Black Holes, Manifolds, and Cohomogeneity-2 Torus Symmetry

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Abstract of the Dissertation

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Abstract

In recent works a method was developed which uses harmonic maps to construct stationary asymptotically flat solutions to the $(4+1)$-dimensional vacuum Einstein equations with symmetry group $\mathbb{R}\times U(1)^2$. This method is characterized by two features: a wide range of domain of outer communication (DOC) topologies, and conical singularities known as struts. In this dissertation we generalize the method to work in all higher dimensions. We find that the range of DOC topologies produced is vastly greater than those found with the $(4+1)$-dimensional method. Unfortunately these topologies are not yet fully understood, as it is deeply connected to the following open question in toric topology:

Which simply connected $(n+2)$-dimensional manifolds admit effective T^n-actions?

In this dissertation we conjecture an answer to the above question and provide a partial proof. In the process we develop tools to study both the homeotype of the manifold and the equivariant homeotype of its torus action. The issue of when solutions are regular, i.e. without conical singularities, is also partially resolved. We find that regular solutions always exist to a Kaluza-Klein reduction of the vacuum Einstein equations, and under certain topological conditions regular solutions of the vacuum equations exist without a dimensional reduction.

Contents

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1 Introduction

1.1 Background and Motivating Problems

For $(3+1)$-dimensional spacetimes, it is expected that any asymptotically flat stationary axially symmetric non-trivial regular solution to the vacuum Einstein equations must have black hole horizon of a single 2-sphere. Some results in this direction have been obtained [7, 17, 51, 53], but a complete resolution is still out of reach. On the other hand, in $(4+1)$ dimensions, there are several known regular solutions other than the S^3-horizon Myers-Perry [38] black holes, namely the Emparan-Reall and Pomeransky-Sen'kov black rings [12, 46] having horizon topology $S^1 \times S^2$, the black Saturns [10] of Elvang-Figueras, as well as the the black bi-rings [11] and di-rings [13, 22] found by Elvang-Rodriguez, Evslin-Krishnan, and Iguchi-Mishima. It is reasonable to expect that many more regular solutions may be found in higher dimensions, other than trivial examples obtained for instance by taking products of known solutions with flat tori. A significant motivation for this dissertation is to expand the availability of candidate regular solutions, as well as to expand the range of topologies exhibited. We accomplish this goal, and indeed provide a plethora of candidates having an increasing variety of topologies, by generalizing a $(4+1)$-dimensional technique known as the harmonic map method.

In recent works the harmonic map method has been developed as a technique to construct stationary asymptotically flat solutions to the $(4+1)$-dimensional vacuum Einstein equations with symmetry group $\mathbb{R} \times U(1)^2$. More precisely, it has been shown that an axially symmetric harmonic map from $\mathbb{R}^3 \setminus \Gamma$ into the symmetric space $SL(3, \mathbb{R})/SO(3)$ produces a stationary $U(1)^2$-symmetric solution to the Einstein equations, provided that the harmonic map satisfies certain asymptotic conditions at infinity and on the subset of the z-axis Γ. This method was used in [28] to produce such solutions which are asymptotically flat, while in [27] a similar approach was applied to obtain solutions with Kaluza-Klein and locally Euclidean asymptotics. The method has also been used in the non-vacuum case [2], where stationary bi-axisymmetric minimal supergravity solutions were produced. The harmonic map method is characterized by two features: a wide range of domain of outer communication (DOC) topologies, and conical singularities known as struts. The topologies of the DOCs were analyzed with the use of plumbing of disk bundles in [25]. In the same paper compactificaitons of the DOCs were classified by using the works of Orlik and Raymond [43, 44]. The absence of conical singularities on the two unbounded axes was also established in [27]. It is important to emphasize, however, that many of these solutions are expected to have conical singularities on at least one of the bounded components of the axis. In [51] it was in fact proven that certain symmetric spacetimes with mutliple black holes cannot exist without conical singularities. However even those solutions which are not regular should still be of interest, since we expect that one could perturb time slices to obtain initial data, satisfying relevant energy conditions, with outermost apparent horizon and DOC having exotic topologies.

In Section 4 we generalize the above method to work in all higher dimensions. Specifically we show that a harmonic map from $\mathbb{R}^3 \setminus \Gamma$ to $SL(n+1, \mathbb{R})/SO(n+1)$ produces a well-behaved, vacuum, asymptotically locally Kaluza-Klein spacetime $(\mathcal{M}^{n+3}, g)$ with symmetry group $\mathbb{R} \times U(1)^n$ (see Theorem F). By *well-behaved* we mean the stationary Killing field are complete, the DOC is globally hyperbolic, and the DOC contains an acausal spacelike connected hypersurface which is asymptotic to the canonical slice in the asymptotic end and whose boundary is a compact cross section of the horizon. These assumptions are used for the reduction of the stationary vacuum equations, and are consistent with [18]. By *asymptotically locally Kaluza-Klein* we refer to a spacetime which asymptotes to the ideal geometry $\left(\mathbb{R}^{4-s,1}/G\right) \times T^{n+s-2}$, where T^{n+s-2} is a flat torus, $G \subset O(4-s)$ is a discrete subgroup of spatial rotations, and $s \in \{0, 1, 2\}$. If G is trivial, then the moniker 'locally' is removed from the terminology.

We find that the range of DOC topologies produced is vastly greater than those found with the $(4+1)$-dimensional method. Unfortunately these topologies are not yet fully understood, as it is deeply connected to the following open question in toric topology:

Which simply connected $(n+2)$-dimensional manifolds admit effective T^n-actions?

This question received a lot of attention in the 70s and 80s when it was solved in the $n = 2$ case by Orlik and Raymond [43, 44] (see Theorem 2.34), in the $n = 3, 4$ case by Oh [41, 42] (see Theorem 2.51), and also in the 2-connected case by McGavran [35] (see Theorem 2.50). In recent years toric topologists have been more focused on understanding higher cohomogeneity torus actions [4]. This research has indirect applications to our cohomogeneity-two case [50, 55], which although less popular remains an active area of research in and of itself [23]. On the other hand, manifolds which admit cohomogeneity-two torus actions are in some sense the simplest of all 'toric objects', and understanding them could serve as a basis for understanding fundamental concepts in toric topology. The concept we are most interested in is *cohomological rigidity*, which roughly states that two 'toric objects' are diffeomorphic if they have the same integral cohomology ring [4, Definition 7.8.29]. Although this is obviously not enough to distinguish the diffeotypes of manifolds in general, at the time of writing this dissertation not a single counter example is known. Cohomological rigidity has been proven for certain classes of 'toric objects' (the so-called Bott manifolds) which admit additional algebraic structure [6], but has not been proven in general even for the cohomogeneity-two case. If an answer to the question of which simply connected $(n + 2)$-dimensional manifolds admit effective T^n-actions is provided, this would in particular resolve cohomological rigidity for the cohomogeneity-two case. In this dissertation we conjecture to this exact question (see Conjecture A) and provide a partial proof (see Theorem D). In the process we develop tools to study both the homeotype of the manifold and the equivariant homeotype of its torus action (see Theorems B and C).

The issue of conical singularities on our spacetimes is also partially resolved. We find topological conditions under which the angular momenta of black holes can be balanced against each other so that struts are not needed (see Theorem G). Surprisingly, we find that this method also works on spacetimes without black holes, producing vacuum soliton solutions. We also show that under no topological restrictions spacetimes can still be balanced (see Theorem H), however they no longer satisfy the vacuum equations. Instead they satisfy the k-reduced Kaluza-Klein equations (see Definition 4.21) which are a higher dimensional generalization of the usual Einstein-Maxwell-Dilaton equations. This can be viewed as evidence towards the idea that times slices of solutions with struts can be perturbed to obtain initial data satisfying relevant energy conditions, thus providing another reason why solutions with struts are still physically relevant.

1.2 Contributions and Main Results

An overarching goal in the study of manifolds with torus actions has always been to understand their topology. This task is far too ambitious in the general case. However restricting attention to $(n + 2)$-dimensional manifolds with effective T^n-actions whose quotient spaces are contractible and without orbifold points, the task becomes manageable. Such manifolds are called *simple T^n-manifolds* (see Definition 2.4). A map between simple T^n-manifolds $F\colon M \to N$ is call *weakly equivariant* if there exists a Lie homomorphism $\varphi\colon T^n \to T^n$ such that $F(\mathbf{t} \cdot \mathbf{p}) = \varphi(\mathbf{t}) \cdot F(\mathbf{p})$ for all $\mathbf{p} \in M$ and $\mathbf{t} \in T^n$ (see Definition 2.6). Such a maps is often denoted by $(F, \varphi)\colon (M, T^n) \to (N, T^n)$. An equivalence class of simple T^n-manifolds up to weakly equivariant homeomorphisms is called a *weak equivariant homeotype.* The classification of simple T^n-manifolds up to weak equivariant homeotype is more or less trivial. Therefore when we speak of 'classification' of simple T^n-manifolds we mean a classification of their homeo or diffeotypes.

In [43] Orlik and Raymond classified all 4-dimensional simple T^2-manifolds (see Theorem 2.34). Later McGavran [35] and Oh [41, 42] examined higher dimensional manifolds, leading to McGavran's classification of 2-connected simple T^n-manifolds (see Theorem 2.50) and Oh's classification of simple T^3 and simple T^4-manifolds (see Theorem 2.51). Inspired by the works of McGavran and Oh, we conjecture a classification of all simple T^n-manifolds for all higher dimensions.

Conjecture A. *For $n > 2$, the manifolds*

$$M(3,3) := S^5 \tag{1.1}$$

$$M(n,k) := \#_{j=0}^{n-3} \left(j \binom{n-2}{j+1} + (k-n) \binom{n-3}{j} \right) S^{2+j} \times S^{n-j} \tag{1.2}$$

$$\widetilde{M}(n,k) := \big(M(n,k) \setminus S^2 \times S^n \big) \cup S^2 \widetilde{\times} S^n \tag{1.3}$$

admit smooth effective T^n-actions making them simple T^n-manifolds. Any closed simple T^n-manifold is diffeomorphic to $T^{n-m} \times M(m,k)$ or $T^{n-m} \times \widetilde{M}(m,k)$, or a quotient of either one by a finite subgroup of T^n.

In the above conjecture the manifold $\widetilde{M}(n,k)$ is created from $M(n,k)$ be removing a single copy of $S^2 \times S^n$ and replacing it with $S^2 \widetilde{\times} S^n$, the unique non-spin S^n-bundle over S^2. Notably $\widetilde{M}(n,k)$ only exists when $k > n$ and $M(n,k)$ only exists when $k \geq n$. This conjecture encompasses McGavran's classification by setting $n = k$ and Oh's classification by setting $n = 3, 4$. Orlik and Raymond's classification is not included in the above conjecture simply because the presence of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ would complicate the statement. Conjecture A also encompasses a conjecture made in a previous work, [24, Conjecture E], which itself is a refinement of the topological portion of a conjecture by Hollands and Ishibash [18, Conjecture 1]. Lastly, if Conjecture A is true then the diffeotype of any $(n+2)$-dimensional simple T^n-manifold will be uniquely determined by its fundamental group, its second Betti number, and whether or not it's spin. This would prove the cohomological rigidity conjecture for all such manifolds and be a useful step towards proving it in general.

The first main result we have is stated in terms of rod structures. For any simple T^n-manifold M^{n+2} the quotient space M/T^n is a contractible 2-manifold with corners, usually a polygon. The action of T^n on M is entirely described by the 1-dimensional stabilizer subgroup on each of the edges of M/T^n. These edges are called *axis rods*, or simply *rods*. Using the isomorphism $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$, the 1-dimensional isotropy subgroup of each rod can be described by a primitive vector $\mathbf{v} \in \mathbb{Z}^n$ which we call a *rod structure*. Theorem B below provides an explicit method to compute the intersection form from the rod structures for any simply connected T^2-manifold.

Theorem B. *Let M^4 be a simply connected T^2-manifold with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{Z}^2$ defining the linear map $A\colon \mathbb{Z}^k \to \mathbb{Z}^2$. There exists an explicit isomorphism*

$$\begin{aligned} \Psi_*\colon \ \ker(A) \subset \mathbb{Z}^k &\to H_2(M;\mathbb{Z}) \cong \mathbb{Z}^{k-2} \\ (\alpha_1, \ldots, \alpha_k) &\mapsto [\alpha] \end{aligned}$$

which computes the intersection form of M in the following way,

$$Q([\alpha],[\beta]) = \sum_{1 \leq i < j \leq k-1} \alpha_i \beta_j \det(\mathbf{v}_i, \mathbf{v}_j). \tag{1.4}$$

This is proved in Theorem 2.40. Note that the apparent lack of symmetry in Equation (1.4) is corrected by hidden symmetry coming from α and β being in $\ker(A)$ which is a $(k-2)$-dimensional primitive sub-lattice of $\mathbb{Z}^k$ (see Remark 2.41). Theorem B is used to give a way to immediately, without any computations, determine whether a manifold is spin or not (see Theorem 2.46). It is also used to prove that any 4-dimensional spin simple T^2-manifold can be extended to a closed spin simple T^2-manifold (see Theorem 2.48). This fills a gap in a previous work, specifically proving that the technical assumption needed for [24, Theorem C] to prove [24, Conjecture D] is always satisfied when $n = 2$.

While writing this dissertation, a paper by Islambouli, Karimi, Lambert-Cole, and Meier was released which contains a result similar to Theorem B (see [23, Prop. 5.6]). However the proof method we use for Theorem B is quite different from theirs and leads to different applications of these two results. In [23] they use a method similar to Melvin in [36] and compute the Euler classes of linearly plumbed disk-bundles over 2-spheres. This method makes for an elegant proof and an easier to read intersection form, although it also requires that all but at most one pair of adjacent rod structures be admissible. For this reason it largely only works for manifolds with at most one boundary component. In order to extend this technique one would need to either develop a theory of Euler classes of linearly plumbed disk-bundles over 2-spheres with orbifold points, or make extensive use of fill-in and excision techniques. By contrast our proof method uses the Hurewicz isomorphism theorem to explicitly represent each homology class as a continuous map from S^2 into the manifold. This allows our proof method to work for any 4-dimensional simply connected T^2-space, including those with orbifold points or many boundary components.

We will see however the true utility of Theorem B is that it can be used to gain intuition for Theorem C below, which is its direct generalization into all dimensions. Theorem B has a statement which is easy to understand and a proof which is geometric. By contrast the statement of Theorem C is in four parts (see below) and the proof requires intense algebraic computations taking up the entirety of Section 3. Nevertheless, Theorem C has proven extremely useful in the effort to prove Conjecture A and thus we believe it is worth the effort of understanding.

Theorem C. *Let M^{n+2} be a simply connected T^n-manifold with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{Z}^n$ defining the linear map*

$$\Lambda^{i-2}(\mathrm{id} \otimes A) \colon \Lambda^{i-2}(\mathbb{Z}^n) \otimes \mathbb{Z}^k \to \Lambda^{i-1}(\mathbb{Z}^n)$$

by sending $\alpha \otimes \mathbf{e}_a$ to $\alpha \wedge \mathbf{v}_a$ for each basis element $\mathbf{e}_a \in \mathbb{Z}^k$ and each $\alpha \in \Lambda^{i-2}(\mathbb{Z}^n)$.

a. For each $2 \leq i \leq n$ there exists a surjective homomorphism

$$\Psi_{i*} \colon \ \ker(\Lambda^{i-2}(\mathrm{id} \otimes A)) \to H_i(M; \mathbb{Z}) \tag{1.5}$$
$$(\alpha_1, \ldots, \alpha_k) \mapsto [\boldsymbol{\alpha}] \tag{1.6}$$

which is described explicitly in terms of the rod structures.

b. The map Ψ_{i*} well-defines a bilinear form, which we refer to as an equivariant intersection form*,*

$$Q \colon H_i(M; \mathbb{Z}) \otimes H_j(M; \mathbb{Z}) \to H_{i+j-2}(T^n; \mathbb{Z}) \tag{1.7}$$

by

$$Q([\boldsymbol{\alpha}], [\boldsymbol{\beta}]) := \sum_{1 \leq a < b \leq k-1} \alpha_a \wedge \mathbf{v}_a \wedge \beta_b \wedge \mathbf{v}_b \in \Lambda^{i+j-2}(\mathbb{Z}^n) \cong H_{i+j-2}(T^n; \mathbb{Z}) \tag{1.8}$$

where $[\boldsymbol{\alpha}]$ and $[\boldsymbol{\beta}]$ are homology classes in $H_i(M; \mathbb{Z})$ and $H_j(M; \mathbb{Z})$ respectively.

c. When $i = 2$ the map $\Psi_{2*} \colon \ker(A) \cong \mathbb{Z}^{k-n} \to H_2(M; \mathbb{Z})$ is an isomorphism. When $i + j = n + 2$ the equivariant intersection form $Q \colon H_i(M; \mathbb{Z}) \otimes H_j(M; \mathbb{Z}) \to H_n(T^n; \mathbb{Z}) \cong \mathbb{Z}$ agrees with the intersection pairing on $H_*(M; \mathbb{Z})$.

d. Let $(F, \varphi) \colon (M^{m+2}, T^m) \to (N^{n+2}, T^n)$ be a weakly equivariant submersion between simply connected T-manifolds with equivariant intersection forms Q_M and Q_N respectively. Then

$$Q_N(F_*[\boldsymbol{\alpha}], F_*[\boldsymbol{\beta}]) = \varphi_* Q_M([\boldsymbol{\alpha}], [\boldsymbol{\beta}]) \tag{1.9}$$

where $[\boldsymbol{\alpha}]$ and $[\boldsymbol{\beta}]$ are homology classes in $H_i(M; \mathbb{Z})$ and $H_j(M; \mathbb{Z})$ respectively.

The entirety of Section 3 is dedicated to proving this theorem, culminating its proof in Theorem 3.5. As stated above, Theorem C is direct generalization of Theorem B. The map $A \colon \mathbb{Z}^k \to \mathbb{Z}^2$ is replaced with a more complicated map $\Lambda^{i-2}(\mathrm{id} \otimes A) \colon \Lambda^{i-2}(\mathbb{Z}^n) \otimes \mathbb{Z}^k \to \Lambda^{i-1}(\mathbb{Z}^n)$ which reduces to $A \colon \mathbb{Z}^k \to \mathbb{Z}^n$ when $i = 2$. Part a generalizes the isomorphism Ψ_* to instead be a collection of surjective homomorphisms Ψ_{i*} onto all the non-trivial homology groups. Part b defines a new bilinear operation which we call the *equivariant intersection form* and shows that Ψ_{i*} can be used to compute it. Part c shows that the two results from Theorem B, the isomorphism and the ability to compute the intersection form, both persist in higher dimensions. However they cannot both happen at the same time unless $n = 2$. Part d shows that the equivariant intersection form transforms naturally with respect to weakly equivariant submersions (see Remark 2.12). In other words, the following diagram commutes.

$$\begin{tikzcd} H_i(M) \otimes H_j(M) \arrow[r, "F_*"] \arrow[d, "Q_M"] & H_i(N) \otimes H_j(N) \arrow[d, "Q_N"] \\ H_{i+j-2}(T^n) \arrow[r, "\varphi_*"] & H_{i+j-2}(T^n) \end{tikzcd}$$

This in particular shows that the equivariant intersection form is an invariant of the weak equivariant homeotype of M.

The existence of an object such as the equivariant intersection form which transforms naturally under certain weakly equivariant maps is interesting in its own right. However Theorem C and the lemmas used in its proof have interesting applications of their own. Part a is used to explicitly compute the integral homology groups of all simply connected closed T^n-manifolds and ends up proving that Conjecture A holds in homology (see Theorem 3.4). A lemma used in the proof of Part d is crucial in understanding the relationship between rod structures and the Euler class in principal torus bundles (see Theorem 2.81). This in turn is used to construct the rod structures for $M(n, k)$ and prove they are indeed simple T^n-manifolds (see Theorem 2.52).

The next main result assembles all of the progress made towards proving Conjecture A.

Theorem D. *Conjecture A is partially proven in the following ways:*

1. *For any $2 < n \leq k$ the manifold $M(n, k)$ admits an effective T^n-action making it a simple T^n-manifold with k rods.*

2. *For all $2 < n < k$ there exist a simply connected non-spin T^n-manifold with exactly k rods.*

3. *If M^{n+2} is a closed simple T^n-manifold with non-trivial fundamental group, then it has a torsion free cover $T^{n-m} \times N$ where N^{m+2} is a simply connected, simple T^m-manifold.*

For the following, let M^{n+2} be a closed, simply connected, simple T^n-manifold with k rods.

4. *If M is 2-connected then it is diffeomorphic to $M(n, n)$.*

5. *If $n \leq 4$ then M is homeomorphic to $M(n, k)$ if it is spin and $\widetilde{M}(n, k)$ if it is not spin.*

6. *M and $M(n, k)$ have the same homotopy groups; $\pi_i(M) \cong \pi_i(M(n, k))$.*

7. *M and $M(n, k)$ have the same integral homology groups; $H_i(M; \mathbb{Z}) \cong H_i(M(n, k); \mathbb{Z})$.*

Part 1 is proven constructively in Theorem 2.52. Part 2 was essentially proven by Oh, though not stated in this way (see Theorem 2.53). Part 3 is proven by Theorem 2.58. Parts 4 and 5 come from the classification theorems of McGavran (see Theorem 2.50) and Oh (see Theorem 2.51) respectively. Part 6 is proven in Theorem 2.76. Note that having the same homotopy groups does not make these manifolds homotopy equivalent because the isomorphsims between $\pi_i(M)$ and $\pi_i(M(n, k))$ do not all come from a single map $M \to M(n, k)$. Part 7 is proven in Theorem 3.1.

In this dissertation we also example simple T^n-manifolds which are not closed. Here classification is possible, albeit in a much less elegant way than Conjecture A. Our goal is to generalize the 4-dimensional classification theorem [25, Theorem 1] which decomposed the manifold into a disjoint union of linearly plumbed disc bundles over 2-spheres, and a few other more simple pieces. There does not seem to be a direct natural generalization of linear plumbing which is applicable to the higher dimensional setting. In fact, a naive approach leads to a construction that is not unique, as there are various ways to glue the neighboring toroidal fibers together (see Figure 2.25 for an example). In order to remedy this issue we define a generalized or *toric plumbing* with additional parameters $\mathfrak{p}_i \in \mathbb{Z}^n$ which are called *plumbing vectors* (see Definition 2.95). In Theorem E, we present a full classification of all one-ended simple T^n-manifolds with boundary. This is the topological portion of a theorem resented in a previous work [24, Theorem B].

We will use the following notation for the building blocks of the decomposition. The axis Γ is a union of intervals $\{\Gamma_{i,j}\}_{i=1}^{I_j+2}$, $j = 1, \dots, \mathfrak{J}$ called *axis rods*, each of which is defined by a particular isotropy subgroup of $U(1)^n$. With each such rod that is flanked on both sides by another axis, we associate $\boldsymbol{\xi}_{i,j} = \xi_{i,j} \times T^{n-3}$ where $\xi_{i,j}$ is a $(\mathbb{D}^2)$ disc-bundle over either the 3-sphere S^3, the ring $S^1 \times S^2$, or a lens space $L(p,q)$ with $p > q$ relatively prime positive integers. A sequence of such product spaces may be glued together, with the help of plumbing vectors, to form the toric plumbing $\mathcal{P}\left(\boldsymbol{\xi}_{1,j}, \dots, \boldsymbol{\xi}_{I_j,j} \middle| \mathfrak{p}_{2,j}, \dots, \mathfrak{p}_{I_j,j}\right)$. The topologies of $\xi_{i,j}$, and the plumbing vectors themselves $\mathfrak{p}_{i,j}$, are completely determined by the rod structures of the axes involved.

Theorem E. *Let M^{n+2} be a one-ended, simple, T^n-manifold with boundary. The manifold M is decomposed as*

$$M^{n+2} = \bigcup_{j=1}^{N_3} \mathcal{P}\left(\boldsymbol{\xi}_{1,j}, \dots, \boldsymbol{\xi}_{I_j,j} \middle| \mathfrak{p}_{2,j}, \dots, \mathfrak{p}_{I_j,j}\right) \bigcup_{m=1}^{N_2} B^4_m \times T^{n-2} \bigcup_{k=1}^{N_1} C^{n+2}_k \bigcup M^{n+2}_{end}, \tag{1.10}$$

in which each constituent is a closed manifold with boundary and all are mutually disjoint expect possibly at the boundaries. Here C^{n+2}_k is $[0,1] \times \mathbb{D}^2 \times T^{n-1}$, B^4_m denotes a 4-dimensional ball, and the asymptotic end M^{n+2}_{end} is given by $\mathbb{R}_+ \times Y \times T^{n-2}$ where Y represents either S^3, or $S^1 \times S^2$. Furthermore N_3, N_2, and N_1 are the number of connected components of the axis which consist of three or more axis rods, two axis rods, and one finite axis rod, respectively.

In the Section 4 we examine the Einstein equations themselves. In doing so we produce a generalization of the harmonic map method introduced in [27] which works in higher dimensions. This generalization was published in a previous work [24, Theorem A] and is presented as Theorem F below.

In the statement of Theorem F we use the phrase rod data set. An *n-dimensional rod data set* $\mathcal{D} = \{(\mathbf{v}_1, \Gamma_1, \mathbf{c}_1), \dots, (\mathbf{v}_k, \Gamma_k, \mathbf{c}_k)\}$ is a collection of primitive vectors $\mathbf{v}_i \in \mathbb{Z}^n$, closed intervals $\Gamma_i \subset \mathbb{R}$ whose interiors do not overlap, and constants $\mathbf{c}_i \in \mathbb{R}^n$. We say that an $(n+2)$-dimensional Riemannian manifold (M^{n+2}, g) *agrees* with the rod data if it admits an effective and isometric T^n-action such that interior of the quotient space is homeomorphic to the half plane $(M/T^n) \setminus \partial(M/T^n) \cong \mathbb{R}^2_+$ and the T^n-action makes M a simple T^n-manifold with rod structures $\mathbf{v}_i$ corresponding to axis rods $\Gamma_i \subset \mathbb{R} \cong \partial\mathbb{R}^2_+$. The constants $\mathbf{c}_i \in \mathbb{R}^n$ are known as *potential constants* and are used to both dictate the asymptotic behavior of the harmonic map near Γ as well as compute the angular momenta of black hole horizons. These horizons correspond to the gaps between the axis rods, $\mathbb{R} \setminus \Gamma$, and in the theorem below are assumed to be non-degenerate closed intervals. A rod data set is said to be *admissible* if $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$ implies a condition on the second determinant divisor of the rod structures, $\mathrm{Det}_2(\mathbf{v}_i, \mathbf{v}_{i+1}) = 1$ (see Section 2.2 and Corollary 2.23). This is a purely topological condition necessary for M to be a manifold. The terms *model map* and *asymptotic* used in the statement of the theorem below are made precise in Definitions 4.3 and 4.7 respectively, but can be understood intuitively as an approximate solution to the harmonic map equations.

Theorem F. *Suppose that $\{(\mathbf{v}_1, \Gamma_1, \mathbf{c}_1), \dots, (\mathbf{v}_k, \Gamma_k, \mathbf{c}_k)\}$ is an n-dimensional admissible rod data set with non-degenerate horizon rods.*

(a) There exists a model map $\varphi_0 : \mathbb{R}^3 \setminus \Gamma \to SL(n+1, \mathbb{R})/SO(n+1)$ which corresponds to the rod data set.

(b) There exists a unique harmonic map $\varphi : \mathbb{R}^3 \setminus \Gamma \to SL(n+1, \mathbb{R})/SO(n+1)$ which is asymptotic to the model map φ_0.

(c) *A well-behaved asymptotically (locally) Kaluza-Klein solution of the $(n+3)$-dimensional vacuum Einstein equations admitting the isometry group $\mathbb{R} \times U(1)^n$ can be constructed from φ. Such a metric is smooth except possibly along the finite axis rods where conical singularities may be present.*

(d) *Any time slice of the spacetime produced is a simple T^n-manifold which agrees with the rod data.*

This is established in Theorem 4.9. Suppose that $(\mathcal{M}, g)$ is a spacetime produced by Theorem F. Strictly speaking g is only known to be smooth on $\mathcal{M} \setminus \Gamma$. There are two obstructions to extending g smoothly across Γ [28, §8.1]. The first obstruction is the presence of conical singularities, the absence of which we call *geometric regularity*. This obstruction is addressed in two separate ways in Theorems G and H. The second obstruction is termed *analytic regularity*. This condition concerns differentiability properties of the metric at Γ and is analogous to the regularity condition for surfaces of revolution at the poles. This analytic regularity condition was treated in the $(3+1)$-dimensional vacuum case by Li-Tian [30, 51] and Weinstein [52], whereas the Einstein-Maxwell setting was addressed more recently by Nguyen [40]. It is generally believed, though not yet proven, that analytic regularity will be satisfied in this higher dimensional setting as well [28, Remark 8.1.1], or at least it is believed that metrics which are geometrically regular will turn out to also be analytically regular. However the analytic regularity condition will not be addressed in this dissertation.

Assuming the analytic regularity condition is satisfied, the presence of conical singularities becomes the only obstruction to smoothly extending the metric across Γ [28, Remark 8.1.2]. It is important to emphasize, however, that many of these solutions are expected to have conical singularities on at least one of the finite axis rods. In [51] it was in fact proven that certain symmetric spacetimes with mutliple black holes cannot exist without conical singularities. Intuitively these conical singularities, or *struts*, provide a non-zero force that prevents the black hole horizons from collapsing and changing topology. This is analogous to how electric charge can be used to balance two black holes which would otherwise fall into each other. Theorem G below shows that under certain topological conditions it is possible to produce vacuum solutions without conical singularities. In the statement of the theorem we use $\mathbf{e}_i \in \mathbb{Z}^n$ to denote the standard basis vectors.

Theorem G. *Given n-dimensional rod data $\{(\mathbf{e}_1, \Gamma_1), \ldots, (\mathbf{e}_k, \Gamma_k)\}$ with $k \leq n$ and non-degenerate horizon rods, there exists a choice of potential constants $\{\mathbf{c}_1, \ldots, \mathbf{c}_k\} \subset \mathbb{R}^n$ such that the spacetime produced by Theorem F is without conical singularities.*

This is proved in Theorem 4.18. Theorem G has not been presented before, but a similar techniques was used in a previous work [24, Proposition 7.2] to produce a counter example to a conjecture by Hollands and Ishibashi [18, Conjecture 1] (see Remark 4.20 for a discussion). Although the topological restrictions in Theorem G appear quite restrictive, it still produces a plethora of candidate examples spacetimes. For instance, the 3-dimensional rod data $\{(\mathbf{e}_1, \Gamma_1), (\mathbf{e}_2, \Gamma_2), (\mathbf{e}_3, \Gamma_3)\}$ with $\Gamma_1 \cap \Gamma_2 = \Gamma_2 \cap \Gamma_3 = \emptyset$ produces a simply connected, asymptotically Kaluza-Klein spacetime with black hole horizon topology $2 \cdot (S^3 \times S^1)$, which we are tentatively calling *tri-rings* (see Example 4.19). Amazingly Theorem G works even when no black holes are present, showing that the physical intuition of counter rotating black holes 'balancing' each other is not the full picture. In reality asymptotically Kaluza-Klein spacetimes, in our case asymptotic to the model geometry on $\mathbb{R}^{1,4} \times T^{n-2}$, have additional freedom compared to the usual asymptotically flat case coming from the choice of flat metric on T^{n-2}. Theorem G uses this additional freedom to 'rescale' the size of individual circles, alleviating conical singularities for certain choices of rod structures. These vacuum solutions without black holes are are solitons, and are believed to be static thus producing complete Ricci flat Riemannian manifolds.

The final main result is Theorem H below which shows that generic rod data can always produce regular spacetimes, though they are no longer vacuum. Instead they satisfy the *k-reduced Kaluza-Klein equations* (also called *axion-dilaton gravity* [45, pg. 349]). These are defined as the field equations on $\mathcal{M}^{n+3}$ which come from solving the vacuum Einstein equations on principal a T^k-bundle $\widetilde{\mathcal{M}}^{n+k+3}$ over $\mathcal{M}$ (see Definition 4.21). When $k = 1$ this reduces to the usual Einstein-Mawxell-Dilaton equations on $\mathcal{M}$, the same equations which resulted from the famously 'almost successful' attempts by Kaluza and Klein to unify general relativity and electro-magnetism. A comprehensive overview of these equations can be found in [47] (see also [45, Ch. 11]). However the technique we use to produce solutions has little to do with the k-reduce Kaluza-Klein field equations themselves, nor much to do with any harmonic map equations. Instead solutions are produced in a topological way by showing that every simple T^n-manifold is covered by a simple T^{n+k}-manifold with rod structures nice enough to satisfy the hypotheses of Theorem G. The proof is seen in Theorem 4.22.

Theorem H. *Suppose that* $\{(\mathbf{v}_1, \Gamma_1), \ldots, (\mathbf{v}_k, \Gamma_k)\}$ *is* n*-dimensional rod data with non-degenerate horizon rods. If the solutions produced in Theorem G are analytically regular, then there exists a well-behaved regular asymptotically Kaluza-Klein stationary solution of the* $(n+3)$*-dimensional* k*-reduced Kaluza-Klein equations with time slice admitting the* $U(1)^n$ *symmetry group, and in particular agreeing with the rod data.*

2 Topology

2.1 Preliminaries

The main objects of study in this thesis are manifolds M^{n+2} equipped with an effective co-dimension 2 torus action, a T^n-action. Recall that a group action is *effective* (or equivalently *faithful*) if $t \in T^n$ and $t \cdot x = x$ for all $x \in M$, then t is the identity element in T^n. This means that any such action is defined by a Lie group embedding $\Phi \colon T^n \hookrightarrow iso(M)$ where $iso(M)$ is the 'isomorphism' group of M. That means $iso(M)$ is the group of homeomorphisms for continuous actions, diffeomorphisms for smooth action, isometries for isometric actions, etc... Because of the rigidity of Lie groups, such torus actions can be described entirely combinatorially. This combinatorial information, paired with topological information about the orbit space allows us to reconstruct the original manifold.

