,1]) := \left\{ a(t) + b(t)dt \mid a(t), b(t) \in \mathcal{A}^0([0,1]) \right\}$$

admits a dg algebra structure $(d, \wedge)$ defined in the usual way. Consider the dg algebra embedding $(\mathbb{R}, 0, \cdot) \xhookrightarrow{\iota} (\mathcal{A}([0,1]), d, \wedge)$ given by inclusion of constant functions, and the evaluation map $\mathcal{A}([0,1]) \xrightarrow{e_{t_0}} \mathbb{R}$ sending $a(t) + b(t)dt$ to $a(t_0)$. It is very easy to see $\big(\mathcal{A}([0,1]), (d, \wedge), \iota, e_0, e_1\big)$ is a model of $[0,1] \times \mathbb{R}$. We remark that the surjectivity of $e_0 \oplus e_1 : \mathcal{A}([0,1]) \to \mathbb{R} \oplus \mathbb{R}$ comes from the existence of $u(t)$.

If $(C, \mathfrak{m})$ is a gapped filtered A_∞ algebra, there is a gapped filtered A_∞ structure $\tilde{\mathfrak{m}}$ on $(\mathcal{A}([0,1]) \otimes \overline{C})\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$ defined in a natural way. In order to define $\tilde{\mathfrak{m}}$, it suffices to look at elements in $\mathcal{A}([0,1]) \otimes \overline{C}[1]$ of the form $a_j(t) \cdot x_j + b_j(t)dt \cdot y_j$, $j = 1, \ldots, k$.

- For $k = 0$, $\tilde{\mathfrak{m}}_0$ is the composition

$$\Lambda_{0,\mathrm{nov}} \xrightarrow{\mathfrak{m}_0} C[1] = (\overline{C}\widehat{\otimes}\Lambda_{0,\mathrm{nov}})[1] \subset ((\mathcal{A}([0,1]) \otimes \overline{C})\widehat{\otimes}\Lambda_{0,\mathrm{nov}})[1];$$

- For $k = 1$, $\tilde{\mathfrak{m}}_1 = d \otimes \mathrm{id}_{C[1]} + \mathrm{id}_{\mathcal{A}([0,1])} \otimes \mathfrak{m}_1$, namely

$$\tilde{\mathfrak{m}}_1(a_1(t) \cdot x_1 + b_1(t)dt \cdot y_1) := \frac{\partial a_1(t)}{\partial t} dt \cdot x_1 + a_1(t) \cdot \mathfrak{m}_1(x_1) - b_1(t)dt \cdot \mathfrak{m}_1(y_1);$$

- For $k \geq 2$, $\tilde{\mathfrak{m}}_k$ is $\mathcal{A}([0,1])$-linear extension of $\mathfrak{m}_k$, namely

$$\begin{aligned} & \tilde{\mathfrak{m}}_k((a_1(t) \cdot x_1 + b_1(t)dt \cdot y_1) \otimes \cdots \otimes (a_k(t) \cdot x_k + b_k(t)dt \cdot y_k)) \\ := \; & a_1(t) \cdots a_k(t) \cdot \mathfrak{m}_k(x_1 \otimes \cdots \otimes x_k) \\ & + \sum_{j=1}^{k} (-1)^{|x_1|' + \cdots + |x_{j-1}|' + 1} a_1(t) \cdots b_j(t) \cdots a_k(t) dt \cdot \mathfrak{m}_k(x_1 \otimes \cdots \otimes y_j \otimes \cdots \otimes x_k). \end{aligned}$$

By [18, Lemma 4.2.13], $\tilde{\mathfrak{m}}$ satisfies A_∞ relation. (See Example 3.6.6 for an alternative proof.) Next, let $\mathrm{Incl}_1 : C[1] \to ((\mathcal{A}([0,1]) \otimes \overline{C})\widehat{\otimes}\Lambda_{0,\mathrm{nov}})[1]$ be obvious inclusion, and $(\mathrm{Eval}_{t=t_0})_1 : ((\mathcal{A}([0,1]) \otimes \overline{C})\widehat{\otimes}\Lambda_{0,\mathrm{nov}})[1] \to C[1]$, $a(t) \cdot x + b(t)dt \cdot y \mapsto a(t_0) \cdot x$ be evaluation at $t = t_0 \in [0,1]$, then

$$\big((\mathcal{A}([0,1]) \otimes \overline{C})\widehat{\otimes}\Lambda_{0,\mathrm{nov}}, \tilde{\mathfrak{m}}, \mathrm{Incl}, \mathrm{Eval}_{t=0}, \mathrm{Eval}_{t=1}\big)$$

is a model of $[0,1] \times C$ ([18, Definition-Proposition 4.2.15]). We remark that $\overline{\mathrm{Incl}}_1$ is a cochain homotopy equivalence simply because $\overline{\mathrm{Incl}}_1 = \iota \otimes \mathrm{id}_{\overline{C}[1]}$ and $\iota : (\mathbb{R}, 0) \hookrightarrow (\mathcal{A}([0,1]), d)$ is a cochain homotopy equivalence. This gives an alternative proof of [18, Lemma 4.2.16].

In the same way, we see $\big(\mathcal{A}([0,1])\otimes\overline{C}, \tilde{\overline{\mathfrak{m}}}, \overline{\mathrm{Incl}}, \overline{\mathrm{Eval}}_{t=0}, \overline{\mathrm{Eval}}_{t=1}\big)$ is a model of $[0,1]\times\overline{C}$.

Example 3.5.9. Consider $(\overline{C},\overline{\mathfrak{m}}) = (\Omega(N), d, \wedge)$, where N is a smooth manifold. Let $\mathrm{pr}_N : [0,1]\times N \to N$, $(t,x)\mapsto x$ be projection onto N and $i_{t_0} : N \to [0,1]\times N$, $x\mapsto (t_0,x)$ be embedding of N at $t=t_0$, then

$$\big(\widetilde{\overline{C}}, \tilde{\overline{\mathfrak{m}}}, \overline{\mathrm{Incl}}, \overline{\mathrm{Eval}}_{t=0}, \overline{\mathrm{Eval}}_{t=1}\big) = \big(\Omega([0,1]\times N), (d,\wedge), \mathrm{pr}_N^*, i_0^*, i_1^*\big)$$

is a model of $[0,1]\times\Omega(N)$. Choose $\mathcal{A}([0,1]) = \Omega([0,1])$, then $\Omega([0,1])\otimes\Omega(N)$ is also a model of $[0,1]\times\Omega(N)$ by Example 3.5.8. Notice that there is a decomposition of $(\mathbb{R}[1,dt],\wedge)$-modules:

$$\begin{aligned}\Omega([0,1]\times N) &= \Gamma([0,1]\times N, \mathrm{pr}_N^*\Omega_N)\oplus dt\wedge\Gamma([0,1]\times N, \mathrm{pr}_N^*\Omega_N)\\ \eta &= \alpha + dt\wedge\gamma, \qquad \alpha,\gamma\in\Gamma([0,1]\times N, \mathrm{pr}_N^*\Omega_N).\end{aligned} \tag{3.5.3}$$

Therefore the obvious dg algebra inclusion $\Omega([0,1])\otimes\Omega(N)\hookrightarrow\Omega([0,1]\times N)$ induces a commutative diagram:

$$\begin{tikzcd}
 & & \Omega(N) \arrow[dll, bend right, "\overline{\mathrm{Incl}}"'] \arrow[dl, "\mathrm{pr}_N^*"] \arrow[d, equal, "\mathrm{id}_{\Omega(N)}"] \\
\Omega([0,1])\otimes\Omega(N) \arrow[r, hook] \arrow[drr, bend right, "\overline{\mathrm{Eval}}_{t=t_0}"'] & \Omega([0,1]\times N) \arrow[dr, "i_{t_0}^*"] & \\
 & & \Omega(N)
\end{tikzcd}$$

In this sense these two models of $[0,1]\times\Omega(N)$ are naturally consistent with each other.

Remark 3.5.10. In Example 3.5.8, 3.5.9, signs can be taken in the following two ways, which are equivalent after sign change (1.8.1). In both ways, Koszul sign rule is applied when commuting $\mathcal{A}([0,1])$ with C.

- (Our choice.) Suspended sign for C and unsuspended sign for $\mathcal{A}([0,1])$, which leads to suspended sign for $\widetilde{C}$. Likewise, suspended sign for $\Omega(N),\Omega([0,1]\times N)$ and unsuspended sign for $\Omega([0,1])$. This choice is convenient for general A_∞ algebras.

- Unsuspended sign for everything, which is convenient for dg algebras like Example 3.5.9

Definition 3.5.11. Let C, C' be gapped filtered A_∞ algebras and $\mathfrak{f}, \mathfrak{g} : C \to C'$ be gapped filtered A_∞ homomorphisms. Let $\widetilde{C}'$ be a model of $[0,1] \times C'$. We say $\mathfrak{f}$ is *homotopic to* $\mathfrak{g}$ *in* $\widetilde{C}'$ and write $\mathfrak{f} \simeq_{\widetilde{C}'} \mathfrak{g}$, if there exists a gapped filtered A_∞ homomorphism $\mathfrak{h} : C \to \widetilde{C}'$ such that $\mathrm{Eval}_{t=0} \circ \mathfrak{h} = \mathfrak{f}$, $\mathrm{Eval}_{t=1} \circ \mathfrak{h} = \mathfrak{g}$.

Lemma 3.5.12. *([18, Lemma 4.2.36, Proposition 4.2.37])* $\simeq_{\widetilde{C}'}$ *is independent of choices of the model* $\widetilde{C}'$ *of* $[0,1] \times C'$, *giving an equivalence relation* $\simeq$ *on the set of gapped filtered* A_∞ *homomorphisms from* C *to* C'.

Definition 3.5.13. Let C, C' be gapped filtered A_∞ algebras.

(i) A gapped filtered A_∞ homomorphism $\mathfrak{f} : C \to C'$ is called a *homotopy equivalence* if there exists a gapped filtered A_∞ homomorphism $\mathfrak{g} : C' \to C$ such that $\mathfrak{f} \circ \mathfrak{g}$ and $\mathfrak{g} \circ \mathfrak{f}$ are homotopic to identity.

(ii) A gapped filtered A_∞ homomorphism $\mathfrak{f} : C \to C'$ is called a *weak homotopy equivalence* if $\bar{\mathfrak{f}}_1 : (\overline{C}, \overline{\mathfrak{m}}_1) \to (\overline{C}', \overline{\mathfrak{m}}'_1)$ is a quasi-isomorphism.

(iii) C, C' are *(weakly) homotopy equivalent* if there exists a (weak) homotopy equivalence between them.

(Weak) homotopy equivalence between unfiltered A_∞ algebras is defined similarly.

The following important result is a homotopical counterpart of Lemma 3.5.4, and is an algebraic analogue of the classical Whitehead theorem in topology.

Proposition 3.5.14. *([18, Theorem 4.2.45]) A weak homotopy equivalence between gapped filtered* A_∞ *algebras is a homotopy equivalence.*

3.5.3 Pseudo-isotopy of (cyclic) A_∞ algebras

In this subsection, N is a closed oriented smooth manifold, and $(\Omega^*(N), d, \wedge, \langle , \rangle_N)$ is the cyclic A_∞ algebra in Example 3.4.3.

The following definition is adapted from [17, Definition 8.5] and [20, Definition 21.25]. In order to make things clear, we introduce two versions (algebraic version and smooth version) of pseudo-isotopy of (cyclic) filtered A_∞ algebras, and use a different notion of smooth dependence compared to [17][20].

Definition 3.5.15. Suppose $(\overline{C}, \overline{\mathfrak{m}})$ is a (cyclic) unfiltered A_∞ algebra, G is a discrete submonoid of $\mathbb{R}_{\geq 0} \times \mathbb{Z}$.

(i) We say a family of linear maps

$$\left\{\mathfrak{m}^t_{k,\beta} \in \mathrm{Hom}^{1-\mu(\beta)}(\overline{C}[1]^{\otimes k}, \overline{C}[1]),\ \mathfrak{c}^t_{k,\beta} \in \mathrm{Hom}^{-\mu(\beta)}(\overline{C}[1]^{\otimes k}, \overline{C}[1])\right\}^{t\in[0,1]}_{k\in\mathbb{Z}_{\geq 0},\beta\in G}$$

is an *algebraic pseudo-isotopy of G-gapped (cyclic) filtered A_∞ deformations of $\overline{\mathfrak{m}}$ (between $\mathfrak{m}^0, \mathfrak{m}^1$)* if:

(a) For $x_1, \ldots, x_k \in \overline{C}[1]$, the assignments

$$t \mapsto \mathfrak{m}^t_{k,\beta}(x_1 \otimes \cdots \otimes x_k), \quad t \mapsto \mathfrak{c}^t_{k,\beta}(x_1 \otimes \cdots \otimes x_k)$$

are elements in $\mathcal{A}^0([0,1]) \otimes \overline{C}[1]$. Here $\mathcal{A}([0,1])$ is as in Example 3.5.8.

(b) $\mathfrak{m}^t = \{\mathfrak{m}^t_{k,\beta}\}$ is a G-gapped (cyclic) filtered A_∞ deformation of $\overline{\mathfrak{m}}$ ($\forall t \in [0,1]$).

(c) $\mathfrak{c}^t_{k,(0,0)} = 0$. (For cyclic pseudo-isotopy, we require $\mathfrak{c}^t_{k,\beta}$ is cyclic for all t, k, β.)

(d) For every $x_1, \ldots, x_k \in \overline{C}[1]$,

$$\begin{aligned}&\frac{d}{dt}\mathfrak{m}^t_{k,\beta}(x_1 \otimes \cdots \otimes x_k)\\ +\sum_{\substack{k_1+k_2=k+1\\ 1\leq i\leq k_1}}\sum_{\beta_1+\beta_2=\beta}&\Big((-1)^{\varepsilon_i}\mathfrak{c}^t_{k_1,\beta_1}(x_1 \otimes \cdots \otimes \mathfrak{m}^t_{k_2,\beta_2}(x_i \otimes \cdots) \otimes \cdots \otimes x_k)\\ &- \mathfrak{m}^t_{k_1,\beta_1}(x_1 \otimes \cdots \otimes \mathfrak{c}^t_{k_2,\beta_2}(x_i \otimes \cdots) \otimes \cdots \otimes x_k)\Big) = 0,\end{aligned}$$

where $\varepsilon_i := |x_1|' + \cdots + |x_{i-1}|'$.

(ii) In case $(\overline{C}, \overline{\mathfrak{m}}) = (\Omega(N), d, \wedge, \langle,\rangle_N)$, we say a family of linear maps

$$\left\{\mathfrak{m}^t_{k,\beta} \in \mathrm{Hom}^{1-\mu(\beta)}(\Omega(N)[1]^{\otimes k}, \Omega(N)[1]),\ \mathfrak{c}^t_{k,\beta} \in \mathrm{Hom}^{-\mu(\beta)}(\Omega(N)[1]^{\otimes k}, \Omega(N)[1])\right\}^{t\in[0,1]}_{k\in\mathbb{Z}_{\geq 0},\beta\in G}$$

is a *smooth pseudo-isotopy of G-gapped (cyclic) filtered A_∞ deformations of* $(d, \wedge)$ *(between* $\mathfrak{m}^0, \mathfrak{m}^1$*)* if:

a) For any C^∞-manifold S and S-parameterized differential forms $\eta_1^s, \dots, \eta_k^s$ on N that are C^∞ in $(s, x) \in S \times N$,

$$\mathfrak{m}_{k,\beta}^t(\eta_1^s \otimes \cdots \otimes \eta_k^s)(x), \quad \mathfrak{c}_{k,\beta}^t(\eta_1^s \otimes \cdots \otimes \eta_k^s)(x)$$

are differential forms on N parameterized by $S \times [0,1]$, and are C^∞ in $(s,t,x) \in S \times [0,1] \times N$.

(b,c,d) The same as conditions (b,c,d) in (i).

We say $\mathfrak{m}^0, \mathfrak{m}^1$ are *algebraically* (resp. *smoothly*) *pseudo-isotopic* if there exists an algebraic (resp. smooth) pseudo-isotopy between them.

Notice that an S-parameterized differential form $\eta^s(x) \in \Omega(N)$ which is C^∞ in $(s, x) \in S \times N$ is the same thing as an element $\eta(s,x) \in \Gamma(S \times N, \mathrm{pr}_N^* \Omega_N)$: simply set $\eta^s = i_s^* \eta$. Here $\mathrm{pr}_N : S \times N \to N$, $i_s : N \to S \times N$ are obvious maps. Similar description applies to $\Gamma(S \times [0,1] \times N, \mathrm{pr}_{[0,1]\times N}^* \Omega_{[0,1]\times N})$.

Now we need the notion of gauge equivalence of (gapped) Maurer-Cartan elements in dg Lie algebras.

Let $\overline{B}$ be a dg Lie algebra over $\mathbb{R}$. If $\widetilde{\overline{B}}$ is a model of $[0,1] \times \overline{B}$, we put $\widetilde{B} := \widetilde{\overline{B}} \widehat{\otimes} \Lambda_{0,\mathrm{nov}}$, and trivially extend $\overline{\mathtt{Incl}}$, $\overline{\mathtt{Eval}}_{t=0}$, $\overline{\mathtt{Eval}}_{t=1}$ over $\Lambda_{0,\mathrm{nov}}$. Clearly $(\widetilde{B}, B, \overline{\mathtt{Incl}}, \overline{\mathtt{Eval}}_{t=0}, \overline{\mathtt{Eval}}_{t=1})$ satisfy similar properties as $(\widetilde{\overline{B}}, \overline{B}, \overline{\mathtt{Incl}}, \overline{\mathtt{Eval}}_{t=0}, \overline{\mathtt{Eval}}_{t=1})$ (Definition 3.3.8). Recall the definition of $\mathrm{MC}_G(B)$ (Definition 3.4.11).

Definition 3.5.16. Suppose $x_0, x_1 \in \mathrm{MC}(B)$ and $\widetilde{\overline{B}}$ is a dg Lie algebra model of $[0,1] \times \overline{B}$. We say x_0 is *gauge equivalent to* x_1 *in* $\widetilde{\overline{B}}$ (via $\tilde{x}$) and write $x_0 \sim_{\widetilde{\overline{B}}} x_1$, if there exists $\tilde{x} \in \mathrm{MC}(\widetilde{B})$ such that $\overline{\mathtt{Eval}}_{t=0}(\tilde{x}) = x_0$, $\overline{\mathtt{Eval}}_{t=1}(\tilde{x}) = x_0$. If $x_0, x_1 \in \mathrm{MC}_G(B)$ for some discrete submonoid G, we also require $\tilde{x} \in \mathrm{MC}_G(\widetilde{B})$.

Lemma-Definition 3.5.17. *$\sim_{\widetilde{\widetilde{B}}}$ is independent of choices of the model $\widetilde{\widetilde{B}}$ of $[0,1] \times \overline{B}$, and induces an equivalence relation $\sim$, called gauge equivalence, on $\mathrm{MC}(B)$. The same holds for $\mathrm{MC}_G(B)$.*

Proof. Viewing $\overline{B}$ as an L_∞ algebra, a dg Lie algebra model $\widetilde{\widetilde{B}}$ of $[0,1] \times \overline{B}$ is also an L_∞ algebra model. Then the result follows from Example 3.6.17 and Lemma-Definition 3.6.19. $\square$

We can now state the main result in this section.

Proposition 3.5.18. *Under the 1-1 correspondence in Lemma 3.4.13, pseudo-isotopy of (cyclic) filtered A_∞ algebras is related to gauge equivalence of Maurer-Cartan elements in dg Lie algebras in the following way.*

(i) Two G-gapped filtered A_∞ deformations $\mathfrak{m}^0, \mathfrak{m}^1$ of $(\overline{C}, \overline{\mathfrak{m}})$ are algebraically pseudo-isotopic if and only if the corresponding G-gapped Maurer-Cartan elements $\hat{\mathfrak{n}}^0, \hat{\mathfrak{n}}^1$ in $\big(\mathrm{Coder}(T(\overline{C}[1]))\widehat{\otimes}\Lambda_{0,\mathrm{nov}}, \mathrm{ad}_{\hat{\overline{\mathfrak{m}}}}, [,]\big)$ are gauge equivalent. The same holds in the cyclic setting, where $\mathrm{Coder}(T(\overline{C}[1]))$ is replaced by $\mathrm{Coder}_{\mathrm{cyc}}(T(\overline{C}[1]))$.

(ii) Two G-gapped filtered A_∞ deformations $\mathfrak{m}^0, \mathfrak{m}^1$ of $(\overline{C}, \overline{\mathfrak{m}}) = (\Omega(N), d, \wedge)$ are smoothly pseudo-isotopic if and only if the corresponding G-gapped Maurer-Cartan elements $\hat{\mathfrak{n}}^0, \hat{\mathfrak{n}}^1$ in $\big(\mathrm{Coder}_\diamond(T(\Omega(N)[1]))\widehat{\otimes}\Lambda_{0,\mathrm{nov}}, \mathrm{ad}_{\hat{\overline{\mathfrak{m}}}}, [,]\big)$ are gauge equivalent. The same holds in the cyclic setting, where $\mathrm{Coder}_\diamond(T(\Omega(N)[1]))$ is replaced by $\mathrm{Coder}_{\diamond,\mathrm{cyc}}(T(\Omega(N)[1]))$.

Corollary 3.5.19. *Algebraic pseudo-isotopy of gapped (cyclic) filtered A_∞ deformations of $(\overline{C}, \overline{\mathfrak{m}})$ is an equivalence relation. The same is true for smooth pseudo-isotopy of gapped (cyclic) filtered A_∞ deformations of $(\Omega(N), d, \wedge, \langle,\rangle_N)$.* $\square$

To prove Proposition 3.5.18, we need some preparation.

Let $(\overline{C},\overline{\mathfrak{m}}),(\overline{C}',\overline{\mathfrak{m}}')$ be unfiltered A_∞ algebras, fix an A_∞ homomorphism $\bar{\mathfrak{f}}:\overline{C}\to\overline{C}'$, and consider the space of coderivations with respect to $\hat{\bar{\mathfrak{f}}}$, see (3.5.1). The map

$$\begin{aligned}\delta_{\overline{\mathfrak{m}},\overline{\mathfrak{m}}'}:\mathrm{Coder}(T(\overline{C}[1]),T(\overline{C}'[1]);\hat{\bar{\mathfrak{f}}})&\to\mathrm{Coder}(T(\overline{C}[1]),T(\overline{C}'[1]);\hat{\bar{\mathfrak{f}}})\\ \hat{\varphi}&\mapsto\hat{\overline{\mathfrak{m}}}'\circ\hat{\varphi}-(-1)^{|\varphi|'}\hat{\varphi}\circ\hat{\overline{\mathfrak{m}}}\end{aligned}$$

is a differential, and gives rise to a version of Hochschild cochain complex ([18, Lemma-Definition 4.4.46]).

In the rest of this subsection, if $(\overline{C},\overline{\mathfrak{m}})$ is an unfiltered A_∞ algebra, then $(\tilde{\overline{C}},\tilde{\overline{\mathfrak{m}}})$ is a model of $[0,1]\times(\overline{C},\overline{\mathfrak{m}})$ discussed in Example 3.5.8 or Example 3.5.9.

Lemma-Definition 3.5.20. (dg Lie algebra model of $[0,1]\times\mathrm{Coder}(T(\overline{C}[1]))$).

(i) If $\tilde{\overline{C}}=\mathcal{A}([0,1])\otimes\overline{C}$*, then there is an injective cochain map*

$$\big(\mathrm{Coder}(T(\overline{C}[1]),T(\tilde{\overline{C}}[1]);\widehat{\mathrm{Incl}}),\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}\big)\to\big(\mathrm{Coder}(T(\tilde{\overline{C}}[1])),\mathrm{ad}_{\hat{\tilde{\overline{\mathfrak{m}}}}}\big)$$

whose image is a dg Lie subalgebra of $\big(\mathrm{Coder}(T(\tilde{\overline{C}}[1])),\mathrm{ad}_{\hat{\tilde{\overline{\mathfrak{m}}}}},[,]\big)$*. This dg Lie algebra, denoted by* $\mathrm{CH}(\overline{C},\tilde{\overline{C}})$*, is a model of* $[0,1]\times\mathrm{Coder}(T(\overline{C}[1]))$.

(ii) If $\tilde{\overline{C}}=\Omega([0,1]\times N)$*, then there is an injective cochain map*

$$\big(\mathrm{Coder}_\diamond(T(\Omega(N)[1]),T(\Omega([0,1]\times N)[1]);\widehat{\mathrm{pr}_N^*}),\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}\big)\to\big(\mathrm{Coder}(T(\Omega([0,1]\times N)[1])),\mathrm{ad}_{\hat{\tilde{\overline{\mathfrak{m}}}}}\big)$$

whose image is a dg Lie subalgebra of $\big(\mathrm{Coder}(T(\Omega([0,1]\times N)[1])),\mathrm{ad}_{\hat{\tilde{\overline{\mathfrak{m}}}}},[,]\big)$*. This dg Lie algebra, denoted by* $\mathrm{CH}_\diamond(\Omega(N),\Omega([0,1]\times N))$*, is a model of* $[0,1]\times\mathrm{Coder}_\diamond(T(\Omega(N)[1]))$.

Proof. (i) $\tilde{\overline{C}}[1]$ is a graded module over the graded $\mathbb{R}$-algebra $\big(\mathcal{A}([0,1]),\wedge\big)$. Let us define the desired $\mathbb{R}$-linear injection $\mathrm{Coder}(T(\overline{C}[1]),T(\tilde{\overline{C}}[1]);\widehat{\mathrm{Incl}})\to\mathrm{Coder}(T(\tilde{\overline{C}}[1]))$, $\hat{\varphi}\mapsto\tilde{\hat{\varphi}}$ by $\mathcal{A}([0,1])$-linear extension:

$$\begin{tikzcd}\mathrm{Coder}(T(\overline{C}[1]),T(\tilde{\overline{C}}[1]);\widehat{\mathrm{Incl}}) \arrow[r, dashed, "\cong", "\text{as }\mathbb{R}\text{-vs}"'] \arrow[d, equal] & \mathrm{Coder}_{\mathcal{A}([0,1])}(T_{\mathcal{A}([0,1])}(\tilde{\overline{C}}[1])) \arrow[r, "\subset"] \arrow[d, equal] & \mathrm{Coder}(T(\tilde{\overline{C}}[1])) \\ \prod_k\mathrm{Hom}(\overline{C}[1]^{\otimes k},\tilde{\overline{C}}[1]) \arrow[r, "\cong", "\text{as }\mathbb{R}\text{-vs}"'] & \prod_k\mathrm{Hom}_{\mathcal{A}([0,1])}(\tilde{\overline{C}}[1]^{\otimes_{\mathcal{A}([0,1])}k},\tilde{\overline{C}}[1]) & \end{tikzcd}$$

$$
\begin{tikzcd}
\hat{\varphi} \arrow[r, dashed, mapsto] \arrow[d, mapsto] & \hat{\tilde{\varphi}} \arrow[r, equal] & :\tilde{\hat{\varphi}} \\
\varphi = (\varphi_k)_{k\geq 0} \arrow[r, mapsto] & \tilde{\varphi} = (\tilde{\varphi}_k)_{k\geq 0} \arrow[u, mapsto] &
\end{tikzcd}
$$

It is clear that the commutator of $\mathcal{A}([0,1])$-linear coderivations is also $\mathcal{A}([0,1])$-linear, so $\mathrm{Coder}_{\mathcal{A}([0,1])}(T_{\mathcal{A}([0,1])}(\widetilde{\overline{C}}[1]))$ is a Lie subalgebra of $\mathrm{Coder}(T(\widetilde{\overline{C}}[1]))$. The induced Lie bracket on $\mathrm{Coder}(T(\overline{C}[1]), T(\widetilde{\overline{C}}[1]); \widehat{\overline{\mathrm{Incl}}})$ is simply

$$\{\hat{\varphi}, \hat{\varphi}'\} := \tilde{\hat{\varphi}} \circ \hat{\varphi}' - (-1)^{|\varphi|'|\varphi'|'} \tilde{\hat{\varphi}}' \circ \hat{\varphi}.$$

We now check $\hat{\varphi} \mapsto \tilde{\hat{\varphi}}$ is a $(\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}, \mathrm{ad}_{\hat{\tilde{\overline{\mathfrak{m}}}}})$-cochain map, namely verify the identity

$$\big(\hat{\tilde{\overline{\mathfrak{m}}}} \circ \hat{\varphi} - (-1)^{|\varphi|'} \hat{\varphi} \circ \hat{\overline{\mathfrak{m}}}\big)^{\sim} = [\hat{\tilde{\overline{\mathfrak{m}}}}, \tilde{\hat{\varphi}}] \in \mathrm{Coder}(T(\widetilde{\overline{C}}[1])).$$

It suffices to check that for each $l \geq 1$,

$$\big(\hat{\tilde{\overline{\mathfrak{m}}}}_l \circ \hat{\varphi} - (-1)^{|\varphi|'} \hat{\varphi} \circ \hat{\overline{\mathfrak{m}}}_l\big)^{\sim} = [\hat{\tilde{\overline{\mathfrak{m}}}}_l, \tilde{\hat{\varphi}}] \ : \ T(\widetilde{\overline{C}}[1]) \to T(\widetilde{\overline{C}}[1]). \tag{3.5.4}$$

Notice that LHS of (3.5.4) is $\mathcal{A}([0,1])$-linear, so it suffices to show RHS of (3.5.4) is also $\mathcal{A}([0,1])$-linear and the two sides have the same restriction to $\mathrm{Hom}(\overline{C}[1]^{\otimes k+l-1}, \widetilde{\overline{C}}[1])$ for each $k \geq 0$. There are two cases.