Example 2.1: As an example consider the quotient map $\mathbb{C}^2 \to \mathbb{R}_+ \times \mathbb{R}_+$ defined by the standard T^2 action on $\mathbb{C}^2$. In Figure 2.1 we imagine the total space, $\mathbb{C}^2$, being reconstructed by placing T^2-fibers over each interior point. The size of each of the two circles depends on the distance the interior point is from each of the axes. On the axes themselves one of the circles vanishes entirely and the fiber is a single S^1. At the origin, or corner, both circles vanish and the fiber is a single point.

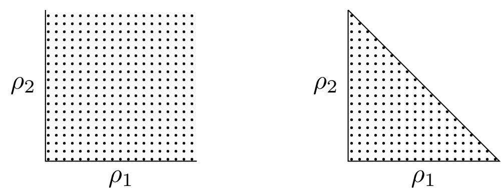


Figure 2.1: On the left is the quotient space of the standard T^2 on $\mathbb{C}^2$, defined by $(\phi_1, \phi_2) \cdot (\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) = (\rho_1 e^{i(\theta_1+\phi_1)}, \rho_2 e^{i(\theta_2+\phi_2)})$, giving the quarter-plane $\{(\rho_1, \rho_2) | \rho_i \geq 0\} = \mathbb{R}_+ \times \mathbb{R}_+$. On the right is the quotient space of the standard T^3 action on $S^5 \subset \mathbb{C}^3$, as described in Example 2.2. lens quotient 2.60 C2 2.1 S5 over CP2 2.64

Example 2.2: For another example, consider the T^3-action on S^5, inherited from the standard T^3-action on $\mathbb{C}^3$. That is, $S^5 = \{(z_1, z_2, z_3) || z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$ and $(\phi^1, \phi^2, \phi^3) \cdot (z_1, z_2, z_3) = (e^{i\phi^1} z_1, e^{i\phi^2} z_2, e^{i\phi^3} z_3)$. Using polar coordinates $z_j = \rho_j e^{i\theta_j}$ we see that the quotient space $S^5/T^3 = \{(\rho_1, \rho_2, \rho_3) | \rho_1^2 + \rho_2^2 + \rho_3^2 = 1\} = \{(\rho_1, \rho_2, \sqrt{\rho_1^2 + \rho_2^2}) | \rho_1^2 + \rho_2^2 \leq 1\}$ is perfectly described by the triangle in Figure 2.1. The total space can then be reconstructed by placing a trivial T^3-fiber bundle over the triangle and collapsing appropriate circles on the boundary. Specifically, the first circle when $\rho_1 = 0$ which is the vertical axis, the second circle collapses when $\rho_2 = 0$ which is on the horizontal axis, and the third circle collapses when $\rho_3 = 0$ or $\rho_1^2 + \rho_2^2 = 1$ which is on the 'diagonal axis'.

For more complicated T^n-actions (as we saw in the previous example) the term 'axis', as in axis of rotation, becomes less and less meaningful. Because of this, term axis rod is defined. For a specific T^n action on an $(n+2)$-manifold M^{n+2}, an *axis rod* is a connected component of the set of points $p \in M$ whose isotropy or stabilizer subgroup $\mathrm{stab}(p) \subset T^n$ is 1-dimensional. In a similar fashion we define a *corner* as a connected component of the set of points whose stabilizer subgroup is 2-dimensional. In Example 2.1 we was that there were two axis rods separated by a single corner. In Example 2.2 there were three axis rods separated by three corners. These terms will be used interchangeably to describe both subsets of M and their images in the orbit space M/T^n.

We now turn our attention to the main theorems. In the main theorems, the total space M^{n+2} is either assumed to be simply connected directly, or it satisfies topological censorship which implies simple connectivity. This results in the quotient space M/T^n being contractible and there being a rather simple description of the topology for M. The following theorem, which originally appeared in [29, Theorem 1.3] but left the proof to the reader, shows exactly how simple the topology for M is.

Theorem 2.3. *Let M^{n+2} be a compact (possibly with boundary), oriented, simply connected space with an effective T^n-action for $n \geq 1$. Either M is the 3-sphere, or its quotient space M/T^n is contractible and the quotient map $M \to M/T^n$ defines a trivial fiber bundle over the interior of the quotient.*

Proof. The fundamental group of a T^n-manifold (a manifold with an effective T^n-action) of dimension $n+2$ can be calculated from the topology of the quotient space and the bundle structure, using the Seifert-Van Kampen Theorem. This was carried out by Orlik and Raymond [44, Page 94] in the case when the quotient space is an orbifold without boundary, yielding the group presentation

$$\begin{aligned}\pi_1(M^{n+2}) \cong \big\langle &\tau_1, \ldots, \tau_n, \alpha_1, \ldots, \alpha_a, \gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g \big| \\ &[\tau_i, \tau_j]; [\tau_i, \alpha_j]; [\tau_i, \gamma_j]; [\tau_i, \delta_j]; \quad \text{for all } i \text{ and } j \\ &[\gamma_1, \delta_1] \cdots [\gamma_g, \delta_g] \cdot \alpha_1 \cdots \alpha_a \cdot \tau_1^{c_1} \cdots \tau_n^{c_n}; \\ &\alpha_l^{q_l} \cdot \tau_1^{p_{l1}} \cdots \tau_n^{p_{ln}}; \quad \text{for } l = 1, \ldots, a \big\rangle.\end{aligned} \tag{2.1}$$

The generators τ arise from the torus fibers, the α's represent loops around each of the a orbifold points, and the γ's and δ's are generators associated with each of the g handles. In the first line of relations we see that the τ's commute with themselves as they are the generators of a torus, and commute with the α's, γ's, and δ's since the former are generators of the fiber and the latter are generators in base space M^{n+2}/T^n. In analogy with the presentation of the fundamental group of a genus g surface, the second line of relations represents the obstruction to contractibility of the circumscribing loop around all of the handles and orbifold points. That loop is homotopic to the loop around the fibers described by $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n \cong \pi_1(T^n)$. The last line of relations indicates how each orbifold point singularity is to be resolved, namely, going around the i-th orbifold point $q_i \neq 1$ times is equivalent to going around each of the torus fibers p_{ij} times.

We wish to show in this case that $M^{n+2} \cong S^3$. To do that, let the list of generators in Equation (2.1) be denoted by $\mathcal{G}$ and the list of relations by $\mathcal{R}$, so that $\pi_1(M^{n+2}) \cong \langle \mathcal{G} | \mathcal{R} \rangle$ is trivial. Clearly then the group $\mathcal{H}_1 = \langle \mathcal{G} | \mathcal{R} \cup \{[\alpha_i, \alpha_j], \gamma_k, \delta_k\} \rangle$ is also trivial. This is an abelian group which can be presented as

$$\mathcal{H}_1 = (\mathbb{Z}^a \oplus \mathbb{Z}^n) \,/\, \mathrm{span}_{\mathbb{Z}}\{(\mathbf{1}, \mathbf{c}), (q_1\mathbf{e}_1, \mathbf{p}_1), \ldots, (q_a\mathbf{e}_a, \mathbf{p}_a)\}, \tag{2.2}$$

where $\mathbf{1} \in \mathbb{Z}^a$ is the vector consisting of all 1's and $\mathbf{p}_l = (p_{l1}, \ldots, p_{ln}) \in \mathbb{Z}^n$. The number of generators is $a+n$, and the number of relations is $a+1$, hence $\mathcal{H}_1$ can only be trivial if $n \leq 1$. If $n = 1$ then M^{n+2} is a simply connected closed 3-manifold, and thus is homeomorphic to S^3.

We now consider the case where the quotient has boundary, that is $\partial\left(M^{n+2}/T^n\right) \neq \emptyset$. The fundamental group in this case was calculated by Hollands and Yazadjiev [20, Theorem 3], and takes the form

$$\begin{aligned}\pi_1(M^{n+2}) \cong \big\langle &\tau_1, \ldots, \tau_n, \alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b, \gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g \big| \\ &[\tau_i, \tau_j]; [\tau_i, \alpha_j]; [\tau_i, \beta_j]; [\tau_i, \gamma_j]; [\tau_i, \delta_j]; \quad \text{for all } i \text{ and } j \\ &[\gamma_1, \delta_1] \cdots [\gamma_g, \delta_g] \cdot \alpha_1 \cdots \alpha_a \cdot \beta_1 \cdots \beta_b; \\ &\alpha_l^{q_l} \cdot \tau_1^{p_{l1}} \cdots \tau_n^{p_{ln}}; \quad \text{for } l = 1, \ldots, a; \\ &\tau_1^{v_k^1} \cdots \tau_n^{v_k^n}; \quad \text{for } k = 1, \ldots, m \big\rangle.\end{aligned} \tag{2.3}$$

The extra generators β represent the b boundary components of the orbit space which are homeomorphic to circles; on these components the torus action may or may not degenerate. Additional relations are included for these generators showing that they commute with the generators of the torus fibers. The last line of relations is defined by vectors $\mathbf{v}_k = (v_k^1, \ldots, v_k^n) \in \mathbb{Z}^n$ known as rod structures. A *rod structure* is a vector assigned to each axis rod which represents the homotopy type (up to sign) of the stabilizer subgroup in $\pi_1(T^n) \cong \mathbb{Z}^n$. As before denote the generators of (2.3) by $\mathcal{G}$ and the list of relations by $\mathcal{R}$. We can immediately determine that $g = 0$ by examining $\langle \mathcal{G} | \mathcal{R} \cup \{\tau_i, \alpha_j, \beta_\ell\} \rangle$, which is in fact the fundamental group of a genus g surface. Next consider the Abelian group $\mathcal{H}_2 = \langle \mathcal{G} \big| \mathcal{R} \cup \{\tau_i, [\alpha_i, \alpha_j], \beta_l\} \rangle$, which may be presented as

$$\mathcal{H}_2 = \mathbb{Z}^a / \operatorname{span}_{\mathbb{Z}}\{\mathbf{1}, q_1\mathbf{e_1}, \dots, q_a\mathbf{e_a}\}. \tag{2.4}$$

This group cannot be trivial unless $q_1 = \cdots = q_a = 1$, however this contradicts the nature of q_i, and thus $a = 0$. Finally consider the subgroup $\langle \mathcal{G} \big| \mathcal{R} \cup \{\tau_i\} \rangle = \langle \beta_1, \dots, \beta_b \big| \beta_1 \cdots \beta_b \rangle$, and observe that it is trivial only when $b = 0$ or 1. However $\partial(M/T^n)$ is compact and non-empty so $b \neq 0$. Therefore quotient space M/T^2 has neither holes nor handles no holes or handles, making it homeomorphic to a disk $\mathbb{D}^2$, and the quotient map $M \to M/T^n$ has no orbifold points, making it a trivial bundle over the interior. $\square$

In the above proof the notion of *rod structures* was introduced. The manifold being simply connected was used to show that there were no holes, handles, and orbifold points and that all the 'topological information' is contained in the rod structures. This means that for simply connected manifolds rod structures completely describe the torus action, and in particular can be used to reconstruct the manifold from its orbit space. In fact rod structures completely describe the torus action even for non-simply connected manifolds, as long as they satisfy the consequences of Theorem 2.3. This leads to the following definition.

Definition 2.4. A *simple T^n-manifold* is an orientable smooth manifold M^{n+k}, $k \geq 0$ equipped with an effective T^n-action, in which the quotient space M^{n+k}/T^n is contractible and the quotient map defines a trivial fiber bundle over the interior of the quotient.

The term *simple T^n-manifold* was chosen for a number of reasons. Firstly, the original objects of study were $(n+2)$-dimensional simply connected T^n-manifolds, which Theorem 2.3 guarantees are simple T^n-manifolds. Secondly, the term intentionally covers more than just the cohomogeneity-2 case, which is the main focus of this thesis, because many of the topological ideas discussed in this thesis are expected to work in higher cohomogeneity. For example the round S^7 can be reconstructed from the standard T^4 action in a manner similar to reconstructing S^5 in Example 2.2. Thirdly, the explicit indication of the dimension of the torus is meant to distinguish the study of these manifolds from the world of 'toric topology' which primarily studies $2n$-manifolds with T^n-actions. Although when referring to a collection of simple T^n-manifolds with various values of n, we will use the term *simple T-manifolds* to avoid confusion. Lastly, the world of toric topology has already claimed the terms: toric manifold, quasitoric manifold, torus manifold, and topological toric manifold [4].

Strictly speaking, a simple T^n-manifold M is a pair (M, Φ) where $\Phi\colon T^n \hookrightarrow iso(M)$ is a Lie group embedding of the torus into the Lie group of isomorphisms of M (homeomorphism for continuous T^n actions, diffeomorphisms for smooth actions, isometries for isometric group actions, etc...). This leads to stricter versions of equivalence between simple T-manifolds than merely being homeomorphic.

Definition 2.5. Suppose M and N as simple T-manifolds with group actions $\Phi\colon T^m \hookrightarrow iso(M)$ and $\Psi\colon T^n \hookrightarrow iso(N)$ respectively. A map $F\colon M \to N$ is *strongly equivariant* if and only if

$$F(\mathbf{t} \cdot p) := F(\Phi(\mathbf{t})(p)) = \Psi(\mathbf{t})(F(p)) =: \mathbf{t} \cdot F(p)) \tag{2.5}$$

for all $\mathbf{t} \in T^m$, $p \in M$. Or equivalently, the following diagram needs to be commutative.

$$\begin{tikzcd}
T^m \times M \arrow[r, "\mathrm{id} \times F"] \arrow[d, "\Phi \times \mathrm{id}"'] & T^n \times N \arrow[d, "\Psi \times \mathrm{id}"] \\
iso(M) \times M \arrow[d, "eval"] & iso(N) \times N \arrow[d, "eval"] \\
M \arrow[r, "F"] & M
\end{tikzcd}$$

If $n = m$ and F is a homeomorphism (diffeomorphism) then the simple T^n-manifolds (M, Φ) and (N, Ψ) are said to be *strongly equivariantly homeomorphic (diffeomorphic).*

Definition 2.6. Suppose M and N as simple T-manifolds with group actions $\Phi\colon T^m \hookrightarrow iso(M)$ and $\Psi\colon T^n \hookrightarrow iso(N)$ respectively. A map $F\colon M \to N$ is *weakly equivariant* if and only if there exists a Lie group homomorphism $\varphi\colon T^m \to T^n$ such that

$$F(\mathbf{t} \cdot p) := F(\Phi(\mathbf{t})(p)) = \Psi(\varphi(\mathbf{t}))(F(p)) =: \varphi(\mathbf{t}) \cdot F(p)) \tag{2.6}$$

for all $\mathbf{t} \in T^n$, $p \in M$. Or equivalently, the following diagram needs to be commutative

$$\begin{tikzcd}
T^m \times M \arrow[r, "\varphi\times F"] \arrow[d, "\Phi\times\mathrm{id}"'] & T^n \times N \arrow[d, "\Psi\times\mathrm{id}"] \\
iso(M) \times M \arrow[d, "eval"] & iso(N) \times N \arrow[d, "eval"] \\
M \arrow[r, "F"] & M
\end{tikzcd}$$

We usually refer to this by saying $(F, \varphi)\colon (M, T^m) \to (N, T^n)$ is a weakly equivariant map. If both φ and F are homeomorphism (diffeomorphism) then the simple T^n-manifolds (M, Φ) and (N, Ψ) are said to be *weakly equivariantly homeomorphic (diffeomorphic).*

Note that what we call 'strongly equivariantly homeomorphic' is traditionally referred to simply as *equivariantly homeomorphic*, without the word 'strongly'. However we add the word strongly here to emphasise the distinction between 'equivariant' and 'weakly equivariant'. The phrase 'weakly equivariant' is the most commonly used term for Definition 2.6, however in some sources the function F in Equation (2.6) is called *φ-equivariant* and sometimes the pair (F, φ) is simply called 'equivariant'. A lack of clear distinction between these terms has caused confusion in the past (compare [42, Theorem 1.6] to [41, Lemma 2.2]). We will see later in Remark 2.14 that this confusion is not merely due to a typo, but do to a subtly in the definition of rod structures.

Remark 2.7. If $(F, \varphi)\colon (M, T^m) \to (N, T^n)$ is a weakly equivariant map between simple T-manifolds, then

$$\varphi(\mathrm{stab}(p)) \subset \mathrm{stab}(F(p)) \tag{2.7}$$

for all $p \in M$. This is seen by using Equation (2.6) since if $\mathbf{t} \in \mathrm{stab}(p)$ then $\varphi(\mathbf{t}) \cdot F(p) = F(\mathbf{t} \cdot p) = F(p)$.

Remark 2.8. Consider a (smooth) manifold M with two distinct T^n-actions $\Phi_0, \Phi_1\colon T^n \hookrightarrow iso(M)$. If the simple T^n-manifolds (Φ_0, M) and (Φ_1, M) are weakly equivariantly homeomorphic (diffeomorphic) then the map $F\colon M \to M$ in Definition 2.6 is itself an element of $iso(M)$. When comparing the embeddings we see

$$F(\Phi_0(\varphi^{-1}(t))(F^{-1}(p))) = \Phi_1(t)(p)$$

which means that as maps $\Phi_1\colon T^n \hookrightarrow iso(M)$ and $\Phi_0 \circ \varphi^{-1}\colon T^n \hookrightarrow iso(M)$ are conjugate by $F \in iso(M)$, that is

$$F(\Phi_0 \circ \varphi^{-1})(t) F^{-1} = \Phi_1(t)$$

for all $t \in T^n$. Stated another way, conjugacy classes of $T^n \subset iso(M)$ are in one-to-one correspondence with weakly equivariant homeo(diffeo)types of M. This bijection was made explicit for smooth 4-dimensional simple T^2-manifolds in [36].

The previous remark has significant consequences when M is given additional structure such as a metric, complex structure, symplectic form, etc... In all of these cases the group of structure preserving isomorphisms is a compact Lie group. Such Lie groups are special because all maximal torus subgroups belong to the same conjugacy class. In effect, there is at most one torus action which respects the additional structure up to weak equivariance.

Theorem 2.9. *Let M^{n+2} be a topological manifold which is homeomorphic to a simple T^n-manifold and not homeomorphic to a simple T^{n+1}-manifold. Endow M with some additional structure κ (such as a complex structure, or an orientation, smooth structure, and a metric) and let $iso(M,\kappa) \subset iso(M)$ denote the Lie subgroup of homeomorphisms which preserve κ. If $iso(M,\kappa)$ is compact, then any two κ-preserving T^n-actions $\Phi_0, \Phi_1 \colon T^n \hookrightarrow iso(M,\kappa)$ are weakly equivariantly homeomorphic via a map $F \in iso(M,\kappa)$ and an automorphism $\varphi \in Aut(T^n)$. Furthermore there exists a homotopy $\Phi \colon [0,1] \times T^n \hookrightarrow iso(M,\kappa)$ which connects Φ_0 and $\Phi_1 \circ \varphi$.*

Proof. The embeddings $\Phi_i \colon T^n \hookrightarrow iso(M,\kappa)$ define Lie subgroups $\Phi_i(T^n) \subset iso(M,\kappa)$ which are both homomorphic to the torus T^n. These are maximal tori since the existence of a larger torus subgroup $T^{n+1} \subset iso(M,\kappa) \subset iso(M)$ would imply that M is homeomorphic to a simple T^{n+1}-manifold. Because $iso(M,\kappa)$ is finite dimensional the Maximal Torus Theorem can be applied, which states that there exists an element in the connected component of the identity $F \in iso(M,\kappa)_0$ so that $F\Phi_0(T^n)F^{-1}$ and $\Phi_1(T^n)$ are identical as subgroups of $iso(M,\kappa)$. Since both subgroups are abstractly homomorphic to T^n, there exists an automorphism $\varphi \in Aut(T^n)$ so that $F\Phi_0(\mathbf{t})F^{-1} = \Phi_1 \circ \varphi(\mathbf{t})$ for all $\mathbf{t} \in T^n$. Lastly F is in the connected component of the identity of $iso(M,\kappa)$ so there exists a path $\gamma \colon [0,1] \to iso(M,\kappa)$ where $\gamma(0) = \mathrm{id}$ and $\gamma(1) = F$. Define the homotopy $\Phi \colon [0,1] \times T^n \hookrightarrow iso(M,\kappa)$ as $\Phi(s,\mathbf{t}) := \gamma(s)\Phi_0(\mathbf{t})\gamma(s)^{-1}$ and observe that Φ connects Φ_0 to $\Phi_1 \circ \varphi$ as desired. □

Remark 2.10. When discussing manifolds with additional structure such as a metric or a complex structure, the torus structure is only defined up to weak equivariance. This makes it somehow the 'correct' category to work in.

The next few lemmas can all in some way be attributed to Orlik and Raymond. They are presented here for completeness with only a brief description of their proofs.

Lemma 2.11 (Cross-Sectioning Theorem [42, 43])**.** *For any simple T^n-manifold M^{n+2}, the quotient map $\pi \colon M \to M/T^n$ admits a global section $\sigma \colon M/T^n \to M$. If M is a smooth manifold with a smooth torus action, then the section can be chosen to be smooth as well.*

Orlik and Raymond originally proved this statement for the $n = 2$ case [43], but their techniques generalize to all $n \geq 2$ [42]. The idea is to first break up the boundary of the orbit space $\partial(M^{n+2}/T^n)$ into separate pieces, which we call axis rods $\Gamma_i \subset \partial(M/T^n)$. Next show that the bundle over these boundary pieces always admits a section. Finally use the triviality of the bundle over the interior to extend the section from the boundary to the entire orbit space.

Remark 2.12. Let M^{m+2} be a simple T^m-manifold with quotient map $\pi_M \colon M \to M/T^m$, section $\sigma_M \colon M/T^m \to M$, and rod structures and axis rods $\{(\mathbf{v}_1^M, \Gamma_1^M), \dots, (\mathbf{v}_k^M, \Gamma_k^M)\}$. Let N^{n+2} be a simple T^n-manifold defined similarly. Any weakly equivariant map $(F,\varphi) \colon (M^{m+2}, T^m) \to (N^{n+2}, T^n)$ defines a map between their quotient spaces $f \colon M/T^m \to N/T^n$ by

$$f(x) := \pi_N(F(\sigma_M(x))). \tag{2.8}$$

If f is a homeomorphism between M/T^m and N/T^n as 'manifolds with corners' then M and N have the same number of rods and up to relabeling

$$f(\Gamma_i^M) = \Gamma_i^N. \tag{2.9}$$

As an abuse of notation we sometimes use Γ_i^M to denote $\pi_M^{-1}(\Gamma_i^M)$, in which case the expressions $F(\Gamma_i^M)$ and $\mathrm{stab}(\Gamma_i^M)$ make sense. Using Equations (2.7) and (2.9) we can see $\varphi(\mathrm{stab}(\Gamma_i^M)) \subset \mathrm{stab}(F(\Gamma_i^M)) = \mathrm{stab}(\Gamma_i^N)$. Since $\mathrm{stab}(\Gamma_i^N)$ is generated by $\mathbf{v}_i^N$ we see

$$\varphi(\mathbf{v}_i^M) = \mathbf{v}_i^N. \tag{2.10}$$

If in addition φ is surjective, then the weakly equivariant map (F, φ) is called a *weakly equivariant submersion.*

The next lemma is our version of Orlik and Raymond's Equivariant Classification Theorem. It was originally stated in terms of the strongly equivariant case, however as noted in Remark 2.10 the weakly equivariant case is more useful. Plus the terminology used in the original presentation of the theorem would no longer be recognized by a modern reader.

Lemma 2.13 (Equivariant Classification Theorem [42, 43])**.** *Let M^{n+2} and N^{n+2} be simple T^n-manifolds with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and $\{\mathbf{w}_1, \dots \mathbf{w}_{k'}\}$ corresponding to axis rods $\{\Gamma_1^M, \dots, \Gamma_k^M\}$ and $\{\Gamma_1^N, \dots, \Gamma_{k'}^N\}$ respectively. There exists an orientation preserving weakly equivariant homeomorphism (diffeomorphism) between M and N if and only if:*

1. *they have the same number of axis rods, that is $k = k'$,*
2. *there exists a matrix $\varphi \in SL(n, \mathbb{Z})$, a fixed constant c, and a permutation* $\rho(i) := \begin{cases} c + i & \det(\varphi) = 1 \\ c - i & \det(\varphi) = -1 \end{cases}$ *such that $\pm\varphi(\mathbf{v}_i) = \mathbf{w}_{\rho(i)}$,*
3. *and there exists a homeomorphism (diffeomorphism) $f\colon M/T^n \to N/T^n$ between their orbit spaces with the property that $f(\Gamma_i^M) = \Gamma_{\rho(i)}^N$.*

If φ is the identity matrix then M and N are strongly equivariantly homeo(diffeo)morphic.

Note that if M and N are closed manifolds, then the third property follows from the first two. The proof of Lemma 2.13 given by Orlik and Raymond relies on the existence of a section $\sigma_M\colon M/T^n \to M$. This allows every point $p \in M$ to be expressed as $p = \phi \cdot \sigma_M(x)$. Therefore the function F can essentially be defined by $F(\phi \cdot \sigma_M(x)) = \phi \cdot \sigma_N(f(x))$. The body of the proof is spent ensuring that F is well defined.

Remark 2.14. The last line of Lemma 2.13 makes it clear that a simple T^n-manifold (Φ, M) is defined up to strongly equivariant homeomorphisms by its rod structures. An extremely subtle point is that rod structures are (up to a sign) the homotopy classes in $\pi_1(T^n)$ of the stabilizer subgroups of the axis rods $\{\Gamma_1, \dots, \Gamma_k\}$. When a basis is chosen for $\pi_1(T^n)$, these rod structures can be represented by integer vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset \mathbb{Z}^n$. However, $\pi_1(T^n)$ has no preferred basis. This means that a different mathematician given the same embedding $\Phi\colon T^n \hookrightarrow iso(M)$ will get the same rod structures, but may represent them by a distinct set of integer vectors $\{\mathbf{v}'_1, \dots, \mathbf{v}'_k\} \subset \mathbb{Z}^n$. These vectors differ exactly by a change of basis $B\mathbf{v}_i = \mathbf{v}'_i$, $B \in SL(n, \mathbb{Z})$ which are in one-to-one correspondence with Lie group homeomorphisms $A\colon T^n \to T^n$. Therefore when one misinterprets the integer vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset \mathbb{Z}^n$ and $\{\mathbf{v}'_1, \dots, \mathbf{v}'_k\} \subset \mathbb{Z}^n$ as being the rod structures themselves instead of merely being a representation of them, it appears that a single embedding $\Phi\colon T^n \hookrightarrow iso(M)$ yields two distinct simple T^n-manifolds which are weakly equivariantly homeomorphic but not strongly equivariantly homeomorphic. This confusion has in at least one [24], and most likely in many more, lead to scrapping the concept of 'strongly equivariantly homeomorphic' all together and erroneously using the term 'equivariantly homeomorphic' to mean 'weakly equivariantly homeomorphic'.

In light of the previous remark, we present the following corollary.

Corollary 2.15. *Every simple T^n-manifold M^{n+2} is strongly equivariantly homeomorphic (diffeomorphic) to*

$$(M/T^n \times T^n)/\sim$$

where $(x, \boldsymbol{\theta}) \sim (x, \boldsymbol{\phi} \cdot \boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in T^n$, $x \in M/T^n$, and $\boldsymbol{\phi} \in \mathrm{stab}(x) \subset T^n$. Upon choosing an ordered basis for $\pi_1(T^n)$ there exists a representative for the strong equivariant homeo(diffeo)type, known as a standard model*, of the form*

$$(B \times \mathbb{R}^n/\mathbb{Z}^n)/\sim$$

where B is the unit ball and $(x, \boldsymbol{\theta}) \sim (x, \lambda \mathbf{v}_i + \boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathbb{R}^n/\mathbb{Z}^n$, $\lambda \in \mathbb{R}$, $x \in \overline{\Gamma_i}$ the closure of the axis rod Γ_i, and the $\mathbf{v}_i \in \mathbb{Z}^n$ represent the rod structures, that is $\mathbf{v}_i = \pm[\mathrm{stab}(\Gamma_i)] \in \pi_1(T^n) \cong \mathbb{Z}^n$. The standard model is unique up to boundary preserving homeomorphism (diffeomorphism) of B, the signs of the integer vectors $\mathbf{v}_i$, and up to at most an action by the dihedral group D_k where k is the number of axis rods.

The proof of this corollary is immediate. Simply realize the constructed spaces as simple T^n-manifolds using the obvious T^n-action and then apply Lemma 2.13.

Remark 2.16. The unit ball in Corollary 2.15 can easily be replaced with any other homeomorphic open subset of $\mathbb{R}^2$. Common choices for this are $\mathbb{R}^2_+$ when two of the axis rods are semi-infinite in length, or a simple k-gon when the manifold is closed.

This last lemma makes clear the distinction between the weak equivariant homeotype and weak equivariant diffeotype of a simple T^n-manifold M^{n+2}; there is none. Orlik and Raymond first proved this in 1970 for closed 4-manifold [43]. However there doesn't appear to be a generalization of this result to higher dimensions in the classic literature. In 2012, Wiemeler proved a similar statement for strongly quasitoric manifolds [55, Corollary 5.7] and more interestingly showed that the diffeotypes of quasitoric manifolds are in one-to-one correspondence with the diffeotype of their quotient space polytope [55, Theorem 5.6]. Wiemeler's proof works equally well for lower cohomogeneity cases and in particular works for $(n+2)$-dimensional simple T^n-manifolds, which themselves can be viewed as intersections of hypersurfaces in quasitoric manifolds. Since the quotient spaces of these manifolds are 2-dimensional and there are no exotic smooth structures in 2 dimensions, we have the following theorem.

Theorem 2.17. *Every closed simple T^n-manifold M^{n+2} has exactly one smooth structure which agrees with the smooth structure of the Lie group T^n.*

The above theorem shows that any continuous effective action of T^n on M^{n+2} defines a unique smooth structure on M. However there may be smooth structures on M where the action of T^n is not smoothable. The following examples shows just that. For instance, although it is currently unknown if exotic smooth structures on S^4 exist, Theorem 2.17 proves that only the standard smooth structure admits a smooth T^2-action. The following example is even more explicit.

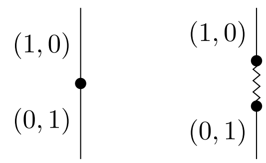
Example 2.18: The 4-manifolds $K3\#\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$ are homeomorphic, however one admits a smooth effective T^2-action and the other admits a complex structure and thus they are not diffeomorphic. The space $K3\#\overline{\mathbb{CP}}^2$ is an algebraic variety which is not rational, ie there does not exist a Zariski open set in $K3\#\overline{\mathbb{CP}}^2$ which is bi-holomorphic to a Zariski open set in $\mathbb{CP}^2$. Since every complex surface with a smooth effective T^2-action is rational we know that $K3\#\overline{\mathbb{CP}}^2$ does not admit a smooth T^2-action. On the hand $3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$ does admit a smooth effective T^2-action as seen in Orlik and Raymond's classification theorem (Theorem 2.34). However it does not satisfy the conditions of Delzant's theorem (Theorem 2.32) meaning it does not admit a complex structure.

2.2 Rod Structures

When studying simple T^n-manifolds M^{n+2} arising from general relativity the quotient space M/T^n is often assumed to be homeomorphic to the half plane $\mathbb{R}^2_+$, or to a lesser extent $\mathbb{D}^2$ which can be considered a compactification of $\mathbb{R}^2_+$. The boundary $\partial\mathbb{R}^2_+$ of the base space is divided into segments separated by finite intervals or *horizon rods* where the fibers do not degenerate. The boundary points of horizon rods are called *poles*. The portion of the boundary not covered by horizon rods is covered by axis rods, intervals separated by corners. Associated to each axis rod interval $\Gamma_i \subset \partial\mathbb{R}^2_+$ is a rod structure $\mathbf{v}_i \in \mathbb{Z}^n$ which defines the 1-dimensional isotropy subgroup $\mathbb{R}/\mathbb{Z} \cdot \mathbf{v}_i \subset \mathbb{R}^n/\mathbb{Z}^n \cong T^n$ for the action of T^n on points that lie over Γ_i. The topology of the total space M^{n+2} is determined by the rod structures, namely

$$M^{n+2} \cong (\mathbb{R}^2_+ \times T^n)/\sim \tag{2.11a}$$

Figure 2.2: Each of these two rod diagram shows the 2-dimensional quotient space as the right-half-plane with the vertical lines being their boundaries. On the left is the standard rod diagram for $\mathbb{R}^4$, and can be seen as a deformation of the image in Figure 2.1. On the right is a rod diagram for $\mathbb{R}^4 \setminus B^4$. The jagged line is a black hole horizon rod, the interior of which correspond to the product of an open interval with T^2. The rod structures flanking the horizon rod yield horizon cross-sectional topology S^3, which is the interior boundary of $\mathbb{R}^4 \setminus B^4$.



where the equivalence relation $\sim$ is given by

$$(p, \boldsymbol{\phi}) \sim (p, \boldsymbol{\phi} + \lambda \mathbf{v}_i) \tag{2.11b}$$

with $p \in \Gamma_i$, $\lambda \in \mathbb{R}/\mathbb{Z}$, and $\boldsymbol{\phi} \in T^n$. A graphical representation of this information is called a *rod diagram*, see Figure 2.2 for simple examples. These are drawn as either a disk or a polygon in the compact case, or a half plane in the noncompact case, in which the boundary is divided into segments with associated rod structure vectors indicating which linear combination of generators that degenerate at the axes. Black dots represent corners or poles where two rods meet, and the segments drawn with jagged lines are horizon rods along which the torus action is free.

Given that M^{n+2} admits an effective T^n action, the quotient map $M^{n+2} \to M^{n+2}/T^n$ exhibits M^{n+2} as a T^n-bundle over a 2-dimensional base space with possibly degenerate fibers on the boundary. Fibers over interior points are n-dimensional, while fibers over points along the boundary can be $(n-1)$ or $(n-2)$-dimensional. The set of points where the fiber is $(n-1)$-dimensional are called *axis rods* while the points with an $(n-2)$-dimensional fiber are called *corners*. The set of corners is always discrete. Associated to each axis rod is a vector $\mathbf{v} \in \mathbb{Z}^n$ called the *rod structure*, that defines the 1-dimensional isotropy subgroup $\mathbb{R}/\mathbb{Z} \cdot \mathbf{v} \subset \mathbb{R}^n/\mathbb{Z}^n \cong T^n$ for the action of T^n upon that axis rod.