- If $l \geq 2$, by definition $\tilde{\overline{\mathfrak{m}}}_l = (\overline{\mathrm{Incl}}_1 \circ \overline{\mathfrak{m}}_l)^{\sim}$ is $\mathcal{A}([0,1])$-linear, so $[\hat{\tilde{\overline{\mathfrak{m}}}}_l, \tilde{\hat{\varphi}}] = [(\overline{\mathrm{Incl}}_1 \circ \overline{\mathfrak{m}}_l)^{\wedge}, \hat{\varphi}]^{\sim}$ is $\mathcal{A}([0,1])$-linear. Clearly both sides of (3.5.4) equal $\tilde{\overline{\mathfrak{m}}}_l \circ \hat{\varphi}_k - (-1)^{|\varphi_k|'} \varphi_k \circ \hat{\overline{\mathfrak{m}}}_l$ on $\overline{C}[1]^{\otimes k+l-1} \to \widetilde{\overline{C}}[1]$.

- If $l = 1$, $\tilde{\overline{\mathfrak{m}}}_1 = \mathrm{id}_{\mathcal{A}([0,1])} \otimes \overline{\mathfrak{m}}_1 + d_{\mathcal{A}([0,1])} \otimes \mathrm{id}_{\overline{C}[1]}$ has two parts. The first part $\mathrm{id}_{\mathcal{A}([0,1])} \otimes \overline{\mathfrak{m}}_1$ is $\mathcal{A}([0,1])$-linear, so by the same reason as the case $l \geq 2$, we only need to care about the second part. Let us write $d_{\mathcal{A}([0,1])} \otimes \mathrm{id}_{\overline{C}[1]}$ as d_t, and write left multiplication by $a \in \mathcal{A}([0,1])$ on $T_{\mathcal{A}([0,1])}(\widetilde{\overline{C}}[1])$ as L_a. By Leibniz rule, $\hat{d}_t$ is a well-defined $\mathbb{R}$-linear operator on $T_{\mathcal{A}([0,1])}(\widetilde{\overline{C}}[1])$, and

$$\hat{d}_t \circ L_a = L_{d_t a} + (-1)^{|a|} L_a \circ \hat{d}_t.$$

Therefore,

$$
\begin{aligned}
[\hat{d}_t, \tilde{\hat{\varphi}}] \circ L_a \;&=\; \hat{d}_t \circ \tilde{\hat{\varphi}} \circ L_a - (-1)^{|\varphi|'} \tilde{\hat{\varphi}} \circ \hat{d}_t \circ L_a \\
&= (-1)^{|\varphi|'|a|} \hat{d}_t \circ L_a \circ \tilde{\hat{\varphi}} - (-1)^{|\varphi|'} \tilde{\hat{\varphi}} \circ (L_{d_t a} + (-1)^{|a|} L_a \circ \hat{d}_t) \\
&= (-1)^{|\varphi|'|a|} (L_{d_t a} + (-1)^{|a|} L_a \circ \hat{d}_t) \circ \tilde{\hat{\varphi}} - (-1)^{|\varphi|'|a|} L_{d_t a} \circ \tilde{\hat{\varphi}} - (-1)^{|\varphi|'+|a|+|\varphi|'|a|} L_a \circ \tilde{\hat{\varphi}} \circ \hat{d}_t \\
&= (-1)^{(1+|\varphi|')|a|} (L_a \circ \hat{d}_t \circ \tilde{\hat{\varphi}} - (-1)^{|\varphi|'} L_a \circ \tilde{\hat{\varphi}} \circ \hat{d}_t) \;=\; (-1)^{(1+|\varphi|')|a|} L_a \circ [\hat{d}_t, \tilde{\hat{\varphi}}].
\end{aligned}
$$

This proves $[\hat{d}_t, \tilde{\hat{\varphi}}]$ is $\mathcal{A}([0,1])$-linear. As for the restrction to $\overline{C}[1]^{\otimes k} \to \tilde{\overline{C}}[1]$, clearly $(\hat{d}_t \circ \hat{\varphi}_k)\tilde{} = d_t \circ \varphi_k = [\hat{d}_t, \tilde{\hat{\varphi}}_k]$.

We have verified $\hat{\varphi} \mapsto \tilde{\hat{\varphi}}$ is a cochain map, so its image, $\mathrm{Coder}_{\mathcal{A}([0,1])}(T_{\mathcal{A}([0,1])}(\tilde{\overline{C}}[1]))$, is a subcomplex. Therefore $\mathrm{CH}(\overline{C}, \tilde{\overline{C}}) := \big(\mathrm{Coder}_{\mathcal{A}([0,1])}(T(\tilde{\overline{C}}[1])), \mathrm{ad}_{\tilde{\hat{\overline{\mathfrak{m}}}}}, [,]\big)$ is a dg Lie subalgebra of $\big(\mathrm{Coder}(T(\tilde{\overline{C}}[1])), \mathrm{ad}_{\tilde{\hat{\overline{\mathfrak{m}}}}}, [,]\big)$.

It remains to show $\mathrm{CH}(\overline{C}, \tilde{\overline{C}})$ is a model of $[0,1] \times \big(\mathrm{Coder}(T(\overline{C}[1])), \mathrm{ad}_{\hat{\overline{\mathfrak{m}}}}, [,]\big)$. Define

$$
\begin{aligned}
\overline{\mathtt{Incl}} : \mathrm{Coder}(T(\overline{C}[1])) &\to \mathrm{Coder}(T(\overline{C}[1]), T(\tilde{\overline{C}}[1]); \hat{\overline{\mathrm{Incl}}}) \\
\hat{\psi} &\mapsto \hat{\overline{\mathrm{Incl}}} \circ \hat{\psi}, \\
\overline{\mathtt{Eval}}_{t=t_0} : \mathrm{Coder}(T(\overline{C}[1]), T(\tilde{\overline{C}}[1]); \hat{\overline{\mathrm{Incl}}}) &\to \mathrm{Coder}(T(\overline{C}[1])) \\
\hat{\varphi} &\mapsto \hat{\overline{\mathrm{Eval}}}_{t=t_0} \circ \hat{\varphi}. \quad (t_0 = 0, 1)
\end{aligned}
$$

Let us check $\overline{\mathtt{Incl}}$, $\overline{\mathtt{Eval}}_{t=t_0} (t_0 = 0, 1)$ satisfy the desired properties.

- They are dg Lie algebra homomorphisms: First, since $\overline{\mathrm{Incl}}$, $\overline{\mathrm{Eval}}_{t=t_0}$ are A_∞ homomorphisms, $\overline{\mathtt{Incl}}$, $\overline{\mathtt{Eval}}_{t=t_0}$ are cochain maps. Next, to see they are Lie algebra homomorphisms, note that for $\hat{\psi}, \hat{\psi}' \in \mathrm{Coder}(T(\overline{C}[1]))$, there holds $\hat{\overline{\mathrm{Incl}}} \circ \hat{\psi} \circ \hat{\psi}' = (\hat{\overline{\mathrm{Incl}}} \circ \hat{\psi})\tilde{} \circ \hat{\overline{\mathrm{Incl}}} \circ \hat{\psi}'$, so

$$\overline{\mathtt{Incl}}([\hat{\psi}, \hat{\psi}']) = \hat{\overline{\mathrm{Incl}}} \circ (\hat{\psi} \circ \hat{\psi}' \pm \hat{\psi}' \circ \hat{\psi}) = \{\hat{\overline{\mathrm{Incl}}} \circ \hat{\psi}, \hat{\overline{\mathrm{Incl}}} \circ \hat{\psi}'\} = \{\overline{\mathtt{Incl}}(\hat{\psi}), \overline{\mathtt{Incl}}(\hat{\psi}')\}.$$

  For $\hat{\varphi}, \hat{\varphi}' \in \mathrm{Coder}(T(\overline{C}[1]), T(\tilde{\overline{C}}[1]); \hat{\overline{\mathrm{Incl}}})$, we have $\hat{\overline{\mathrm{Eval}}}_{t=t_0} \circ \tilde{\hat{\varphi}} = \hat{\overline{\mathrm{Eval}}}_{t=t_0} \circ \hat{\varphi} \circ \hat{\overline{\mathrm{Eval}}}_{t=t_0}$, so

$$\overline{\mathtt{Eval}}_{t=t_0}(\{\hat{\varphi}, \hat{\varphi}'\}) = \hat{\overline{\mathrm{Eval}}}_{t=t_0} \circ (\tilde{\hat{\varphi}} \circ \hat{\varphi}' \pm \tilde{\hat{\varphi}}' \circ \hat{\varphi}) = [\hat{\overline{\mathrm{Eval}}}_{t=t_0} \circ \hat{\varphi}, \hat{\overline{\mathrm{Eval}}}_{t=t_0} \circ \hat{\varphi}'].$$

- $\overline{\mathtt{Eval}}_{t=t_0} \circ \overline{\mathtt{Incl}} = \mathrm{id}_{\mathrm{Coder}(T(\overline{C}[1]))}$ since $\overline{\mathrm{Eval}}_{t=t_0} \circ \overline{\mathrm{Incl}} = \mathfrak{id}_{\overline{C}}$.

- To show $\overline{\mathtt{Eval}}_{t=t_0}$ is a quasi-isomorphism, simply notice that $\overline{\mathtt{Incl}}$ is a quasi-isomorphism by [18, Lemma 4.4.49][2], and $\overline{\mathtt{Eval}}_{t=t_0} \circ \overline{\mathtt{Incl}} = \mathrm{id}$.

- To show $\overline{\mathtt{Eval}}_{t=0} \oplus \overline{\mathtt{Eval}}_{t=1} : \mathrm{Coder}(T(\overline{C}[1]), T(\tilde{\overline{C}}[1]); \hat{\overline{\mathrm{Incl}}}) \to \mathrm{Coder}(T(\overline{C}[1]))^{\oplus 2}$ is surjective, it suffices to show that for each $k \geq 0$, the induced map

$$\mathrm{Hom}(\overline{C}^{\otimes k}, \tilde{\overline{C}}) \to \mathrm{Hom}(\overline{C}^{\otimes k}, \overline{C})^{\oplus 2}, \quad \varphi_k \mapsto ((\overline{\mathrm{Eval}}_{t=0})_1 \circ \varphi_k, (\overline{\mathrm{Eval}}_{t=1})_1 \circ \varphi_k)$$

  is surjective. Over a field, the surjection $(\overline{\mathrm{Eval}}_{t=0})_1 \oplus (\overline{\mathrm{Eval}}_{t=1})_1 : \tilde{\overline{C}}[1] \to \overline{C}[1]^{\oplus 2}$ is split surjective, so $\mathrm{Hom}(\overline{C}^{\otimes k}, \tilde{\overline{C}}) \to \mathrm{Hom}(\overline{C}^{\otimes k}, \overline{C})^{\oplus 2}$ is also split surjective.

(ii) Proof of this part is formally the same as part (i), while technically one has to be careful with smooth extendability. First, it is easy to see $\mathrm{Coder}_\diamond(T(\Omega(N)[1]), T(\Omega([0,1]\times N)[1]); \widehat{\mathrm{pr}_N^*})$ is a $\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}$-subcomplex of $\mathrm{Coder}(T(\Omega(N)[1]), T(\Omega([0,1]\times N)[1]); \widehat{\mathrm{pr}_N^*})$, since $\overline{\mathfrak{m}}, \tilde{\overline{\mathfrak{m}}}$ are made up of exterior differential and wedge product. Next, by the decomposition (3.5.3), there are linear maps

$$p^0, p^1 : \Omega([0,1]\times N) \to \Gamma([0,1]\times N, \mathrm{pr}_N^*\Omega_N)$$

such that $\eta = p^0\eta + dt \wedge p^1\eta$ for any $\eta \in \Omega([0,1]\times N)$. Then we define a linear map

$$\begin{aligned}
\mathrm{Ext}_\diamond : \mathrm{Hom}_\diamond(T(\Omega(N)[1]), \Omega([0,1]\times N)[1]) &\to \mathrm{Hom}(T(\Omega([0,1]\times N)[1]), \Omega([0,1]\times N)[1]) \\
\varphi = (\varphi_k)_{k\geq 0} &\mapsto \tilde{\varphi} = (\tilde{\varphi}_k)_{k\geq 0}
\end{aligned}$$

$$\begin{aligned}
\tilde{\varphi}_k(\eta_1 \otimes \cdots \otimes \eta_k)(t,x) &:= \varphi_k(i_t^* p^0\eta_1 \otimes \cdots \otimes i_t^* p^0\eta_k)(t,x) \\
&+ dt \wedge \sum_{1\leq j\leq k} (-1)^{|\varphi_k|' + \sum_{l=1}^{j-1} |p^0\eta_l|'} \varphi_k(i_t^* p^0\eta_1 \otimes \cdots \otimes i_t^* p^1\eta_j \otimes \cdots \otimes i_t^* p^0\eta_k)(t,x).
\end{aligned} \tag{3.5.5}$$

Notice that $\varphi_k(i_t^* p^0\eta_1 \otimes \cdots \otimes i_t^* p^0\eta_k)(t,x)$ may have dt component. By Remark 3.3.10(iii), $\tilde{\varphi}$ is well-defined. Clearly $\varphi_k = \tilde{\varphi}_k \circ (\mathrm{pr}_N^*)^{\otimes k}$, so $\mathrm{Ext}_\diamond$ is an injective. We call $\tilde{\varphi} = \mathrm{Ext}_\diamond(\varphi)$

[2] There is a typo in the proof of [18, Lemma 4.4.49], which makes the proof there shorter than it should be. Namely, the equation $\overline{\mathfrak{m}}_1 \circ \psi - (-1)^{\deg \psi} \psi \circ \hat{\overline{d}} = 0$ in the 4th-to-last line on [18, page 229] should really be $\overline{\mathfrak{m}} \circ \hat{\psi} - (-1)^{\deg \psi} \psi \circ \hat{\overline{d}} = 0$ (see [18, Lemma & Definition 4.4.46]). We will fix the error in part (ii) below.

the *smooth extension* of φ. It is easy to see $\tilde{\varphi}_k$ is $\mathbb{R}[dt]$-linear, in the sense that for any $\eta_1, \ldots, \eta_k \in \Omega([0,1] \times N)[1]$ and $j \in \{1, \ldots, k\}$,

$$\tilde{\varphi}_k(\eta_1 \otimes \cdots \otimes (dt \wedge \eta_j) \otimes \cdots \otimes \eta_k) = (-1)^{|\varphi_k|' + \sum_{l=1}^{j-1} |\eta_l|'} dt \wedge \tilde{\varphi}_k(\eta_1 \otimes \cdots \otimes \eta_j \otimes \cdots \otimes \eta_k).$$

Now we show that for any $\hat{\varphi}, \hat{\varphi}' \in \mathrm{Coder}_\diamond(T(\Omega(N)[1]), T(\Omega([0,1] \times N)[1]); \widehat{\mathrm{pr}_N^*})$, the commutator $[\tilde{\hat{\varphi}}, \tilde{\hat{\varphi}}'] \in \mathrm{Coder}(T(\Omega([0,1] \times N)[1]))$ also lies in the image of $\mathrm{Ext}_\diamond$.

Firstly, for $k \in \mathbb{Z}_{\geq 1}, k' \in \mathbb{Z}_{\geq 0}, j \in \{1, \ldots, k\}$, define

$$\varphi_k \tilde{\circ}_j \varphi'_{k'} \in \mathrm{Hom}(\Omega(N)[1]^{\otimes k+k'-1}, \Omega([0,1] \times N)[1])$$

as follows. For $\eta_1, \ldots, \eta_{k+k'-1} \in \Omega(N)[1]$,

$$\begin{aligned}
&(\varphi_k \tilde{\circ}_j \varphi'_{k'})(\eta_1 \otimes \cdots \otimes \eta_{k+k'-1})(t,x) \\
:= {}& (-1)^{|\varphi'_{k'}|' \sum_{l=1}^{j-1} |\eta_l|'} \tilde{\varphi}_k(\mathrm{pr}_N^* \eta_1 \otimes \cdots \otimes \varphi'_{k'}(\eta_j \otimes \cdots) \otimes \mathrm{pr}_N^* \eta_{j+k'} \otimes \cdots)(t,x) \\
= {}& (-1)^{|\varphi'_{k'}|' \sum_{l=1}^{j-1} |\eta_l|'} \varphi_k(\eta_1 \otimes \cdots \otimes i_t^* p^0 \varphi'_{k'}(\eta_j \otimes \cdots) \otimes \eta_{j+k'} \otimes \cdots)(t,x) \\
& + (-1)^{|\varphi_k|' + (|\varphi'_{k'}|'+1) \sum_{l=1}^{j-1} |\eta_l|'} dt \wedge \varphi_k(\eta_1 \otimes \cdots \otimes i_t^* p^1 \varphi'_{k'}(\eta_j \otimes \cdots) \otimes \eta_{j+k'} \otimes \cdots)(t,x).
\end{aligned}$$

We claim $\varphi_k \tilde{\circ}_j \varphi_{k'}$ is smoothly extendable. Indeed, similar to Remark 3.3.10(iii), for $m = 0, 1$ and any C^∞-manifold S,

$$\varphi_k(\eta_1^s \otimes \cdots \otimes i_t^* p^m \varphi'_{k'}(\eta_j^s \otimes \cdots \otimes \eta_{j+k'-1}^s)) \otimes \cdots \otimes \eta_{k+k'-1}^s)(t,x)$$

is C^∞ in $(s,t,x) \in S \times [0,1] \times N$ because of smooth extendability of $\varphi'_{k'}$ (with respect to S) and φ_k (with respect to $S \times [0,1]$), taking into account pull-back via the diagonal map $S \times [0,1] \times N \to S \times [0,1]^2 \times N$, $(s,t,x) \mapsto (s,t,t,x)$.

Secondly, for $k, k' \in \mathbb{Z}_{\geq 0}$, we claim that $[\tilde{\hat{\varphi}}_k, \tilde{\hat{\varphi}}'_{k'}]$ is the smooth extension of

$$\{\varphi_k, \varphi'_{k'}\} := \sum_{1 \leq j \leq k'} \varphi_k \tilde{\circ}_j \varphi'_{k'} - (-1)^{|\varphi_k|' |\varphi'_{k'}|'} \sum_{1 \leq j \leq k} \varphi'_{k'} \tilde{\circ}_j \varphi_k.$$

It suffices to check that for any $\eta_1, \ldots, \eta_{k+k'-1} \in \Omega([0,1] \times N)$, $j \in \{1, \ldots, k\}$,

$$\tilde{\varphi}_k(\eta_1 \otimes \cdots \otimes \tilde{\varphi}'_{k'}(\eta_j \otimes \cdots) \otimes \cdots)(t,x) = (-1)^{|\varphi'_{k'}|' \sum_{l=1}^{j-1} |\eta_l|'} (\varphi_k \tilde{\circ}_j \varphi'_{k'})\tilde{}(\eta_1 \otimes \cdots \otimes \eta_{k+k'-1})(t,x).$$

Since both sides are $\mathbb{R}[dt]$-linear, we may assume $p^1\eta_1 = \cdots = p^1\eta_{k+k'-1} = 0$. But then

$$
\begin{aligned}
&\tilde{\varphi}_k(\eta_1 \otimes \cdots \otimes \tilde{\varphi}'_{k'}(\eta_j \otimes \cdots) \otimes \cdots \otimes \eta_{k+k'-1})(t,x) \\
&= \varphi_k(i_t^* p^0 \eta_1 \otimes \cdots \otimes i_t^* p^0 \tilde{\varphi}'_{k'}(\eta_j \otimes \cdots) \otimes \cdots \otimes i_t^* p^0 \eta_{k+k'-1})(t,x) \\
&\quad + dt \wedge (-1)^{|\varphi_k|' + \sum_{l=1}^{j-1} |p^0\eta_l|'} \varphi_k(i_t^* p^0 \eta_1 \otimes \cdots \otimes i_t^* p^1 \tilde{\varphi}'_{k'}(\eta_j \otimes \cdots) \otimes \cdots \otimes i_t^* p^0 \eta_{k+k'-1})(t,x) \\
&= \varphi_k(i_t^* p^0 \eta_1 \otimes \cdots \otimes i_t^* p^0 \varphi'_{k'}(i_t^* p^0 \eta_j \otimes \cdots) \otimes \cdots \otimes i_t^* p^0 \eta_{k+k'-1})(t,x) \\
&\quad + dt \wedge (-1)^{|\varphi_k|' + \sum_{l=1}^{j-1} |p^0\eta_l|'} \varphi_k(i_t^* p^0 \eta_1 \otimes \cdots \otimes i_t^* p^1 \varphi'_{k'}(i_t^* p^0 \eta_j \otimes \cdots) \otimes \cdots \otimes i_t^* p^0 \eta_{k+k'-1})(t,x) \\
&= (-1)^{|\varphi'_{k'}|' \sum_{l=1}^{j-1} |\eta_l|'} \widetilde{(\varphi_k \tilde{\circ}_j \varphi'_{k'})}(\eta_1 \otimes \cdots \otimes \eta_{k+k'-1})(t,x).
\end{aligned}
$$

We have thus proved the image of $\mathrm{Ext}_\diamond$ is a Lie subalgebra of $\mathrm{Coder}(T(\Omega([0,1]\times N)[1]))$. Next we need to show $\mathrm{Ext}_\diamond$ is a $(\delta_{\bar{\mathfrak{m}},\tilde{\bar{\mathfrak{m}}}}, \mathrm{ad}_{\hat{\bar{\mathfrak{m}}}})$-cochain map. It suffices to prove a counterpart of (3.5.4), which is even easier, since here we are dealing with $\mathbb{R}[dt]$-linearity instead of $\mathcal{A}([0,1])$-linearity, and dg algebras instead of A_∞ algebras. Therefore we omit the details.

It remains to show $\mathrm{CH}_\diamond(\Omega(N), \Omega([0,1]\times N))$ is a model of $[0,1]\times \mathrm{CH}_\diamond(\Omega(N),\Omega(N))$. For $t_0 = 0, 1$, because of Remark 3.3.10(i)(ii), we can define

$$
\begin{gathered}
\overline{\mathtt{Incl}} : \mathrm{Coder}_\diamond(T(\Omega(N)[1])) \to \mathrm{Coder}_\diamond(T(\Omega(N)[1]), T(\Omega([0,1]\times N)[1]); \widehat{\mathrm{pr}_N^*}) \\
\hat{\psi} \mapsto \widehat{\mathrm{pr}_N^*} \circ \hat{\psi}, \\
\overline{\mathtt{Eval}}_{t=t_0} : \mathrm{Coder}_\diamond(T(\Omega(N)[1]), T(\Omega([0,1]\times N)[1]); \widehat{\mathrm{pr}_N^*}) \to \mathrm{Coder}_\diamond(T(\Omega(N)[1])) \\
\hat{\varphi} \mapsto \widehat{i_{t_0}^*} \circ \hat{\varphi}.
\end{gathered}
$$

Most of the rest is literally the same as part (i), except for the following two things.

First, we need to prove that for each $k \geq 0$, the map

$$
\mathrm{Hom}_\diamond(\Omega(N)^{\otimes k}, \Omega([0,1]\times N)) \to \mathrm{Hom}_\diamond(\Omega(N)^{\otimes k}, \Omega(N))^{\oplus 2}, \quad \varphi_k \mapsto (i_0^* \circ \varphi_k, i_1^* \circ \varphi_k).
$$

is surjective. If $\psi_k, \psi'_k \in \mathrm{Hom}_\diamond(\Omega(N)^{\otimes k}, \Omega(N))$, define $\varphi_k \in \mathrm{Hom}(\Omega(N)^{\otimes k}, \Omega([0,1]\times N))$ by

$$
\varphi_k(\eta_1 \otimes \cdots \otimes \eta_k)(t,x) := (1-t)\cdot \mathrm{pr}_N^*(\psi_k(\eta_1 \otimes \cdots \otimes \eta_k))(t,x) + t \cdot \mathrm{pr}_N^*(\psi'_k(\eta_1 \otimes \cdots \otimes \eta_k))(t,x).
$$

Then φ_k is clearly smoothly extendable and $i_0^* \circ \varphi_k = \psi_k$, $i_1^* \circ \varphi_k = \psi'_k$.

It remains to prove $\overline{\texttt{Incl}}$ is a quasi-isomorphism. Since $\overline{\texttt{Eval}}_{t=0} \circ \overline{\texttt{Incl}} = \mathrm{id}_{\mathrm{Coder}_\diamond(T(\Omega(N)[1]))}$, the induced map $\overline{\texttt{Incl}}_*$ between cohomology groups is injective, so it suffices to prove its surjectivity. Recall the differential $\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}$ on $\mathrm{Hom}_\diamond(T(\Omega(N)[1]), \Omega([0,1]\times N)[1])$ is defined by $\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}(\varphi) = \hat{\tilde{\overline{\mathfrak{m}}}} \circ \hat{\varphi} - (-1)^{|\varphi|'}\varphi \circ \hat{\overline{\mathfrak{m}}}$. Write

$$\delta_1(\varphi) := \delta_{\overline{\mathfrak{m}}_1,\tilde{\overline{\mathfrak{m}}}_1}(\varphi) = \tilde{\overline{\mathfrak{m}}}_1 \circ \varphi - (-1)^{|\varphi|'}\varphi \circ \hat{\overline{\mathfrak{m}}}_1.$$

Notice that $\mathrm{pr}_N^* \circ i_0^*$ is $\tilde{\overline{\mathfrak{m}}}_1$-cochain homotopic to identity:

$$\mathrm{id}_{\Omega([0,1]\times N)} - \mathrm{pr}_N^* \circ i_0^* = d \circ h + h \circ d, \qquad h(\eta) := \int_0^t dt \wedge p^1\eta. \tag{3.5.6}$$

Moreover,

$$\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}(\mathrm{pr}_N^* \circ i_0^* \circ \varphi) = \mathrm{pr}_N^* \circ i_0^* \circ \delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}(\varphi). \tag{3.5.7}$$

Let $\varphi \in \mathrm{Hom}_\diamond(T(\Omega(N)[1]), \Omega([0,1]\times N)[1])$ be a $\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}$-cocycle. Define a sequence $\{\varphi^{(i)}\}_{i\in\mathbb{Z}_{\geq 0}}$ in $\mathrm{Hom}_\diamond(T(\Omega(N)[1]), \Omega([0,1]\times N)[1])$ by

$$\varphi^{(0)} := \varphi, \quad \varphi^{(i+1)} := \varphi^{(i)} - \mathrm{pr}_N^* \circ i_0^* \circ \varphi^{(i)} - \delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}(h \circ \varphi_i^{(i)}). \tag{3.5.8}$$

We shall prove by induction that for any i,

$$\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}(\varphi^{(i)}) = 0, \qquad \varphi_k^{(i)} = 0 \ (\forall k < i). \tag{3.5.9}$$

The case $i = 0$ is obvious. If (3.5.9) holds for i, then by (3.5.8) and (3.5.7), we have $\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}(\varphi^{(i+1)}) = 0$ and $\varphi_k^{(i+1)} = 0$ $(\forall k < i)$. Moreover,

$$\delta_1(\varphi_i^{(i)}) = \tilde{\overline{\mathfrak{m}}}_1 \circ \varphi_i^{(i)} - (-1)^{|\varphi|'}\varphi_i^{(i)} \circ \hat{\overline{\mathfrak{m}}}_1 = \delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}(\varphi^{(i)})\big|_{\Omega(N)[1]^{\otimes i}} = 0.$$

It follows that

$$\begin{aligned}
\varphi_i^{(i+1)} &= \varphi_i^{(i)} - \mathrm{pr}_N^* \circ i_0^* \circ \varphi_i^{(i)} - \delta_1(h \circ \varphi_i^{(i)}) \\
&\overset{(3.5.6)}{=\!=} \tilde{\overline{\mathfrak{m}}}_1 \circ h \circ \varphi_i^{(i)} + h \circ \tilde{\overline{\mathfrak{m}}}_1 \circ \varphi_i^{(i)} - \delta_1(h \circ \varphi_i^{(i)}) \\
&= \tilde{\overline{\mathfrak{m}}}_1 \circ h \circ \varphi_i^{(i)} + (-1)^{|\varphi|'} h \circ \varphi_i^{(i)} \circ \hat{\overline{\mathfrak{m}}}_1 - \delta_1(h \circ \varphi_i^{(i)}) = 0.
\end{aligned}$$

So (3.5.9) holds for $i+1$. Thus (3.5.9) holds for all i. Now we set $\varphi' := \sum_{i\geq 0} \varphi^{(i)}$ and $\varphi'' := \sum_{i\geq 0} h\circ\varphi_i^{(i)}$. By construction, φ', φ'' are elements in $\mathrm{Hom}_\diamond(T(\Omega(N)[1]), \Omega([0,1]\times N)[1])$, satisfying

$$\varphi = \mathrm{pr}_N^* \circ i_0^* \circ \varphi' + \delta_{\overline{\mathfrak{m}},\tilde{\tilde{\mathfrak{m}}}}(\varphi'') = \overline{\mathtt{Incl}}(\overline{\mathtt{Eval}}_{t=0}(\varphi')) + \delta_{\overline{\mathfrak{m}},\tilde{\tilde{\mathfrak{m}}}}(\varphi'').$$

This shows $\overline{\mathtt{Incl}}_*$ is surjective. $\square$

Next, we discuss cyclic A_∞ algebras.

Definition 3.5.21. Let $(\overline{C}, \overline{\mathfrak{m}}, \langle,\rangle)$ be an unfiltered cyclic A_∞ algebra.

(i) If $\tilde{\tilde{\overline{C}}} = \mathcal{A}([0,1]) \otimes \overline{C}$, then $\langle,\rangle$ extends $\mathcal{A}([0,1])$-linearly to a $\mathcal{A}([0,1])$-valued graded symmetric bilinear form on $\tilde{\tilde{\overline{C}}}$. We say $\varphi_k \in \mathrm{Hom}(\overline{C}[1]^{\otimes k}, \tilde{\tilde{\overline{C}}}[1])$ is *cyclic*, if for any $x_1, \dots, x_k, x_0 \in \tilde{\tilde{\overline{C}}}[1]$, there holds

$$\langle \tilde{\varphi}_k(x_1 \otimes \cdots \otimes x_k), x_0\rangle' = -(-1)^{|x_0|'(|x_1|'+\cdots+|x_k|')}\langle \tilde{\varphi}_k(x_0 \otimes x_1 \otimes \cdots \otimes x_{k-1}), x_k\rangle',$$

where $\tilde{\varphi}_k$ is the $\mathcal{A}([0,1])$-linear extension of φ_k.