It should be noted that the notion of rod structures given so far does not guarantee a unique presentation. Indeed, the vectors $\mathbf{v}$ and $2\mathbf{v}$ both generate the same isotropy subgroup $\mathbb{R}/\mathbb{Z} \cdot \mathbf{v}$, and thus both can be used to describe the same rod structure. In order reduce the number of presentations, it is natural to restrict attention to primitive elements. A vector $\mathbf{v}$ is *primitive* or a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{Z}^n$ forms a *primitive set*, if they are linearly independent and

$$\mathbb{Z}^n \cap \operatorname{span}_{\mathbb{R}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}. \tag{2.12}$$

For a single vector $\mathbf{v} = (v^1, \ldots, v^n)$, this is equivalent to the components being relatively prime, that is $\gcd\{v^1, \ldots, v^n\} = 1$. Note however that even demanding rod structures be primitive vectors, we cannot distinguish between one choice of rod structure $\mathbf{v}$ and its negative $-\mathbf{v}$ since both generate the same isotropy subgroup. This fact will be used when it is advantageous to replace $\mathbf{v}$ with $-\mathbf{v}$.

Next, observe that the group of unimodular matrices $GL(n, \mathbb{Z})$ (which is equal to $SL(n, \mathbb{Z})$) provides the group of coordinate transformations for $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Two rod diagrams are equivalent, and thus the T^n-spaces they describe are equivariantly homeomorphic, if every rod structure of one is obtained from the corresponding rod structure of the other by the action of the same unimodular matrix. Figure **??** shows an example. This means quantities depending only on the T^n-structure will be invariant under $GL(n, \mathbb{Z})$ transformations. The following proposition exhibits an example of such a quantity, Det_k, referred to as the k^{th} *determinant divisor* [39, Chapter II, Section 14]. In the statement we will use the multi-index notation I^n_k, for $k \le n$, to denote the set of k-tuples $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{Z}^k$ such that $1 \le i_1 < \cdots < i_k \le n$.

Proposition 2.19. *Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{Z}^n$, $k \leq \min\{m,n\}$, and set*

$$\mathrm{Det}_k(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \gcd\{Q^{\mathbf{i}}_{\mathbf{j}} \mid \mathbf{i} \in I^n_k,\ \mathbf{j} \in I^m_k\}, \tag{2.13}$$

where $Q^{\mathbf{i}}_{\mathbf{j}}$ is the determinant of the $k \times k$ minor obtained from the matrix defined by the column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, by picking columns $\mathbf{j}$ and rows $\mathbf{i}$. Then Det_k is invariant under $GL(n,\mathbb{Z})$, that is

$$\mathrm{Det}_k(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \mathrm{Det}_k(A\mathbf{v}_1, \ldots, A\mathbf{v}_m) \tag{2.14}$$

for all $A \in GL(n,\mathbb{Z})$.

Proof. Let $\omega \in \bigwedge^k \mathbb{Z}^n$ be a k-form on $\mathbb{Z}^n$. Each such form can be written as a linear combination of the basis elements $\{\mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_k} \mid \mathbf{i} \in I^n_k\}$, where $\{\mathbf{e}^i\}$ is the basis of covectors dual to the standard basis $\{\mathbf{e}_j\}$ of $\mathbb{Z}^n$, so that $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$. Thus

$$\omega = \sum_{\mathbf{i} \in I^n_k} a_{i_1 \ldots i_k} \mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_k}, \qquad a_{\mathbf{i}} \in \mathbb{Z}, \tag{2.15}$$

where by definition $\mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_k}(\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_k})$ is the minor determinant $Q^{\mathbf{i}}_{\mathbf{j}}$. Consider the $k \times k$ minor determinant $Q'^{\mathbf{i}}_{\mathbf{j}}$ of the matrix formed from the column vectors $A\mathbf{v}_{j_1}, \ldots, A\mathbf{v}_{j_k}$, and observe that $Q'^{\mathbf{i}}_{\mathbf{j}}$ is multilinear and antisymmetric in $\{\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_k}\}$. Therefore it is a linear combination as in (2.15), and may be expressed as

$$Q'^{\mathbf{i}}_{\mathbf{j}} = \sum_{\mathbf{i}' \in I^n_k} a^{\mathbf{i}}_{\mathbf{i}'} Q^{\mathbf{i}'}_{\mathbf{j}}. \tag{2.16}$$

Observe that if $p \in \mathbb{Z}$ divides $Q^{\mathbf{i}'}_{\mathbf{j}}$ for all $\mathbf{i}' \in I^n_k$, then p also divides $Q'^{\mathbf{i}}_{\mathbf{j}}$ and hence

$$\mathrm{Det}_k(A\mathbf{v}_1, \ldots, A\mathbf{v}_m) = \gcd\{Q'^{\mathbf{i}}_{\mathbf{j}} \mid \mathbf{i} \in I^n_k,\ \mathbf{j} \in I^m_k\} \geq \gcd\{Q^{\mathbf{j}'}_{\mathbf{i}} \mid \mathbf{i}' \in I^n_k,\ \mathbf{j} \in I^m_k\} = \mathrm{Det}_k(\mathbf{v}_1, \ldots, \mathbf{v}_m). \tag{2.17}$$

Furthermore since $A^{-1} \in GL(n,\mathbb{Z})$, the same reasoning shows that

$$\mathrm{Det}_k(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \mathrm{Det}_k(A^{-1}(A\mathbf{v}_1), \ldots, A^{-1}(A\mathbf{v}_m)) \geq \mathrm{Det}_k(A\mathbf{v}_1, \ldots, A\mathbf{v}_m). \tag{2.18}$$

The desired invariance follows from these two inequalities. ☐

A corner point between two adjacent axis rods is *admissible* if the total space over a neighborhood of the corner is a manifold. The importance of the second determinant divisor in the current context arises from the fact that it determines whether or not a corner is admissible. Since the corner point represents an $(n-2)$-torus within the total space, a tubular neighborhood will be a manifold if and only if it is homeomorphic to $B^4 \times T^{n-2}$, or equivalently if its boundary is $S^3 \times T^{n-2}$. This last criteria occurs precisely when there is a matrix $Q \in GL(n,\mathbb{Z})$ such that $Q\mathbf{v} = \mathbf{e}_1$ and $Q\mathbf{w} = \mathbf{e}_2$, where $\mathbf{v}$, $\mathbf{w}$ are the rod structures of the axis rods forming the corner, and $\mathbf{e}_1$, $\mathbf{e}_2$ are members of the standard basis for $\mathbb{Z}^n$. Corollary 2.23 below, guarantees that such a Q exists if and only if $\mathrm{Det}_2(\mathbf{v},\mathbf{w}) = 1$. The statement of this result uses the *Hermite normal form*, whose properties are listed in the next lemma. A proof of this lemma can be found in [31]. The Hermite normal form may be viewed as the integer version of the reduced echelon form, or as the integer version of the QR decomposition for real matrices.

Lemma 2.20. *Let A be a $n \times k$ integer matrix. There exist integer matrices Q and H such that $QA = H$, where Q is unimodular and $H = (h_{ij})$ has the following properties.*

1. *For some integer m, the rows 1 through m of H are non-zero, and the rows $m+1$ through n are rows of zeros.*

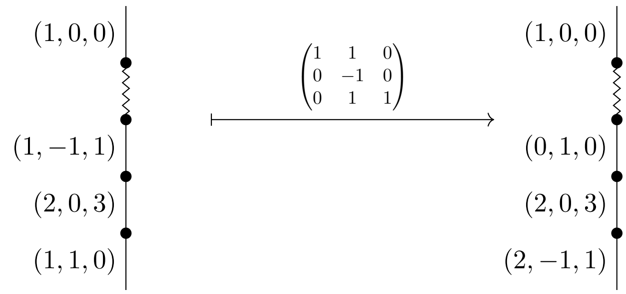


Figure 2.3: This figure shows two rod diagrams, separated by an arrow, both depicting $(5+1)$-dimensional spacetimes with a single black hole. Each rod diagram shows the 2-dimensional quotient space as the right-half-plane with the vertical lines being their boundaries. The jagged lines are black hole horizon rods, the interior of which correspond to the product of an open interval with T^3. The rod structures flanking the horizon rod yield horizon cross-sectional topology $S^1 \times S^3$. The two rod diagrams depict the same spacetime. The unimodular matrix in the middle represents a coordinate change on T^n. In particular, it is the transformation matrix from Lemma 2.20 which sends the rod structures on the left to their Hermite normal form on the right.

2. *There is a sequence of integers* $1 \leq r_1 < r_2 < \cdots < r_m \leq r = \operatorname{rank} A$ *such that the entries* h_{ir_i} *of* H*, called* pivots*, are positive for* $i = 1, \ldots, m$*. The pivot* h_{ir_i} *is the first non-zero element in the row* i*, that is,* $h_{ij} = 0$ *for* $1 \leq j < r_i$*.*

3. *In each column of* H *that contains a pivot, the entries of the column are bounded between* 0 *and the pivot, that is, for* $i = 1, \ldots, m$ *and* $1 \leq j < i$ *we have* $0 \leq h_{jr_i} < h_{ir_i}$*.*

The matrix H *is unique and is known as the* Hermite normal form *of* A*. Furthermore, the Hermite normal form of* BA *is equal to the Hermite normal form of* A *whenever* B *is a unimodular matrix. Finally, the unimodular matrix* Q*, known as the* transformation matrix *of* A*, is unique when* A *is an invertible square matrix.*

It should be noted that if the first l columns of A are linearly independent, then the upper-left $l \times l$ block of the Hermite normal form of A is upper triangular with nonzero diagonal entries, namely $r_i = i$ for $i = 1, \ldots, l$. For our purposes, the matrix A will typically consist of a collection of k rod structures for rods which are not necessarily adjacent. An example of this is shown in Figure 2.21, where the 3×4 matrix A is assembled from the rod structures on the left (treated as column vectors), and sent to its Hermite normal form consisting of the rod structures on the right, via the transformation matrix that appears in the middle of the diagram.

Remark 2.21. If rod structures $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ arise from three consecutive rods with admissible corners, then more information is known about their Hermite normal form $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. In particular $\mathbf{w}_1 = \mathbf{e}_1$, $\mathbf{w}_2 = \mathbf{e}_2$, and $\mathbf{w}_3 = (q, r, p, 0, \ldots, 0)$ with $0 \leq q, r < p$, $p = \mathrm{Det}_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, and $\gcd\{q, p\} = 1$ if the set of vectors is linearly independent. In the case of a linearly dependent triple, we have $p = 0$ and $q = 1$, while r is unconstrained. Furthermore, given any integers $\mu, \lambda \in \mathbb{Z}$ there exists a coordinate change which sends $\mathbf{v}_i$ to $\mathbf{w}'_i$ where

$$\begin{aligned} \mathbf{w}'_1 &= (1, 0, \ldots, 0) \\ \mathbf{w}'_2 &= (0, 1, 0, \ldots, 0) \\ \mathbf{w}'_3 &= (q + \mu p, r + \lambda p, p, 0, \ldots, 0). \end{aligned} \tag{2.19}$$

In order to establish the relationship between the admissibility condition for corners and the 2nd determinant divisor, we recall the *Smith normal form.* This may be considered as the integer matrix analog of the singular value decomposition, and is utilized in the classification of finitely generated Abelian groups.

This latter fact will be employed when we compute the fundamental group of the DOC in Corollary 2.25. A proof of the following result can be found in [39].

Lemma 2.22. *Let A be an $n \times k$ integer matrix of rank l. There exist integer matrices U, V, and S such that $UAV = S$. The matrices U and V are unimodular, and S is diagonal with entries s_i such that $s_i | s_{i+1}$ for $1 \leq i < l$. These entries, referred to as elementary divisors, satisfy $s_i = 0$ for $i > l$ with all others computed by*

$$s_i = \frac{\mathrm{Det}_i(A)}{\mathrm{Det}_{i-1}(A)}, \qquad i \leq l, \tag{2.20}$$

where we have set $\mathrm{Det}_0(A) = 1$. The matrix S is unique and is known as the Smith normal form *of A.*

The distinction between the Hermite and Smith normal forms, in the context of rod structures, is as follows. The transformations used to obtain Hermite normal form are always actions by $n \times n$ matrices on the left. Such an action corresponds to shuffling the Killing vectors around by linear combinations. This does not affect the topology of the total space nor its toric structure, only the representation of the torus $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ and thus the rod structures. By contrast, Smith normal form also includes actions on the right by $k \times k$ matrices. These actions correspond to shuffling the axis rods themselves. This changes the topology of our space, possibly no longer making it a manifold. Consequently, when seeking out a simpler presentation of the rod structures we will invoke the Hermite normal form in order to avoid changing the topology. Two exceptions to this are in the proof of Corollary 2.25, where only the integer span of the rod structures is significant and not their order, and in the proof of Corollary 2.23 below, where the Hermite and Smith normal forms coincide.

Corollary 2.23. *Let A be an $n \times k$ integer matrix of rank k. Then $\mathrm{Det}_k(A) = 1$ if and only if the upper $k \times k$ block of the the Hermite normal form of A is the identity matrix.*

Proof. Assume that the upper $k \times k$ block of the Hermite normal form is the identity. By uniqueness, this matrix is also the Smith normal form. The diagonal entries are then $1 = s_i = \mathrm{Det}_i(A)/\mathrm{Det}_{i-1}(A)$, which implies that $\mathrm{Det}_k(A) = \mathrm{Det}_{k-1}(A) = \cdots = \mathrm{Det}_0(A) = 1$.

Conversely, assume that $\mathrm{Det}_k(A) = 1$ and let

$$\begin{bmatrix} S \\ 0 \end{bmatrix} = UAV \tag{2.21}$$

be the Smith normal form of A, where $S = \mathrm{diag}(s_1, \ldots, s_k)$. Consider the $n \times n$ matrix

$$B = U^{-1} \begin{bmatrix} S & 0 \\ 0 & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} V^{-1} & 0 \\ 0 & \mathbf{I}_{n-k} \end{bmatrix} = \begin{bmatrix} A & E \end{bmatrix}, \tag{2.22}$$

where E consists of the last $n-k$ columns of U^{-1}. It follows that

$$\det(B) = \det(U^{-1})\det(S)\det(V^{-1}) = s_1 \cdots s_k = \frac{\mathrm{Det}_1(A)}{\mathrm{Det}_0(A)} \cdots \frac{\mathrm{Det}_k(A)}{\mathrm{Det}_{k-1}(A)} = \mathrm{Det}_k(A). \tag{2.23}$$

By assumption $\mathrm{Det}_k(A) = 1$, and thus B is invertible. Therefore

$$B^{-1}A = \begin{bmatrix} \mathbf{I}_k \\ 0 \end{bmatrix}, \tag{2.24}$$

and by uniqueness this must be the Hermite normal form of A. □

As mentioned after the proof of Proposition 2.19, this corollary shows that a pair of adjacent rod structures $\mathbf{v}, \mathbf{w}$ is admissible if and only if $\mathrm{Det}_2(\mathbf{v}, \mathbf{w}) = 1$. Moreover, in a similar manner, a collection of k rod structures

$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ can be sent to the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$, and thus forms a primitive set, if and only if $\mathrm{Det}_k(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 1$. Another application of the Hermite normal form is to give a variant proof of Hollands and Yazadjiev's horizon topology theorem [20, Theorem 2]. It states that for $n \geq 2$, all closed $(n+1)$-manifolds with an effective T^n-action, whose quotient is not a circle, must be a product of T^{n-2} and either S^3, a lens space $L(p,q)$, or $S^1 \times S^2$. This is a generalization of a result by Orlik and Raymond for 3-manifolds, see [43, Section 2]. Observe that the $(n+1)$-dimensional case can be reduced to the 3-dimensional case by applying the transformation matrix from Lemma 2.20 to the matrix of rod structures defining the horizon, which we assume to be primitive vectors. In particular, the resulting Hermite normal form consists of the new rod structures $(1, 0, \ldots, 0)$ and $(q, p, 0, \ldots, 0)$, with $0 \leq q < p$. With this representation of the T^n-action, the last $n-2$ coordinate Killing fields clearly never vanish. Therefore the total space is homeomorphic to a product of T^{n-2}, and a 3-manifold Σ with an effective T^2 action. According to the possibilities given for the 3-dimensional case, we find that Σ is either S^3 if $p = 1$, $S^1 \times S^2$ if $p = 0$, or the lens space $L(p,q)$ if $p > 1$.

Remark 2.24. Given a horizon topology $\Sigma \times T^{n-2}$, it is possible to determine the topology of Σ directly from the 2nd determinant divisor. Let $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ be primitive vectors that describe the flanking rod structures of the horizon, and compute $\mathrm{Det}_2(\mathbf{v}, \mathbf{w})$. If this value is 0, then $\mathbf{v} = \pm\mathbf{w}$ and $\Sigma = S^1 \times S^2$. If it is 1, then the pair is admissible and $\Sigma = S^3$. If $\mathrm{Det}_2(\mathbf{v}, \mathbf{w}) = p > 1$ then $\Sigma = L(p,q)$ for some $q < p$. Moreover, q may be found from the relation $\mathbf{w} = q\mathbf{v} \mod p$.

Note that the asymptotic end of a manifold can be thought of as a 'horizon at infinity' and the rod structures of the two semi-infinite rods can be used to calculate the topology of the asymptotic end in a similar manner. The last result in this subsection will show that rod structures can also be used to calculate information about the global topology, in this case the fundamental group.

Corollary 2.25. *Let M^{n+2} be a connected, simple T^n-space, possibly with boundary, with rod structures $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subset \mathbb{Z}^n$. The fundamental group of M is*

$$\pi_1(M) \cong \frac{\mathbb{Z}^n}{\mathrm{span}_{\mathbb{Z}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}} \cong \mathbb{Z}^{n-l} \oplus \mathbb{Z}_{s_1} \oplus \cdots \oplus \mathbb{Z}_{s_l}, \tag{2.25}$$

where $s_i | s_{i+1}$, s_i is the i^{th} entry in the Smith normal form of the matrix composed of the column vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ as in (2.20)*, and $l = \dim \mathrm{span}_{\mathbb{R}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.*

Proof. In the proof of Theorem 2.3 it is revealed that the fundamental group of a T^n-manifold M^{n+2} takes the form of (2.3). This reduces to the first equality in Equation (2.25) in the case that M is a simple T^n-space, since it then has no holes, handles, or orbifold points. Furthermore, recall that the Smith normal form of the matrix $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ is obtained by both left and right actions using unimodular matrices. This does not alter the integral span of the columns. Thus, as in the classification of finitely generated abelian groups, by a change of basis given by these unimodular matrices, we obtain the second equality in (2.25). ☐

2.3 4 Dimensions

This subsection will be dedicated exclusively to 4-dimensional simple T^2-manifolds. First several spacetime topologies will be reviewed, such as the famous Black Ring of Emparan-Reall [12] and the more recent Khuri-Weinstein-Yamada Black Lens [28]. Following that, the works of Orlik and Raymond will be discussed, including a proof of their classification theorem (Theorem 2.34) for closed simply connected 4-manifolds with effective T^2-actions.

The rod structures of several well known $(4+1)$-dimensional stationary spacetimes are depicted in Figure 2.4. These rod diagrams show the topology of a Cauchy surface intersected with the domain of outer communication. Note these can all be constructed by adding horizons to the $\{(1,0),(0,1)\}$ rod diagram for Minkowski spacetime. This is equivalent to excising regions from $\mathbb{R}^4$, the spatial part of Minkowski spacetime. Indeed, as seen in Figure 2.2, the Myers-Perry rod diagram dipicts $\mathbb{R}^4 \setminus B^4$. Similarly the Black Ring shows $\mathbb{R}^4 \setminus (S^1 \times B^3)$ whereas the Black Saturn, Bi-Rings, and Di-Rings are combinations of these two.

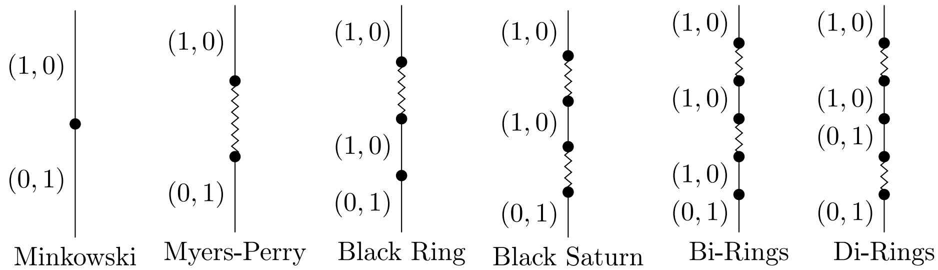


Figure 2.4: Shown above are the rod diagrams for several well known examples of (4 + 1)-dimensional stationary bi-axially symmetric spacetimes which exist without conical singularities. These are Minkowski space, Myers-Perry [38] with horizon S^3, The Emparan-Reall and Pomeransky-Sen'kov black rings [12, 46] with horizons $S^1 \times S^2$, the black Saturns [10] of Elvang-Figueras with horizons $S^1 \times S^2$, and the the black bi-rings [11] and di-rings [13, 22] found by Elvang-Rodriguez, Evslin-Krishnan, and Iguchi-Mishima with horizons $2 \cdot S^1 \times S^2$.

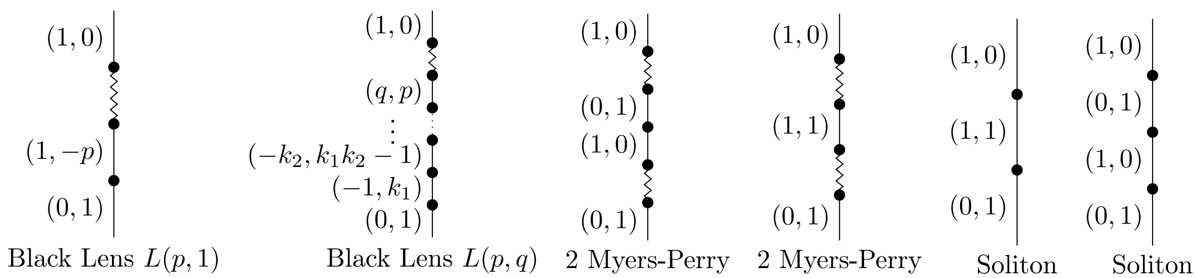


Figure 2.5: Shown above are the rod diagrams for spacetimes that are known or suspected to always require struts. The left two are which produce the Black Lens studied in [28]. The middle two show two different ways to produce asymptotically flat spacetimes with black hole horizons $2 \cdot S^3$. Because of symmetry both of these spacetime are known to have conical singularities [51]. The last two show spacetimes without black holes, which are known as solitons.

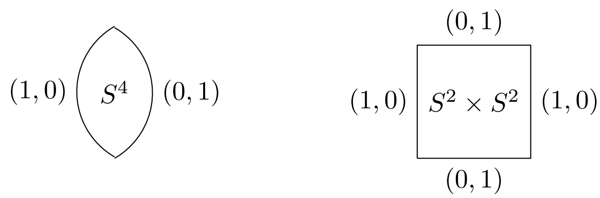


Figure 2.6: Shown above are the rod diagrams for the simplest examples of a T^2-action on a closed 4-manifold.

Figure 2.5 shows spacetimes with slightly more complicated topologies. These are all still asymptotically flat because the second determinate divisor of the two semi-infinite rods, $\mathrm{Det}_2\{(1,0),(0,1)\}$, is 1. However these spacetimes are all known to [51], or suspected to, have conical singularities known as *struts* present along the finite axis rods. We will examine these struts later in Sections 4.4 and 4.5. To construct the $L(p,q)$ Black Lens spacetime one needs to be able to join the $(1,0)$ and (q,p) rods by a finite number of axis rods in such a way that every corner is admissible. This was proven possible in [25] (see also [1] for an exploration of efficient fill-ins). The following theorem is a slight generalization which works in higher dimensions.

Theorem 2.26. *Given any two (primitive) rod structures $\mathbf{v},\mathbf{w}\in\mathbb{Z}^n$, it is always possible to find a finite number of additional rod structures that connect $\mathbf{v}$ to $\mathbf{w}$ in such a way that each corner in the resulting sequence of rods is admissible. That is, there exists a sequence of rod structures $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$, with $\mathbf{v}_1=\mathbf{v}$ and $\mathbf{v}_k=\mathbf{w}$, having the property that* $\mathrm{Det}_2(\mathbf{v}_i,\mathbf{v}_{i+1})=1$ *for* $i=1,\ldots,k-1$.

Proof. By Lemma 2.20 there exists a unimodular matrix Q which transforms $\mathbf{v}$ and $\mathbf{w}$ into Hermite normal form, in particular $Q\mathbf{v}=(1,0,\ldots,0)$ and $Q\mathbf{w}=(q,p,0,\ldots,0)$ where $0\leq q<p$. If $q=0$, then $p=1$ since $\mathbf{w}$ is primitive, and hence $\mathrm{Det}_2(\mathbf{v},\mathbf{w})=1$. So assume that $q\geq 1$. In [25, Section 3] (see also Lemma 2.47) an algorithm is presented that is based on the continued fraction decomposition of p/q, which produces a sequence of rod structures in $\mathbb{Z}^2$ connecting $(1,0)$ to (q,p) such that each corner is admissible. We may then append zeros to each of the rod structures in this sequence, to obtain a sequence in $\mathbb{Z}^n$ that connects $(1,0,\ldots,0)$ to $(q,p,0,\ldots,0)$ with the same property. Applying Q^{-1} then produces the desired sequence. $\square$

Remark 2.27. This result was also used in [25], for $(4+1)$-dimensional spacetimes, to construct simply connected fill-ins for horizons. The simple connectivity of the fill-ins preserves the fundamental group of the DOC, and is not difficult to achieve since in this low dimensional setting admissible rod structures cannot contribute to the fundamental group, as can be seen with Corollary 2.25. In higher dimensions this is not the case, and a more careful choice of rod structures is needed to achieve simply connected fill-ins. Moreover, since the boundary between the filled in region and the DOC in the higher dimensional case has a much larger fundamental group, there is a more complicated relation between the topologies of these regions.

The topology of closed simply connected 4-manifolds with an effective T^2-action is known do to the works of Orlik and Raymond. Below are some examples.

Example 2.28: The rod diagram $\{(1,0),(0,1),(1,0),(0,1)\}$ in Figure 2.6 depicts the standard Cartesian product $S^2\times S^2$. The fiber over each interior point is $S^1\times S^1$. The first S^1 degenerates on the left and the right sides, the $(1,0)$ rods, thus the fiber over a horizontal line is $S^2\times S^1$. That second S^1 degenerates on the top and bottom sides, the $(0,1)$ rods, making the total space $S^2\times S^2$.

Example 2.29: The rod diagram in Figure 2.7 depicts $\mathbb{CP}^2$. First observe that removing the $(1,1)$ rod turns this into a diagram for $\mathbb{R}^4$, which is homeomorphic to B^4. The $(1,1)$ rod itself has an S^1 fiber on each interior point, which shrinks to a single point when the rod meets either the $(0,1)$ or $(1,0)$ rods. This makes the total space over the $(1,1)$ rod homeomorphic to S^2. Therefore M can be constructed from B^4 and S^2 with a single gluing map $f\colon \partial B^4\to S^2$. Such maps are classified by $\pi_3(S^2)$ which is the free group generated by the Hopf map $S^3\to S^2$. Recall that the homomorphism $\pi_3(S^2)\to\mathbb{Z}$ is given by

Figure 2.7: Shown above are the rod diagrams for $\mathbb{CP}^2$, and its opposite orientation manifold $\overline{\mathbb{CP}^2}$.

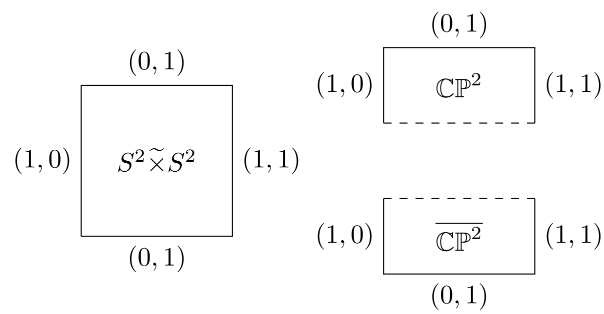


Figure 2.8: The rod diagram above shows how $S^2\widetilde{\times}S^2$ is $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.

calculating the linking number of any two fibers of the map $S^3 \to S^2$. The two obvious points to look at the fibers over at the $\{(1,0),(1,1)\}$ corner and the $\{(1,1),(0,1)\}$ corner. Thinking of B^4 as the unit ball in $\mathbb{C}^2$ shows that the fiber over the first corner is $\{(0,z) : |z|^2 = 1\}$ and the fiber of over the second corner is $\{(w,0) : |w|^2 = 1\}$. These obvious have linking number 1 and thus, up to a choice of orientation, the map $f\colon \partial B^4 \to S^2$ is the Hopf map which by definition makes M homeomorphic to $\mathbb{CP}^2$. The opposite orientation is chosen when $(1,1)$ is replaces with $(1,-1)$, resulting in $\overline{\mathbb{CP}^2}$.

Example 2.30: The manifold $S^2\widetilde{\times}S^2$ in Figure 2.8 is the unique non-trivial S^2 bundle over S^2. This is diffeomorphic to $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, the blow up of $\mathbb{CP}^2$. To view this we draw a horizontal line across the square and note again that sitting above that line is an S^3. However now we look at each half of the square that the horizontal line separates. The top half looks similar to the first triangle representing $\mathbb{CP}^2$ if we remove a neighborhood of the bottom left corner. In fact, the 4-manifold sitting above the top half of our square is homeomorphic to $\mathbb{CP}^2$ with a ball removed. Similarly the bottom half looks like same triangle but with the position of the $(0,1)$ and the $(1,0)$ edges switched. This switching induces a change in orientation, and the 4-manifold sitting above the bottom half of our square is $\overline{\mathbb{CP}^2}$ with a ball removed. These two manifolds are glued along their shared boundary and thus we get $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.

Example 2.31: Rod diagram A in Figure 2.10 depicts the m^{th} Hirzebruch surface $\mathbb{F}_m$. To see this, first use Equation (2.11) to express A as

$$A = ([0,\infty]^2 \times T^2)/\sim \tag{2.26}$$

where

$$(\phi_1,\phi_2)\cdot(x,y,\theta_1,\theta_2) = (x,y,\phi_1+\theta_1,\phi_2+\theta_2) \tag{2.27}$$

and

$$\begin{aligned}(0,y,\theta_1,\theta_2) &\sim (0,y,\phi+\theta_1,\theta_2)\\ (x,0,\theta_1,\theta_2) &\sim (x,0,\theta_1,\phi+\theta_2)\\ (\infty,y,\theta_1,\theta_2) &\sim (\infty,y,\phi+\theta_1,\theta_2)\\ (x,\infty,\theta_1,\theta_2) &\sim (x,\infty,m\phi+\theta_1,\phi+\theta_2).\end{aligned} \tag{2.28}$$

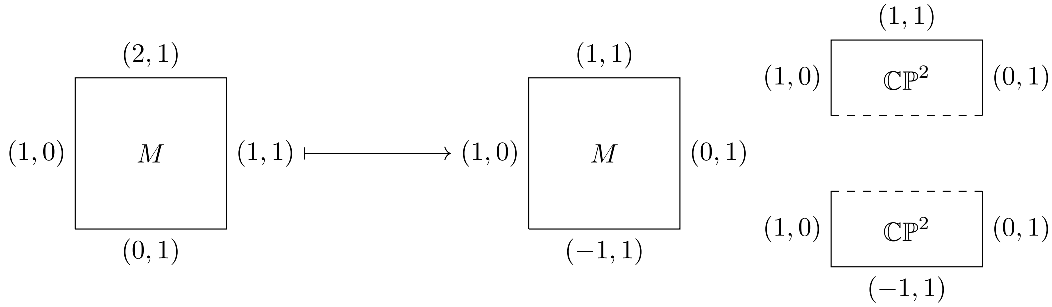


Figure 2.9: The rod diagram above shows that the manifold M described in Example 2.33 is $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.

Next, split A into $A_- = \{(x, y, \theta_1, \theta_2) \in A : y \leq 1\}$ and $A_+ = \{(x, y, \theta_1, \theta_2) \in A : y \geq 1\}$. Define B, B_-, and B_+ similarly. Both A and B define the same T^2-manifold, in particular $S\colon A \to B$ defined by

$$S(x, y, \theta_1, \theta_2) = (x, y, \theta_1 - m\theta_2, \theta_2) \tag{2.29}$$

is an isomorphism. Notice that S fixes the y coordinate, so A_+ is isomorphic to B_+. This means A can be constructed by gluing A_- and B_+ together, using S as a transition function. Now consider the map $(x, y, \theta_1, \theta_2) \mapsto (xe^{i\theta_1}, ye^{i\theta_2})$ and notice that is defines an isomorphism between A_- and $\hat{\mathbb{C}} \times \{|z| \leq 1\}$ and between B_+ and $\hat{\mathbb{C}} \times \{|z| \geq 1\}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In these new complex coordinates S acts as a clutching function, an automorphism of the bounary $\hat{\mathbb{C}} \times \{|z| = 1\}$ which sends $S(w, z) = (z^{-m}w, z)$. This makes

$$A = \frac{\hat{\mathbb{C}} \times \{|z| \leq 1\} \sqcup \hat{\mathbb{C}} \times \{|z| \geq 1\}}{(w, e^{i\theta}) \sim (e^{-im\theta}w, e^{i\theta})} \tag{2.30}$$

which is a standard description of the m^{th} Hirzebruch surface $\mathbb{F}_m$.

Both $\mathbb{CP}^2$ and all of the Hirzebruch surfaces belong to a family of complex surfaces known as *projective toric varieties*. These complex manifolds come equipped with a natural torus action which makes them an excellent source of examples of simple T^2-manifolds. Indeed the *blow up* of any projective toric variety is itself a projective toric variety and thus also a simple T^2-manifold. However not all simple T^2-manifolds admit a complex structure. The relation between these two areas is best seen in Delzant's theorem [9, ch. 28] which we will state below without proof.