(ii) If $(\overline{C}, \overline{\mathfrak{m}}, \langle,\rangle) = (\Omega(N), d, \wedge, \langle,\rangle_N)$ where N is a closed oriented smooth manifold, and $\tilde{\tilde{\overline{C}}} = \Omega([0,1]\times N)$. Then $\langle,\rangle_N$ extends to a $\Omega([0,1])$-valued graded symmetric bilinear form on $\Omega([0,1]\times N)$ by

$$\langle \alpha_1 + dt\wedge\gamma_1, \alpha_2 + dt\wedge\gamma_2\rangle_N := \int_N \alpha_1\wedge\alpha_2 + dt\int_N (-1)^{|\alpha_1|}\alpha_1\wedge\gamma_2 + \gamma_1\wedge\alpha_2,$$

where $\alpha_i, \gamma_i \in \Gamma([0,1]\times N, \mathrm{pr}_N^*\Omega_N)$. We say $\varphi_k \in \mathrm{Hom}_\diamond(\Omega(N)[1]^{\otimes k}, \Omega([0,1]\times N)[1])$ is *cyclic*, if for any $\eta_1, \dots, \eta_k, \eta_0 \in \Omega([0,1]\times N)[1]$, there holds

$$\langle \tilde{\varphi}_k(\eta_1 \otimes \cdots \otimes \eta_k), v_0\rangle_N' = -(-1)^{|\eta_0|'(|\eta_1|'+\cdots+|\eta_k|')}\langle \tilde{\varphi}_k(\eta_0 \otimes \eta_1 \otimes \cdots \otimes \eta_{k-1}), \eta_k\rangle_N',$$

where $\tilde{\varphi}_k = \mathrm{Ext}_\diamond(\varphi_k)$ (3.5.5).

Lemma-Definition 3.5.22. (dg Lie algebra model of $[0,1]\times \mathrm{Coder}_{\mathrm{cyc}}(T(\overline{C}[1]))$).

(i) For $(\overline{C}, \overline{\mathfrak{m}}, \langle,\rangle)$, the subspace of cyclic elements in $\mathrm{CH}(\overline{C}, \widetilde{\widetilde{C}})$ *is a dg Lie subalgebra, and is a model of* $[0,1] \times \mathrm{Coder}_{\mathrm{cyc}}(T(\overline{C}[1]))$*, where* $\overline{\mathtt{Incl}}$, $\overline{\mathtt{Eval}}_{t=0}$, $\overline{\mathtt{Eval}}_{t=1}$ *are restrictions of those of* $\mathrm{CH}(\overline{C}, \widetilde{\widetilde{C}})$*.*

(ii) For $(\Omega(N), d, \wedge, \langle,\rangle_N)$, the subspace of cyclic elements in $CH_\diamond(\Omega(N), \Omega([0,1] \times N))$ is a dg Lie subalgebra, and is a model of $[0,1] \times \mathrm{CH}_{\diamond,\mathrm{cyc}}(\Omega(N), \Omega(N))$*, where* $\overline{\mathtt{Incl}}$, $\overline{\mathtt{Eval}}_{t=0}$, $\overline{\mathtt{Eval}}_{t=1}$ *are restrictions of those of* $\mathrm{CH}_\diamond(\Omega(N), \Omega([0,1] \times N))$*.*

Here, cyclic elements are defined in Definition 3.5.21, $\mathrm{CH}(\overline{C}, \widetilde{\widetilde{C}})$, $\mathrm{CH}_\diamond(\Omega(N), \Omega([0,1] \times N))$ *are defined in Lemma-Definition 3.5.20.*

Proof. It is easy to see the proof of Lemma-Definition 3.5.20 also works for cyclic elements. □

Proof of Proposition 3.5.18. The proofs of (i)(ii) are literally the same, so we only write the proof of (i). Let $(\widetilde{\widetilde{C}} = \mathcal{A}([0,1]) \otimes \overline{C}, \tilde{\overline{\mathfrak{m}}})$ be the model of $[0,1] \times (\overline{C}, \overline{\mathfrak{m}})$ (Example 3.5.8). We will show that there is a bijection between:

- Algebraic G-gapped (cyclic) pseudo-isotopies $\{\mathfrak{m}^t_{k,\beta}, \mathfrak{c}^t_{k,\beta}\}^{t\in[0,1]}_{k\in\mathbb{Z}_{\geq 0},\beta\in G}$ of G-gapped filtered A_∞ deformations of $(\overline{C}, \overline{\mathfrak{m}})$ between $\mathfrak{m}^0, \mathfrak{m}^1$;
- G-gapped (cyclic) Maurer-Carten elements $\{\mathfrak{N} = \mathfrak{N}_{k,\beta}\}^{\beta\in G\setminus\{(0,0)\}}_{k\in\mathbb{Z}_{\geq 0}}$ in $\mathrm{CH}(\overline{C}, \widetilde{\widetilde{C}})\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$ ($\mathrm{CH}_{\mathrm{cyc}}(\overline{C}, \widetilde{\widetilde{C}})\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$ in the cyclic setting) satisfying $\overline{\mathtt{Eval}}_{t=t_0}(\hat{\mathfrak{N}}) = \hat{\overline{\mathfrak{m}}} - \hat{\mathfrak{m}}^{t_0}$ ($t_0 = 0, 1$).

In the following, Conditions (a)(b)(c)(d) refer to the conditions in Definition 3.5.15. For the sake of convenience, we divide Condition (b) into two parts: (b1) $\mathfrak{m}^t_{k,(0,0)} = \overline{\mathfrak{m}}_k$ for all $k \in \mathbb{Z}_{\geq 0}, t \in [0,1]$, and (b2) $\mathfrak{m}^t$ satisfies A_∞ relations $\forall t \in [0,1]$.

First, if $\big\{\mathfrak{m}^t_{k,\beta} \in \mathrm{Hom}^{1-\mu(\beta)}(\overline{C}[1]^{\otimes k}, \overline{C}[1]), \mathfrak{c}^t_{k,\beta} \in \mathrm{Hom}^{-\mu(\beta)}(\overline{C}[1]^{\otimes k}, \overline{C}[1])\big\}^{t\in[0,1]}_{k\in\mathbb{Z}_{\geq 0},\beta\in G}$ satisfy Conditions (a)(c), then they determine $\big\{\mathfrak{N}_{k,\beta} \in \mathrm{Hom}^{1-\mu(\beta)}(\overline{C}[1]^{\otimes k}, \widetilde{\widetilde{C}}[1])\big\}_{k\in\mathbb{Z}_{\geq 0},\beta\in G\setminus\{(0,0)\}}$ by

$$\mathfrak{m}^t_{k,\beta} + dt \wedge \mathfrak{c}^t_{k,\beta} = -\mathfrak{N}_{k,\beta} \quad (\beta \neq (0,0)), \tag{3.5.10}$$

and vice versa. Moreover, if $\mathfrak{m}^t_{k,\beta}$ is cyclic $\forall t \in [0,1]$, then $\mathfrak{c}^t_{k,\beta}$ is cyclic $\forall t \in [0,1]$ iff $\mathfrak{N}_{k,\beta}$ is cyclic. Next, (3.5.10) implies $\overline{\mathtt{Eval}}_{t=t_0}(\mathfrak{N}_{k,\beta}) = -\mathfrak{m}^{t_0}_{k,\beta}$ ($\beta \neq (0,0)$), so Condition (b1) just says

$\overline{\texttt{Eval}}_{t=t_0}(\hat{\mathfrak{N}}) = \hat{\overline{\mathfrak{m}}} - \hat{\mathfrak{m}}^{t_0} =: \hat{\mathfrak{n}}^{t_0}$. It remains to show that $\{\mathfrak{m}^t_{k,\beta}, \mathfrak{c}^t_{k,\beta}\}^{t\in[0,1]}_{k\in\mathbb{Z}_{\geq 0},\beta\in G}$ satisfy Conditions (b2)(d) iff $\hat{\mathfrak{N}} = \sum_\beta T^{E(\beta)} e^{\frac{\mu(\beta)}{2}}(\hat{\mathfrak{N}}_{k,\beta})_k$ is Maurer-Cartan in $\mathrm{CH}(\overline{C}, \tilde{\overline{C}})\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$, namely

$$\delta_{\overline{\mathfrak{m}},\tilde{\overline{\mathfrak{m}}}}\mathfrak{N} - \frac{1}{2}\{\mathfrak{N},\mathfrak{N}\} = \tilde{\overline{\mathfrak{m}}} \circ \hat{\mathfrak{N}} + \mathfrak{N} \circ \hat{\overline{\mathfrak{m}}} + \tilde{\mathfrak{N}} \circ \hat{\mathfrak{N}} = 0 \in \mathrm{Hom}(T(\overline{C}[1]), \tilde{\overline{C}})\widehat{\otimes}\Lambda_{0,\mathrm{nov}}, \tag{3.5.11}$$

where $\tilde{\mathfrak{N}}$ is the $\mathcal{A}([0,1])$-linear extension of $\mathfrak{N}$. By (3.5.10), that $\hat{\mathfrak{N}}$ satisfies (3.5.11) means for each $k \in \mathbb{Z}_{\geq 0}$, $\beta \in G \setminus \{(0,0)\}$, $x_1, \ldots, x_k \in \overline{C}[1]$, the following summation is zero (here $\varepsilon_i := |x_1|' + \cdots + |x_{i-1}|'$):

$$\sum_{\substack{k_1+k_2=k+1\\1\leq i\leq k_1}} \Big((-1)^{\varepsilon_i}\tilde{\overline{\mathfrak{m}}}_{k_1}(x_1 \otimes \cdots \otimes \mathfrak{m}^t_{k_2,\beta}(x_i \otimes \cdots) \otimes \cdots \otimes x_k) \tag{3.5.12a}$$

$$+ dt \wedge \tilde{\overline{\mathfrak{m}}}_{k_1}(x_1 \otimes \cdots \otimes \mathfrak{c}^t_{k_2,\beta}(x_i \otimes \cdots) \otimes \cdots \otimes x_k) \tag{3.5.12b}$$

$$+ (-1)^{\varepsilon_i}\mathfrak{m}^t_{k_1,\beta}(x_1 \otimes \cdots \otimes \overline{\mathfrak{m}}_{k_2}(x_i \otimes \cdots) \otimes \cdots \otimes x_k) \tag{3.5.12c}$$

$$+ (-1)^{\varepsilon_i} dt \wedge \mathfrak{c}^t_{k_1,\beta}(x_1 \otimes \cdots \otimes \overline{\mathfrak{m}}_{k_2}(x_i \otimes \cdots) \otimes \cdots \otimes x_k)\Big) \tag{3.5.12d}$$

$$+ \sum_{\substack{k_1+k_2=k+1\\1\leq i\leq k_1}} \sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_1,\beta_2\neq(0,0)}} \Big((-1)^{\varepsilon_i}\mathfrak{m}^t_{k_1,\beta_1}(x_1 \otimes \cdots \otimes \mathfrak{m}^t_{k_2,\beta_2}(x_i \otimes \cdots) \otimes \cdots \otimes x_k) \tag{3.5.12e}$$

$$+ dt \wedge \mathfrak{m}^t_{k_1,\beta_1}(x_1 \otimes \cdots \otimes \mathfrak{c}^t_{k_2,\beta_2}(x_i \otimes \cdots) \otimes \cdots \otimes x_k) \tag{3.5.12f}$$

$$+ (-1)^{\varepsilon_i} dt \wedge \mathfrak{c}^t_{k_1,\beta_1}(x_1 \otimes \cdots \otimes \mathfrak{m}^t_{k_2,\beta_2}(x_i \otimes \cdots) \otimes \cdots \otimes x_k)\Big). \tag{3.5.12g}$$

Since $\tilde{\overline{\mathfrak{m}}}_1 = d_{\mathcal{A}([0,1])} \otimes \mathrm{id}_{\overline{C}[1]} + \mathrm{id}_{\mathcal{A}([0,1])} \otimes \overline{\mathfrak{m}}_1$ and $\tilde{\overline{\mathfrak{m}}}_k = (\overline{\mathrm{Incl}}_1 \circ \overline{\mathfrak{m}}_k)\tilde{}\ (k \geq 2)$,

$$\begin{aligned}
&(3.5.12\mathrm{a}) + (3.5.12\mathrm{c}) + (3.5.12\mathrm{e})\\
&= dt \wedge \frac{d}{dt}(\mathfrak{m}^t_{k,\beta}(x_1 \otimes \cdots)) + \sum_{\substack{k_1+k_2=k+1\\1\leq i\leq k_1}} \sum_{\beta_1+\beta_2=\beta} (-1)^{\varepsilon_i}\mathfrak{m}^t_{k_1,\beta_1}(x_1 \otimes \cdots \otimes \mathfrak{m}^t_{k_2,\beta_2}(x_i \otimes \cdots) \otimes \cdots),
\end{aligned}$$

$$(3.5.12\mathrm{b}) + (3.5.12\mathrm{f}) = \sum_{\substack{k_1+k_2=k+1\\1\leq i\leq k_1}} \sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_2\neq(0,0)}} dt \wedge \mathfrak{m}^t_{k_1,\beta_1}(x_1 \otimes \cdots \otimes \mathfrak{c}^t_{k_2,\beta_2}(x_i \otimes \cdots) \otimes \cdots),$$

$$(3.5.12\mathrm{d}) + (3.5.12\mathrm{g}) = \sum_{\substack{k_1+k_2=k+1\\1\leq i\leq k_1}} \sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_1\neq(0,0)}} (-1)^{\varepsilon_i} dt \wedge \mathfrak{c}^t_{k_1,\beta_1}(x_1 \otimes \cdots \otimes \mathfrak{m}^t_{k_2,\beta_2}(x_i \otimes \cdots) \otimes \cdots).$$

By looking at the terms containing dt or not separately, we conclude that $\hat{\mathfrak{N}}$ is a Maurer-Cartan element iff Conditions (b2)(d) are satisfied. $\square$

3.6 Homological algebra of L_∞ algebras

The theory of L_∞ algebras ia parallel to that of A_∞ algebras. It is sufficient for us to deal with filtered L_∞ algebras that are filtered in the simplest way, i.e. the (completed) tensor product of an unfiltered L_∞ algebra with $\Lambda_{0,\mathrm{nov}}$. So we restrict discussion to this special case.

3.6.1 Coderivations on the reduced symmetric coalgebra

Let V be a graded vector space, the *reduced symmetric algebra* of V, denoted by $S_+(V)$, is the quotient of the reduced tensor algebra $T_+(V)$ by the homogeneous ideal generated by elements of the form

$$u \otimes v - (-1)^{|u||v|} v \otimes v, \qquad u, v \in V \text{ homogeneous.}$$

Equivalently, $S_+(V) = \bigoplus_{k\geq 1}(V^{\otimes k}/\mathfrak{S}_k)$, where the symmetric group $\mathfrak{S}_k$ acts on $V^{\otimes k}$ by

$$\sigma(v_1 \otimes \cdots \otimes v_k) := \epsilon(\sigma; v_1, \ldots, v_k)\ v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}, \quad \sigma \in \mathfrak{S}_k,$$

where $\epsilon(\sigma; v_1, \ldots, v_k)$ is Koszul sign:

$$\epsilon(\sigma; v_1, \ldots, v_k) := (-1)^{\sum_{i<j, \sigma(i)>\sigma(j)} |v_i||v_j|}.$$

There is a natural identification between $V^{\otimes k}/\mathfrak{S}_k$ and the fixed point set $(V^{\otimes k})_{\mathfrak{S}_k}$, say

$$[v_1 \otimes \cdots \otimes v_k] \mapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma(v_1 \otimes \cdots \otimes v_k) =: v_1 \odot \cdots \odot v_k.$$

$v_1 \odot \cdots \odot v_k$ is the *symmetric product* of $v_1, \ldots, v_n$. Let us write $V^{\odot k} := V^{\otimes k}/\mathfrak{S}_k \cong (V^{\otimes k})_{\mathfrak{S}_k}$.

Before defining a graded coalgebra structure on $S_+(V)$, we recall the notion of (un)shuffles.

Definition 3.6.1. Let $r \in \mathbb{Z}_{\geq 1}$, $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$ and $k = k_1 + \cdots + k_r \geq 1$. The set of *$r$-unshuffles* of type $(k_1, \ldots, k_r)$, denoted by $\mathrm{Sh}(k_1, \ldots, k_r)$, consists of permutations $\sigma \in \mathfrak{S}_k$ such that

$$\sigma(i) < \sigma(i+1) \ \ \forall i \in \{1, \ldots, k\} \setminus \{k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_{r-1}\}.$$

Since an r-unshuffle of type $(k_1, \ldots, k_r)$ is uniquely determined by successive choices of $k_1, \ldots, k_r$ elements out of $\{1, \ldots, k_1 + \cdots + k_r\}$, the cardinality of $\mathrm{Sh}(k_1, \ldots, k_r)$ is $(k_1 + \cdots + k_r)!/(k_1! \cdots k_r!)$.

We are now ready to define a degree 0 linear map $\Delta : S_+(V) \to S_+(V) \otimes S_+(V)$ by

$$\Delta(v_1 \odot \cdots \odot v_n) := \sum_{\substack{1 \le k \le n-1 \\ \sigma \in \mathrm{Sh}(k, n-k)}} \epsilon(\sigma; v_{1\ldots n}) \ (v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}),$$

where $\epsilon(\sigma; v_{1\ldots n})$ is abbreviation for $\epsilon(\sigma; v_1, \ldots, v_n)$. Then

$$(1 \otimes \Delta)(\Delta(v_1 \odot \cdots \odot v_n)) \ = \ (\Delta \otimes 1)(\Delta(v_1 \odot \cdots \odot v_n))$$
$$= \sum_{\substack{k_1+k_2+k_3=n \\ \sigma \in \mathrm{Sh}(k_1,k_2,k_3)}} \epsilon(\sigma; v_{1\ldots n}) \ (v_{\sigma_1} \odot \cdots \odot v_{\sigma_{k_1}}) \otimes (v_{\sigma_{k_1+1}} \odot \cdots \odot v_{\sigma_{k_1+k_2}}) \otimes (v_{\sigma_{k_1+k_2+1}} \odot \cdots \odot v_{\sigma_n}),$$

so $(S_+(V), \Delta)$ is a graded coalgebra (without counit). Moreover, there is a natural correspondence between $\mathrm{Hom}(S_+(V), V)$ and $\mathrm{Coder}(S_+(V))$ as follows.

- For any $k \ge 1$, a graded linear map $\varphi_k : V^{\odot k} \to V$ extends to a coderivation $\hat{\varphi}_k$:
  - If $n \ge k$, $\hat{\varphi}_k(v_1 \odot \cdots \odot v_n) :=$
  $$\sum_{\sigma \in \mathrm{Sh}(k,n-k)} \epsilon(\sigma; v_{1\ldots n}) \ \varphi_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}.$$
  - If $n < k$, $\hat{\varphi}_k(v_1 \odot \cdots \odot v_n) := 0$.
- For any coderivation $\hat{\varphi}$ on $S_+(V)$, denote its $\mathrm{Hom}(V^{\odot k}, V)$ component by φ_k, then $\hat{\varphi}$ can be recovered as $\hat{\varphi} = \sum_{k \ge 1} \hat{\varphi}_k$. This is a finite sum when evaluating on $S_+(V)$.

If we include $\varphi_0 = 0 \in \mathrm{Hom}(\mathbb{R}, V)$ into $(\varphi_k)_{k \ge 1}$, then $\varphi(1) = \varphi_0(1) = 0$, $\hat{\varphi}(1) = \hat{\varphi}_0(1) = 0$.

Let V, W be two graded vector spaces, there is a natural correspondence between $\mathrm{Hom}(S_+(V), W)$ and $\mathrm{Hom}((S_+(V), \Delta), (S_+(W), \Delta))$: Any $f = (f_k)_{k \ge 1} \in \prod_{k \ge 1} \mathrm{Hom}(V^{\odot k}, W)$ can be uniquely extended to a coalgebra homomorphism $\hat{f} : S_+(V) \to S_+(W)$ by

$$\hat{f}(v_1 \odot \cdots \odot v_n) := \sum_{\substack{m \ge 1, \ k_1 + \cdots + k_m = n \\ \sigma \in \mathrm{Sh}(k_1, \ldots, k_m)}} \frac{\epsilon(\sigma; v_{1\ldots n})}{m!} f_{k_1}(\odot_{j_1=1}^{k_1} v_{\sigma(j_1)}) \odot \cdots \odot f_{k_m}(\odot_{j_m=n-k_m+1}^{n} v_{\sigma(j_m)}).$$

If we include $f_0 = 0 \in \mathrm{Hom}(\mathbb{R}, W)$ into $(f_k)_{k\geq 1}$, then $f(1) = f_0(1) = 0$, $\hat{f}(1) = \hat{f}_0(1) = 1$.

For $m_1 \in \mathbb{Z}_{\geq 1}$, $m_2 \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, $m_1 \leq m_2$, denote $S_{m_1}^{m_2}(V) := \bigoplus_{k=m_1}^{m_2} V^{\odot k}$, then $S_1^{m_2}(V)$ is a subcoalgebra of $S_+(V)$, and $S_{m_1}^{m_2}(V) \cong S_1^{m_2}(V)/S_1^{m_1-1}(V)$ is a quotient coalgebra of $S_1^{m_2}(V)$. Moreover:

- Every $\hat{\varphi} = \sum_{k\geq 1} \hat{\varphi}_k \in \mathrm{Coder}(S_+(V))$ restricts to a coderivation on $S_{m_1}^{m_2}(V)$, which is determined by $\varphi_1, \ldots, \varphi_{m_2-m_1+1}$. We denote it by $\hat{\varphi}|_{S_{m_1}^{m_2}(V)\to S_{m_1}^{m_2}(V)}$.
- Every $\hat{f} \in \mathrm{Hom}((S_+(V), \Delta), (S_+(W), \Delta))$ restricts to a coalgebra homomorphism from $S_{m_1}^{m_2}(V)$ to $S_{m_1}^{m_2}(W)$, which is determined by $f_1, \ldots, f_{m_2-m_1+1}$. We denote it by $\hat{f}|_{S_{m_1}^{m_2}(V)\to S_{m_1}^{m_2}(W)}$.

All discussion above extends to filtered $\Lambda_{0,\mathrm{nov}}$-modules.

Definition 3.6.2. Let $\overline{C}$ be a graded vector space and $C = \overline{C}\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$.

(i) A structure of (unfiltered) L_∞ *algebra* on $\overline{C}$ is a sequence of linear maps $\{\bar{\mathfrak{l}}_k : \overline{C}[1]^{\odot k} \to \overline{C}[1]\}_{k\geq 1}$ of degree 1 such that the coderivation $\hat{\bar{\mathfrak{l}}} = \sum_{k\geq 1} \hat{\bar{\mathfrak{l}}}_k$ on $(S_+(\overline{C}[1]), \Delta)$ is a codifferential, i.e. $\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}} = 0$.

(ii) A structure of (trivially filtered) L_∞ *algebra* on C is a sequence of degree 1 filtered homomorphisms $\{\mathfrak{l}_k : C[1]^{\odot_{\Lambda_{0,\mathrm{nov}}} k} \to C[1]\}_{k\geq 1}$ such that $\hat{\mathfrak{l}} = \hat{\bar{\mathfrak{l}}}\otimes \mathrm{id}_{\Lambda_{0,\mathrm{nov}}}$ for an L_∞ algebra structure $\{\bar{\mathfrak{l}}_k\}_{k\geq 1}$ on $\overline{C}$. By abuse of notation we also write such $(C, \mathfrak{l})$ as $(C, \bar{\mathfrak{l}})$.

The condition $\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}} = 0$ (L_∞ relation) is equivalent to $\bar{\mathfrak{l}} \circ \hat{\bar{\mathfrak{l}}} = (\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}})|_{S_+(\overline{C}[1])\to\overline{C}[1]} = 0$, which explicitly says that for each $n \geq 1$ and $x_1, \ldots, x_n \in \overline{C}[1]$,

$$\sum_{\substack{1\leq k\leq n\\ \sigma\in\mathrm{Sh}(k,n-k)}} \epsilon(\sigma; x_{1\ldots n})\ \bar{\mathfrak{l}}_{n-k+1}(\bar{\mathfrak{l}}_k(x_{\sigma(1)} \odot \cdots \odot x_{\sigma(k)}) \odot x_{\sigma(k+1)} \odot \cdots \odot x_{\sigma(n)}) = 0.$$

In particular $\bar{\mathfrak{l}}_1 \circ \bar{\mathfrak{l}}_1 = 0$, so one can discuss cohomology of $(\overline{C}, \bar{\mathfrak{l}}_1)$.

Definition 3.6.3. Let $(\overline{C}, \bar{\mathfrak{l}}), (\overline{C}', \bar{\mathfrak{l}}')$ be L_∞ algebras. An L_∞ homomorphism from $(\overline{C}, \bar{\mathfrak{l}})$ to $(\overline{C}', \bar{\mathfrak{l}}')$ is a degree 0 coalgebra homomorphism $\hat{\bar{\mathfrak{f}}} : S_+(\overline{C}[1]) \to S_+(\overline{C}'[1])$ such that $\hat{\bar{\mathfrak{l}}}' \circ \hat{\bar{\mathfrak{f}}} = \hat{\bar{\mathfrak{f}}} \circ \hat{\bar{\mathfrak{l}}}$.

The condition $\hat{\bar{\mathfrak{l}}}' \circ \hat{\bar{\mathfrak{f}}} = \hat{\bar{\mathfrak{f}}} \circ \hat{\bar{\mathfrak{l}}}$ is equivalent to $\bar{\mathfrak{l}}' \circ \hat{\bar{\mathfrak{f}}} = \bar{\mathfrak{f}} \circ \hat{\bar{\mathfrak{l}}}$. Thus an L_∞ homomorphism is given by a sequence of linear maps (or filtered homomorphisms) $\{\bar{\mathfrak{f}}_k : \overline{C}[1]^{\odot k} \to \overline{C}'[1]\}_{k\geq 1}$ of degree 0 such that for each $n \geq 1$ and $x_1, \ldots, x_n \in \overline{C}[1]$,

$$\sum_{\substack{r\geq 1,\ k_1+\cdots+k_r=n \\ \sigma\in \mathrm{Sh}(k_1,\ldots,k_r)}} \epsilon(\sigma; x_{1\ldots n})\ \bar{\mathfrak{l}}'_r(\bar{\mathfrak{f}}_{k_1}(x_{\sigma(1)} \odot \cdots \odot x_{\sigma(k_1)}) \odot \cdots \odot \bar{\mathfrak{f}}_{k_r}(x_{\sigma(n-k_r+1)} \odot \cdots \odot x_{\sigma(n)}))$$
$$= \sum_{\substack{m_1+m_2=n+1 \\ \tau\in \mathrm{Sh}(m_2,n-m_2)}} \epsilon(\tau; x_{1\ldots n})\ \bar{\mathfrak{f}}_{m_1}(\bar{\mathfrak{l}}_{m_2}(x_{\tau(1)} \odot \cdots \odot x_{\tau(m_2)}) \odot x_{\tau(m_2+1)} \odot \cdots \odot x_{\tau(n)}).$$

An L_∞ homomorphism $\{\bar{\mathfrak{f}}_k\}$ is said to be *linear* if $\bar{\mathfrak{f}}_k = 0$ for all $k \neq 1$.

The L_∞ *composition* $\bar{\mathfrak{f}}^2 \circ \bar{\mathfrak{f}}^1 = \big((\bar{\mathfrak{f}}^2 \circ \bar{\mathfrak{f}}^1)_k\big)$ of L_∞ homomorphisms $\bar{\mathfrak{f}}^1, \bar{\mathfrak{f}}^2$ is given by composition of corresponding coalgebra homomorphisms.

Example 3.6.4. Let $(B, d, \{,\})$ be a dg Lie algebra. In view of sign change rule (1.8.1), let us set $(\bar{\mathfrak{l}}_k)_{k\geq 1}$ as

$$\bar{\mathfrak{l}}_1(x) = dx, \quad \bar{\mathfrak{l}}_2(x \odot y) = (-1)^{|x|}\{x, y\},$$

and $\bar{\mathfrak{l}}_k = 0$ for $k \geq 3$. Here $x \odot y$ respects grading in $\overline{B}[1]$. Then L_∞ relations for $\bar{\mathfrak{l}}$ are equivalent to the defining relations of a dg Lie algebra. Likewise, a dg Lie algebra homomorphism is exactly an L_∞ homomorphism that is linear.

3.6.2 Homotopy equivalence of L_∞ algebras

In this subsection, let $(\overline{C}, \bar{\mathfrak{l}}), (\overline{C}', \bar{\mathfrak{l}}')$ be L_∞ algebras, and $C = \overline{C}\widehat{\otimes}\Lambda_{0,\mathrm{nov}}, C' = \overline{C}'\widehat{\otimes}\Lambda_{0,\mathrm{nov}}$.