Theorem 2.32 (Delzant)**.** *A closed simple T^2-manifold M^4 with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ admits a complex structure which agree with the T^2-action if and only if it is possible to construct a simple, convex k-gon in $\mathbb{R}^2$ where the normal vector to the i^{th} edge is $\pm\mathbf{v}_i$ for all i.*

Example 2.33: The manifold M given by the rod structures $\{(1, 0), (0, 1), (1, 1), (2, 1)\}$, seen in Figure 2.9, does not admit a complex structure and is in fact homeomorphic to $\mathbb{CP}^2\#\mathbb{CP}^2$. We know M does not admit a complex structure because any simple quadrilateral with normal vectors equal to $(1, 0)$, $(0, 1)$, $(1, 1)$, and $(2, 1)$ is not convex. To see that M is homeomorphic to $\mathbb{CP}^2\#\mathbb{CP}^2$ we note that the first and third rod structures form an admissible pair. This allows a change of coordinates to make the first and third rods structures $(1, 0)$ and $(0, 1)$ respectively, resulting in an equivalent rod diagram $\{(1, 0), (-1, 1), (0, 1), (1, 1)\}$. Consider a curve in M/T^2 which connects the first and third rods. The fiber over each point on the interior of the curve is T^2, collapsing to $\{pt.\} \times S^1$ at the $(1, 0)$ and to $S^1 \times \{pt.\}$ at the $(0, 1)$ end. This makes the total space of the curve into an S^3 which separates the rod diagram into to half. In particular $\{(1, 0), (0, 1), (1, 1), (2, 1)\}$ is composed of the diagrams $\{(1, 0), (0, 1), (1, 1)\}$ and $\{(1, 0), (1, 1), (2, 1)\}$ which are both rod diagrams for $\mathbb{CP}^2$ as seen in Example 2.29, meaning $M \cong \mathbb{CP}^2\#\mathbb{CP}^2$.

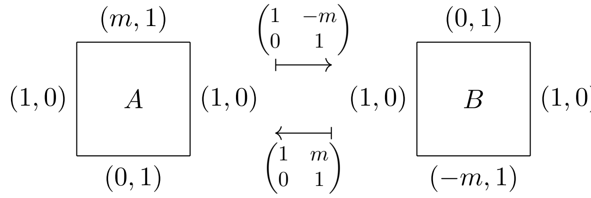


Figure 2.10: Both A and B represent the same 4-manifold and the same T^2-structure. They differ only by a change of coordinates $S = \begin{pmatrix}1 & -m\\ 0 & 1\end{pmatrix}$.

Theorem 2.34 (Orlik & Raymond)**.** *A simply connected, closed* 4*-manifold which admits an effected T^2 action is either S^4 or the connect sum of $S^2 \times S^2$'s, $\mathbb{CP}^2$'s, and $\overline{\mathbb{CP}^2}$'s. This connect sum respects the toric action in the sense that there exist* 3*-spheres breaking up the manifold into its constituent parts which inherits the toric action* [43]*.*

The proof of this theorem is presented below for both completeness, and because it is a useful example of how topology can be read off from the rod structures. It is slightly different than the original proof published by Orlik and Raymond in [43] do to the use of Hermite normal form and other direct computations. Because of this, the proof is shorter and arguably easier to read.

Proof. The first step is to prove that any 4-dimensional simply connected closed T^2-manifold must be a simple T^2-manifold with at least two axis rods, no horizons, and a base-space homeomorphic to $\mathbb{D}^2$. The fact that it is a simple T^2-manifold was proven in Theorem 2.3 and the fact that it contains at least two axis rods was proven in Corollary 2.25.

The next step is to classify all closed 4-dimensional simple T^2-manifolds with 2, 3, or 4 axis rods. These can be seen in Figures 2.6, 2.7, 2.8, 2.9, and 2.10. Let M be such a manifold with 2 rods. Since M is a manifold and the rods are adjacent, separated by a corner, the pair of rod structures must be admissible. Thus their second determinent divisor is 1 and their unique Hermite normal form is $\{(1,0),(0,1)\}$. Without loss of generality assume that these are the rod structures. Removing one of the two corners is equivalent to removing one of the two fixed points. This yields the standard rod structure for $\mathbb{R}^4$ and thus M is its one-point compactification, S^4.

Now assume M has 3 rods. Remark 2.21 say that any three consecutive admissible rods can be assumed to be in the form $\{(1,0),(0,1),(1,r)\}$. The first and third rods must also form an admissible pair, and thus $r = \pm 1$. Since there are only two choices for r there must be at most two distinct simply connected, closed 4-manifolds with effective T^2-actions and exactly 3 fixed points (recall that the fixed points are corners). The manifold $\mathbb{CP}^2 = \{[z_1 : z_2 : z_3]\}$ inherits an effective T^2 action from $\mathbb{C}^3$, the only fixed points of which are $[z : 0 : 0]$, $[0 : z : 0]$, and $[0 : 0 : z]$. Choosing the opposite orientation gives $\overline{\mathbb{CP}^2}$ with the exact same action. Thus any rod diagram with 3 rods must be either $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$.

Now assume M has 4 rods. Again, using Remark 2.21 assume that the rod structures are of the form $\{(1,0),(0,1),(1,a),(b,c)\}$. The admissibility of the first and fourth rods forces $c = 1$. Similarly the admissibility of the third and fourth rods forced $ab - 1 = \pm 1$. If $ab = 2$ then without loss of generality assume the rod structures are of the form $\{(1,0),(0,1),(1,1),(2,1)\}$. In this case the first and third rods also form an admissible pair. This allows a change of coordinates to make the first and third rods structures $(1,0)$ and $(0,1)$ respectively, resulting in an equivalent rod diagram $\{(1,0),(-1,1),(0,1),(1,1)\}$. Consider a curve in M/T^2 which connects the first and third rods. The fiber over each point on the interior of the curve is T^2, collapsing to $\{pt.\} \times S^1$ at the $(1,0)$ and to $S^1 \times \{pt.\}$ at the $(0,1)$ end. This makes the total space of the curve into an S^3 which separates the rod diagram into to half. In particular $\{(1,0),(0,1),(1,1),(2,1)\}$ is composed of the diagrams $\{(1,0),(0,1),(1,1)\}$ and $\{(1,0),(1,1),(2,1)\}$ which are both rod diagrams for $\mathbb{CP}^2$, meaning $M \cong \mathbb{CP}^2 \# \mathbb{CP}^2$.

In the other case where $ab = 0$, assume without loss of generality that the rod structures are of the form $\{(1,0),(0,1),(1,0),(b,1)\}$. From Equation (2.11) the total space can be described as

$$M \cong ([0,1]^2 \times T^2)/\sim \tag{2.31}$$

where $\sim$ is defined by the rod structures, in particular

$$\begin{aligned}(0,y,\theta_1,\theta_2) &\sim (0,y,\phi+\theta_1,\theta_2)\\ (x,0,\theta_1,\theta_2) &\sim (x,0,\theta_1,\phi+\theta_2)\\ (1,y,\theta_1,\theta_2) &\sim (1,y,\phi+\theta_1,\theta_2)\\ (x,1,\theta_1,\theta_2) &\sim (x,1,b\phi+\theta_1,\phi+\theta_2)\end{aligned} \tag{2.32}$$

for all $\phi \in S^1$. Now consider the description of S^2 as $([0,1]\times S^1)/\sim$ where $(0,\theta) \sim (0,\phi+\theta)$ and $(1,\theta) \sim (1,\phi+\theta)$. Using these descriptions, define the projection map $P\colon M \to S^2$ by $P(x,y,\theta_1,\theta_2) = (y,\theta_2)$. By construction this defines a fiber bundle $P\colon M \to S^2$ with fibers $P^{-1}(y_0,\theta_0) = \{(x,y_0,\theta,\theta_0)\}$. Equation (2.32) shows $(0,y_0,\theta,\theta_0) \sim (0,y_0,0,\theta_0)$ and $(1,y_0,\theta,\theta_0) \sim (1,y_0,0,\theta_0)$, meaning the fiber over each point (y_0,θ_0) is S^2, thus M is an S^2-bundle over S^2. Up to homeomorphism these are classified by maps from S^1 to $Diff(S^2)$. The latter space deformation retracts to $SO(3)$ and $\pi_1(SO(3)) = \mathbb{Z}_2$, meaning there are two distinct homeotypes possible for M. These are the trivial bundle $S^2 \times S^2$, and the non-trivial bundle $S^2 \widetilde{\times} S^2$ which is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

Now that the manifolds with 4 or fewer rods have been classified, let M have $k > 4$ rods and proceed by induction on k. Consider a rod diagram with rod structures (m_1,n_1) through (m_k,n_k). Following the notation of Orlik and Raymond, let L_{ij} denote a curve connecting the i^{th} and j^{th} rods and let W_{ij} be the region bounded by this curve containing all the rods in-between i and j.

Without loss of generality assume $(m_1,n_1) = (1,0)$ and $(m_k,n_k) = (0,1)$. Because of the admissibility of adjacent rods, there exists $\varepsilon_2 = \pm 1$ and $\varepsilon_{k-1} = \pm 1$ such that $(m_2,n_2) = (m_2,\varepsilon_2)$ and $(m_{k-1},n_{k-1}) = (\varepsilon_{k-1},n_{k-1})$. Now proceed by cases.

Case 0: $m_2 = 0$ or $n_{k-1} = 0$. If $m_2 = 0$ then $\det \begin{bmatrix} 0 & \varepsilon_2 \\ \varepsilon_{k-1} & n_{k-1}\end{bmatrix} = \pm 1$ and thus the curve $L_{2,k-1}$ represents an S^3 separating M into $W_{2,k-1} \# W_{k-1,2}$. The same holds true if instead $n_{k-1} = 0$.

Case 1: $m_j = 0$ for $2 < j < k-1$. Here we see $n_j = \pm 1$ and the curve L_{1j} separates M into $W_{1j} \# W_{j1}$.

Case 2: $n_j = 0$ for $2 < j < k-1$. Very similarly $m_j = \pm 1$ and the curve L_{kj} separates M.

Case 3: $\left|\frac{m_j}{n_j}\right| = 1$. Now $|n_j| = |m_j| = 1$ and both L_{1j} and L_{kj} represent an S^3. If $j = k-1$ then $L_{1,k-1}$ separates M. If $j = 2$ then $L_{k,2}$ separates M. For all other j either L_{1j} or L_{kj} can separate M.

Case 4: None of the above happen. This case is impossible. To show that we list the ratios of each edge in order

$$\left|\frac{1}{0}\right|, \left|\frac{m_2}{\varepsilon_2}\right|, \dots, \left|\frac{m_j}{n_j}\right|, \dots, \left|\frac{\varepsilon_{k-1}}{n_{k-1}}\right|, \left|\frac{0}{1}\right|$$

where $\frac{1}{0} = \infty$. This is shown in Figure 2.11. Let $j+1$ the first time that $\left|\frac{m_{j+1}}{n_{j+1}}\right| < 1$. Since we are not in cases 1, 2 or 3, we know that $2 \le j < k-1$. Because the determinant of consecutive edges must be ± 1, we have $m_j n_{j+1} = \pm 1 + m_{j+1} n_j$ or $|m_j n_{j+1}| \le 1 + |m_{j+1} n_j|$. Since everything is an integer, we also have $|m_j| \ge 1 + |n_j|$ and $|n_{j+1}| \ge 1 + |m_{j+1}|$ to give us $(1+|n_j|)(1+|m_{j+1}|) \le 1 + |m_{j+1} n_j|$. This is a contradiction, thus the proof is complete. □

Remark 2.35. The decomposition of our 4-manifold into connect sums is not unique. Indeed in case 3 of the above proof we can see there are typically two different way to split the manifold. In Figure 2.12 we see a more concrete example. Take each region separated by a dashed line and connect the boundary edges

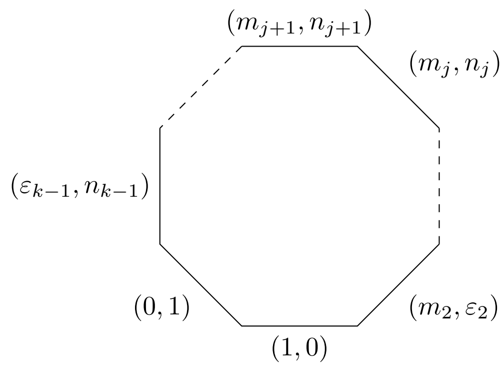


Figure 2.11: Above we see a rod diagram accompanying Case 4 in the proof of Orlik and Raymond's classification theorem (Theorem 2.34).

together at a new corner. Now refer to Examples 2.29 and 2.28 to confirm that each region is homeomorphic to its label in the diagram in Figure 2.12.

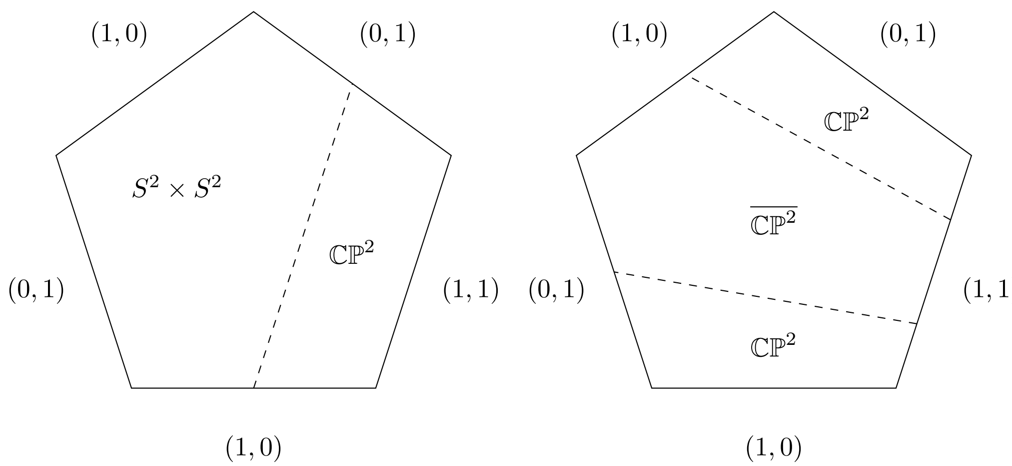


Figure 2.12: A toric diagram showing $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}\#\mathbb{CP}^2 \cong S^2 \times S^2\#\mathbb{CP}^2$.

We can generalize the above a bit more, as seen in Figure 2.13.

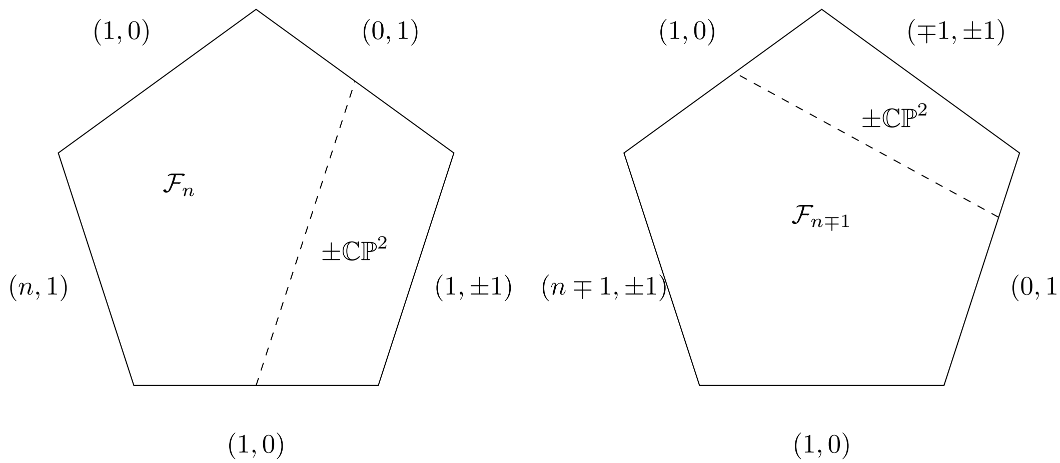


Figure 2.13: These two toric diagrams are related by the change of basis $\begin{bmatrix} 1 & \mp 1 \\ 0 & \pm 1 \end{bmatrix}$. The lines show that $\mathcal{F}_n \# \pm \mathbb{CP}^2 \cong \mathcal{F}_{n\mp 1} \# \pm \mathbb{CP}^2$.

Remark 2.36. In the proof of Theorem 2.34 it is noted that there are only two distinct homeotypes of S^2-bundles over S^2, despite there being an integers worth of choices of rod structures and therefore distinct T^2-structures. This is because the clutching functions $f\colon S^1 \to Aut(S^2)$ used to define the S^2-bundle must itself respect the S^1-structure of S^2, and thus the automorphism $f(\theta) \in Aut(S^2)$ is actually described by an element in $Aut(S^1)$. So instead of $\pi_1(SO(3)) = \mathbb{Z}_2$ classifying S^2-bundles over S^2 up to homeomorphism, the group $\pi_1(SO(2)) = \mathbb{Z}$ classifies effective T^2-actions on S^2-bundles over S^2 up to equivariant homeomorphism.

2.4 Intersection Form

The classification theorem of Orlik and Raymond is a powerful tool for the study of 4-dimensional simple T^2-manifolds. However this tool does not provide us with a method to actually compute topology from rod structures. For example, a manifold with 4 rods may be homeomorphic to $S^2 \times S^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, or $\mathbb{CP}^2 \# \mathbb{CP}^2$. In fact the number of distinct homeotypes grows linearly with the number of rods. Fortunately all of these homeotypes can be distinguished by their intersection forms. In this section we give a formula to compute the intersection form and prove Theorem B. Following that we will demonstrate some immediate consequences of Theorem B.

Recall that the intersection form of a 4-manifold, $Q\colon H_2(M;\mathbb{Z}) \otimes H_2(M;\mathbb{Z}) \to \mathbb{Z}$, is a bilinear form on second integral homology. Calculating $H_2(M;\mathbb{Z})$ is easy in the present case since Theorem 2.34 has M decomposed as a connected sum of $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, and $S^2 \times S^2$.

Proposition 2.37. *If M is a simply connected, closed 4-manifold with an effective T^2-action and k axis rods, then*

$$H_2(M;\mathbb{Z}) \cong \mathbb{Z}^{k-2}. \tag{2.33}$$

Proof. If M is indecomposable, then it either has 2 rods and is S^4, 3 rods and is $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$, or 4 rods and is $S^2 \times S^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. In all cases Equation (2.33) is satisfies. Now assume by induction that the statement has been proven for all such manifolds with fewer than k rods. If M is decomposable, then there exists closed simple T^2-manifolds N_1 and N_2 such that $N_1 \# N_2 \cong M$ where the connected sum is equivariant. In the proof of Theorem 2.34 is is apparent that N_i has $3 \le k_i < k$ rods and in particular $k_1 + k_2 - 2 = k$. Therefore $H_2(N_i;\mathbb{Z}) \cong \mathbb{Z}^{k_i - 2}$ and by Mayer-Viatoris $H_2(M;\mathbb{Z}) = H_2(N_1) \oplus H_2(N_2) \cong \mathbb{Z}^{k_1+k_2-4} = \mathbb{Z}^{k-2}$. ☐

The above proof is perhaps the simplest way to compute $H_2(M;\mathbb{Z})$, although it uses the powerful Orlik and Raymond classification theorem (see Theorem 2.34). Other methods of computing the second homology group exist which don't rely on the classification theorem. One such method in the case when M is not closed is shown in [26, §5.2]. There the authors compute the generators of the deRham cohomology group $H^2(M;\mathbb{R})$, which turns out to be equivalent to computing $H_2(M;\mathbb{Z})$ when M all horizons are flanked by admissible rod structures. However in order to compute the intersection form directly from the rod structures we would like a geometric picture of $H_2(M;\mathbb{Z})$ which doesn't rely on Poincaré duality. This brings us to the following Lemma 2.38 which computes an isomorphism between $\ker(A\colon \mathbb{Z}^k \to \mathbb{Z}^2)$ and $H_2(M;\mathbb{Z})$. Strictly speaking Lemma 2.38 contains no more information than Proposition 2.37. Indeed, since M is simply connected Theorem 2.25 shows A has full rank and thus $\ker(A) \cong \mathbb{Z}^{k-2}$. However the explicit construction of the isomorphism Ψ_* will be crucial in the proof of Theorem 2.40 and thus Theorem B.

Lemma 2.38. *For any simple T^2-manifold M^4 with rod structures forming the matrix $A\colon \mathbb{Z}^k \to \mathbb{Z}^2$, there exists an isomorphism*

$$\Psi_*\colon \ker(A) \to H_2(M;\mathbb{Z}) \tag{2.34}$$

defined explicitly in terms of the rod structures.

Proof. It turns out to be easier to construct $\Psi_*^{-1}\colon H_2(M;\mathbb{Z}) \to \ker(A)$, so that is what we will do. Recall the Hurewicz Isomorphism Theorem; the first non-trivial homotopy group induces an isomorphism between its Abelianization and the associated homology group [16, Theorem 4.37]. In particular, since M is simply connected, every class $[\alpha] \in H_2(M;\mathbb{Z})$ is assigned a unique $[f] \in \pi_2(M)$. The homology class is realized by the push forward $[\alpha] = f_*([S^2])$ where $[S^2]$ is the canonical generator of $H_2(S^2;\mathbb{Z})$) and $f\colon S^2 \to M$ is a representative of $[f] \in \pi_2(M)$. The homotopy class $[f]$ assigned to $[\alpha] \in H_2(M;\mathbb{Z})$ will be used to compute the value $\Psi_*^{-1}([\alpha]) \in \ker(A)$.

Pick a representative of the homotopy class $f \in [f]$. Denote its image by $f(S^2) \subset M$. Assume that $f(S^2)$ is disjoint from the corners of M/T^2 (ie, the fixed points of the T^2-action). This is possible by choosing a generic $f \in [f]$ so that $f(S^2)$ is transverse to the corners. Since the corners are 0-dimensional and $f(S^2)$ is 2-dimensional their intersection must be empty. Define $\widetilde{N} := T^2(f(S^2)) \subset M$ to be the orbit of $f(S^2)$ under the T^2 action. Since $\widetilde{N}$ is invariant under the T^2 action, it can be described as $\pi^{-1}(\widetilde{R}) = \widetilde{N}$ where $\widetilde{R} \subset M/T^2$ is some subset of the orbit space which misses the corners. This is shown as the shaded region in Figure 2.14.

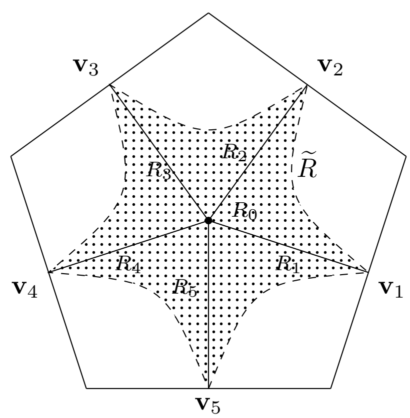


Figure 2.14: The shaded region shows the image of S^2 under $\pi \circ f$, i.e. $\widetilde{R} := \pi(f(S^2))$. This region is then homotoped down to the 'spider-like' region $R := R_0 \cup R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$, which is equivalent to the image of S^2 under $\pi \circ f_1$, i.e. $R = \pi(f_1(S^2))$.

We now wish to deform $\widetilde{R} \subset M/T^2$ into a spider-like shape $R \subset \widetilde{R}$ composed of a body R_0 and k legs $R_1, \ldots, R_k$. Each leg R_i is a single line segment which connects its associated axis rod Γ_i to the small contactable region which is the body R_0. These pieces form $R = R_0 \cup R_1 \cup \cdots \cup R_k$ where the union is disjoint except at their boundaries, as depicted in Figure 2.14. The explicit homotopy $G\colon \widetilde{R} \times [0,1] \to M/T^2$

which deforms $\widetilde{R}$ into R induces another homotopy $F\colon \widetilde{N} \times [0,1] \to M$ which deforms $\widetilde{N}$ to $N := \pi^{-1}(R)$. Similarly let N_i denote $\pi^{-1}(R_i)$ and N_0 denote $\pi^{-1}(R_0)$. By restricting our attention to $f(S^2)$, we see that this defines a homotopy $f_t(x) := F(f(x),t)$ between the map $f = f_0$ and a new map $f_1\colon S^2 \to M$ where $f_1(S^2) \subset N$. Since f_1 is homotopic to f, it is clear that $f_1 \in [f]$.

Choose a leg R_i and parameterize it by $t \in [0,1]$ where $t = 0$ is the point on the edge $\Gamma_i \subset \partial(M/T^2)$. Define $B_i := f_1^{-1}(N_i) \subset S^2$ and $\Sigma_i := f_1(B_i) \subset N_i$ for $i = 0,1,\ldots,k$. Without loss of generality assume that 1 is a regular value of the map $\pi|_{\Sigma_i}\colon \Sigma_i \to [0,1]$. This can be accomplished by possibly homotoping f_1 or slightly pushing the boundary of R_0 to contain an ε-portion of the curve. We therefore see the boundary $\partial\Sigma_i = \pi|_{\Sigma_i}^{-1}(\{1\})$ as a 1-dimensional submanifold with orientation induced from $\Sigma_i = f_1(B_i)$ in the standard way. The space $\pi|_{\Sigma_i}^{-1}(\{1\})$ is a closed, oriented submanifold of $\pi^{-1}(R_0) = T^2 \times R_0$ so define $\mathbf{a}_i \in H_1(T^2 \times R_0;\mathbb{Z}) \cong H_1(T^2;\mathbb{Z})$ to be its homology class. Now take the push forward $\iota_*(\mathbf{a}_i)$ via the inclusion map $\iota\colon \pi|_{\Sigma_i}^{-1}(\{1\}) \hookrightarrow \pi^{-1}(R_i)$. Since $\pi|_{\Sigma_i}^{-1}(\{1\})$ is the boundary of $\Sigma_i \subset \pi^{-1}(R_i)$ we see that $\iota_*(\mathbf{a}_i) \in H_1(\pi^{-1}(R_i);\mathbb{Z})$ is trivial. Recall that $\pi^{-1}(R_i)$ is homeomorphic to the product of an interval and the torus $\mathbb{R}^2/\mathbb{Z}^2$, with the $\mathbf{v}_i\mathbb{R}/\mathbb{Z} \subset \mathbb{R}^2/\mathbb{Z}^2$ circle on the torus collapsing at one of the end points. Therefore the first homology group of $\pi^{-1}(R_i)$ is isomorphic to $H_1([0,1] \times T^2;\mathbb{Z})/\mathbf{v}_i\mathbb{Z} \cong \mathbb{Z}$ and thus $\mathbf{a}_i$ must be an integer multiple of $\mathbf{v}_i \in H_1(T^2;\mathbb{Z})$. We will denote this integer by α_i so that

$$\mathbf{a}_i = \alpha_i \mathbf{v}_i. \tag{2.35}$$

This allows us to finally define the inverse function Ψ_*^{-1} as

$$\Psi_*^{-1}([\alpha]) := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n. \tag{2.36}$$

Now that Ψ_*^{-1} has been described, we must show three things: that it is well-defined, that its image is contained in $\ker(A)$, and that it is in fact an isomorphism. Let us start with the proof that $A \circ \Psi_*^{-1} = 0$ since it is the most technical. The definition of A shows $A(\alpha_i \mathbf{e}_i) = \alpha_i \mathbf{v}_i = \mathbf{a}_i$. Of course $\mathbf{a}_i$ is the homology class of $\pi|_{\Sigma_i}^{-1}(\{1\})$ which coincides with $f_1(\partial B_i)$. Recall that $S^2 = B_0 \cup B_1 \cup \cdots \cup B_k$ and $f_1(S^2) \subset M$ is a closed cycle, thus $-[f_1(\partial B_0)] = [f_1(\partial B_1)] + \cdots + [f_1(\partial B_k)] \in H_1(T^2 \times R_0;\mathbb{Z})$. Since $f_1(\partial B_0) \subset T^2 \times R_0$ is exact in $H_*(T^2 \times R_0;\mathbb{Z})$ we conclude that

$$\begin{aligned}
A(\Psi_*^{-1}([\alpha])) &= A((\alpha_1, \ldots, \alpha_n)) \\
&= \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \\
&= \mathbf{a}_1 + \cdots + \mathbf{a}_n \\
&= [f_1(\partial B_1)] + \cdots + [f_1(\partial B_k)] \\
&= 0 \in H_1(T^2;\mathbb{Z}).
\end{aligned}$$

We will now show that $\Psi_*^{-1}([\alpha])$ is well defined. Recall that in the construction of $\Psi_*^{-1}([\alpha])$ a map $f \in [f]$ was chosen with the property that it missed the corners of M/T^2. Let X denote the set of all possible maps $g \in [f]$ which miss the corners of M/T^2. Define $\Phi(g) \in \ker(A)$ to be the value of $\Psi_*^{-1}([\alpha])$ using the map $g \in [f]$. Since $\ker(A)$ is a discrete space and $\Phi\colon X \to \ker(A)$ is clearly continuous, we only need to show that X is path connected to prove that Φ is a constant map. Let $g_0, g_1 \in X$ and let $G\colon S^2 \times [0,1] \to M$ be a homotopy between them. Without loss of generality we can assume that G is generic and thus its image $G(S^2 \times [0,1]) \subset M$ will be transverse to the corners. Since this is 3-dimensional and the corners are 0-dimensional, we conclude that $G(S^2 \times [0,1])$ is disjoint from the corners. In particular $g_t := G(\cdot,t) \in X$ for all t, which means that X is path connected and Φ is a constant map. This means $\Psi_*^{-1}([\alpha])$ does not depend on any choice and $\Psi_*^{-1}\colon H_2(M;\mathbb{Z}) \to \ker(A)$ is a well defined map.

Since Ψ_*^{-1} is a linear map between two free $\mathbb{Z}$-modules of the same finite dimension, proving surjectivity will be sufficient to show that Ψ_*^{-1} is an isomorphism. To that end, let $R_0, R_1, \ldots, R_k \subset M/T^2$ be subsets defined as above and choose some vector $(\alpha_1, \ldots, \alpha_n) \in \ker(A)$. For each $i = 1, \ldots, k$ we define a map

$f_i \colon D^2 \to \pi^{-1}(R_i) \subset M$ from the disk with orientation chosen so that the homology class $[f_i(\partial D^2)] \in H_1(T^2 \times R_0; \mathbb{Z})$ is equal to $\alpha_i \mathbf{v}_i$. Pick a point $p_0 \in \pi^{-1}(R_0)$ and perturb f_i so that $p_0 \in f_i(\partial D^2)$ for all $i = 1, \ldots, k$. Note that $f_i(\partial D^2)$ is a representative of the trivial (and only) element in the based homotopy group $\pi_1(M, p_0)$ and that $f_i \colon D^2 \to M$ can be interpreted as a null-homotopy. Now consider the bouquet of circles $C := f_1(\partial D^2) \cup \cdots \cup f_k(\partial D^2)$ with the opposite orientation. By construction its homology class is $[C] = -(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n) = 0 \in H_1(T^2 \times R_0; \mathbb{Z})$ and thus C represents a trivial element in the homotopy group $\pi_1(T^2 \times R_0, p_0)$. Define the map $f_0 \colon D^2 \to T^2 \times R_0 \subset M$ with $f_0(\partial D^2) = C$ to be a null-homotopy of C. By attaching these null-homotopies along their boundaries in the obvious way, we construct an element of $\pi_2(M, p_0)$, which we denote by $[f]$. Letting $[\alpha] \in H_2(M; \mathbb{Z})$ denote the element associated to $[f] \in \pi_2(M)$ via the Hurewicz isomorphism, we see that $\Psi_*^{-1}([\alpha]) = (\alpha_1, \ldots, \alpha_n)$ as desired. Hence our construction of a well defined isomorphism $\Psi_*^{-1} \colon H_2(M; \mathbb{Z}) \to \ker(A)$ is complete.

□

Note that since $\Psi_* \colon \ker(A) \to H_2(M; \mathbb{Z})$ is an isomorphism its dual map $\Psi^* \colon H^2(M; \mathbb{Z}) \to \mathrm{Hom}(\ker(A), \mathbb{Z})$ is also an isomorphism. Using these maps and the natural sublattice structure of $\ker(A) \subset \mathbb{Z}^k$ allows us to represent important topological quantities as vectors in $\mathbb{Z}^k$. This is seen in the following example.

Example 2.39: Consider the closed simple T^2-manifold M given by the rod structures $\mathbf{v}_1 = (1,1)$, $\mathbf{v}_2 = \mathbf{v}_4 - (1,0)$, and $\mathbf{v}_3 = \mathbf{v}_5 = (0,1)$. From Theorem 2.34 we know that $M \cong \mathbb{CP}^2 \# S^2 \times S^2$ and can see the decomposition explicitly in Figure 2.15. Define $\iota^1 \colon \mathbb{CP}^2 \setminus \{pt.\} \hookrightarrow M$ and $\iota^2 \colon S^2 \times S^2 \setminus \{pt.\} \hookrightarrow M$ to be the embeddings which give this decomposition. Similarly define the maps A^i, Ψ_*^i, and j^i according to the following commutative diagram

$$\mathbb{CP}^2 \setminus \{pt.\} \xhookrightarrow{\iota^1} M \xhookleftarrow{\iota^2} S^2 \times S^2 \setminus \{pt.\}$$

$$\begin{array}{ccccc} \ker(A^1) & \xhookrightarrow{j^1} & \ker(A) & \xhookleftarrow{j^2} & \ker(A^2) \\ \big\downarrow{\scriptstyle \Psi_*^1} & & \big\downarrow{\scriptstyle \Psi_*} & & \big\downarrow{\scriptstyle \Psi_*^2} \\ H_2(\mathbb{CP}^2) & \xhookrightarrow{\iota_*^1} & H_2(M) & \xhookleftarrow{\iota_*^2} & H_2(S^2 \times S^2) \end{array}$$

where the maps j^i are actually restrictions of the identity on $\mathbb{Z}^5$. Observe that $\ker(A^1)$ is defined by a single vector, which we will denote by $\underline{\mathbb{CP}^1} := \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_5$. Similarly $\ker(A^2)$ is generated by two vectors, $\underline{S_\alpha^2} := \mathbf{e}_2 - \mathbf{e}_4$ and $\underline{S_\beta^2} := \mathbf{e}_3 - \mathbf{e}_5$. Using the above commutative diagram it is clear that image of these vectors under Ψ_* generate $H_2(M; \mathbb{Z})$. Take the second Stiefel-Whitney class $w_2 \in H^2(M; \mathbb{Z}_2)$ and pair it with any one of these generators $\mathbf{x}$ to see $\langle w_2, \Psi_* \mathbf{x} \rangle = \langle w_2, \iota_*^i \Psi_*^i \mathbf{x} \rangle = \langle \iota_i^* w_2, \Psi_*^i \mathbf{x} \rangle$. By naturality $\iota_1^* w_2 = w_2(\mathbb{CP}^2) \neq 0$ and $\iota_2^*(w_2) = w_2(S^2 \times S^2) = 0$, hence w_2 is described by sending $\underline{\mathbb{CP}^1}$ to 1 and $\underline{S_\alpha^2}$ and $\underline{S_\beta^2}$ to 0.

If we wish to see a 'vector representation' of w_2, then choose a $\overline{w} \in H^2(M; \mathbb{Z})$ whose mod-2 reduction is w_2 and consider the linear map $L \in \mathrm{Hom}(\ker(A), \mathbb{Z})$ which satisfies the equation

$$L(\mathbf{u}) = \langle \overline{w}, \Psi_* \mathbf{u} \rangle \tag{2.37}$$

for all $\mathbf{u} \in \ker(A)$. By an integral version of the Riesz Representation Theorem, there exists a (non-unique) vector $\boldsymbol{\eta} \in \mathbb{Z}^5$ such that

$$\boldsymbol{\eta} \cdot \mathbf{u} = \langle \overline{w}, \Psi_* \mathbf{u} \rangle, \tag{2.38}$$

where the dot product is inherited from $\mathbb{Z}^5$ using the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_5\}$. If we choose the representative $\overline{w} \in [w_2]$ which sends $\underline{\mathbb{CP}^1} = \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_5$ to 1 and both $\underline{S_\alpha^2} = \mathbf{e}_2 - \mathbf{e}_4$ and $\underline{S_\beta^2} = \mathbf{e}_3 - \mathbf{e}_5$, we see that $\boldsymbol{\eta} = \mathbf{e}_1$ satisfies Equation (2.38) and thus can be considered a 'vector representation' of w_2.

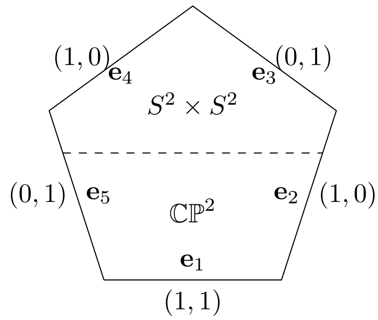


Figure 2.15: In the diagram above the integer vectors on the outside of the pentagon represent the 5 rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ of the manifold $M \cong \mathbb{CP}^2 \# S^2 \times S^2$ described in Example 2.39. The vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_5\}$ are the basis vectors of $\mathbb{Z}^5$ used to define the linear map $A\colon \mathbb{Z}^5 \to \mathbb{Z}^2$ by $A(\mathbf{e}_i) = \mathbf{v}_i$, or$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$. The kernel of A is generated by three vectors. The first vector $\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_5 \in \mathbb{Z}^5$ comes from $\mathbb{CP}^1 \subset \mathbb{CP}^2$. The second and third vectors, $\mathbf{e}_2 - \mathbf{e}_4$ and $\mathbf{e}_3 - \mathbf{e}_5$, come from the spheres $S^2_\alpha := S^2 \times \{pt.\}$ and $S^2_\beta := \{pt.\} \times S^2$ respectively.

Theorem 2.40 (Theorem B)**.** *Let M^4 be a simply connected T^2-manifold with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{Z}^2$ defining the linear map $A\colon \mathbb{Z}^k \to \mathbb{Z}^2$. There exists an explicit isomorphism*

$$\begin{aligned} \Psi_*\colon \ker(A) \subset \mathbb{Z}^k &\to H_2(M;\mathbb{Z}) \cong \mathbb{Z}^{k-2} \\ (\alpha_1, \ldots, \alpha_k) &\mapsto [\alpha] \end{aligned}$$

which computes the intersection form of M in the following way,

$$Q([\alpha],[\beta]) = \sum_{1 \le i < j \le k-1} \alpha_i \beta_j \det(\mathbf{v}_i, \mathbf{v}_j). \tag{2.39}$$

Before the proof of this theorem we need to make some remarks on Equation (2.39).

Remark 2.41. At first glance, Equation (2.39) does not appear to be symmetric since it contains the term $\det(\mathbf{v}_i, \mathbf{v}_j)$ which is antisymmetric. However the symmetry is hidden in the fact that $(\alpha_1, \ldots, \alpha_k) \in \ker(A)$ and thus $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0} \in \mathbb{Z}^2$. Using this and the fact that $\det(\mathbf{v}_i, \mathbf{v}_i) = 0$ we can express Equation (2.39) as

$$Q([\alpha],[\beta]) = \sum_{1 \le i \le j \le k} \alpha_i \beta_j \det(\mathbf{v}_i, \mathbf{v}_j). \tag{2.40}$$

From here a simple calculation will prove the symmetry.

$$\begin{aligned}
Q([\alpha],[\beta]) - Q([\beta],[\alpha]) &= \left(\sum_{1\le i\le j\le k} \det(\alpha_i \mathbf{v}_i, \beta_j \mathbf{v}_j)\right) - \left(\sum_{1\le i\le j\le k} \det(\beta_i \mathbf{v}_i, \alpha_j \mathbf{v}_j)\right) \\
&= \left(\sum_{1\le i\le j\le k} \det(\alpha_i \mathbf{v}_i, \beta_j \mathbf{v}_j)\right) - \left(\sum_{1\le i< j\le k} \det(\beta_i \mathbf{v}_i, \alpha_j \mathbf{v}_j)\right) \\
&= \left(\sum_{1\le i\le j\le k} \det(\alpha_i \mathbf{v}_i, \beta_j \mathbf{v}_j)\right) + \left(\sum_{1\le i< j\le k} \det(\alpha_j \mathbf{v}_j, \beta_i \mathbf{v}_i)\right) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} \det(\alpha_i \mathbf{v}_i, \beta_j \mathbf{v}_j) \\
&= \det\left(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k, \beta_1 \mathbf{v}_1 + \cdots + \beta_k \mathbf{v}_k\right) \\
&= \det(\mathbf{0}, \mathbf{0}) \\
&= 0
\end{aligned}$$

Remark 2.42. The sign of the left hand side Equation (2.39) depends on a choice of orientation of M while the sign of the right hand side depends on if the rod structures are labeled clockwise or counterclockwise. The existing convention is to label diagrams counterclockwise, and for the manifold associated to $\{(1,0),(0,1),(1,1)\}$ (oriented counterclockwise as seen in Example 2.29) to be oriented as $\mathbb{CP}^2$ and not $\overline{\mathbb{CP}^2}$. In order to preserve these conventions, we choose to define

$$\omega := dx \wedge d\phi^1 \wedge dy \wedge d\phi^2 \tag{2.41}$$

to be a positively oriented top form on $M \cong \frac{M/T^2 \times T^2}{\sim}$, where $(x,y) \in M/T^2$ are coordinates on M/T^2 and $(\phi^1, \phi^2) \in T^2$ are coordinates on T^2.

Proof of Theorem 2.40. The isomorphism $\Psi_*\colon \ker(A) \to H_2(M;\mathbb{Z})$ was given in Lemma 2.38. Choose homology classes $[\alpha],[\beta] \in H_2(M;\mathbb{Z})$ and define $(\alpha_1,\ldots,\alpha_k),(\beta_1,\ldots,\beta_k) \in \ker(A)$ so that $\Psi_*((\alpha_1,\ldots,\alpha_k)) = [\alpha]$ and $\Psi_*((\beta_1,\ldots,\beta_k)) = [\beta]$. Following the construction in Lemma 2.38, let $[f] \in \pi_2(M)$ represent $[\alpha]$ and define $f\colon S^2 \to M$ so that $\pi \circ f(S^2) \subset M/T^2$ is a union of k line segments. Denote π^{-1} of each of those line segments as A_i and π^{-1} of the center A_0, so that $A := A_0 \cup A_1 \cup \cdots \cup A_k = f(S^2)$. Similarly define $g\colon S^2 \to M$ and $B := B_0 \cup B_1 \cup \cdots \cup B_k = g(S^2)$ for $[\beta]$.

Perturb B slightly so that A and B intersect transversely. Now that they are transverse, the intersection pairing $Q([\alpha],[\beta])$ can be computed by summing up the signed intersection numbers of A_i and B_j, denoted by $\#(A_i \cap B_j)$;

$$Q([\alpha],[\beta]) = \sum_{i,j\in\{1,\ldots,k\}} \#(A_i \cap B_j). \tag{2.42}$$

However many of these intersections may be empty. In fact Figure 2.16 shows that it is possible to homotope B so that the intersection $A_i \cap B_j$ is only nonempty when $1 \le i < j \le k-1$;

$$Q([\alpha],[\beta]) = \sum_{1\le i<j\le k-1} \#(A_i \cap B_j). \tag{2.43}$$

The signed intersection $\#(A_i \cap B_j)$ is quite easy to compute. The curves $\pi(A_i), \pi(B_j) \subset M/T^2$ intersect in a single point $p_0 \in M/T^2$, so the intersection of the surfaces is contained in a single fiber, $A_i \cap B_i \subset \pi^{-1}(p_0) \cong T^2$. Choose coordinates on M/T^2 so that $p_0 = (x_0, y_0)$ and that the curves A_i and B_j are tangent to the vectors $\mathbf{u}^A = (u^A_x, u^A_y)$ and $\mathbf{u}^B = (u^B_x, u^B_y)$ respectively. Rescale $\mathbf{u}^A$ and $\mathbf{u}^B$ so that $\det(\mathbf{u}^A, \mathbf{u}^B) = \pm 1$.

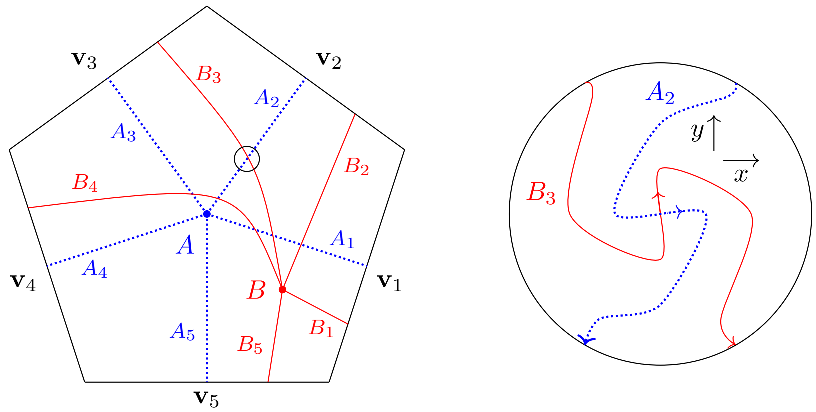


Figure 2.16: The left diagram shows the quotient space M/T^2. In it are the images of a representative of each of the homology classes $[\alpha] \in H_2(M;\mathbb{Z})$ (represented by A and depicted by the blue dashed lines) and $[\beta] \in H_2(M;\mathbb{Z})$ (represented by B and depicted by the red solid lines). Both A and B are broken up into line segments which are individually labeled according to which axis rod they meet; $A = A_1 \cup \cdots \cup A_5$ and $B = B_1 \cup \cdots \cup B_5$. A small circle is drawn around the intersection of A_2 and B_3 with a blown up image to the right. This shows how each intersection can be deformed so that A_i is pointing in the positive x direction and B_j pointing in the positive y direction.

We can then assume that A_i is parameterized by $f_i\colon (-\varepsilon,\varepsilon) \times S^1 \to M/T^2 \times T^2$ where

$$f_i(t,\theta) := (p_0 + t\mathbf{u}^A, \theta\alpha_i\mathbf{v}_i) = (x_0 + tu_x^A, y_0 + tu_y^A, \theta\alpha_i v_i^1, \theta\alpha_i v_i^2) \tag{2.44}$$

and the positive t direction is pointing toward the center, away from the edge Γ_i. Similarly assume

$$g_j(t,\theta) := (p_0 + t\mathbf{u}^B, \theta\beta_j\mathbf{v}_j) = (x_0 + tu_x^B, y_0 + tu_y^B, \theta\beta_i v_j^1, \theta\beta_i v_j^2) \tag{2.45}$$

parameterizes B_j. To determine the signed interesection number, we have to take the wedge product of the volume forms of A_i and B_j and compare it to the volume form of M, $\omega = dx \wedge d\phi^1 \wedge dy \wedge d\phi^2$. This requires knowing the sign of $(u_x^A dx + u_y^A dy) \wedge (u_x^B dx + u_y^B dy) = \det(\mathbf{u}^A, \mathbf{u}^B) dx \wedge dy$. As seen in Figure 2.16, the fact that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are labeled counterclockwise and $i < j$ means that $(u_x^A dx + u_y^A dy) \wedge (u_x^B dx + u_y^B dy) = dx \wedge dy$. We can now examine the wedge product of the volume forms.

$$\begin{aligned}
\omega_{A_i} \wedge \omega_{B_j} &= (u_x^A dx + u_y^A dy) \wedge \alpha_i(v_i^1 d\phi^1 + v_i^2 d\phi^2) \wedge (u_x^B dx + u_y^B dy) \wedge \beta_i(v_j^1 d\phi^1 + v_j^2 d\phi^2) && (2.46)\\
&= -\alpha_i\beta_j dx \wedge dy \wedge (v_i^1 d\phi^1 + v_i^2 d\phi^2) \wedge (v_j^1 d\phi^1 + v_j^2 d\phi^2) && (2.47)\\
&= -\alpha_i\beta_j \det(\mathbf{v}_i, \mathbf{v}_j) dx \wedge dy \wedge d\phi^1 \wedge d\phi^2 && (2.48)\\
&= \alpha_i\beta_j \det(\mathbf{v}_i, \mathbf{v}_j)\omega. && (2.49)
\end{aligned}$$

Therefore the signed intersection number is

$$\#(A_i \cap B_j) = \alpha_i\beta_j \det(\mathbf{v}_i, \mathbf{v}_j) \tag{2.50}$$

which confirms Equation (2.39) and completes the proof. □

Remark 2.43. Theorem 2.40 gives a formula to compute the intersection pairing $Q([\alpha],[\beta])$ of any homology classes. If we wish to see the intersection form as a $(k-2) \times (k-2)$ matrix we must first compute the

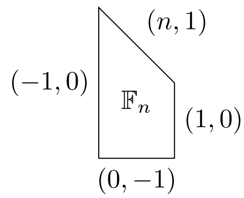


Figure 2.17: The Hirzebruch surface $\mathbb{F}_n$ is shown above as a Delzant polytope. The rod structures are defined from the embedding into $\mathbb{R}^2$ as the outward pointing normal vector.

following $k \times k$ matrix

$$D^{ij} := \begin{cases} \det(\mathbf{v}_i, \mathbf{v}_j) & i \leq j \\ \det(\mathbf{v}_j, \mathbf{v}_i) & i > j \end{cases}. \tag{2.51}$$

and then compute a surjection $R\colon \mathbb{Z}^{k-2} \to \ker(A) \subset \mathbb{Z}^k$. Once these are computed, we see the matrix representation of the intersection form Q as

$$[Q] = R^T DR. \tag{2.52}$$

The above equation can be verified by computations done in Remark 2.41.

Example 2.44: Let us compute the intersection form of the n^{th} Hirzebruch surfaces $\mathbb{F}_n$, a manifold which was previously examined in Example 2.31. Since algebraic geometers may be interested in this example, we will use rod structures coming from the Delzant polytope construction of $\mathbb{F}_m$; $\{(-1,0),(0,-1),(1,0),(n,1)\}$. Since rod structures are only defined up to sign this is equivalent to the rod structures seen in Example 2.31. It also has the added advantage that $\det(\mathbf{v}_i, \mathbf{v}_{i+1}) = 1$ for all i. This is apparent when looking at the off-diagonals in the matrix

$$D = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & n \\ 0 & 1 & 0 & 1 \\ -1 & n & 1 & 0 \end{bmatrix}. \tag{2.53}$$

The rod structures show

$$\mathbf{0} = \mathbf{v}_1 + \mathbf{v}_3 \tag{2.54}$$

$$\mathbf{0} = n\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4 \tag{2.55}$$

and in particular $\operatorname{span}_{\mathbb{Z}}\{(n,1,0,1),(1,0,1,0)\} = \ker(A) \subset \mathbb{Z}^4$. Therefore the surjection $R\colon \mathbb{Z}^2 \to \ker(A) \subset \mathbb{Z}^4$ has a matrix form of

$$R = \begin{bmatrix} n & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.56}$$

which is used to compute the intersection form

$$[Q] = R^T DR = \begin{bmatrix} n & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.57}$$

If instead different generators of $\ker(A)$ were chosen, giving a different matrix

$$R' = \begin{bmatrix} n-l & 1 \\ 1 & 0 \\ -l & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.58}$$

then it would simply lead to a different presentation of Q;

$$R'^T D R' = \begin{bmatrix} n-2l & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.59}$$

In the above example we see the intersection form itself is not enough to distinguish the Hirzebruch surfaces $\mathbb{F}_n$ from $\mathbb{F}_{n-2l}$, as seen in Equations (2.57) and (2.59). This is because $\mathbb{F}_n$ and $\mathbb{F}_{n-2l}$ have the same underlying smooth manifold structure and are only distinguished by their torus actions (or equivalently their complex structures, see Theorem 2.9). On the other hand, the bilinear form D in Equation (2.53) clearly comes from $\mathbb{F}_n$ and not $\mathbb{F}_{n-2l}$. This is because in general the matrix of determinants D defined in Equation (2.51) is an invariant of the weak equivariant homeotype.

Remark 2.45. The Hermite normal form of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ can be read off from the top two lines of the matrix of determinants D defined by Equation (2.51) in Remark 2.43. Specifically $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is the Hermite normal form of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ where

$$\mathbf{w}_i := \begin{cases} \mathbf{e}_i & i \leq 2 \\ (-D^{2i}, D^{1i}) & i > 2. \end{cases} \tag{2.60}$$

The following theorem gives a method to read off whether or not a manifold is spin from the rod structures without preforming any computations. While writing this dissertation a different proof of the same result was published [23, Prop. 6.1]. Their proof is much shorter than ours but utilizes a more sophisticated characterization of spin.

Theorem 2.46. *Let M^4 be a simple T^2-manifold with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{Z}^2$. The manifold M is non-spin if and only if there exists three rod structures $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ (in no particular order) such that*

$$\begin{aligned} \mathbf{u} &\equiv (1,0) \mod 2 \\ \mathbf{v} &\equiv (0,1) \mod 2 \\ \mathbf{w} &\equiv (1,1) \mod 2. \end{aligned} \tag{2.61}$$

Proof. Recall that a simply connected 4-manifold M is spin if and only if its intersection form is even, meaning $Q([\alpha],[\alpha])$ is always an even integer. Thus to prove that M is non-spin we only need to find a homology class $[\alpha] \in H_2(M;\mathbb{Z})$ such that $Q([\alpha],[\alpha]) \equiv 1 \mod 2$. As a short hand for this proof, let $m \equiv a$ denote $m \equiv a \mod 2$ for $m \in \mathbb{Z}$ and let $\mathbf{v} \equiv (a,b)$ denote that $v^1 \equiv a$ and $v^2 \equiv b$ for $\mathbf{v} \in \mathbb{Z}^2$.

We begin with proving the "if" direction. Let $i_1, i_2, i_3 \in \{1, \ldots, k\}$ be three distinct integers so that the rod structures $\{\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \mathbf{v}_{i_3}\}$ satisfy Equations (2.61) (in no particular order). Consider the intersection of the primitive submodule $\mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \mathbf{e}_{i_3}\} \cap \ker(A) \subset \mathbb{Z}^k$. Since $\dim(\mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \mathbf{e}_{i_3}\}) = 3$ and $\dim(\ker(A)) = k-2$, we know that their intersection is a non-trivial primitive submodule of $\mathbb{Z}^k$. Let $\alpha \in \mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \mathbf{e}_{i_3}\} \cap \ker(A) \subset \mathbb{Z}^k$ be a primitive element in their intersection. Since $\alpha = (\alpha_1, \ldots, \alpha_k)$ is primitive, at least one of its three non-zero entries must be odd. So without loss of generality assume $\alpha_{i_2} \equiv 1$. Next, by possibly relabeling the rod structures, we can assume that $i_3 = k$. This gives us the following equations:

$$\alpha_k \mathbf{v}_k + \alpha_{i_1} \mathbf{v}_{i_1} \equiv \mathbf{v}_{i_2}, \tag{2.62}$$

and

$$Q([\alpha],[\alpha]) = \sum_{1\le i<j\le k-1} \det(\alpha_i \mathbf{v}_i, \alpha_j \mathbf{v}_j) = \pm\det(\alpha_{i_1}\mathbf{v}_{i_1}, \alpha_{i_2}\mathbf{v}_{i_2}) \equiv \alpha_{i_1}\det(\mathbf{v}_{i_1},\mathbf{v}_{i_2}) \equiv \alpha_{i_1}. \tag{2.63}$$

Hence if $\alpha_{i_1} \equiv 1$, then M is non-spin.

Assume by contradiction that $\alpha_{i_1} \equiv 0$. Next assume without loss of generality that $v_2^1 \equiv 1$. With these two assumptions we see $\alpha_k v_k^1 \equiv \alpha_k v_k^1 + \alpha_{i_1} v_{i_1}^1 \equiv v_{i_2}^1 \equiv 1$ which means $\alpha_k \equiv 1$ and $v_k^1 \equiv 1$. Now we have two cases. In the first case $\mathbf{v}_k \equiv (1,0)$. By looking at the determinant $\det(\mathbf{v}_k, \mathbf{v}_{i_1}) \equiv 1$ we see $v_{i_1}^2 \equiv 1$. Plug this into the second equation of (2.62) we see $v_{i_2}^2 \equiv \alpha_k v_k^2 + \alpha_{i_1} v_{i_1}^2 \equiv 0$. This means $\mathbf{v}_{i_2} \equiv (1,0)$ which is impossible since $\mathbf{v}_k$ is already congruent to $(1,0)$. In the second case $v_k \equiv (1,1)$. Since we already know $v_{i_2}^1 \equiv 1$ the vector $\mathbf{v}_{i_2}$ must be congruent to $(1,0)$. This leave only one equivalence class remaining and thus $\mathbf{v}_{i_1} \equiv (0,1)$. Of course doesn't agree with Equation (2.62) either since $(1,1) \equiv \mathbf{v}_k \equiv \alpha_k\mathbf{v}_k + \alpha_{i_1}\mathbf{v}_{i_1} \equiv \mathbf{v}_{i_2} \equiv (1,0)$. Therefore it is impossible for $\alpha_{i_1} \equiv 0$ which means $\alpha_{i_1} \equiv 1$ and thus the self-intersection of $[\alpha]$ is odd, proving M is non-spin.

For the other half of the proof, assume that Equation (2.61) is never satisfied, that is at least one of these three equivalence classes is never achieved by any rod structure on M. Without loss of generality, assume $\mathbf{v}_i \not\equiv (1,1)$ for all $i = 1,\ldots,k$. Next, assume by contradiction that there exists a homology class $[\alpha]$ such that $Q([\alpha],[\alpha]) \equiv 1$. Let $\Phi([\alpha]) = \alpha = (\alpha_1,\ldots,\alpha_k) \in \ker(A) \subset \mathbb{Z}^k$ and define the subset $\{i_1,\ldots,i_m\} \subset \{1,\ldots,k\}$ so that $\alpha_j \equiv 1$ if and only if $j \in \{i_1,\ldots,i_m\}$. Reducing Q mod 2 shows

$$Q([\alpha],[\alpha]) \equiv \sum_{\substack{1\le i<j\le k-1\\ i,j\in\{i_1,\ldots,i_m\}}} \det(\mathbf{v}_i,\mathbf{v}_j) \tag{2.64}$$

which means if $Q([\alpha],[\alpha]) \equiv 1$ then the summand on the right must be nonzero an odd number of times. Of course $\det(\mathbf{v}_{i_j},\mathbf{v}_{i_l}) \equiv 1$ if and only if $\mathbf{v}_{i_j} \not\equiv \mathbf{v}_{i_l}$. By hypothesis there are only two equivalence classes for rod structures, $(1,0)$ and $(0,1)$, so let $1 \le n \le m-1$ be the number of rod structures in $\{\mathbf{v}_{i_1},\ldots,\mathbf{v}_{i_m}\}$ which are congruent to $(1,0)$ and $m-n$ be the number of rod structures congruent to $(0,1)$. Without loss of generality assume that $\mathbf{v}_{i_1} \equiv (1,0)$ and $\mathbf{v}_{i_m} \equiv (0,1)$. Next we use the fact that by ordering the rod structures, we can force $i_m = k$ and thus

$$Q([\alpha],[\alpha]) \equiv \sum_{1\le l<j\le m-1} \det(\mathbf{v}_l,\mathbf{v}_j) \equiv n(m-n-1) \tag{2.65}$$

because the set $\{\mathbf{v}_{i_1},\ldots,\mathbf{v}_{i_{m-1}}\}$ contains exactly n vectors congruent to $(1,0)$ and $m-n-1$ vectors congruent to $(0,1)$. Similarly when we reorder the rod structures to make $i_1 = k$, we see

$$Q([\alpha],[\alpha]) \equiv \sum_{2\le l<j\le m} \det(\mathbf{v}_l,\mathbf{v}_j) \equiv (n-1)(m-n). \tag{2.66}$$

It is impossible for both $n(m-n-1)$ and $(n-1)(m-n)$ to be odd which means we have reached a contradiction and the proof is complete.

□

The previous lemma shows the parity of rod structures is what determines whether or not a manifold is spin. The following lemma shows that we can control the parity of rod structures when preforming fill-in operations, like those done in Theorem 2.26.

Lemma 2.47. *The rods $(1,0)$ and (q,p) can always be connected by $\{\mathbf{w}_1,\ldots,\mathbf{w}_k\}$ such that $\mathbf{w}_1 = (1,0)$, $\mathbf{w}_k = (q,p)$, and $w_i^1 w_i^2 \equiv 0 \mod 2$ for all $i < k$.*

Proof. The proof of this is very similar to the proof in [25], but with a slight complication to guarantee

$w_1^1 w_2^2 \equiv 0 \mod 2$. First, let A_i and B_i denote the entries of the i^{th} rod, that is $\mathbf{w}_i = (A_i, B_i)$. Notice that

$$(A_1, B_1) = (1, 0) \tag{2.67}$$
$$(A_2, B_2) = (b_2, 1) \tag{2.68}$$

for some unknown integer $b_2 \in \mathbb{Z}$. Since each pairs of consecutive of rods $\{\mathbf{w}_i, \mathbf{w}_{i+1}\}$ is admissible they must span all of $\mathbb{Z}^2 = \text{span}\{\mathbf{w}_i, \mathbf{w}_{i+1}\}$. Therefore it is possible to define integers a_i and b_i for $i > 2$ such that $\mathbf{w}_i = b_i \mathbf{w}_{i-1} + a_i \mathbf{w}_{i-2}$, or written in terms of A and B

$$A_i = b_i A_{i-1} + a_i A_{i-2} \tag{2.69}$$
$$B_i = b_i B_{i-1} + a_i B_{i-2}. \tag{2.70}$$

This recursion equation has the unique property that the coefficients a_i and b_i are related to a continued fraction expansion, namely

$$\frac{A_n}{B_n} = b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots + \frac{a_k}{b_k}}}}. \tag{2.71}$$

Now a simple continued fraction for the rational number $\frac{q}{p}$ of the form

$$\frac{q}{p} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \frac{1}{b_k}}}} \tag{2.72}$$

can be computed using the Euclidean algorithm. The algorithm is as follows. Let $q = x_1$ and $p = x_2$ and find the unique integers $b_i, r_i \in \mathbb{Z}$ where $0 \leq r_i < x_i$ which solve the following

$$x_1 = b_1 x_2 + r_2 \tag{2.73}$$
$$x_2 = b_2 x_3 + r_3 \tag{2.74}$$
$$\vdots \tag{2.75}$$
$$x_{k-1} = b_{k-1} x_k + r_k \tag{2.76}$$
$$x_k = b_k x_{k+1} \tag{2.77}$$

where $r_i = x_{i+1}$. Note that the algorithm terminates at when $r_{k+1} = 0$, which is always achieved since $\{r_i\}$ is a strictly decreasing sequence of non-negative integers. This was the algorithm used in [25] to compute the rod structures connecting $(1, 0)$ and (q, p).

To prove the current theorem, the Euclidean algorithm needs to be modified slightly by adding an additional term a_i. Like before, set $q = x_1$ and $p = x_2$. At each step, define b'_i and r'_{i+1} using the Euclidean algorithm, that is

$$x_i = b'_i x_{i+1} + r'_{i+1} \tag{2.78}$$

for $0 \leq r'_{i+1} < x_{i+1}$. If b'_i is a multiple of 2, then let $b_i = b'_i$, $r_{i+1} = r'_{i+1}$, $a_{i+1} = 1$ and proceed to the next step. If not, then let $b_i = b'_i + 1$ and define $r_{i+1} = \left| r'_{i+1} - x_{i+1} \right|$ and $a_{i+1} = \text{sgn}\left(r'_{i+1} - x_{i+1} \right)$ so that the equation

$$x_i = b_i x_{i+1} + a_{i+1} r_{i+1} \tag{2.79}$$

is satisfied. Notice that $\{r_i\}$ is still a strictly decreasing sequence of non-negative integers, and thus reaches 0 in finite time. The algorithm terminates as soon as this happens, meaning the final b_k may or may not be a multiple of 2. All together, the algorithm appears as

$$x_0 = b_0 x_1 + a_1 r_1 \tag{2.80}$$
$$x_1 = b_1 x_2 + a_2 r_2 \tag{2.81}$$
$$\vdots \tag{2.82}$$
$$x_{k-1} = b_{k-1} x_k + a_k r_k \tag{2.83}$$
$$x_k = b_k x_{k+1}. \tag{2.84}$$

This gives the coefficients for the continued fraction expansion of $\frac{q}{p}$ in the form of Equation (2.71).

Finally we need to check that $A_i B_i \equiv 0 \mod 2$ for all $i < k$. The $i = 1$ case is satisfied by our initial conditions $(A_1, B_1) = (1, 0)$ and for the $i = 2$ case, $(A_2, B_2) = (b_2, 1)$, the modified Euclidean algorithm guaranties that $b_2 \equiv 0 \mod 2$. The remaining cases are proven by induction.

$$A_i B_i = (b_i A_{i-1} + a_i A_{i-2})(b_i B_{i-1} + a_i B_{i-2}) \tag{2.85}$$
$$\equiv a_i^2 A_{i-2} B_{i-2} \mod 2 \tag{2.86}$$
$$\equiv 0 \mod 2 \tag{2.87}$$

□

The next theorem is an application of the previous two lemmas. It shows that any 4-dimensional spin simple T^2-manifold can be extended to a closed spin simple T^2-manifold. This was hypothesised to be true in a previous work. Specifically Theorem 2.48 proves that in the $n = 2$ case the technical assumption needed for [24, Theorem C] to prove [24, Conjecture D] is always satisfied.

Theorem 2.48. *Let M^4 be a simply connected, asymptotically flat, spin, T^2-manifold with boundary. There exists an extension $M \subset \overline{M}$ to a complete, asymptotically flat, spin, simple T^2-manifold.*

Proof. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be rod structures for M. Since M is spin Theorem 2.46 applies and we can assume without loss of generality that for all i, $\mathbf{v}_i \not\equiv (1, 1) \mod 2$. If M does not have any horizons then it is closed and the proof is complete. If there is a horizon between the i^{th} and $(i+1)^{st}$ then we first find a $U \in SL(2, \mathbb{Z})$ so that $U(\mathbf{v}_i) = (1, 0)$ and $U(\mathbf{v}_{i+1}) \not\equiv (1, 1) \mod 2$. Then apply Lemma 2.47 to $U(\mathbf{v}_i)$ and $U(\mathbf{v}_{i+1})$ to produce a string of admissible vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_l\}$ where $\mathbf{w}_1 = U(\mathbf{v}_i)$ and $\mathbf{w}_l = U(\mathbf{v}_{i+1})$. We now insert rods with rod structures $\{U^{-1}(\mathbf{w}_2), \ldots, U^{-1}(\mathbf{w}_{l-1})\}$ between the $i^t h$ and $(i+1)^{st}$ rod in a manner similar to Theorem 2.26. Repeat this process for every horizon. □

The following proposition gives a slight spin to a previously known result (see [50, Theorem 4.4]).

Proposition 2.49. *Every 3-dimensional lens spaces $L(p; q)$ is T^2-equivariantly spin cobordant to zero.*

Proof. A lens space $L(p; q)$ is the boundary of a horizon flanked on either side by the rods $\mathbf{v} = (1, 0)$ and $\mathbf{w} = (-q, p)$. Without loss of generality assume $\mathbf{w} \not\equiv (1, 1) \mod 2$ possibly by applying the change of coordinates $U = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in SL(2, \mathbb{Z})$ for some $a \in \mathbb{Z}$. Now apply Lemma 2.47 to produce admissible rod structures $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ where $\mathbf{w}_1 = \mathbf{v}$ and $\mathbf{w}_k = \mathbf{w}$. These rod structures form a simple T^2-manifold M^4 with boundary $\partial M = L(p; q)$. Since $\mathbf{w}_i \not\equiv (1, 1) \mod 2$ for all i, we use Theorem 2.46 to conclude that M is spin. Thus $L(p; q)$ is T^2-equivariantly spin cobordant to zero. □

2.5 Higher Dimensions

In this subsection we overview previous work that has been done by McGavran and Oh towards proving Conjecture A. It does not appear that this conjecture has previously been recorded in the literature. However, it

n	$M(n,n)$
2	S^4
3	S^5
4	$S^3 \times S^3$
5	$\#5(S^3 \times S^4)$
6	$\#9(S^3 \times S^5)\#8(S^4 \times S^4)$
7	$\#14(S^3 \times S^6)\#35(S^4 \times S^5)$
8	$\#20(S^3 \times S^7)\#64(S^4 \times S^6)\#45(S^5 \times S^5)$
9	$\#27(S^3 \times S^8)\#105(S^4 \times S^7)\#189(S^5 \times S^6)$
10	$\#35(S^3 \times S^9)\#160(S^4 \times S^8)\#350(S^5 \times S^7)\#224(S^6 \times S^6)$

Table 2.18: Listed are the first few manifolds described in Theorem 2.50.

should be noted that McGavran claimed in [35, Theorem 3.6] (see also [34]) to have proven a similar statement. Specifically McGavran claimed to have proven that the manifolds $M(n,k)$ described in Conjecture A are the *only* simply connected manifolds which admit an effective T^n-action. Oh [42] pointed out flaws in McGavran's argument, the least of which being the existence of simply connected non-spin manifolds admitting effective T^n-actions. Nevertheless a portion McGavran's argument (Theorem 3.4 in [35]) was found to be without error and in effect proves Conjecture A for the 2-connected case. This argument has in fact been generalized to higher cohomogeneity and been shown to work up to diffeomorphism (see [3, Theorem 6.3] and [4, Theorem 4.6.12] which both reduce to the cohomogeneity 2 case when their simplex/polytope is 2-dimensional). The theorem presented below is Theorem 3.4 in [35], modified slightly to include the $n = 2$ and 3 cases which were certainly known to McGavran, and set in the language of rod structures.