Definition 3.6.5. An L_∞ algebra $(\widetilde{\overline{C}}, \tilde{\bar{\mathfrak{l}}})$ together with L_∞ homomorphisms

$$\overline{\mathtt{Incl}} : (\overline{C}, \bar{\mathfrak{l}}) \to (\widetilde{\overline{C}}, \tilde{\bar{\mathfrak{l}}}), \quad \overline{\mathtt{Eval}}_{t=0} : (\widetilde{\overline{C}}, \tilde{\bar{\mathfrak{l}}}) \to (\overline{C}, \bar{\mathfrak{l}}), \quad \overline{\mathtt{Eval}}_{t=1} : (\widetilde{\overline{C}}, \tilde{\bar{\mathfrak{l}}}) \to (\overline{C}, \bar{\mathfrak{l}})$$

is said to be an *(L_∞ algebra) model of* $[0, 1] \times \overline{C}$ if the following holds:

- $\overline{\mathtt{Incl}}$, $\overline{\mathtt{Eval}}_{t=0}$, $\overline{\mathtt{Eval}}_{t=1}$ are linear L_∞ homomorphisms;
- $\overline{\mathtt{Eval}}_{t=0} \circ \overline{\mathtt{Incl}} = \overline{\mathtt{Eval}}_{t=1} \circ \overline{\mathtt{Incl}} = \mathfrak{id}_{\overline{C}}$, the identity on $\overline{C}$;

- $\overline{\texttt{Incl}}_1$, $(\overline{\texttt{Eval}}_{t=0})_1$, $(\overline{\texttt{Eval}}_{t=1})_1$ are quasi-isomorphisms between the cochain complexes $(\overline{C}, \bar{\mathfrak{l}}_1), (\tilde{\overline{C}}, \tilde{\bar{\mathfrak{l}}}_1)$;

- $(\overline{\texttt{Eval}}_{t=0})_1 \oplus (\overline{\texttt{Eval}}_{t=1})_1 : \tilde{\overline{C}} \to \overline{C} \oplus \overline{C}$ is surjective.

Example 3.6.6. Let $\big(\mathcal{A}([0,1]), d, \wedge\big)$ be the *commutative* dg algebra introduced in Example 3.5.8, then $\mathcal{A}([0,1]) \otimes \overline{C}$ is a model of $[0,1] \times \overline{C}$, where $\tilde{\bar{\mathfrak{l}}}_1 = d \otimes \mathrm{id}_{\overline{C}[1]} + \mathrm{id}_{\mathcal{A}([0,1])} \otimes \bar{\mathfrak{l}}_1$, $\tilde{\bar{\mathfrak{l}}}_k = \mathcal{A}([0,1])$-linear extension of $\bar{\mathfrak{l}}_k$ for $k \geq 2$, and $\overline{\texttt{Incl}}_1, (\overline{\texttt{Eval}}_{t=0})_1, (\overline{\texttt{Eval}}_{t=1})_1$ are obtained by tensoring $\mathrm{id}_{\overline{C}[1]}$ with obvious maps between $\mathbb{R}$ and $\mathcal{A}([0,1])$. Let us check $\tilde{\bar{\mathfrak{l}}}$ satisfies L_∞ relations. First, denote $d_t = d \otimes \mathrm{id}_{\overline{C}[1]}$, $\tilde{\bar{\mathfrak{l}}}'_1 = \mathrm{id}_{\mathcal{A}([0,1])} \otimes \bar{\mathfrak{l}}_1$ and $\tilde{\bar{\mathfrak{l}}}'_k = \tilde{\bar{\mathfrak{l}}}_k$ for $k \geq 2$, then $\hat{\tilde{\bar{\mathfrak{l}}}}' \circ \hat{\tilde{\bar{\mathfrak{l}}}}' = 0$ on $S_+(\tilde{\overline{C}})$ since it is the $\mathcal{A}([0,1])$-linear extension of $\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}}$. Notice that the commutativity of $\wedge$ on $\mathcal{A}([0,1])$ is required here. Second, as in the proof of Lemma-Definition 3.5.20, we can show $[\hat{d}_t, \hat{\tilde{\bar{\mathfrak{l}}}}']$ is $\mathcal{A}([0,1])$-linear on $S_{\mathcal{A}([0,1]),+}(\tilde{\overline{C}})$. Then $[\hat{d}_t, \hat{\tilde{\bar{\mathfrak{l}}}}'] = 0$ since $[\hat{d}_t, \hat{\tilde{\bar{\mathfrak{l}}}}']\big|_{S_+(\overline{C})} = 0$. This verifies $\hat{\tilde{\bar{\mathfrak{l}}}} \circ \hat{\tilde{\bar{\mathfrak{l}}}} = 0$. We finally remark that if $\overline{C}$ is a dg Lie algebra as in Example 3.6.4, then $\mathcal{A}([0,1]) \otimes \overline{C}$ is also a dg Lie algebra.

Definition 3.6.7. Let $\bar{\mathfrak{f}}, \bar{\mathfrak{g}} : \overline{C} \to \overline{C}'$ be L_∞ homomorphisms, and let $\tilde{\overline{C}}'$ be a model of $[0,1] \times \overline{C}'$. We say $\bar{\mathfrak{f}}$ is *homotopic to* $\bar{\mathfrak{g}}$ *in* $\tilde{\overline{C}}'$ and write $\bar{\mathfrak{f}} \simeq_{\tilde{\overline{C}}'} \bar{\mathfrak{g}}$, if there exists an L_∞ homomorphism $\bar{\mathfrak{h}} : \overline{C} \to \tilde{\overline{C}}'$ such that $\overline{\texttt{Eval}}_{t=0} \circ \bar{\mathfrak{h}} = \bar{\mathfrak{f}}$, $\overline{\texttt{Eval}}_{t=1} \circ \bar{\mathfrak{h}} = \bar{\mathfrak{g}}$. Such an $\bar{\mathfrak{h}}$ is called an *homotopy* between $\bar{\mathfrak{f}}$ and $\bar{\mathfrak{g}}$ in $\tilde{\overline{C}}'$.

Lemma-Definition 3.6.8. *$\simeq_{\tilde{\overline{C}}'}$ is independent of choices of the model $\tilde{\overline{C}}'$ of $[0,1] \times \overline{C}'$, and gives an equivalence relation $\simeq$ on the set of L_∞ homomorphisms from $\overline{C}$ to $\overline{C}'$.*

The proof of Lemma-Definition 3.6.8 relies on the following lifting result.

Theorem 3.6.9. *Let $\tilde{\overline{C}}, \tilde{\overline{C}}'$ be any L_∞ algebra models of $[0,1] \times \overline{C}, [0,1] \times \overline{C}'$, respectively, and let $\bar{\mathfrak{f}} : \overline{C} \to \overline{C}'$ be an L_∞ homomorphism. Then there exists an L_∞ homomorphism $\tilde{\bar{\mathfrak{f}}} : \tilde{\overline{C}} \to \tilde{\overline{C}}'$ which lifts $\bar{\mathfrak{f}}$, in the sense that*

$$\overline{\texttt{Incl}}' \circ \bar{\mathfrak{f}} = \tilde{\bar{\mathfrak{f}}} \circ \overline{\texttt{Incl}}, \quad \overline{\texttt{Eval}}'_{t=t_0} \circ \tilde{\bar{\mathfrak{f}}} = \bar{\mathfrak{f}} \circ \overline{\texttt{Eval}}_{t=t_0} \ (t_0 = 0, 1).$$

For the proof of Theorem 3.6.9, see Remark 3.6.13.

Proof of Lemma-Definition 3.6.8. Let $\overline{\widetilde{C}}'_1$, $\overline{\widetilde{C}}'_2$ be two models of $[0,1] \times \overline{C}'$, and suppose $\bar{\mathfrak{f}} \simeq_{\overline{\widetilde{C}}'_1} \bar{\mathfrak{g}} : \overline{C} \to \overline{C}'$. Let $\bar{\mathfrak{h}} : \overline{C} \to \overline{\widetilde{C}}'_1$ be a homotopy between $\bar{\mathfrak{f}}, \bar{\mathfrak{g}}$ in $\overline{\widetilde{C}}'_1$. By Theorem 3.6.9, there is an L_∞ homomorphism $\widetilde{\mathfrak{id}} : \overline{\widetilde{C}}'_1 \to \overline{\widetilde{C}}'_2$ which lifts the identity on $\overline{C}'$. Then $\widetilde{\mathfrak{id}} \circ \bar{\mathfrak{h}}$ is a homotopy between $\bar{\mathfrak{f}}, \bar{\mathfrak{g}}$ in $\overline{\widetilde{C}}'_2$, so $\bar{\mathfrak{f}} \simeq_{\overline{\widetilde{C}}'_2} \bar{\mathfrak{g}}$. This proves $\simeq_{\overline{\widetilde{C}}'}$ does not depend on the choice of $\overline{\widetilde{C}}'$. Next we show $\simeq$ is an equivalence relation.

- To see $\simeq$ is reflexive, simply notice that $\overline{\mathtt{Incl}}' \circ \bar{\mathfrak{f}}$ is a homotopy between $\bar{\mathfrak{f}}$ and $\bar{\mathfrak{f}}$.

- To see $\simeq$ is symmetric, notice that if $\bar{\mathfrak{f}} \simeq \bar{\mathfrak{g}}$ in $\overline{\widetilde{C}}'$, then $\bar{\mathfrak{g}} \simeq \bar{\mathfrak{f}}$ in $\overline{\widetilde{C}}'_{\mathrm{op}}$ where
$$\big(\overline{\widetilde{C}}'_{\mathrm{op}}, \bar{\tilde{\mathfrak{l}}}', \overline{\mathtt{Incl}}', \overline{\mathtt{Eval}}'_{t=0}, \overline{\mathtt{Eval}}'_{t=1}\big) = \big(\overline{\widetilde{C}}', \bar{\tilde{\mathfrak{l}}}', \overline{\mathtt{Incl}}', \overline{\mathtt{Eval}}'_{t=1}, \overline{\mathtt{Eval}}'_{t=0}\big).$$

- To see $\simeq$ is transitive, choose a specific model $\mathcal{A}([0,1]) \otimes \overline{C}'$ of $[0,1] \times \overline{C}'$ as in Example 3.6.6, and choose $\mathcal{A}([0,1])$ as the space of *piecewise* smooth differential forms on $[0,1]$. Notice that $[0,1]$ can be replaced by any other closed interval $[a,b]$ $(a < b)$, with $\overline{\mathtt{Eval}}^{[a,b]}_{t=a}, \overline{\mathtt{Eval}}^{[a,b]}_{t=b}$ instead of $\overline{\mathtt{Eval}}_{t=0}, \overline{\mathtt{Eval}}_{t=1}$. For $i = 0,1$, suppose $\bar{\mathfrak{h}}^{(i)} : \overline{C} \to \mathcal{A}([i,i{+}1]) \otimes \overline{C}'$ is a homotopy between $\bar{\mathfrak{f}}^{(i)}, \bar{\mathfrak{f}}^{(i+1)} : \overline{C} \to \overline{C}'$, then $\bar{\mathfrak{h}} = (\bar{\mathfrak{h}}^{(0)}, \bar{\mathfrak{h}}^{(1)})$ is an L_∞ homomorphism from $\overline{C}$ to
$$\big\{(x,y) \in (\mathcal{A}([0,1]) \oplus \mathcal{A}([1,2])) \otimes \overline{C}' \;\; | \, \overline{\mathtt{Eval}}'^{[0,1]}_{t=1}(x) = \overline{\mathtt{Eval}}'^{[1,2]}_{t=1}(y)\big\} = \mathcal{A}([0,2]) \otimes \overline{C}',$$
such that $\overline{\mathtt{Eval}}'^{[0,2]}_{t=0} \circ \bar{\mathfrak{h}} = \bar{\mathfrak{f}}^{(0)}$, $\overline{\mathtt{Eval}}'^{[0,2]}_{t=2} \circ \bar{\mathfrak{h}} = \bar{\mathfrak{f}}^{(2)}$. This shows $\bar{\mathfrak{f}}^{(0)} \simeq \bar{\mathfrak{f}}^{(2)}$.

The proof is complete. □

Corollary 3.6.10. *If* $\bar{\mathfrak{f}} \simeq \bar{\mathfrak{g}} : \overline{C} \to \overline{C}'$ *and* $\bar{\mathfrak{f}}' \simeq \bar{\mathfrak{g}}' : \overline{C}' \to \overline{C}''$, *then* $\bar{\mathfrak{f}}' \circ \bar{\mathfrak{f}} \simeq \bar{\mathfrak{g}}' \circ \bar{\mathfrak{g}} : \overline{C} \to \overline{C}''$.

Proof. It follows easily from Theorem 3.6.9. □

Definition 3.6.11. Let $\bar{\mathfrak{f}} : \overline{C} \to \overline{C}'$ be an L_∞ homomorphism.

(i) $\bar{\mathfrak{f}}$ is called a *homotopy equivalence* if there exists an L_∞ homomorphism $\bar{\mathfrak{g}} : \overline{C}' \to \overline{C}$ such that $\bar{\mathfrak{f}} \circ \bar{\mathfrak{g}}$ and $\bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}$ are homotopic to identity.

(ii) $\bar{\mathfrak{f}}$ is called a *weak homotopy equivalence* if $\bar{\mathfrak{f}}_1 : (\overline{C}, \bar{\mathfrak{l}}_1) \to (\overline{C}', \bar{\mathfrak{l}}'_1)$ is a quasi-isomorphism.

Theorem 3.6.12 (Whitehead Theorem for L_∞ algebras)**.** *A weak homotopy equivalence between L_∞ algebras is a homotopy equivalence.*

Remark 3.6.13. Lifting theorem (Theorem 3.6.9) and Whitehead theorem (Theorem 3.6.12) for L_∞ algebras can be proved in the same way as [18, Theorem 4.2.34, Theorem 4.2.45] for A_∞ algebras, where the proofs are written in an inductive, obstruction-theoretic way. The only difference is to replace tensor coalgebra (for A_∞ algebras) by symmetric coalgebra (for L_∞ algebras). What is more, we only need the unfiltered version. Therefore we omit the proofs of Theorem 3.6.9 and Theorem 3.6.12. Nonetheless, we prove the following lemma, which is useful in inductive proof of these theorems, as well as Theorem **??**.

Lemma 3.6.14. *(Compare [18, Lemma 4.4.9, Lemma 4.4.10]). Let $\overline{C}, \overline{C}'$ be graded vector spaces, $m_1, m_2, K \in \mathbb{Z}_{\geq 1}$, $0 \leq m_2 - m_1 < K$. Consider $\bar{\mathfrak{l}} = (\bar{\mathfrak{l}}_k)_{k=1}^K \in \mathrm{Hom}(S_1^K(\overline{C}[1]), \overline{C}[1])$, $\bar{\mathfrak{l}}' = (\bar{\mathfrak{l}}'_k)_{k=1}^K \in \mathrm{Hom}(S_1^K(\overline{C}'[1]), \overline{C}'[1])$, $\bar{\mathfrak{f}} = (\bar{\mathfrak{f}}_k)_{k=1}^K \in \mathrm{Hom}(S_1^K(\overline{C}[1]), \overline{C}'[1])$.*

(i) If $\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}} = 0$ on $S_1^K(\overline{C}[1])$, then $\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}} = 0$ on $S_{m_1}^{m_2}(\overline{C}[1]) \to S_{m_1}^{m_2}(\overline{C}[1])$.

(ii) If $\hat{\bar{\mathfrak{f}}} \circ \hat{\bar{\mathfrak{l}}} = \hat{\bar{\mathfrak{l}}}' \circ \hat{\bar{\mathfrak{f}}}$ on $S_1^K(\overline{C}[1])$, then $\hat{\bar{\mathfrak{f}}} \circ \hat{\bar{\mathfrak{l}}} = \hat{\bar{\mathfrak{l}}}' \circ \hat{\bar{\mathfrak{f}}}$ on $S_{m_1}^{m_2}(\overline{C}[1]) \to S_{m_1}^{m_2}(\overline{C}'[1])$.

Proof. We prove by induction on $K \geq 1$. The case $K = 1$ is obvious. To perform the induction, observe that $\odot : S_+(V) \otimes S_+(V) \to S_+(V)$, $(v_1 \odot \cdots \odot v_k) \otimes (v_{k+1} \odot \cdots \odot v_{k+l}) \mapsto v_1 \odot \cdots \odot v_{k+l}$ is left inverse to $\Delta : S_2^\infty(V) \to S_+(V) \otimes S_+(V)$ up to constant multiples, namely they satisfy

$$\odot \circ \Delta = \bigoplus_{k \geq 2} (2^k - 2)\mathrm{id}_{V^{\odot k}}.$$

Therefore, to prove (i), it suffices to assume $m_1 \geq 2$ and prove $\Delta \circ (\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}})|_{S_{m_1}^{m_2}(\overline{C}[1]) \to S_{m_1}^{m_2}(\overline{C}[1])} = 0$ provided $\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}} = 0$ on $S_1^K(V)$. Since $\hat{\bar{\mathfrak{l}}}$ is a coderivation, on $S_+(V)$ we have

$$\Delta \circ \hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}} = (\mathrm{id} \otimes \hat{\bar{\mathfrak{l}}} + \hat{\bar{\mathfrak{l}}} \otimes \mathrm{id}) \circ \Delta \circ \hat{\bar{\mathfrak{l}}} = (\mathrm{id} \otimes \hat{\bar{\mathfrak{l}}} + \hat{\bar{\mathfrak{l}}} \otimes \mathrm{id})^2 \circ \Delta = \big(\mathrm{id} \otimes (\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}}) + (\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}}) \otimes \mathrm{id}\big) \circ \Delta. \quad (3.6.1)$$

Inductive assumption says $\hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}} = 0$ on $S^{m_2}_{m_1}(\overline{C}[1]) \to S^{m_2}_{m_1}(\overline{C}[1])$ whenever $m_2 - m_1 < K$. In case $m_2 - m_1 < K+1$, Since Δ maps $S^{m_2}_{m_1}(\overline{C}[1])$ into $S^{m_2-1}_{1}(\overline{C}[1]) \otimes S^{m_2-1}_{1}(\overline{C}[1])$, we have by (3.6.1) that $\Delta \circ \hat{\bar{\mathfrak{l}}} \circ \hat{\bar{\mathfrak{l}}} = 0$ on $S^{m_2}_{m_1}(\overline{C}[1]) \to S^{m_2}_{m_1}(\overline{C}[1])$. The proof of (ii) is similar. $\square$

3.6.3 Maurer-Cartan elements in L_∞ algebras

Definition 3.6.15. The set of *Maurer-Cartan elements* in an L_∞ algebra $(C, \bar{\mathfrak{l}})$ is

$$\mathrm{MC}(C) := \{x \in C[1]^0 = C^1 \;\mid \|x\| < 1,\ \bar{\mathfrak{l}}(\exp(x)) = 0\},$$

where

$$\exp(x) := \sum_{k \geq 0} \frac{x^{\odot k}}{k!} = 1 + \frac{x}{1!} + \frac{x \odot x}{2!} + \cdots, \qquad \bar{\mathfrak{l}}(\exp(x)) = \sum_{k \geq 1} \frac{1}{k!} \bar{\mathfrak{l}}_k(x^{\odot k}). \tag{3.6.2}$$

Let $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ be a discrete submonoid. The set of *G-gapped Maurer Cartan elements* in $(C, \bar{\mathfrak{l}})$, deonted by $\mathrm{MC}_G(C)$, consists of those $x \in \mathrm{MC}(\overline{C})$ of the form $x = \sum_{\beta \in G} T^{E(\beta)} e^{\frac{\mu(\beta)}{2}} x_\beta$ where each $x_\beta \in \overline{C}[1]^{-\mu(\beta)}$.

Note that (3.6.2) converges in C since $\|x\| < 1$ i.e. $x \in \overline{C}[1]\widehat{\otimes}\Lambda^+_{0,\mathrm{nov}}$. If x is G-gapped, then $x_{(0,0)} = 0$.

Lemma-Definition 3.6.16. *If $\bar{\mathfrak{f}} : (\overline{C}, \bar{\mathfrak{l}}) \to (\overline{C}', \bar{\mathfrak{l}}')$ is an L_∞ homomorphism, then $\bar{\mathfrak{f}}$ induces a map*

$$\bar{\mathfrak{f}}_* : \mathrm{MC}(C) \to \mathrm{MC}(C'), \quad \bar{\mathfrak{f}}_*(x) := \bar{\mathfrak{f}}(\exp(x)) = \sum_{k \geq 1} \frac{\bar{\mathfrak{f}}_k(x^{\odot k})}{k!}.$$

The assignment $\bar{\mathfrak{f}} \mapsto \bar{\mathfrak{f}}_*$ is covariant. Moreover, $\bar{\mathfrak{f}}_*$ maps $\mathrm{MC}_G(C)$ into $\mathrm{MC}_G(C')$.

Proof. First, for any $x \in C[1]^0$, $\|x\| < 1$,

$$\begin{aligned}
\hat{\bar{\mathfrak{l}}}(\exp(x)) &= \Big(\sum_{k \geq 1} \hat{\bar{\mathfrak{l}}}_k\Big)\Big(\sum_{n \geq 0} \frac{x^{\odot n}}{n!}\Big) = \sum_{n \geq k \geq 1} \sum_{\mathrm{Sh}(k, n-k)} \frac{1}{n!} \bar{\mathfrak{l}}_k(x^{\odot k}) \odot x^{\odot n-k} \\
&= \sum_{n \geq k \geq 1} \frac{\bar{\mathfrak{l}}_k(x^{\odot k})}{k!} \odot \frac{x^{\odot n-k}}{(n-k)!} = \bar{\mathfrak{l}}(\exp(x)) \odot \exp(x).
\end{aligned}$$

It follows easily that $\bar{\mathfrak{l}}(\exp(x)) = 0$ iff $\hat{\bar{\mathfrak{l}}}(\exp(x)) = 0$. Next,

$$\begin{aligned}\exp(\bar{\mathfrak{f}}_*(x)) &= \sum_{l\geq 0}\frac{1}{l!}\Big(\sum_{k\geq 1}\frac{1}{k!}\bar{\mathfrak{f}}_k(x^{\odot k})\Big)^{\odot l} = \sum_{l\geq 0}\sum_{k_1,\dots,k_l\geq 1}\frac{1}{l!}\frac{\bar{\mathfrak{f}}_{k_1}(x^{\odot k_1})}{k_1!}\odot\cdots\odot\frac{\bar{\mathfrak{f}}_{k_l}(x^{\odot k_l})}{k_l!} \\ &= \sum_{\substack{k\geq 0\\ l\geq 0}}\sum_{\substack{k_1+\cdots+k_l=k\\ \mathrm{Sh}(k_1,\dots,k_l)}}\frac{1}{k!l!}\bar{\mathfrak{f}}_{k_1}(x^{\odot k_1})\odot\cdots\odot\bar{\mathfrak{f}}_{k_l}(x^{\odot k_l}) = \sum_{k\geq 0}\frac{\hat{\bar{\mathfrak{f}}}(x^{\odot k})}{k!} = \hat{\bar{\mathfrak{f}}}(\exp(x)).\end{aligned} \tag{3.6.3}$$

Therefore if $x \in \mathrm{MC}(C)$, then $\hat{\bar{\mathfrak{l}}}(\exp(\bar{\mathfrak{f}}_*(x))) = \hat{\bar{\mathfrak{l}}}(\hat{\bar{\mathfrak{f}}}(\exp(x))) = \hat{\bar{\mathfrak{f}}}(\hat{\bar{\mathfrak{l}}}(\exp(x))) = 0$, and so $\bar{\mathfrak{f}}_*(x) \in \mathrm{MC}(C)$. It also follows from (3.6.3) that $\bar{\mathfrak{g}}_*(\bar{\mathfrak{f}}_*(x)) = \bar{\mathfrak{g}}(\exp(\bar{\mathfrak{f}}_*(x))) = \bar{\mathfrak{g}}(\hat{\bar{\mathfrak{f}}}(\exp(x))) = (\bar{\mathfrak{g}}\circ\bar{\mathfrak{f}})(\exp(x)) = (\bar{\mathfrak{g}}\circ\bar{\mathfrak{f}})_*(x)$, so $\bar{\mathfrak{f}} \mapsto \bar{\mathfrak{f}}_*$ is covariant. Finally, if $x = \sum_{\beta\in G} T^{E(\beta)}e^{\frac{\mu(\beta)}{2}}x_\beta \in \mathrm{MC}_G(C)$, then $\bar{\mathfrak{f}}_*(x)$ is also G-gapped:

$$\bar{\mathfrak{f}}_*(x) = \sum_{\beta\in G} T^{E(\beta)}e^{\frac{\mu(\beta)}{2}}\bar{\mathfrak{f}}_*(x)_\beta, \qquad \bar{\mathfrak{f}}_*(x)_\beta = \sum_{k\geq 1}\sum_{\beta_1+\cdots+\beta_k=\beta}\frac{\bar{\mathfrak{f}}_k(x_{\beta_1}\odot\cdots\odot x_{\beta_k})}{k!}, \tag{3.6.4}$$

where each $\bar{\mathfrak{f}}_*(x)_\beta$ is a finite sum because $x_{(0,0)} = 0$ and G is a discrete submonoid. $\square$

Example 3.6.17. Let $(\overline{C}, \bar{\mathfrak{l}}_1, \bar{\mathfrak{l}}_2), (\overline{C}', \bar{\mathfrak{l}}_1', \bar{\mathfrak{l}}_2')$ be dg Lie algebras as in Example 3.6.4, then the the notion of (gapped) Maurer-Cartan elements coincides with Definition 3.4.11. Let $\bar{\mathfrak{f}} : \overline{C} \to \overline{C}'$ be an L_∞ homomorphism. If $\bar{\mathfrak{f}}$ is linear, then $\bar{\mathfrak{f}}_* : \mathrm{MC}(C) \to \mathrm{MC}(C')$ is the same as that for dg Lie algebras.

Definition 3.6.18. Suppose $x_0, x_1 \in \mathrm{MC}(C)$ and $\widetilde{\overline{C}}$ is a model of $[0,1]\times\overline{C}$. We say x_0 is *gauge equivalent to* x_1 *in* $\widetilde{\overline{C}}$ (via $\tilde{x}$) and write $x_0 \sim_{\widetilde{\overline{C}}} x_1$, if there exists $\tilde{x} \in \mathrm{MC}(\widetilde{C})$ such that $\overline{\mathtt{Eval}}_{t=0}(\tilde{x}) = x_0$, $\overline{\mathtt{Eval}}_{t=1}(\tilde{x}) = x_0$. If $x_0, x_1 \in \mathrm{MC}_G(C)$ for some discrete submonoid G, we also require $\tilde{x} \in \mathrm{MC}_G(\widetilde{C})$.

Lemma-Definition 3.6.19. $\sim_{\widetilde{\overline{C}}}$ *is independent of choices of the model* $\widetilde{\overline{C}}$ *of* $[0,1]\times\widetilde{\overline{C}}$*, and thus induces an equivalence relation* $\sim$*, called gauge equivalence, on* $\mathrm{MC}(C)$*. The same result holds for* $\mathrm{MC}_G(C)$*.*

Proof. The proof is similar to that of Lemma-Definition 3.6.8. To see $\sim_{\widetilde{\overline{C}}}$ does not depend on the choice of $\widetilde{\overline{C}}$, suppose $\widetilde{\overline{C}}_1$, $\widetilde{\overline{C}}_2$ are two models of $[0,1]\times\overline{C}$, and $x_0 \sim_{\widetilde{\overline{C}}_1} x_1$ via $\tilde{x} \in \mathrm{MC}(\widetilde{\overline{C}}_1)$.

By Theorem 3.6.9, there is an L_∞ homomorphism $\widetilde{\mathfrak{io}} : \widetilde{\overline{C}}_1 \to \widetilde{\overline{C}}_2$ lifting the identity on $\overline{C}$. Then $x_0 \sim_{\widetilde{\overline{C}}_2} x_1$ via $\widetilde{\mathfrak{io}}_*(\tilde{x}) \in \mathrm{MC}(\widetilde{\overline{C}}_2)$ by Lemma-Definition 3.6.16. The rest of the proof is also a copy of the proof of Lemma-Definition 3.6.8. $\square$

Corollary 3.6.20. *Suppose $\bar{\mathfrak{f}}, \bar{\mathfrak{g}} : \overline{C} \to \overline{C}'$ are L_∞ homomorphisms and $x, y \in \mathrm{MC}(C)$. If $\bar{\mathfrak{f}} \simeq \bar{\mathfrak{g}}$ and $x_0 \sim x_1$, then $\bar{\mathfrak{f}}_*(x_0) \sim \bar{\mathfrak{g}}_*(x_1)$. Then same holds if $x, y \in \mathrm{MC}_G(C)$.*

Proof. Choose specific models of $[0,1] \times \overline{C}, [0,1] \times \overline{C}'$ as in Example 3.6.6. Let $\bar{\mathfrak{h}} : \overline{C} \to \mathcal{A}([0,1]) \otimes \overline{C}'$ be a homotopy between $\bar{\mathfrak{f}}, \bar{\mathfrak{g}}$, which extends $\mathcal{A}([0,1])$-linearly to $\tilde{\bar{\mathfrak{h}}} : \mathcal{A}([0,1]) \otimes \overline{C} \to \mathcal{A}([0,1]) \otimes \overline{C}'$. Then one checks $\tilde{\bar{\mathfrak{h}}}$ is an L_∞ homomorphism by similar arguments as in Example 3.6.6. It is easy to see $\overline{\mathtt{Eval}}'_{t=t_0} \circ \tilde{\bar{\mathfrak{h}}} = \overline{\mathtt{Eval}}'_{t=t_0} \circ \bar{\mathfrak{h}} \circ \overline{\mathtt{Eval}}_{t=t_0}$, so $\overline{\mathtt{Eval}}'_{t=0} \circ \tilde{\bar{\mathfrak{h}}} = \bar{\mathfrak{f}} \circ \overline{\mathtt{Eval}}_{t=0}$, $\overline{\mathtt{Eval}}'_{t=1} \circ \tilde{\bar{\mathfrak{h}}} = \bar{\mathfrak{g}} \circ \overline{\mathtt{Eval}}_{t=1}$. Suppose $x_0 \sim x_1$ via $\tilde{x} \in \mathrm{MC}(C)$, then clearly $\bar{\mathfrak{f}}_*(x_0) \sim \bar{\mathfrak{g}}_*(x_1)$ via $\tilde{\bar{\mathfrak{h}}}_*(\tilde{x}) \in \mathrm{MC}((\mathcal{A}([0,1]) \otimes C')\widehat{\otimes}\Lambda_{0,\mathrm{nov}})$. $\square$

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