Theorem 2.50 (McGavran)**.** *For all $n \geq 2$, there is a unique closed 2-connected $(n+2)$-manifold which admits an effective T^n-action. This manifold admits rod structures $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subset \mathbb{Z}^n$ and is diffeomorphic to*

$$M(n,n) := \begin{cases} S^4 & n = 2 \\ S^5 & n = 3 \\ \#_{j=1}^{n-3} j\binom{n-2}{j+1} S^{2+j} \times S^{n-j} & n \geq 4. \end{cases} \tag{2.88}$$

In [42] and [41] Oh studied the 5 and 6 dimensional cases and ended up proving Conjecture A for all simply connected 5 and 6-manifolds. His classification argument relies heavily on surgery theory, Barden's classification of simply connected 5-manifolds, and Jupp's classification of simply connected 6-manifolds. Unfortunately these specific classification results do not reach higher dimensions in the way that is needed, and thus Oh's techniques cannot be used to classify simple T^n-manifolds of dimensions greater than 6.

Theorem 2.51 (Oh)**.** *For $n = 3, 4$ the homeotype of a closed simply connected T^n-manifold M^{n+2} is uniquely determined by the number of rods and whether or not it is spin. In particular is M has 3 rods then it is S^5, and if it has more than 3 rods it is determined by the table below;*

	$n = 3$	$n = 4$
spin	$\#(k-3)(S^2 \times S^3)$	$\#(k-4)(S^2 \times S^4)\#(k-3)(S^3 \times S^3)$
non-spin	$(S^2 \widetilde{\times} S^3)\#(k-4)(S^2 \times S^3)$	$(S^2 \widetilde{\times} S^4)\#(k-5)(S^2 \times S^4)\#(k-3)(S^3 \times S^3)$

where k is the number of rods.

In [42, §5] Oh discusses the errors in McGavran's classification. A key step in the classification is a surgery procedure which Oh refers to as *equivariant replacement*. In terms of rod structures, preforming an equivariant replacement on a closed simple T^n-manifold M at the corner $\{\mathbf{v}_i, \mathbf{v}_{i+1}\}$ would be equivalent to inserting an additional rod with rod structure $\mathbf{w}$ between them, so long as $\det_2\{\mathbf{v}_i, \mathbf{w}, \mathbf{v}_{i+1}\} = 1$. The resulting space M' is a closed simple T^n-manifold with one more rod than M. McGavran's classification relied on the assumption that if M is spin, then any manifold M' resulting from equivariant replacement operations on M would also be spin. The issue with this procedure is that without knowledge of the remaining rod structures of M, it is in general impossible to choose a $\mathbf{w}$ so that the resulting manifold is still spin. However if we are restricted to a very limited choice of rod structures, then this problem disappears. This is shown in Theorem 2.52 below. A complete proof of this theorem requires tools we will developed in Sections 2.7 and 2.8. Nevertheless we will present the theorem and proof here for completeness.

Theorem 2.52 (Theorem D, Part 2). *The manifolds $M(n,k)$ as described in Conjecture A admit effective T^n-actions making them simple T^n-manifolds with k rods.*

Proof. From Theorem 2.50 we know that $M(n,n)$ is a simple T^n-manifold with structure $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$. By preforming $k-n$ equivariant replacements on $M(n,n)$ we can form the simply connected T^n-manifold M' with rod structures $\{\mathbf{e}_1,\dots,\mathbf{e}_n,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_3,\dots,\mathbf{e}_{2/3}\}$ where the last rod structure is $\mathbf{e}_2$ if $k-n$ is odd and $\mathbf{e}_3$ if $k-n$ is even. The arguments in [35, Lemma 3.5] and [35, Theorem 3.6] show that M' is homeomorphic to $M(n,k)$ if we can show M' is spin. Lemma 2.82 in Section 2.8 shows that M' is spin and completes the proof. □

In [42, Remark 5.8] Oh proved that for all $n>2$ there exists an $(n+2)$-dimensional non-spin, simply connected T^n- manifold. His proof is by induction on n. Starting with a simply connected, non-spin T^{n-1}-manifold M^{n+1}, he creates a new manifold $\widetilde{M}^{n+2}$ by attaching n additional rods with rod structures $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ to the simple T^n-manifold $S^1\times M$. Oh then proves that $\widetilde{M}$ is also non-spin. However the point of attaching n rods to $S^1\times M$ is only to ensure that $\widetilde{M}$ is simply connected. This can however be accomplished by only a single rod, assuming that M is simply connected. Hence by starting with a simply connected non-spin T^2-manifold with $m>2$ rods, Oh's construction produces a simply connected non-spin T^n manifold with $m+n-2$ rods. We shall record this observation we just proved in the following Theorem.

Theorem 2.53. *For all $2<n<k$ there exists an $(n+2)$-dimensional closed, non-spin, simply connected T^n-manifold with exactly k rods.*

The remainder of this section is dedicated to examples.

Example 2.54: Since S^3 admits an effective T^2 and S^2 admits an effective S^1 action, the product $S^3\times S^2$ admits an effective T^3-action. This can easily be seen with the following presentation

$$S^3\times S^2=\{(e^{i\theta^1}\cos(x),e^{i\theta^2}\sin(x),e^{i\theta^3}y)|(x,y)\in[0,\pi/2]\times[0,\infty],\theta^i\in[0,2\pi]\}. \tag{2.89}$$

The rod structures for $S^3\times S^2$ with the standard torus action are shown in Figure 2.19.

Example 2.55: The total space of a S^3-bundle over an orbifold $S^2/\mathbb{Z}_p$ with a single orbifold point of order p admits an effective T^3-action. This can be see in Figure 2.20. Since M is simply connected, Theorem 2.51 tells us that M is diffeomorphic to either $S^2\times S^3$ or $S^2\widetilde{\times}S^3$. To determine which one would require computing the cohomology ring which is equivalent to the intersection pairs $H_2(M)\otimes H_3(M)\to\mathbb{Z}$.

2.6 Covering Spaces and Orbifolds

Let us recall the definition of an orbifold. A topological space X is an *orbifold* if it is second countable, Hausdorff, and locally covered by charts homeomorphic to a quotient of a Euclidean ball by a finite group. Note that all topological manifolds are orbifolds, but the underlying topological space of an orbifold may or may not be a manifold. For example the quotient of $\mathbb{R}^2$ by any finite subgroup of $SO(2)$ is homeomorphic to $\mathbb{R}^2$, while the quotient of $\mathbb{R}^4$ by certain finite subgroups of $SO(4)$ is no longer homeomorphic to $\mathbb{R}^4$.

Definition 2.56. A *simple T^n-orbifold* is an orientable smooth orbifold M^{n+2}, $k\geq 0$ equipped with an effective T^n-action, in which the quotient space M/T^n is contractible and the quotient map defines a trivial fiber bundle over the interior of the quotient.

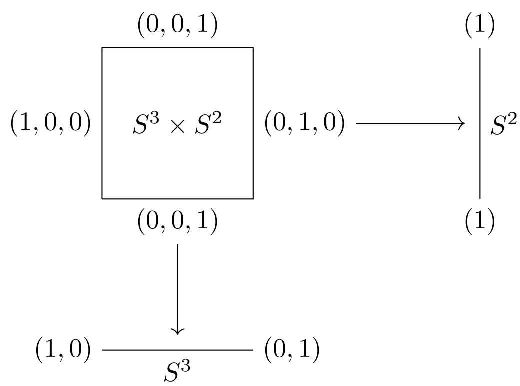


Figure 2.19: We see the toric diagram for $S^3 \times S^2$. The projection map down to S^3 is equivalent to collapsing the third circle. The projection map over to S^2 is equivalent to collapsing the first two circles.

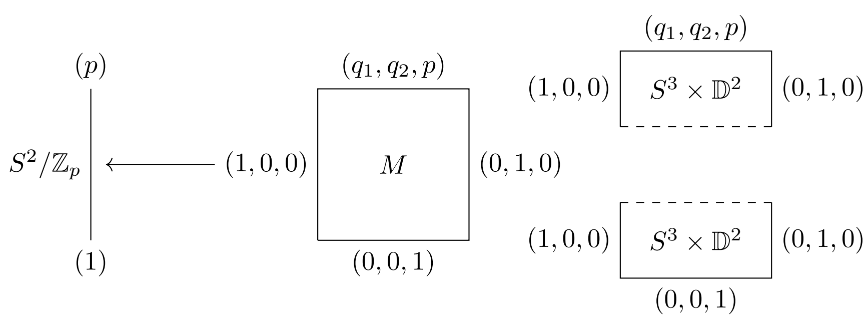


Figure 2.20: M is the total space of an S^3-bundle over the orbifold $S^2/\mathbb{Z}_p$. To the left we see the projection map which collapses the first two circles entirely.

Recall the definition of a subaction. An action of group G on a topological space X is defined by a continuous map $G \times X \to X$ satisfying certain properties. Any subgroup $H \subset G$ defines a groups action of H on X, known as a *subaction*, by restricting the map $G \times X \to X$ to $H \times X \to X$. For a simple T^n-manifold M^{n+2}, any subgroup of the torus $H \subset T^n$ defines a subaction. These subactions are automatically effective because the action of T^n on M is effective. If in addition to being effective the subgroup action is free, then the quotient M/H defines a manifold. If the subaction is only *almost free*, meaning that every isotropy subgroup is discrete, then the quotient M/H defines an orbifold [5, Proposition 1.5.1].

Any discrete subgroup $H \subset T^n$ is finite and therefore defines an effective almost free action of H on M^{n+2}. This gives M/H the natural structure of an orbifold. Moreover the action of T^n on M descends to an action of T^n/H on M/H. It is easy to see that the inherited action is effective and that T^n/H is abstractly homomorphic to an n-dimensional torus. This gives an effective action of T^n on M/H and gives it the structure of a simple T^n-orbifold. If M/H is in fact a manifold, then it is of course a simple T^n-manifold.

The quotient map $M \to M/H$ defines M as a covering space of M/H. When M/H is also a manifold then the usual theory of covering spaces relates the fundamental groups of M and M/H. This relationship is explored in the following theorem.

Theorem 2.57. *Let M^{n+2} be a T^n-manifold with k axis rods and with fundamental group*

$$\pi_1(M) = \mathbb{Z}^{n-l} \oplus \mathbb{Z}_{s_1} \oplus \cdots \oplus \mathbb{Z}_{s_l},$$

where $s_i | s_{i+1}$. For each s_i and for each q which divides s_i there exists a T^n-manifold $\widetilde{M}^{n+2}$ with a free $\mathbb{Z}_q \subset T^n$-subaction making $\widetilde{M}$ a q-fold cover over M. Furthermore there exists a single set of rods structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for M which describes the rod structures of each $\widetilde{M}$ as $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_k\}$ where

$$\tilde{\mathbf{v}}_j = (v_j^1, \ldots, v_j^{i-1}, \frac{v_j^i}{q}, v_j^{i+1}, \ldots, v_j^n). \tag{2.90}$$

Proof. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be any set of rod structures for M, and let $A = [\mathbf{w}_1 \cdots \mathbf{w}_k]$ be the $n \times k$ matrix where the $\mathbf{w}_i$ are column vectors. Using Lemma 2.22 there exists unimodular integral matricies U and V such that $S = UAV$ where S is the unique Smith normal form of A. Since $U \in SL(n, \mathbb{Z})$, the columns of UA represent rod structures of M, which we denote as $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

Note that $\mathrm{span}_{\mathbb{Z}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is equal to the span of the columns of UAV, and thus the columns of S. Using Corollary 2.25 to calculate the fundamental group of M reveals

$$\pi_1(M) = \mathbb{Z}^{n-l'} \oplus \mathbb{Z}_{s'_1} \oplus \cdots \oplus \mathbb{Z}_{s'_{l'}}, \tag{2.91}$$

where S is rank l' with diagonal entries s'_i. The fundamental group being unique easily shows that $l = l'$ and $s'_i = s_i$ and $\mathrm{span}_{\mathbb{Z}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{span}_{\mathbb{Z}}\{s_1\mathbf{e}_1, \ldots, s_l\mathbf{e}_l\}$.

This is used to show that the $\tilde{\mathbf{v}}_j$ defined by Equation (2.90) are integral. Indeed, observe

$$\begin{aligned} \mathbf{v}_j &\in \mathrm{span}_{\mathbb{Z}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \\ &= \mathrm{span}_{\mathbb{Z}}\{s_1\mathbf{e}_1, \ldots, s_l\mathbf{e}_l\} \\ &\subset \mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_1, \ldots, \mathbf{e}_{i-1}, q\mathbf{e}_i, \mathbf{e}_{i+1}, \ldots, \mathbf{e}_n\} \end{aligned}$$

which shows q divides the i^{th} entry of $\mathbf{v}_j$. We can also wee that $\tilde{\mathbf{v}}_j$ are primitive since $\gcd\{\tilde{v}_j^1, \ldots, \tilde{v}_j^n\}$ divides $\gcd\{\tilde{v}_j^1, \ldots, \tilde{v}_j^{i-1}, q\tilde{v}_j^i, \tilde{v}_j^{i+1}, \ldots, \tilde{v}_j^n\} = \gcd\{v_j^1, \ldots, v_j^n\} = 1$. In a similar vein the fact that $\mathrm{Det}_2\{\tilde{\mathbf{v}}_j, \tilde{\mathbf{v}}_{j+1}\}$ divides $\mathrm{Det}_2\{\mathbf{v}_j, \mathbf{v}_{j+1}\} = 1$ proves admissibility. Putting this all together confirms that $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_k\}$ is a set of admissible rod structures.

These rod structures define a T^n-manifold $\widetilde{M} := \frac{D^2 \times \mathbb{R}^n/\mathbb{Z}^n}{\approx}$ where $(p, \boldsymbol{\theta}) \approx (p, \boldsymbol{\theta} + \lambda\tilde{\mathbf{v}}_j)$ for all $p \in \Gamma_j \subset \partial D^2$. We can similarly view M as $\frac{D^2 \times \mathbb{R}^n/\mathbb{Z}^n}{\sim}$ where $(p, \boldsymbol{\theta}) \sim (p, \boldsymbol{\theta} + \lambda\mathbf{v}_j)$. With these representations of our

T^n-manifolds we define the covering map

$$P\colon \frac{D^2 \times \mathbb{R}^n/\mathbb{Z}^n}{\approx} \to \frac{D^2 \times \mathbb{R}^n/\mathbb{Z}^n}{\sim}$$

given by $P(p, \boldsymbol{\theta}) = (p, Q(\boldsymbol{\theta}))$, where Q is an endomorphism of $\mathbb{R}^n/\mathbb{Z}^n$ given by

$$Q(\theta^1, \ldots, \theta^n) = (\theta^1, \ldots, \theta^{i-1}, q\theta^i, \theta^{i+1}, \ldots, \theta^n). \tag{2.92}$$

Since $\widetilde{M}$ is defined as a quotient space, we need to ensure this function is well defined. Note that Q is linear and that $Q(\tilde{\mathbf{v}}_j) = \mathbf{v}_j$ by construction. Letting $p \in \Gamma_j$ we see

$$\begin{aligned} P(p, \boldsymbol{\theta} + \lambda\tilde{\mathbf{v}}_j) &= (p, Q(\boldsymbol{\theta} + \lambda\tilde{\mathbf{v}}_j)) \\ &= (p, Q(\boldsymbol{\theta}) + \lambda Q(\tilde{\mathbf{v}}_j)) \\ &= (p, Q(\boldsymbol{\theta}) + \lambda \mathbf{v}_j) \\ &\sim (p, Q(\boldsymbol{\theta})) \\ &= P(p, \boldsymbol{\theta}), \end{aligned}$$

thus P is well-defined.

Lastly we need to show that $P\colon \widetilde{M} \to M$ is a covereing map, i.e. the deck transformations act properly discontinuously. To that end, consider a non-trivial element of the deck transformation group $\boldsymbol{\theta} \mapsto \boldsymbol{\theta} + \frac{d}{q}\mathbf{e}_i$ where $1 \leq d < q$. Suppose this fixes a point in $\widetilde{M}$. It clearly cannot fix a point on the interior of $\widetilde{M}/T^n$ and must instead fix a point on the boundary. If this sub-action fixes a point in Γ_j it will have to fix all points in the rod, and thus will also fix the points in $\Gamma_j \cap \Gamma_{j+1}$. Supposing that this action fixes a point p on the $\{\tilde{\mathbf{v}}_j, \tilde{\mathbf{v}}_{j+1}\}$ corner means that $(p, \boldsymbol{\theta}) \approx (p, \boldsymbol{\theta} + \frac{d}{q}\mathbf{e}_i)$. This can only happen when there exists rational numbers $\lambda, \mu \in \mathbb{Q}$ such that

$$\frac{d}{q}\mathbf{e}_i = \lambda\tilde{\mathbf{v}}_j, +\mu\tilde{\mathbf{v}}_{j+1} \in \mathbb{R}^n/\mathbb{Z}^n.$$

It is easier to work in the vector space $\mathbb{R}^n$ than the quotient space $\mathbb{R}^n/\mathbb{Z}^n$, so find a $\mathbf{w} \in \mathbb{Z}^n$ such that

$$\frac{d}{q}\mathbf{e}_i + \mathbf{w} = \lambda\tilde{\mathbf{v}}_j, +\mu\tilde{\mathbf{v}}_{j+1} \in \mathbb{R}^n.$$

Since λ and μ are rational numbers, there exists integers $a, b, c \in \mathbb{Z}$ such that

$$c\frac{d}{q}\mathbf{e}_i + c\mathbf{w} = a\tilde{\mathbf{v}}_j + b\tilde{\mathbf{v}}_{j+1}, \tag{2.93}$$

or equivalently

$$cd\mathbf{e}_i + cQ(\mathbf{w}) = a\mathbf{v}_j + b\mathbf{v}_{j+1}. \tag{2.94}$$

Looking at the second determinant divisor, we see that

$$\begin{aligned} b &= b\,\mathrm{Det}_2\{\mathbf{v}_j, \mathbf{v}_{j+1}\} \\ &= \mathrm{Det}_2\{\mathbf{v}_j, b\mathbf{v}_{j+1}\} \\ &= \mathrm{Det}_2\{\mathbf{v}_j, a\mathbf{v}_j + b\mathbf{v}_{j+1}\} \\ &= \mathrm{Det}_2\{\mathbf{v}_j, cd\mathbf{e}_i + cQ(\mathbf{w})\} \\ &= c\,\mathrm{Det}_2\{\mathbf{v}_j, d\mathbf{e}_i + Q(\mathbf{w})\}, \end{aligned}$$

thus $c|b$. Similarly, we can see that $c|a$, so without loss of generality, let $c = 1$. This means that $d\mathbf{e}_i + Q(\mathbf{w}) = a\mathbf{v}_j + b\mathbf{v}_{j+1} \in \mathrm{span}_{\mathbb{Z}}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{span}_{\mathbb{Z}}\{s_1\mathbf{e}_1, \ldots, s_l\mathbf{e}_l\}$. In particular, the i^{th} entry of $d\mathbf{e}_i + Q(\mathbf{w})$ is divisible by s_i. However $q|s_i$ combined with $s_i|d+qw^i$ gives $q|d+qw^i$, which contradicts our assumption of $1 \le d < q$. Therefore there are no fixed points and the q-to-1 map $P\colon \widetilde{M} \to M$ is indeed a covering map. $\square$

Corollary 2.58 (Theorem D, Part 3)**.** *Every simple T^n-manifold M^{n+2} has a torsion-free covering space $\widetilde{M}$ with a weakly equivariant covering map $(P, Q)\colon (\widetilde{M}, T^n) \to (M, T^n)$.*

Corollary 2.59. *Let M^{n+2} be a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Choose an integer $q > 1$ and an $i \in \{1, \dots, k\}$ to define*

$$\mathbf{v}'_j := (v^1_j, \dots, v^{i-1}_j, qv^i_j, v^{i+1}_j, \dots, v^n_j) \tag{2.95}$$

for all $j = 1, \dots, k$. The vectors $\{\mathbf{v}'_1, \dots, \mathbf{v}'_k\}$ define a simple T^n-orbifold

$$M' := \frac{M/T^n \times T^n}{\sim} \tag{2.96}$$

where $(p, \boldsymbol{\theta}) \sim (p, \boldsymbol{\theta} + \lambda \mathbf{v}'_i)$ for all $p \in \Gamma_i \subset \partial(M/T^n)$, $\boldsymbol{\theta} \in \mathbb{R}^n/\mathbb{Z}^n$, and $\lambda \in \mathbb{R}/\mathbb{Z}$. The space M is a q-fold orbifold covering of M' with orbifold covering map

$$P\colon M \to M' \tag{2.97}$$

$$(p, \theta^1, \dots, \theta^n) \mapsto (p, \theta^1, \dots, \theta^{i-1}, q\theta^i, \theta^{i+1}, \dots, \theta). \tag{2.98}$$

If each $\mathbf{v}'_j$ is a primitive vector, and if each $\{\mathbf{v}'_j, \mathbf{v}'_{j+1}\}$ is admissible whenever $\{\mathbf{v}_j, \mathbf{v}_{j+1}\}$ is, then M' is a simple T^n-manifold and $P\colon M \to M'$ is a covering map.

Proof. Consider the effective almost free $\mathbb{Z}_q$-action on M given by

$$m \cdot (p, \theta^1, \dots, \theta^n) = (p, \theta_1, \dots, \theta^{i-1}, \frac{m}{q} + \theta^i, \theta^{i+1}, \dots, \theta^n) \tag{2.99}$$

where $\frac{m}{q} \in \mathbb{Z}_q \cong (q^{-1}\mathbb{Z})/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$, $p \in M/T^n$, and $\theta^i \in \mathbb{R}/\mathbb{Z}$. Since $\mathbb{Z}_q$ acts effectively on M there exists a q-to-1 map $P\colon M \to M/\mathbb{Z}_q$. Furthermore $\mathbb{Z}_q$ is finite and thus acts both almost freely and properly discontinuously on M. Almost freeness guarantees that $M/\mathbb{Z}_q$ is an orbifold [5, Proposition 1.5.1], while proper discontinuity means $P\colon M \to M/\mathbb{Z}_q$ is an orbifold covering map [5, Example 2.3.2]. Lastly notice that $\mathbb{Z}_q \cong (q^{-1}\mathbb{Z})/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$ is a subgroup of the i^{th} circle in $T^n = \mathbb{R}/\mathbb{Z} \times \dots \mathbb{R}/\mathbb{Z}$ and is therefore a subgroup of T^n. Since $\mathbb{Z}_q \subset T^n$ is finite, its quotient space $M/\mathbb{Z}_q$ is automatically a simple T^n-orbifold by definition.

The topology of the quotient space can be described by the topology of M and the quotient map P. Using Corollary 2.15 we express M as

$$M \cong \frac{M/T^n \times T^n}{\sim_1} \tag{2.100}$$

where $(p, \boldsymbol{\theta}) \sim_1 (p, \boldsymbol{\theta} + \lambda \mathbf{v}_j)$ for all $p \in \Gamma_j \subset \partial(M/T^n)$, $\boldsymbol{\theta} \in \mathbb{R}^n/\mathbb{Z}^n$, and $\lambda \in \mathbb{R}/\mathbb{Z}$. The $\mathbb{Z}_q$ action described in Equation (2.99) naturally gives $M/\mathbb{Z}_q$ the quotient topology of

$$M/\mathbb{Z}_q \cong \left(\frac{M/T^n \times T^n}{\sim_1}\right) \Big/ \sim_2 \tag{2.101}$$

where $(p, \theta^1, \dots, \theta^n) \sim_2 (p, \theta^1, \dots, \theta^{i-1}, \theta^i + \frac{1}{q}, \theta^{i+1}, \dots, \theta^n)$ for all $p \in M/T^n$ and $\boldsymbol{\theta} \in \mathbb{R}^n/\mathbb{Z}^n$. Of course these two relations commute and we can see

$$M/\mathbb{Z}_q \cong \frac{M/(T^n/\sim_2) \times (T^n/\sim_2)}{\sim_1}. \tag{2.102}$$

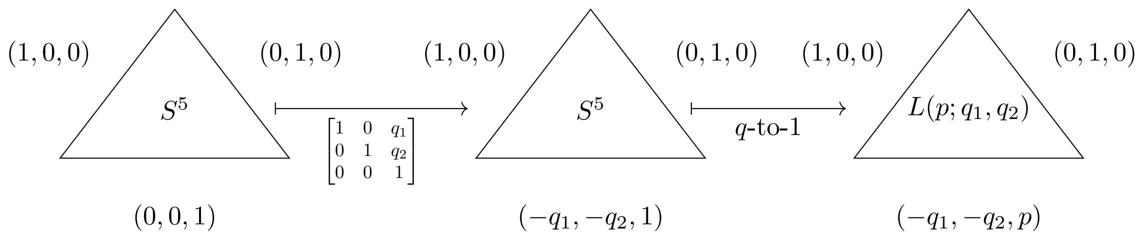


Figure 2.21: The transition from S^5 to $L(p; q_1, q_2)$

The relation $\sim_2$ comes from a $\mathbb{Z}_q$-action on the i^{th} circle in T^n. This means we can express T^n as a product of n copies of $\mathbb{R}/\mathbb{Z}$ to see that $(T^n/\sim_2)$ is a product of n copies of $\mathbb{R}/\mathbb{Z}$ where the i^{th} copy is replaces by $\mathbb{R}/q^{-1}\mathbb{Z}$. There is a natural homomorphism $\mathbb{R}/q^{-1}\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $\theta \mapsto q\theta$. This defines a natural homomorphism $(T^n/\sim_2) \to T^n$ via $(\theta^1, \dots, \theta^n) \mapsto (\theta^1, \dots, \theta^{i-1}, q\theta^i, \theta^{i+1}, \dots, \theta)$ where $\theta^j \in \mathbb{R}/\mathbb{Z}$ for $j \neq i$ and $\theta^i \in \mathbb{R}/q^{-1}\mathbb{Z}$. This gives us the expression

$$M/\mathbb{Z}_q \cong \frac{M/T^n \times T^n}{\sim_1} \tag{2.103}$$

Where $(p, \theta^1, \dots, q\theta^i, \dots, \theta^n) \sim_1 (p, \theta^1 + \lambda v_j^1, \dots, q(\theta^i + \lambda v_j^i), \dots, \theta^n + \lambda v_j^n)$ for all $p \in \Gamma_j$, $\theta^l \in \mathbb{R}/\mathbb{Z}$ for $l \neq i$, and $\theta^i \in \mathbb{R}/q^{-1}\mathbb{Z}$. Finally we replace $q\theta^i \in q(\mathbb{R}/q^{-1}\mathbb{Z})$ with $\theta^i \in \mathbb{R}/\mathbb{Z}$ and denote the relation by $\sim$ so that

$$M/\mathbb{Z}_q \cong \frac{M/T^n \times T^n}{\sim} \tag{2.104}$$

where $(p, \boldsymbol{\theta}) \sim (p, \boldsymbol{\theta} + \lambda \mathbf{v}'_j)$ for all $p \in \Gamma_j$, $\boldsymbol{\theta} \in (\mathbb{R}/\mathbb{Z})^n$, and $\lambda \in \mathbb{R}/\mathbb{Z}$ as desired.

Let $M' := M/\mathbb{Z}_q$. We now see that M' is a simple T^n-orbifold defined by the vectors $\{\mathbf{v}'_1, \dots, \mathbf{v}'_k\}$ with a q-fold orbifold covering map $P\colon M \to M/\mathbb{Z}_q$, just as required. The last thing to check are the conditions under which M' is a manifold. First we must assume that each vector $\mathbf{v}'_j \in \mathbb{Z}^n$ is primitive and therefore agree with the definition of rod structures as primitive vectors. The only other thing to check is that M' is a manifold at the corners, i.e. a tubular neighborhood of each corner is homeomorphic to $T^{n-2} \times B^4$. This is equivalent to the rod structures flanking the corner forming an admissible pair. Since M is a manifold, whenever there is a corner at $\Gamma_j \cap \Gamma_{j+1} \subset M/T^n$ the rod structures $\{\mathbf{v}_j, \mathbf{v}_{j+1}\}$ must be admissible. Therefore if we assume that $\{\mathbf{v}'_j, \mathbf{v}_{j+1}\}$ is admissible whenever $\{\mathbf{v}_j, \mathbf{v}_{j+1}\}$ is, we know that M' has admissible rod structures at every corner. This means that M' must be a manifold and is therefore a simple T^n-manifold. □

Example 2.60: Corollary 2.59 can be used to construct the 5-dimensional lens space $L(p; q_1, q_2)$ as a simple T^3-manifold. We begin with a model for $S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \subset \mathbb{C}^3$. Next consider the free S^1 action on S^5 defined by rotating the third circle once and the first and second circles q_1 and q_2 times respectively, that is $\theta \cdot (z_1, z_2, z_3) := (e^{iq_1\theta} z_1, e^{iq_2\theta} z_2, e^{i\theta} z_3)$. The lens space $L(p; q_1, q_2)$ is defined to be the quotient of S^5 by the discrete subgroup $\mathbb{Z}_p \subset S^1$. This S^1 is itself an embedded subgroup of T^3 which acts effectively on S^5. By choosing the 'standard' coordinates on $T^3 \cong \mathbb{R}^3/\mathbb{Z}^3$ (as seen in Example 2.2), S^5 has rod structures $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $L(p; q_1, q_2)$ is the quotient of a discrete subgroup of the $(q_1, q_2, 1)\mathbb{R}/\mathbb{Z} \subset \mathbb{R}^3/\mathbb{Z}^3$ subtorus action. Of course we can choose a different set of coordinates on T^3. Consider the matrix $U \in SL(3, \mathbb{Z})$ which sends $U(\mathbf{e}_1) = \mathbf{e}_1$, $U(\mathbf{e}_2) = \mathbf{e}_2$, $U(\mathbf{e}_3) = (-q_1, -q_2, 1)$, and $U(q_1, q_2, 1) = \mathbf{e}_3$. This linear transformation gives S^5 the rod structures of $\{(1, 0, 0), (0, 1, 0), (-q_1, -q_2, 1)\}$ and shows that $L(p; q_1, q_2)$ is the quotient of a discrete subgroup of the $\mathbf{e}_3\mathbb{R}/\mathbb{Z} \subset \mathbb{R}^3/\mathbb{Z}^3$ subtorus action. With these rod structures we can apply Corollary 2.59 and see that S^5 is a q-fold cover of $L(p; q_1, q_2)$ with rod structures $\{(1, 0, 0), (0, 1, 0), (-q_1, -q_2, p)\}$.

2.7 Subtorus Actions

One useful fact about higher dimensional simple T^n-manifolds is that they admit *almost free* circle actions. Recall that an action $S^1 \hookrightarrow iso(M)$ is almost free if for any point $p \in M$ the stabilizer subgroup $\text{stab}(p) \subset S^1$ is finite.

Lemma 2.61. *Let M^{n+2} be a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Let $\mathbf{u} \in \mathbb{Z}^n$ be a primitive vector such that*

$$\mathbf{u} \notin \text{span}_{\mathbb{Z}}\{\mathbf{v}_i, \mathbf{v}_{i+1}\} \tag{2.105}$$

for all $i = 1, \dots, k$. The subtorus action $\mathbf{u}\mathbb{R}/\mathbb{Z} \subset \mathbb{R}^n/\mathbb{Z}^n \cong T^n$ on M is almost free, the quotient space $M' := M/S^1$ is a simple T^{n-1}-orbifold, and the quotient map $P\colon M \to M'$ is weakly equivariant. Furthermore if $Q\colon \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ is a surjective homomorphism with $\mathbf{u} \in \ker(Q)$, then $\{Q(\mathbf{v}_1), \dots, Q(\mathbf{v}_k)\}$ are rod structures for M'.

Proof. Note that every vector $\mathbf{u} \in \mathbb{Z}^n$ defines a subtorus action $\mathbf{u}\mathbb{R}/\mathbb{Z}$ on M. Any such action clearly preserves the fibers of the map $\pi\colon M \to M/T^n$. Thus the subtorus action is almost free if and only if it acts freely or almost freely on each of the fibers. Since any subtorus action acts freely on fibers over interior points, we must only check if $\mathbf{u}\mathbb{R}/\mathbb{Z}$ acts almost freely on $\pi^{-1}(\Gamma_i)$ and $\pi^{-1}(\Gamma_i \cap \Gamma_{i+1})$.

To determine the isotropy subgroup of $\mathbf{u}\mathbb{R}/\mathbb{Z}$ acting on $\pi^{-1}(\Gamma_i)$, let $p \in \Gamma_i$, $\boldsymbol{\phi}_0 \in \mathbb{R}^n/\mathbb{Z}^n$ and $t \in [0,1)$. The action is defined as $t \cdot (p, \boldsymbol{\phi}_0) = (p, \boldsymbol{\phi}_0 + t\mathbf{u})$. Since $p \in \Gamma_i$, we know that $(p, \boldsymbol{\phi}_0 + t\mathbf{u}) \sim (p, \boldsymbol{\phi}_0 + t\mathbf{u} + \lambda\mathbf{v}_i)$ for any $\lambda \in \mathbb{R}$. Therefore the action by $t \in \mathbb{R}/\mathbb{Z}$ fixes a point if and only if there exists a $\lambda \in \mathbb{R}$ such that $t\mathbf{u} + \lambda\mathbf{v}_i = \mathbf{0} \in \mathbb{R}^n/\mathbb{Z}^n$, or equivalently if there exists a $\mathbf{w} \in \mathbb{Z}^n$ such that

$$t\mathbf{u} + \mathbf{w} \in \text{span}_{\mathbb{R}}\{\mathbf{v}_i\}. \tag{2.106}$$

Similarly the action of $t \in \mathbb{R}/\mathbb{Z}$ on $\pi^{-1}(\Gamma_i \cap \Gamma_{i+1})$ fixes a point if and only if there exists a $\mathbf{w} \in \mathbb{Z}^n$ such that

$$t\mathbf{u} + \mathbf{w} \in \text{span}_{\mathbb{R}}\{\mathbf{v}_i, \mathbf{v}_{i+1}\}. \tag{2.107}$$

Of course the only way the action $\mathbf{u}\mathbb{R}/\mathbb{Z}$ can fail to be almost free is if there is an $i \in \{1, \dots, k\}$ and a $\mathbf{w} \in \mathbb{Z}^n$ such that either Equation (2.106) or (2.107) is satisfied for all t. This can only happen when $\mathbf{w} = \mathbf{0} \in \mathbb{Z}^n$ and thus only happens when $\mathbf{u} \in \text{span}_{\mathbb{R}}\{\mathbf{v}_i, \mathbf{v}_{i+1}\}$ for some i. By hypothesis $\mathbf{u}$ satisfies Equation (2.105) and therefore must always induce an almost free subtorus action.

Since the action is almost free, the quotient space $M' = M'/S^1$ is automatically an orbifold [5, Proposition 1.5.1]. It also inherits the T^n-action from M with the change that the stabilizer group $\text{stab}(p')$ for every point in $p' \in M'$ now contains the subgroup $\mathbf{u}\mathbb{R}/\mathbb{Z}$. By construction $\mathbf{u}\mathbb{R}/\mathbb{Z} \cap \text{stab}(p) = \{0\}$ for any point $p \in M$ and thus $\text{stab}(p') = \text{stab}(p) \oplus \mathbf{u}\mathbb{R}/\mathbb{Z}$ when p' is the image of p. Now consider the linear surjection $Q\colon \mathbb{Z}^n \to \mathbb{Z}^n/\mathbf{u}\mathbb{Z} \cong \mathbb{Z}^{n-1}$. Observe that it induces a Lie group surjective homomorphism $Q\colon \mathbb{R}^n/\mathbb{Z}^n \to (\mathbb{R}^n/\mathbb{Z}^n)/(\mathbf{u}\mathbb{R}/\mathbb{Z})$. In particular this has the property that $\text{stab}(p') = Q(\text{stab}(p))$. We can now describe the quotient space as

$$M' \cong \frac{M/T^n \times Q(\mathbb{R}^n/\mathbb{Z}^n)}{\sim} \tag{2.108}$$

where $(p, Q(\boldsymbol{\theta})) \sim (p, Q(\boldsymbol{\theta}) + \lambda Q(\mathbf{v}_j)) = (p, Q(\boldsymbol{\theta} + \lambda\mathbf{v}_j))$ for all $p \in \Gamma_j$, $\boldsymbol{\theta} \in \mathbb{R}^n/\mathbb{Z}^n$, and $\lambda \in \mathbb{R}/\mathbb{Z}$. Observe that $Q(\mathbb{R}^n/\mathbb{Z}^n) \cong \mathbb{R}^{n-1}/Z^{n-1}$ and thus the T^n-action on M descends to a T^{n-1}-action on M'. Moreover the action is effective, the quotient space $M'/T^{n-1} \cong M/T^n$ is contractible, and quotient map $M' \to M'/T^{n-1}$ defines a trivial fiber bundle over the interior. Hence M' is a simple T^{n-1}-orbifold. Since for each j the stabilizer subgroup $\text{stab}(\Gamma_j) = \text{span}_{\mathbb{Z}}\{Q(\mathbf{v}_j)\}$ is not trivial we conclude that $\{Q(\mathbf{v}_1), \dots, Q(\mathbf{v}_k)\}$ is the set of rod structures for M'. However since M' is in general only a simple T^n-orbifold, these rod structures need not be primitive nor satisfy admissibility.

The presentation of M' above allows us to express the quotient map $P\colon M \to M'$ as $P(p, \boldsymbol{\theta}) = (p, Q(\boldsymbol{\theta}))$. In this form it is clear that P is weakly equivariant since $Q\colon T^n \to T^{n-1}$ is a Lie group homomorphism and thus $Q(\mathbf{t}+\boldsymbol{\theta}) = Q(\mathbf{t}) + Q(\boldsymbol{\theta})$ for all $\mathbf{t}, \boldsymbol{\theta} \in T^n$. In particular letting $\mathbf{p} \in M$ denote $(p, \boldsymbol{\theta})$ we see $P(\mathbf{t} \cdot \mathbf{p}) = P(p, \mathbf{t} + \boldsymbol{\theta}) = (p, Q(\mathbf{t}) + Q(\boldsymbol{\theta})) = Q(\mathbf{t}) \cdot P(\mathbf{p})$. □

Remark 2.62. For any simple T^n-manifold M^{n+2} with $n > 2$ there always exists infinitely many primitive vectors $\mathbf{u} \in \mathbb{Z}^n$ such that $\mathbf{u}\mathbb{R}/\mathbb{Z}$ defines an almost free subtorus action on M. This is because the number of primitive vectors in $B_r(0) \cap \mathbb{Z}^n$ grows at a rate of $O(r^n)$ while the number of vectors in $B_r(0) \cap \operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_i, \mathbf{v}_{i+1}\}$ grows at a rate of $O(r^2)$. Thus for large enough $r > 0$ there always exists a primitive $\mathbf{u} \in \mathbb{Z}^n$ which satisfies Equation (2.105) for all i. In fact, exists $n-2$ linearly independent primitive vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_{n-2}\}$ which generated an almost free subtorus action of $\mathbf{u}_1\mathbb{R}/\mathbb{Z} \times \cdots \times \mathbf{u}_{n-2}\mathbb{R}/\mathbb{Z}$.

Even though almost free actions are plentiful we will later see with Lemma 2.66 that free actions are not. One useful way to construct free actions is with the following corollary.

Corollary 2.63. *Let M^{n+2} be a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Any collection of vectors $\{\mathbf{w}_1, \dots, \mathbf{w}_k\} \subset \mathbb{Z}^l$ defines a simple T^{n+l}-manifold $\widetilde{M}^{n+l+2}$ with rod structures $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_k\} \subset \mathbb{Z}^{n+l}$ where*

$$\tilde{\mathbf{v}}_i := (\mathbf{v}_i, \mathbf{w}_i). \tag{2.109}$$

The manifold $\widetilde{M}$ admits a free $T^l \subset T^{n+l}$ subtorus action of

$$T^l \cong \mathbf{e}_{n+1}\mathbb{R}/\mathbb{Z} \times \cdots \times \mathbf{e}_{n+l}\mathbb{R}/\mathbb{Z} \subset \mathbb{R}^{n+l}/\mathbb{Z}^{n+l} \cong T^{n+l}. \tag{2.110}$$

The quotient space is M and the quotient map $P\colon \widetilde{M} \to M$ is

$$P(p, \theta^1, \dots, \theta^{n+l}) = (p, \theta^1, \dots, \theta^n). \tag{2.111}$$

Proof. The first step of the proof is to construct the space $\widetilde{M}$. Let

$$\widetilde{M} := \frac{M/T^n \times T^{n+l}}{\sim} \tag{2.112}$$

where $(p, \boldsymbol{\theta}) \sim (p, \boldsymbol{\theta} + \lambda\tilde{\mathbf{v}}_i)$ for all $p \in \Gamma_i \subset \partial(M/T^n)$, $\boldsymbol{\theta} \in \mathbb{R}^{n+l}/\mathbb{Z}^{n+l}$, and $\lambda \in \mathbb{R}$. Now we must show that $\widetilde{M}$ is indeed a simple T^{n+l}-manifold, that is $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_k\}$ are admissible rod structures. Note that each vector is primitive since $\gcd(v_i^1, \dots, v_i^n, w_i^1, \dots, w_i^l)$ divides $\gcd(\mathbf{v}_i^1, \dots, \mathbf{v}_i^n) = 1$. Similarly the fact that $\det_2(\tilde{\mathbf{v}}_i, \tilde{\mathbf{v}}_j)$ divides $\det_2(\mathbf{v}_i, \mathbf{v}_j)$ shows that $\{\tilde{\mathbf{v}}_i, \tilde{\mathbf{v}}_j\}$ is admissible whenever $\{\mathbf{v}_i, \mathbf{v}_j\}$ is. Therefore $\widetilde{M}$ is a simple T^{n+l}-manifold with rod structures $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_k\}$.

Next let us show that the subtorus action defined by $\operatorname{span}_{\mathbb{R}}\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{n+l}\}/\operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{n+l}\}$ is in fact free. To do this we will show that the subtorus action of $\mathbf{u}\mathbb{R}/\mathbb{Z} \subset \mathbb{R}^{n+l}/\mathbb{Z}^{n+l}$ is free for any primitive vector $\mathbf{u} \in \operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{n+l}\}$. Choose $\mathbf{u} \in \operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{n+l}\}$ and assume by contradiction that such an action is not free. In particular this means there exists an $i \in \{1, \dots, k\}$, a vector $\mathbf{z} \in \mathbb{Z}^{n+l}$, and integers $1 < d < q$ such that

$$\frac{d}{q}\mathbf{u} + \mathbf{z} \in \operatorname{span}_{\mathbb{R}}\{\tilde{\mathbf{v}}_i, \tilde{\mathbf{v}}_{i+1}\}. \tag{2.113}$$

Following the proof of Theorem 2.59 this gives relatively prime integers $a, b, c \in \mathbb{Z}$ such that

$$c\frac{d}{q}\mathbf{u} + c\mathbf{z} = a\tilde{\mathbf{v}}_i + b\tilde{\mathbf{v}}_{i+1}. \tag{2.114}$$

Let $\mathbf{x} \in \mathbb{Z}^n$, $\mathbf{y} \in \mathbb{Z}^l$ so that $\mathbf{z} = (\mathbf{x}, \mathbf{y})$. We now split up Equation (2.114) into two parts

$$c\mathbf{x} = a\mathbf{v}_i + b\mathbf{v}_{i+1} \tag{2.115}$$

$$c\frac{d}{q}\mathbf{u} + c\mathbf{y} = a\mathbf{w}_i + b\mathbf{w}_i. \tag{2.116}$$

Rearranging terms we see that

$$cd\mathbf{u} = q(a\mathbf{w}_i + b\mathbf{w}_i - c\mathbf{y}). \tag{2.117}$$

We know that q cannot divide d since $1 < d < q$. We also know that q cannot divide each term of $\mathbf{u} = (u^1, \dots, u^l)$ since $\mathbf{u}$ is primitive. Thus q must divide c. This means there exists an $m \in \mathbb{Z}$ such that $c = mq$. This allows us to express b in the following way

$$\begin{aligned} b &= b\,\mathrm{Det}_2(\mathbf{v}_i, \mathbf{v}_{i+1}) \\ &= \mathrm{Det}_2(\mathbf{v}_i, b\mathbf{v}_{i+1}) \\ &= \mathrm{Det}_2(\mathbf{v}_i, a\mathbf{v}_i + b\mathbf{v}_{i+1}) \\ &= \mathrm{Det}_2(\mathbf{v}_i, c\mathbf{x}) \\ &= q\,\mathrm{Det}_2(\mathbf{v}_i, m\mathbf{x}). \end{aligned}$$

Hence q divides b. By a similar argument we can see that q divides a as well. We have now reached a contradiction since $q > 1$ divides a, b, and c which are relatively prime. Therefore the subtorus action must be free.

This subtorus action freely rotates the last l circles of $\widetilde{M} \cong (M/T^n \times T^{n+1})/\sim$ while leaving the first n circles untouched. Hence the projection map $P\colon \widetilde{M} \to \widetilde{M}/T^l$ is described by

$$P(p, \theta^1, \dots, \theta^{n+l}) = (p, \theta^1, \dots, \theta^n). \tag{2.118}$$

The last step of this proof is to show that the quotient space $\widetilde{M}/T^l$ is indeed M. By repeated applications of Lemma 2.61 we can see that $\widetilde{M}/T^l$ is a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Finally we use the face that $\widetilde{M}/T^{n+l} \cong M/T^n$ to see

$$\widetilde{M}/T^l \cong \left(\frac{M/T^n \times T^{n+l}}{\sim}\right)\Big/ T^l \tag{2.119}$$

$$\cong \frac{M/T^n \times (T^{n+l}/T^l)}{\sim} \tag{2.120}$$

$$\cong \frac{M/T^n \times T^n}{\sim} \tag{2.121}$$

where $\sim$ in the first line is using the rod structures $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_k\}$ and $\sim$ in the last two lines is using the rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. This is the standard model of M as described in Corollary 2.15, thus $\widetilde{M}/T^l \cong M$ as desired. □

Example 2.64: Consider $\mathbb{CP}^2$ with rod structures $\{(1,0),(0,1),(1,1)\}$ as in Example 2.29. Using Corollary 2.63 with $l = 1$ let $\mathbf{w}_1 = 0$, $\mathbf{w}_2 = 0$, and $\mathbf{w}_3 = 1$. This constructs the simple T^3-manifold $\widetilde{M}$ with rod structures $\{(1,0,0),(0,1,0),(1,1,1)\}$ and defines the projection map $P\colon \widetilde{M} \to \mathbb{CP}^2$ as $P(p, \boldsymbol{\theta}) = (p, Q(\boldsymbol{\theta}))$ where $Q\colon \mathbb{R}^3/\mathbb{Z}^3 \to (\mathbb{R}^3/\mathbb{Z}^3)/(\mathbf{e}_3\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}^2/\mathbb{Z}^2$, or in coordinates

$$Q(\theta^1, \theta^2, \theta^3) = (\theta^1, \theta^2). \tag{2.122}$$

This projection map is depicted by $\mathrm{proj}_{\{(0,0,1)\}^\perp}$ in Figure 2.22. Now apply the change of coordinates $U \in SL(3, \mathbb{Z})$,

$$U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.123}$$

which sends $\{(1,0,0),(0,1,0),(1,1,1)\}$ to its Hermite normal form $\{(1,0,0),(0,1,0),(0,0,1)\}$. These are the standard rod structures for S^5 as seen in Example, thus $\widetilde{M} \cong S^5$. The free S^1-action on $\widetilde{M}$ generated by $\mathbf{e}_3$ in the old coordinate system now becomes a free S^1-action generated by $U(\mathbf{e}_3) = (-1,-1,1)$ on S^5 in the new coordinate system. Similarly the projection map $P\colon S^5 \to \mathbb{CP}^2$ is now described by $P(p,\boldsymbol{\theta}) = (p, Q'(\boldsymbol{\theta}))$ where $Q'\colon \mathbb{R}^3/\mathbb{Z}^3 \to (\mathbb{R}^3/\mathbb{Z}^3)/((-1,-1,1)\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}^2/\mathbb{Z}^2$, or in coordinates

$$Q'(\theta^1,\theta^2,\theta^3) = (\theta^1-\theta^3, \theta^2-\theta^3). \tag{2.124}$$

This projection map is depicted by $\mathrm{proj}_{\{(-1,-1,1)\}^\perp}$ in Figure 2.22.

Example 2.65: In this example we will show that the T^n-manifold $M(n,n)$ described in Conjecture A is the total space of a principal T^{n-2}-bundle over $\#(\frac{n-4}{2})(S^2\times S^2)$ if n is even or $\mathbb{CP}^2\#\lfloor\frac{n-4}{2}\rfloor(S^2\times S^2)$ if n is odd. From Theorem 2.50 we know that $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ are rod structures for $M(n,n)$. We now define the rod structures

$$\tilde{\mathbf{v}}_i := \begin{cases} \mathbf{e}_i & i \le 2 \\ \mathbf{e}_i + \mathbf{e}_1 & 2 < i < n, i \equiv 1 \mod 2 \\ \mathbf{e}_i + \mathbf{e}_2 & 2 < i, i \equiv 0 \mod 2 \\ \mathbf{e}_i + \mathbf{e}_1 + \mathbf{e}_2 & i = n \equiv 1 \mod 2 \end{cases} \tag{2.125}$$

By construction $\{\tilde{\mathbf{v}}_1,\dots,\tilde{\mathbf{v}}_n\} \subset \mathbb{Z}^n$ is a primitive set, and in particular there exists a $U \in SL(n,\mathbb{Z})$ such that $U(\mathbf{e}_i) = \tilde{\mathbf{v}}_i$ making $\{\tilde{\mathbf{v}}_1,\dots,\tilde{\mathbf{v}}_n\}$ rod structures for $M(n,n)$. Now for each i define $\mathbf{v}_i \in \mathbb{Z}^2$ and $\mathbf{w}_i \in \mathbb{Z}^{n-2}$ by the equation

$$\tilde{\mathbf{v}}_i = (\mathbf{v}_i, \mathbf{w}_i). \tag{2.126}$$

From Equation (2.125) we see that $\{\mathbf{v}_1,\dots,\mathbf{v}_n\} = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_2,\dots,\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2\} \subset \mathbb{Z}^2$, where the last vector is only equal to $\mathbf{e}_1+\mathbf{e}_2$ is n is odd. Observe that $\{\mathbf{v}_1,\dots,\mathbf{v}_n\} \subset \mathbb{Z}^2$ is an admissible set of rod structures. Using these rod structure we define the T^2-manifold M. Now apply Corollary 2.63 to see that $M(n,n)$ admits a free T^{n-2} subtorus action with quotient M. This is equivalent to saying $M(n,n)$ is the total space of a principal T^{n-2}-bundle over M. Finally we use the Orlik and Raymond classification theorem, Theorem 2.34, to show $M \cong \#(\frac{n-4}{2})(S^2\times S^2)$ if n is even or $\mathbb{CP}^2\#\lfloor(n-4)/2\rfloor(S^2\times S^2)$ as desired.

Lemma 2.66. *Let M^{n+2} be a simple T^n-manifold with $n > 2$ and with rod structures $\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$. For each $i = 1,\dots,k$ such that the $\Gamma_i \cap \Gamma_{i+1}$ corner exists, define a linear surjection*

$$P_i\colon \mathbb{Z}^n \to \frac{\mathbb{Z}^n}{\mathrm{span}_{\mathbb{Z}}\{\mathbf{v}_i,\mathbf{v}_{i+1}\}} \cong \mathbb{Z}^{n-2}. \tag{2.127}$$

Let $\mathbf{u} \in \mathbb{Z}^n$ be a primitive vector that defines an almost free subtorus action $\mathbf{u}\mathbb{R}/\mathbb{Z} \subset \mathbb{R}^n/\mathbb{Z}^n$ on M. This action is free if and only if $P_i(\mathbf{u}) \in \mathbb{Z}^{n-2}$ is primitive for all $i = 1,\dots,k$.

Proof. First assume that the action $\mathbf{u}\mathbb{R}/\mathbb{Z}$ on M is only almost free. This means there exists an integer $m > 1$ where the finite subgroup $\mathbb{Z}_m \subset \mathbf{u}\mathbb{R}/\mathbb{Z} \cong S^1$ fixes a point $(p,\boldsymbol{\phi}_0)$ in $\pi^{-1}(\Gamma_i)$ or $\pi^{-1}(\Gamma_i\cap\Gamma_{i+1})$ for some i. Without loss of generality assume that the point $(p,\boldsymbol{\phi}_0) \in \pi^{-1}(\Gamma_i\cap\Gamma_{i+1})$ is fixed by the subgroup $\mathbb{Z}_m \cong (\frac{1}{m}\mathbb{Z})/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$. This means that $(p,\boldsymbol{\phi}_0) \sim \frac{1}{m}\cdot(p,\boldsymbol{\phi}_0) = (p, \boldsymbol{\phi}_0 + \frac{1}{m}\mathbf{u})$. We see from Equation (2.107) that this means there exists a $\mathbf{w} \in \mathbb{Z}^n$ such that $\mathbf{u} + m\mathbf{w} \in \mathrm{span}_{\mathbb{R}}\{\mathbf{v}_i,\mathbf{v}_{i+1}\}$, or in particular

$$P_i(\mathbf{u}) = -mP_i(\mathbf{w}). \tag{2.128}$$

Therefore $P_i(\mathbf{u})$ is not primitive.

Now assume that there exists an i such that $P_i(\mathbf{u})$ is not primitive. In this case there exists $\mathbf{w}' \in \mathbb{Z}^{n-2}$ and an integer $m > 1$ such that $P_i(\mathbf{u}) = m\mathbf{w}'$. Choose $\mathbf{w} \in P_i^{-1}(\mathbf{w}')$ and we see that $P_i(\mathbf{u} - m\mathbf{w}) = 0$, thus satisfying Equation (2.107). Therefore the action is not free. $\square$

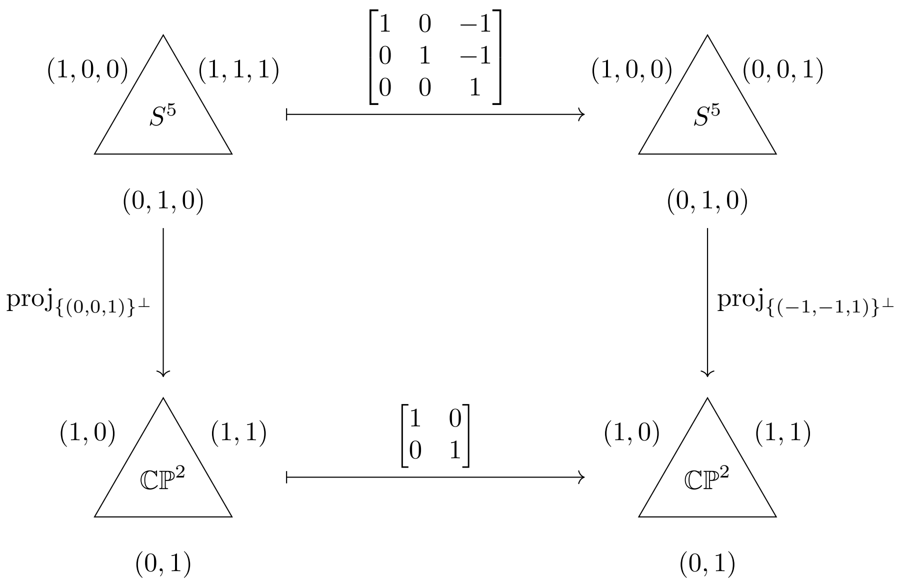


Figure 2.22: The diagram above depicts the process described in Example 2.64. By presenting the projection maps $S^5 \to \mathbb{CP}^2$ as 'collapsing' certain vectors, we can see a clear relation between the rod structures and the vector that generates the circle action. In particular, letting $\tilde{\mathbf{v}}_i$ and $\mathbf{v}_i$ be rod structures for S^5 and $\mathbb{CP}^2$ respectively, and letting $\mathbf{u}$ be the vector that describes the projection map we see that for all i there exists a $\lambda_i \in \mathbb{Z}$ such that $\tilde{\mathbf{v}}_i + \lambda \mathbf{u} = (\mathbf{v}_i, 0)$.

Corollary 2.67. *Let M^{n+2} be a simple T^n-manifold with with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and let $E \subset \{1, \dots, k\}$ denote the set for which the corner $\Gamma_i \cap \Gamma_{i+1}$ exists. For each $i \in E$ define a linear surjection*

$$P_i \colon \mathbb{Z}^n \to \frac{\mathbb{Z}^n}{\mathrm{span}_{\mathbb{Z}}\{\mathbf{v}_i, \mathbf{v}_{i+1}\}} \cong \mathbb{Z}^{n-2}. \tag{2.129}$$

Any primitive set $\{\mathbf{u}_1, \dots, \mathbf{u}_l\} \subset \mathbb{Z}^n$ defines an almost free subtorus action of $T^l \subset T^n$ on M if

$$\mathrm{span}_{\mathbb{Z}}\{\mathbf{u}_1, \dots, \mathbf{u}_l\} \cap \mathrm{span}_{\mathbb{Z}}\{\mathbf{v}_i, \mathbf{v}_{i+1}\} = \{0\} \tag{2.130}$$

for all i. This action is free if $\{P_i(\mathbf{u}_1), \dots, P_i(\mathbf{u}_l)\}$ is a primitive set for all i, that is if

$$\mathrm{Det}_l\{P_i(\mathbf{u}_1), \dots, P_i(\mathbf{u}_l)\} = 1 \tag{2.131}$$

for all i.

The following example shows that simple T^n-manifolds which admit almost free subtorus actions need not admit free subtorus action.

Example 2.68: Consider the 5-dimensional lens space $L(p; q_1, q_2)$ as described in Example 2.60 and shown in Figure **??**. We will show that not every lens space admits free subtorus actions. The argument follows a similar approach to that used in [50] to distinguish the homeotype of higher dimensional lens spaces. Let $\mathbf{u} \in \mathbb{Z}^3$ be a primitive vector which describes an almost free action on $L(p; q_1, q_2)$. Applying Lemma 2.66 we construct the maps P_i as

$$P_1(\mathbf{u}) = u^3 \tag{2.132}$$
$$P_2(\mathbf{u}) = pu^1 + q_1 u^3 \tag{2.133}$$
$$P_3(\mathbf{u}) = pu^2 + q_2 u^3. \tag{2.134}$$

We now assert that $P_i(\mathbf{u})$ is primitive. Note that $\pm 1 \in \mathbb{Z}$ are the only primitive elements in $\mathbb{Z}$ which gives us a manageable system of equations, namely $P_i(\mathbf{u}) = \pm_i 1$. This reduces to

$$u^1 = \frac{\pm_2 1 \mp_1 q_1}{p} \tag{2.135}$$

$$u^2 = \frac{\pm_3 1 \mp_1 q_2}{p} \tag{2.136}$$

$$u^3 = \pm_1 1. \tag{2.137}$$

The fact that $\mathbf{u}$ must be an integer vector mean $L(p; q_1, q_2)$ admits a free subtorus action if and only if

$$q_1 \equiv \pm_2 1 \mod p \tag{2.138}$$

$$q_2 \equiv \pm_3 1 \mod p \tag{2.139}$$

for some choice of $\pm_2$ and $\pm_3$. In particular $L(5; 4, 1)$ admits the free circle action coming from $\mathbf{u} = (-1, 0, 1)$, while $L(5; 3, 1)$ does not admit any free subtorus action.

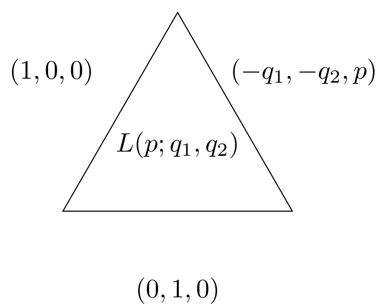


Figure 2.23: Above we see the rod diagram for the 5-dimensional lens space $L(p; q_1, q_2)$.

The existence of free and almost free subtorus actions on simple T^n-manifolds has several immediate consequences. We will list them as propositions here. Most of these results are known, however the proofs presented here are simpler than the ones cited.

This first proposition discusses the *toral rank conjecture*, generally attributed to S. Halperin, (see [14, pg. 271] and [14, Remark 7.11]) which states that any closed manifold M which admits an almost free torus action of rank r satisfies the inequality

$$\dim(M; \mathbb{Q}) \geq \dim(T^r; \mathbb{Q}) = 2^r. \tag{2.140}$$

As of the date of this dissertation, this conjecture is unsolved in the general case and remains one of the most prominent questions in toric topology.

Proposition 2.69. *Assuming Conjecture A is true, any simply connected T^n-manifolds M^{n+2} satisfies the toral rank conjecture. Specifically M admits an almost free action of T^{n-2} and the sum of the Betti numbers of M is*

$$\dim(H(M; \mathbb{Q})) = 4 + (k-4)2^{n-2} \tag{2.141}$$

where $k \geq n$ is the number of rods.

Proof. Conjecture A asserts that $M \cong M(n, k)$. The proof of this proposition is a simple calculation of the total Betti numbers of $M(n, k)$. To avoid repeated computations, we use Lemma 2.86 which computes the

Betti numbers of $M(n,k)$ as

$$b_j = \begin{cases} 1 & j = 0, n+2 \\ 0 & j = 1, n+1 \\ (k-2)\binom{n-2}{j-2} - \binom{n-2}{j-3} - \binom{n-2}{j-1} & 2 \le j \le n. \end{cases} \tag{2.142}$$

Now observe the following computations.

$$\begin{aligned}
\dim(H(M;\mathbb{Q})) &= \sum_{j=0}^{n+2} b_j && (2.143)\\
&= 2 + \sum_{j=2}^{n} b_j && (2.144)\\
&= 2 + \sum_{j=2}^{n} \left((k-2)\binom{n-2}{j-2} - \binom{n-2}{j-3} - \binom{n-2}{j-1} \right) && (2.145)\\
&= 2 + \sum_{j=0}^{n-2} \left((k-2)\binom{n-2}{j} - \binom{n-2}{j-1} - \binom{n-2}{j+1} \right) && (2.146)\\
&= 2 + \binom{n-2}{n-2} + \binom{n-2}{0} + \sum_{j=0}^{n-2} \left((k-2)\binom{n-2}{j} - \binom{n-2}{j} - \binom{n-2}{j} \right) && (2.147)\\
&= 4 + (k-2)2^{n-2} - 2 \cdot 2^{n-2} && (2.148)\\
&= 4 + (k-4)2^{n-2} && (2.149)
\end{aligned}$$

□

This next proposition is a special case of [14, Theorem 7.33], however with a much simpler proof.

Proposition 2.70. *Let M^{n+2} be a closed simple T^n-manifold with k rods. Its Euler characteristic is*

$$\chi(M) = \begin{cases} k & n = 2 \\ 0 & n > 2. \end{cases} \tag{2.150}$$

Proof. For the $n = 2$ case let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be rod structures for M. Choose a primitive vector $\mathbf{u} \in \mathbb{Z}^2 \setminus \bigcup_{i=1}^{k} \mathrm{span}_{\mathbb{Z}}\{\mathbf{v}_i\}$. By design the subtorus action $\mathbf{u}\mathbb{R}/\mathbb{Z} \subset \mathbb{R}^2/\mathbb{Z}^2$ has fixed points only on the k corners. We now apply the Poincaré-Hopf index theorem which states that the Euler characteristic of a closed manifold is equal to the Euler characteristic of the fixed point set of an S^1 action on that manifold. The fixed point set of the $\mathbf{u}\mathbb{R}/\mathbb{Z}$ action is k points thus $\chi(M) = k$.

When $n > 2$ Lemma 2.61 guarantees the existence of an almost free S^1-action. In particular this means the action has no fixed points. Hence by the Poincaré-Hopf index theorem $\chi(M) = 0$. □

Proposition 2.71. *Let M^{n+2} be a simple T^n-manifold and let $w_i(TM)$ be the i^{th} Stiefel-Whitney class of its tangent bundle. Then*

$$w_{n+2}(TM) = \cdots = w_5(TM) = 0. \tag{2.151}$$

Proof. The statement is trivial if $n \le 2$. When $n > 2$ the existence of an almost free T^{n-2}-action on M means there are $n-2$ linearly independent non-vanishing sections of the tangent bundle. This forces the top $n-2$ Stiefel-Whitney classes to vanish [37, §2, Prop. 4]. □

Proposition 2.72. *Let M^{n+2} be a simply connected closed T^n-manifold. If $n+2 \neq 4k$ and M is spin, then it is oriented null-cobordant.*

Proof. Since M is simply connected and spin we know $w_1 = w_2 = 0$. A simple application of the Wu formula [37, Problem 8-A] shows $Sq^1(w_2) = w_1 w_2 + w_3$, hence $w_3 = 0$ as well. Proposition 2.71 shows that $w_i = 0$ for all $i > 0$ as well, thus the only possibly non-trivial Stiefel-Whitney class is w_4.

Any closed oriented manifold is oriented null-cobordant if and only if all of their Stiefel-Whitney and Pontryagin numbers vanish [37, pg. 217]. These numbers are pairings of the the fundamental class of M with polynomials of characteristic classes. But all Pointryagin classes are in degree $4i$ for some i. Similarly the only possibly non-vanishing Stiefel-Whitney class is in degree 4. Hence any non-vanishing characteristic class polynomial must be of degree $4k$ for some k. By hypothesis $\dim(M) \neq 4k$ and thus all Stiefel-Whitney and Pontryagin numbers vanish. $\square$

Proposition 2.73. *Let M^{n+2} be a simple T^n-manifold admitting a free subtorus action of $T^l \subset T^n$ with submersion map $f\colon M \to N := M/T^l$. Then*

$$TM \cong f^*(TN) \oplus E \tag{2.152}$$

where E is the trivial rank l vector bundle.

Proof. Recall that the submersion $f\colon M \to N$ splits the tangent space T_pM splits into a vertical and horizontal part. Further recall that the pullback vector bundle is defined as $f^*(TN) := \bigsqcup_{p\in M} \{p\} \times T_{f(p)}N$ using the transition functions of TN over N. This means that the tangent bundle can be split as

$$T_pM \cong T_{f(p)}N \oplus \ker(df_p).$$

Now let $\{X_1, \dots, X_l\}$ be linearly independent, non-vanishing sections of TM defined by the free T^l action. These form a basis for $\ker(df)$. Choose any inner product on TM and define the map $F\colon TM \to f^*(TN) \oplus E$ by

$$F(p, v) := (p, (df_p(v), \langle X_1|_p, v\rangle, \dots, \langle X_l|_p, v\rangle)).$$

Note that if $df_p(v) = 0$ then $v \in \ker(df_p)$. If $df_p(v) = 0$ and $\langle X_i|_p, v\rangle = 0$ for all i, then $v = 0$. This means $v \mapsto F(p, v)$ is injective and therefore an isomorphism. Hence $F\colon TM \to f^*(TN) \oplus E$ is a bundle isomorphism and TM and $f^*(TN) \oplus E$ are isomorphic as vector bundles [37, §2, Lemma 2.3]. $\square$

Note that in general, having isomorphic tangent bundles is not enough to conclude that two homeomorphic smooth manifolds are diffeomorphic (consider exotic smooth structures on $\mathbb{R}^4$). This means that Proposition 2.73 on its own is not strong enough to prove that two smooth simple T^n-manifolds are diffeomorphic, even when they are homeomorphic and are both torus bundles over the same 4-manifold.

We can see that having a free subtorus action is a useful property, unfortunately not every simple T^n-manifold admits one. The following lemma and theorem offer a slight work-around.

Lemma 2.74. *Let M^{n+2} be a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. For each $i \in \{1, \dots, k\}$ there exists a simple T^{n+1}-manifold $\widetilde{M}^{n+3}$ with rod structures*

$$\tilde{\mathbf{v}}_j = \begin{cases} (\mathbf{v}_j, 0) & j \neq i \\ \mathbf{e}_{n+1} & j = i. \end{cases} \tag{2.153}$$

Assuming that $\mathrm{Det}_2\{\mathbf{v}_i, \mathbf{v}_{i\pm1}\} = 1$*, then the subtorus action defined by $\mathbf{u} = (\mathbf{v}_i, -1) \in \mathbb{Z}^{n+1}$ is free and the quotient space is M, that is*

$$M \cong \widetilde{M}/(\mathbf{u}\mathbb{R}/\mathbb{Z}). \tag{2.154}$$

Proof. We must first check that the vectors $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_k\} \subset \mathbb{Z}^{n+1}$ are indeed rod structures. Clearly each $\tilde{\mathbf{v}}_j$ is a primitive vector. For admissibility observe that $\mathrm{Det}_2\{\tilde{\mathbf{v}}_{i\pm1}, \tilde{\mathbf{v}}_i\} = \mathrm{Det}_1\{\mathbf{v}_{i\pm1}\} = 1$ and $\mathrm{Det}_2\{\tilde{\mathbf{v}}_j, \tilde{\mathbf{v}}_{j+1}\} =$

$\text{Det}_2\{\mathbf{v}_j, \mathbf{v}_{j+1}\}$ whenever $j, j+1 \neq i$. Thus $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_k\} \subset \mathbb{Z}^{n+1}$ are indeed rod structures for a simple T^{n+1}-manifold $\widetilde{M}$.

Next we need to check that the action defined by $\mathbf{u} = (\mathbf{v}_i, -1)$ is almost free. Since $\mathbf{v}_{i\pm1} \neq \pm\mathbf{v}_i$, we see $(\mathbf{v}_i, -1) \notin \text{span}_{\mathbb{Z}}\{(\mathbf{v}_{i\pm1}, 0), \mathbf{e}_{n+1}\} = \text{span}_{\mathbb{Z}}\{\tilde{\mathbf{v}}_{i\pm1}, \tilde{\mathbf{v}}_i\}$. For $j, j-1 \neq i$ we have $(\mathbf{v}_i, -1) \notin \mathbb{Z}^n \times \{0\} = \text{span}_{\mathbb{Z}}\{(\mathbf{v}_j, 0), (\mathbf{v}_{j+1}, 0)\} = \text{span}_{\mathbb{Z}}\{\tilde{\mathbf{v}}_j, \tilde{\mathbf{v}}_{j+1}\}$.

To see that the action is free, assume by contradiction that the action has a non-trivial isotropy subgroup on the $\Gamma_j \cap \Gamma_{j+1}$ corner. If $j, j+1 \neq i$ then following the argument in Lemma 2.61 there exits a $t \in (0,1)$ and a $\tilde{\mathbf{w}} \in \mathbb{Z}^{n+1}$ such that $t(\mathbf{v}_i, -1) + \tilde{\mathbf{w}} \in \text{span}_{\mathbb{R}}\{\tilde{\mathbf{v}}_j, \tilde{\mathbf{v}}_{j+1}\} \subset \mathbb{R}^n \times \{0\}$. This is impossible since the $n+1^{st}$ component of $t(\mathbf{v}_i, -1) + \tilde{\mathbf{w}}$ will always be non-zero. If the action has a non-trivial isotropy subgroup on the $\Gamma_i \cap \Gamma_{i\pm1}$ corner, then $t(\mathbf{v}_i, 0) + \tilde{w} \in \text{span}_{\mathbb{R}}\{(\mathbf{v}_{i\pm1}, 0), \mathbf{e}_{n+1}\}$. Letting $\tilde{\mathbf{w}} = (\mathbf{w}, w^{n+1})$ we see that this reduces to $t\mathbf{v}_i + \mathbf{w} \in \text{span}_{\mathbb{R}}\{\mathbf{v}_{i\pm1}\}$. Now by hypothesis $\text{Det}_2\{\mathbf{v}_i, \mathbf{v}_{i\pm1}\} = 1$ which means there exists a $U \in SL(n, \mathbb{Z})$ such that $U(\mathbf{v}_i) = \mathbf{e}_1$ and $U(\mathbf{v}_{i\pm1}) = \mathbf{e}_2$. Applying this transformation to the equation we see $t\mathbf{e}_1 + U(\mathbf{w}) \in \text{span}_{\mathbb{R}}\{\mathbf{e}_2\}$. This is impossible to satisfy because the first component of $t\mathbf{e}_1 + U(\mathbf{w})$ will always be non-zero.

The last step is to check that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are indeed the rod structures for $\widetilde{M}/(\mathbf{u}\mathbb{R}/\mathbb{Z})$. This is accomplished by defining the map $Q\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ with $Q(\mathbf{e}_j) = \mathbf{e}_j$ for all $j \leq n$ and $Q(\mathbf{e}_{n+1}) = \mathbf{v}_i$. Observe that $\mathbf{u} \in \ker(Q)$ since $Q(\mathbf{u}) = Q((\mathbf{v}_i, 0)) - Q(\mathbf{e}_{n+1}) = 0$. Lemma 2.61 now shows that $\{Q(\tilde{\mathbf{v}}_1), \ldots, Q(\tilde{\mathbf{v}}_k)\}$ are the rod structures for $\widetilde{M}/(\mathbf{u}\mathbb{R}/\mathbb{Z})$. By construction $Q(\tilde{\mathbf{v}}_j) = Q((\mathbf{v}_j, 0)) = \mathbf{v}_j$ for $j \neq i$ and $Q(\tilde{\mathbf{v}}_i) = Q(\mathbf{e}_{n+1}) = \mathbf{v}_i$. The proof is now complete. ☐

Theorem 2.75. *Every closed simple T^n-manifold with k rods is homeom(diffeo)morphic to a quotient of $M(k,k) \times T^{n-2}$ by a free T^{k-2} subtorus action, that is*

$$M \cong \left(M(k,k) \times T^{n-2}\right)/T^{k-2} \tag{2.155}$$

where $M(k,k)$ is the manifold described in Theorem 2.50.

Proof. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be the rod structures for M. By applying Hermite normal form without loss of generality we can assume that $\mathbf{v}_1 = \mathbf{e}_1$ and $\mathbf{v}_2 = \mathbf{e}_2$. We now apply Lemma 2.74 repeatedly on 3^{rd} through k^{th} rod structure. This gives us a simple T^{n+k-2}-manifold $\widetilde{M}^{n+k}$ with rod structures $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_k\}$ defined as

$$\tilde{\mathbf{v}}_j = \begin{cases} \mathbf{e}_j & i = 1,2 \\ \mathbf{e}_{n+j-2} & j > 2, \end{cases} \tag{2.156}$$

and a free $T^{k-2} \subset T^{n+k-2}$ subtorus action so that $\widetilde{M}/T^{k-2} \cong M$. After applying a change of basis so that $\tilde{\mathbf{v}}_j = \mathbf{e}_j$ for all $j = 1, \ldots, k$, we can clearly see $\widetilde{M}$ splits as the product of T^{n-2} and a simple T^k-manifold with k rods. In Theorem 2.50 this particular manifold was shown to be $M(k,k)$ as described in Conjecture A. ☐

Theorem 2.76. *Let M^{n+2} be a closed simple T^n-manifold with rod structures $\{1, \ldots, k\}$ forming a rank l matrix $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$. The higher homotopy groups of M are*

$$\pi_i(M) \cong \begin{cases} \mathbb{Z}^{k-l} & i = 2 \\ \pi_i(M(k,k)) & i > 2 \end{cases} \tag{2.157}$$

where $M(k,k)$ is the manifold described in Theorem 2.50.

Proof. First note that since A has rank l, M must be homeomorphic to the product of T^{n-l} and a simple T^l-manifold N'^{l+2} with k rods and a finite fundamental group. This combined with the fact that $\pi_i(M) \cong \pi_i(T^l \times N') \cong \pi_i(N')$ for all $i \geq 2$ allows us to compute the homotopy groups of M in terms of N'. Now apply Corollary 2.58 to produce a simply connected covering space of N' which we will denote by N. Since the higher homotopy groups of covering spaces agree, i.e. $\pi_i(N') \cong \pi_i(N)$ for all $i \geq 2$, we can compute homotopy groups of M in terms of N.

Applying Theorem 2.75 to N creates a fiber bundle

$$T^{k-2} \to M(k,k) \times T^{l-2} \to N \tag{2.158}$$

which induces the following long exact sequence in homotopy

$$\cdots \to \pi_*(T^{k-2}) \to \pi_*(M(k,k) \times T^{l-2}) \to \pi_*(N) \to \pi_{*-1}(T^{k-2}) \to \cdots \tag{2.159}$$

Again recall that $\pi_i(T^{k-2}) = 0$ and $\pi_i(N \times T^{k-2}) \cong \pi_i(N)$ for all $i \geq 2$. This simplifies Equation (2.159) into

$$0 \to \pi_i(M(k,k)) \to \pi_i(N) \to 0 \tag{2.160}$$

for $i > 2$ and

$$\pi_2(M(k,k)) \to \pi_2(N) \to \pi_1(T^{k-2}) \to \pi_1(M(k,k) \times T^{l-2}) \to \pi_1(N) \tag{2.161}$$

for $i = 2$. We immediately see from Equation (2.160) that $\pi_i(N) \cong \pi_i(M(k,k))$ for $i > 2$. For the $i = 2$ case we recall that N is simply connected and from Theorem 2.50 $M(k,k)$ is 2-connected. This reduces Equation (2.161) into the short exact sequence

$$0 \to \pi_2(N) \to \mathbb{Z}^{k-2} \to \mathbb{Z}^{l-2} \to 0 \tag{2.162}$$

Now recall that any short exact sequence of Abelian groups splits and we see $\pi_2(N) \cong \mathbb{Z}^{k-l}$ as desired. ☐

The next result concerns the rational homotopy groups of simple T^n-manifolds. These are the groups $\pi_i(M) \otimes \mathbb{Q}$ for large i. The study of these groups is closely related to the study of the rational cohomology ring, and for manifolds with torus action is examined in [14, Remark 1.105]. A simple dichotomy in rational homotopy theory is the distinction between elliptic and hyperbolic. A manifold M is *rationally elliptic* if there exists an $N > 1$ such that $\pi_i(M) \otimes \mathbb{Q} = 0$ for all $i > N$. A manifold which is not rationally elliptic is called *rationally hyperbolic*. The proposition below distinguishes between these two cases.

Proposition 2.77. *Any closed simple T^n-manifold M^{n+2} with k rods is rationally elliptic if and only if $k \leq 4$.*

Proof. Theorem 2.76 shows that $\pi_i(M) \cong \pi_i(M(k,k))$ which means we need to determine when $M(k,k)$ is rationally elliptic. Let us recall some facts from rational homotopy theory; All spheres are rationally elliptic [14, Example 2.44] and the product to two rationally elliptic manifolds is rationally elliptic [14, Example 2.45]. Theorem 2.50 shows $M(2,2) = S^4$, $M(3,3) = S^5$, and $M(4,4) = S^3 \times S^3$. Therefore M is rationally elliptic if $k \leq 4$. For the other direction recall that the connected sum of two manifolds whose cohomologies have at least two generators each is rationally hyperbolic [14, Remark 3.5]. Theorem 2.50 shows that $M(k,k) \cong \#_{j=1}^{k-3} j\binom{k-2}{j+1} S^{2+j} \times S^{k-j}$ is a non-trivial connected sum of products of spheres when $k > 4$. Since the cohomology of a product of spheres has at least to generators, M must be rationally hyperbolic when $k > 4$. ☐

2.8 Torus Bundles

In the previous section the existence of free and almost free subtorus actions was used to gather information about the topology of M^{n+2}. In the case where the action is free this defines a fiber-bundle over a manifold. Specifically, if $T^l \subset T^n$ acts freely on M^{n+2} then M is the total space of a principal T^l-bundle, or *torus bundle*, over a simple T^{n-l}-manifold B^{n-l+2}

$$T^l \to M^{n+2} \to B^{n-l+2}.$$

In this section we will examine the topological information contained in the bundle structure itself.

Recall that a rank 2 real vector bundle over B, along with its associated principal S^1-bundle, are completely determined by a characteristic class $\tilde{e} \in H^2(B;\mathbb{Z})$ known as the *Euler class*. In fact, there is a one-to-one correspondence between $H^2(B;\mathbb{Z}^l)$ and principal T^l-bundles over B. In Theorem 2.80 and Corollary 2.81 this correspondence will be used to relate the rods structures of B and M. However in order to state and prove Theorem 2.80 we will need the following lemmas, which will be proved in Section 3.

Lemma 2.78 (Corollary 3.14)**.** *For any simply connected T^n-manifold M^{n+2} with rod structures forming the matrix $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$, there exists an isomorphism*

$$\Psi_*\colon \ker(A) \to H_2(M;\mathbb{Z}) \cong \mathbb{Z}^{k-n} \tag{2.163}$$

defined explicitly in terms of the rod structures.

Lemma 2.79 (Corollary 3.20)**.** *Let M^{m+2} be a simply connected T^m-manifold with projection map $\pi_M\colon M \to M/T^m$, rods $\{\Gamma_1^M,\dots,\Gamma_k^M\}$, rod structures $\{\mathbf{v}_1^M,\dots,\mathbf{v}_k^M\}$ forming the matrix $A^M\colon \mathbb{Z}^k \to \mathbb{Z}^m$ and isomorphism $\Psi_*^M\colon \ker(A^M) \to H_2(M;\mathbb{Z})$. Similarly define N^{n+2} to be a simply connected T^n-manifold. Suppose there exists a weakly equivariant map $(F,\varphi)\colon (M,T^m) \to (N,T^n)$ which induces a homeomorphism between the quotient spaces M/T^m and N/T^n with the property that*

$$\pi_N(F(\pi_M^{-1}(\Gamma_i^M))) = \Gamma_i^N \tag{2.164}$$

$$\varphi(\mathbf{v}_i^M) = \mathbf{v}_i^N \tag{2.165}$$

for all $i = 1,\dots,k$. Then $\ker(A^M) \subset \ker(A^N) \subset \mathbb{Z}^k$ and

$$F_*(\Psi_*^M(\mathbf{w})) = \Psi_*^N(\mathbf{w}) \tag{2.166}$$

for all $\mathbf{w} \in \ker(A^M)$.

Note that Lemma 2.78 has already been proven for the $n = 2$ case in Lemma 2.38. Interestingly the proof of Lemma 2.38 generalizes to a proof of Lemma 2.78 without issue. In Section 3 we however prove a much more general statement, Lemma 3.13, which Corollary 3.14 is merely a special case of. Lemma 2.79 can either be viewed as a technical lemma relating $\ker(A^M)$ to $\ker(A^N)$, or be viewed in a geometric way as an embedding of $H_2(M;\mathbb{Z})$ into $H_2(N;\mathbb{Z})$. This most likely also has a geometric proof, however we will not attempt to write such a proof (as it is not needed and this dissertation is hundreds of pages long already).

In Theorem 2.80 we examine a principal S^1-bundle $\widetilde{M} \to M$ over a simply connected T^n-manifold M^{n+2}. Lemma 2.78 is important because it allows us to express the Euler class of this bundle, $\tilde{e} \in H^2(M;\mathbb{Z})$, as a vector in $\mathbb{Z}^k$. Consider the domain of the isomorphism $\Psi_*\colon \ker(A\colon \mathbb{Z}^k \to \mathbb{Z}^n) \to H_2(M;\mathbb{Z})$ as a subspace of $\mathbb{Z}^k$ and equip it with the dot product inherited from $\mathbb{Z}^k$ using the standard basis $\{\mathbf{e}_1,\dots,\mathbf{e}_k\}$. Using the definition of cohomology as the dual space of homology, and the fact that $H_2(M;\mathbb{Z})$ is torsion free, we can define the 'dual' of the Euler class $\tilde{e}^* \in H_2(M;\mathbb{Z})$ by the equation

$$\Psi_*^{-1}(\tilde{e}^*)\cdot\mathbf{u} = \tilde{e}(\Psi_*\mathbf{u}) \tag{2.167}$$

for all $\mathbf{u} \in \ker(A) \subset \mathbb{Z}^k$.

Theorem 2.80. *Let M^{n+2} be a simply connected T^n-manifold with rod structures $\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$ forming the matrix $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$. Let $\widetilde{M}^{n+3}$ be simply connected and let $P\colon \widetilde{M} \to M$ be a principal S^1-bundle over M with Euler class $\tilde{e} \in H^2(M;\mathbb{Z})$. The total space $\widetilde{M}$ is a simple T^{n+1}-manifold which admits rod structures $\{\tilde{\mathbf{v}}_1,\dots,\tilde{\mathbf{v}}_k\}$ defined by*

$$\tilde{\mathbf{v}}_i := (\mathbf{v}_i,\eta_i) \in \mathbb{Z}^{n+1} \tag{2.168}$$

for any vector $\boldsymbol{\eta} = (\eta_1,\dots,\eta_k) \in \mathbb{Z}^k$ satisfying the equation

$$\boldsymbol{\eta}\cdot\mathbf{u} = \tilde{e}(\Psi_*\mathbf{u}) \tag{2.169}$$

for all $\mathbf{u} \in \ker(A)$*. The projection map* P *is weakly equivariant and expressed as*

$$P(p, \theta^1, \ldots, \theta^{n+1}) = (p, \theta^1, \ldots, \theta^n). \tag{2.170}$$

Proof. Applying Corollary 2.63 with $l = 1$ and $\mathbf{w}_i = \eta_i$ shows that $\widetilde{M}$ is a simple T^{n+1}-manifold admitting a free subtorus action of $S^1 \cong \mathbf{e}_{n+1}\mathbb{R}/\mathbb{Z} \subset \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ with quotient map $P\colon \widetilde{M} \to M$ defined by Equation (2.170). The fact that the S^1-action is free means $P\colon \widetilde{M} \to M$ is a principal S^1-bundle. Using Equation (2.170) it is clear that P is weakly equivariant. The only thing left to prove is that the Euler class of $P\colon \widetilde{M} \to M$ is indeed $\tilde{e}$.

By definition the Euler class of a bundle is the generator of the kernel of $P^*\colon H^2(M;\mathbb{Z}) \to H^2(\widetilde{M};\mathbb{Z})$. To express this as a vector, let $\widetilde{A}\colon \mathbb{Z}^k \to \mathbb{Z}^{n+1}$ denote the matrix composed of the rod structures $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_k\}$. Then let $\Psi_*\colon \ker(A) \to H_2(M;\mathbb{Z})$ and $\widetilde{\Psi}_*\colon \ker(\widetilde{A}) \to H_2(\widetilde{M};\mathbb{Z})$ be the isomorphism described in Lemma 2.78.

Suppose $\vartheta \in H^2(M;\mathbb{Z})$. Then using the definition of $H^2(M;\mathbb{Z})$ as a the dual of $H_2(M;\mathbb{Z})$, and using Lemma 2.79 to reduce $P_*\widetilde{\Psi}_*$ to Ψ_*, we see

$$P^*(\vartheta)(\widetilde{\Psi}_*\mathbf{w}) = \vartheta(\Psi_*\mathbf{w}) \tag{2.171}$$

for all $\mathbf{w} \in \ker(\widetilde{A})$. We now replace ϑ with $\tilde{e}$ and use Equation (2.169) to see

$$P^*(\tilde{e})(\widetilde{\Psi}_*\mathbf{w}) = \tilde{e}(\Psi_*\mathbf{w}) = \boldsymbol{\eta} \cdot \mathbf{w}. \tag{2.172}$$

By construction $\mathbf{w} \in \ker(\widetilde{A})$ which means $\mathbf{w} \cdot \tilde{\mathbf{v}} = 0$ for all i. Since $\boldsymbol{\eta} = \tilde{\mathbf{v}}_{i+1}$ we conclude that

$$P^*(\tilde{e})(\widetilde{\Psi}_*\mathbf{w}) = 0 \tag{2.173}$$

for all $\mathbf{w} \in \ker(\widetilde{A})$ and thus $\tilde{e} \in \ker(P^*)$.

We now recall a fact from topology. If $P'\colon M' \to M$ is a principal S^1-bundle with Euler class $x \in H^2(M;\mathbb{Z})$ and $P\colon \widetilde{M} \to M$ is a principal S^1-bundle with Euler class $\tilde{e} = qx \in H^2(M;\mathbb{Z})$, then there exists a q-to-1 bundle covering map $\widetilde{M} \to M'$. In particular this means there exists a q-to-1 covering map $\widetilde{M} \to M'$ as manifolds. The fact that $\widetilde{M}$ is simply connected means this cannot happen and thus $\tilde{e}$ must be primitive. Finally we note from Lemma 2.79 that $P^*\colon H^2(M;\mathbb{Z}) \to H^2(\widetilde{M};\mathbb{Z})$ is injective and from Lemma 2.78 and that $\dim(H^2(M;\mathbb{Z})) = \dim(H^2(\widetilde{M};\mathbb{Z})) + 1$ meaning $\ker(P^*)$ has only one generator. Since $\tilde{e} \in \ker(P^*)$ is primitive we know it must generate the space and therefore is the Euler class.

□

Theorem 2.81. *Let* M^{n+2} *be a simply connected* T^n*-manifold with rod structures* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *forming the matrix* $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$*. Let* $\widetilde{M}^{n+l+2}$ *be the total space of a principal* T^l*-bundle* $P\colon \widetilde{M} \to M$ *with Euler class* $\tilde{e} = (\tilde{e}^1, \ldots, \tilde{e}^l) \in H^2(M;\mathbb{Z}^l) \cong H^2(M;\mathbb{Z}) \otimes \mathbb{Z}^l$*.* $\widetilde{M}$ *is a simple* T^{n+l}*-manifold which admits rod structures* $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_k\}$ *defined by*

$$\tilde{\mathbf{v}}_i := (\mathbf{v}_i, \eta_i^1, \ldots, \eta_i^l) \in \mathbb{Z}^{n+l} \tag{2.174}$$

where for each $j = 1, \ldots, l$*, the vector* $\boldsymbol{\eta}^j := (\eta_1^j, \ldots, \eta_k^j) \in \ker(A) \subset \mathbb{Z}^k$ *is defined by the equation*

$$\boldsymbol{\eta}^j \cdot \mathbf{w} = \tilde{e}^j\,(\Psi_*\mathbf{w}) \tag{2.175}$$

for all $\mathbf{w} \in \ker(A) \subset \mathbb{Z}^k$*. The projection map* P *is weakly equivariant and expressed as*

$$P(p, \theta^1, \ldots, \theta^{n+l}) = (p, \theta^1, \ldots, \theta^n). \tag{2.176}$$

Proof. The proof is a direct consequence of repeated applications of Theorem 2.80, followed by applications of Corollary 2.59 to deal with non-primitive Euler classes.

□

We now have the tools needed to finish the proof of Theorem 2.52, stated earlier in Section 2.5. The final piece of the proof requires proving the following lemma.

Lemma 2.82. *Let $n > 2$ and $N(n,k)$ be a closed, simply connected T^n-manifold with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ defined by $\mathbf{v}_i = \mathbf{e}_i$ for $i \le n-1$ and $\mathbf{v}_i \in \{\mathbf{e}_2, \mathbf{e}_3\}$ for $i > n-1$. $N(n,k)$ is spin.*

Proof. The proof relies on induction. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be rod structures for $N(n-1,k)$ and define the T^n-manifold $\widetilde{N}$ by the rod structures $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_k\} \subset \mathbb{Z}^n$ where $\tilde{\mathbf{v}}_i := (\mathbf{v}_i, 0) \in \mathbb{Z}^n$ for $i \neq n$ and $\tilde{\mathbf{v}}_n := (\mathbf{v}_i, 1)$. Using Corollary 2.63 we see that $\widetilde{N}$ is the total space of a principal S^1-bundle over $N(n-1,k)$. Applying the change of coordinates $U \in SL(n, \mathbb{Z})$ defined by $U(\mathbf{e}_i) = \mathbf{e}_i$ for $i \neq n$ and $U(\tilde{\mathbf{v}}_n) = \mathbf{e}_n$ to $N(n,k)$ we see that $\widetilde{N}$ is homeomorphic to $N(n,k)$ and thus $N(n,k)$ is also the total space of a principal S^1-bundle over $N(n-1,k)$. We now assume by induction that $N(n-1,k)$ is spin. Since all principal circle bundles over spin manifolds are spin, we know that M' is spin. Therefore if $N(3,k)$ is spin for all $k > 3$ then the proof is complete.

By construction $N(3,k)$ has rod structures $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_{2/3}\}$ where the last rod structure is $\mathbf{e}_2$ if k is even and $\mathbf{e}_3$ is k is odd. If k is even then we preform a similar trick as before and define a change of basis $U \in SL(3, \mathbb{Z})$ so that $U(\mathbf{e}_1) = \mathbf{e}_1$, $U(\mathbf{e}_2) = \mathbf{e}_2$, and $U(\mathbf{e}_3) = \mathbf{e}_1 + \mathbf{e}_3$. Using Corollary 2.63 we see $N(3,k)$ is a principal S^1-bundle over a simple T^2-manifold with rod structures $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_1, \mathbf{e}_2\}$. From Theorem 2.46 we know this is spin, and thus $N(3,k)$ is spin.

If k is odd the argument must be modified as any simple T^2-manifold with an odd number of rods is non-spin. We will prove the statement for $k = 5$ since the manifold $N(3,5)$ was already studies in Example 2.39. However the arguments presented here and in Example 2.39 works perfectly fine for any odd $k \ge 3$. Define the change of basis $U \in SL(3, \mathbb{Z})$ by $U(\mathbf{e}_1) = (1,1,1)$, $U(\mathbf{e}_2) = \mathbf{e}_1$, and $U(\mathbf{e}_3) = \mathbf{e}_2$ to get rod structures $\{(1,1,1), \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2\}$. We apply Corollary 2.63 with $l = 1$, $\mathbf{w}_1 = 1$, and $\mathbf{w}_i = 0$ for $i \neq 1$ to see $N = N(3,k)$ is a principal S^1-bundle over a simple T^2-manifold W with rod structures $\{(1,1), \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2\}$. Now apply Theorem 2.80 to see that the Euler class $\tilde{e} \in H^2(W; \mathbb{Z})$ of the bundle $P \colon N \to W$ satisfies the equation

$$\mathbf{e}_1 \cdot \mathbf{u} = \tilde{e}(\Psi_* \mathbf{u}) \tag{2.177}$$

for all $\mathbf{u} \in \ker(A)$ where A is generated by the rod structures of W.

The manifold W was studied in Example 2.39. In this example it was found that an integral representative of the second Stiefel-Whitney class $\overline{w} \in [w_2(TW)] \subset H^2(W; \mathbb{Z})$ also satisfies the equation

$$\mathbf{e}_1 \cdot \mathbf{u} = \overline{w}(\Psi_* \mathbf{u}) \tag{2.178}$$

for all $\mathbf{u} \in \ker(A)$. From Equations (2.177) and (2.178) it is clear that $\overline{w} = \tilde{e}$. By definition $\tilde{e} \in H^2(W; \mathbb{Z})$ generates the kernel of $P^* \colon H^2(W; \mathbb{Z}) \to H^2(N; \mathbb{Z})$ which means $P^*(\overline{w})$ vanishes, and thus its mod-2 reduction $P^*(w_2(TW))$ vanishes as well. Finally we use Proposition 2.73 to see $TN \cong P^*(TW) \oplus E$ where E is a trivial line bundle. The additivity property of Stiefel-Whitney classes shows $w_2(TN) = w_2(P^*(TW)) \oplus w_2(E) = w_2(P^*(TW))$ and the naturality property shows $w_2(P^*(TW)) = P^*(w_2(TW)) = 0$. Hence N is spin and the proof is complete.

□

We end this section with a simple lemma which is vital to the proof of Theorem H.

Lemma 2.83. *For any simple T^n-manifold M^{n+2} with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, there exists a principal T^k-bundle $P \colon \widetilde{M} \to M$ where $\widetilde{M}^{n+k+2}$ is a simple T^{n+k}-manifold admitting rod structures $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$.*

Proof. We first apply Corollary 2.63 to M with $l = k$ and with $\mathbf{w}_i = \mathbf{e}_i$. This produces a simple T^{n+k+2}-manifold $\widetilde{M}$ with rod structures $\tilde{\mathbf{v}}_i := (\mathbf{v}_i, \mathbf{w}_i) \in \mathbb{Z}^{n+k}$. By construction the top determinant divisor of these new rod structures is one, $\mathrm{Det}_k\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_k\} = 1$. This means the Hermite normal form of $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_k\}$ is $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ (see Corollary 2.23). Therefore there exists a change of basis $U \in SL(n+k, \mathbb{Z})$ which sends $U(\tilde{\mathbf{v}}_i) = \mathbf{e}_i$ making $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ rod structures for $\widetilde{M}$ as desired. □

2.9 Spectral Sequence

For the remainder of this section we will be examining a special class of T^n-manifolds which are the total spaces of principal T^{n-2}-bundles over 4-dimensional simple T^2-manifolds. This is equivalent to the existence of a free T^{n-2} subtorus action.

Lemma 2.84. *Let M^{n+2} for $n > 2$ be a closed simply connected T^n-manifold with k rods. Suppose M is the total space of a principal T^{n-2}-bundle over a T^2-manifold B^4;*

$$T^{n-2} \to M \to B.$$

The E_∞ page of the Serre spectral sequence is

$$E_\infty^{p,q} = \begin{cases} \mathbb{Z} & (p,q) = (0,0) \\ \mathbb{Z}^{b_{2+q}} & p = 2 \\ \mathbb{Z} & (p,q) = (4, n+2) \\ 0 & \textit{else} \end{cases} \tag{2.179}$$

where

$$b_j = (k-2)\binom{n-2}{j-2} - \binom{n-2}{j-3} - \binom{n-2}{j-1}. \tag{2.180}$$

Proof. Recall that Serre spectral sequences have various pages. When doing calculations for a general fiber bundle $F \to X \to B$, we start with the second page $E_2^{**} = H^*(B) \otimes H^*(F)$. Note that cohomology without coefficients is assumed to have $\mathbb{Z}$ coefficients. This page inherits a product structure from the cup products on the two cohomology rings. Namely

$$(b \otimes f) \cup (b' \otimes f') = (-1)^{|f||b'|}(b \cup b') \otimes (f \cup f') \tag{2.181}$$

for any $b, b' \in H^*(B)$ and $f, f' \in H^*(F)$, where $|f|$ and $|b'|$ denote the degrees of f and b' in their respective cohomology rings. In our specific case the cohomology rings are quite simple,

$$\Lambda^*(\mathbb{Z}^{n-2}) = H^*(T^{n-2}) \tag{2.182}$$
$$\mathbb{Z} \oplus \mathbb{Z}^{k-2} \oplus \mathbb{Z} = H^*(B), \tag{2.183}$$

where $\mathbb{Z}^{k-2} = H^2(B)$ and the product structure is given by some full rank, symmetric bi-linear form $Q\colon \mathbb{Z}^{k-2} \otimes \mathbb{Z}^{k-2} \to \mathbb{Z}$. Thus the groups on this second page are

$$E_2^{p,q} = H^p(B; H^q(T^{n-2})) = H^p(B) \otimes \Lambda^q(\mathbb{Z}^{n-2}) = \begin{cases} \Lambda^q(\mathbb{Z}^{n-2}) & p = 0, 4 \\ \mathbb{Z}^{k-2} \otimes \Lambda^q(\mathbb{Z}^{n-2}) & p = 2 \\ 0 & \text{else} \end{cases} \tag{2.184}$$

and the differential

$$d_2\colon E_2^{p,q} \to E_2^{p+2,q-1} \tag{2.185}$$

respects this product structure in the sense that is follows the graded-Leibniz rule. The E_2 page is depicted in Figure 2.24 which immediately shows

$$E_3^{0,0} = \mathbb{Z} \tag{2.186}$$
$$E_3^{4,n-2} = \mathbb{Z}. \tag{2.187}$$

Notice that $E_2^{p,q} = 0$ for odd p. This passes to the third page where $E_3^{p,q} = 0$ for odd p as well. In

$$
\begin{tikzcd}[column sep=small]
\Lambda^q(\mathbb{Z}^{n-2}) & & & & & & \\
n-2 & \Lambda^{n-2}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \mathbb{Z}^{k-2}\otimes\Lambda^{n-2}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \Lambda^{n-2}(\mathbb{Z}^{n-2}) & \\
n-3 & \Lambda^{n-3}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \mathbb{Z}^{k-2}\otimes\Lambda^{n-3}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \Lambda^{n-3}(\mathbb{Z}^{n-2}) & \\
n-4 & \Lambda^{n-4}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \mathbb{Z}^{k-2}\otimes\Lambda^{n-4}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \Lambda^{n-4}(\mathbb{Z}^{n-2}) & \\
\vdots & & \vdots & & \vdots & & \\
\vdots & {} \arrow[drr] & \vdots & {} \arrow[drr] & \vdots & & \\
3 & \Lambda^{3}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \mathbb{Z}^{k-2}\otimes\Lambda^{3}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \Lambda^{3}(\mathbb{Z}^{n-2}) & \\
2 & \Lambda^{2}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \mathbb{Z}^{k-2}\otimes\Lambda^{2}(\mathbb{Z}^{n-2}) \arrow[drr] & 0 & \Lambda^{2}(\mathbb{Z}^{n-2}) & \\
1 & \mathbb{Z}^{n-2} \arrow[drr] & 0 & \mathbb{Z}^{k-2}\otimes\mathbb{Z}^{k-2} \arrow[drr] & 0 & \mathbb{Z}^{n-2} & \\
0 & \mathbb{Z} & 0 & \mathbb{Z}^{k-2} & 0 & \mathbb{Z} & \\
E_2^{p,q} & 0 & 1 & 2 & 3 & 4 & H^p(B)
\end{tikzcd}
$$

Figure 2.24: Here is the image of the second page of the Serre spectral sequence.

particular

$$E_3^{1,q} = 0 = E_3^{3,q} \tag{2.188}$$

for all q. But since the differential on the third page $d_3 \colon E_3^{p,q} \to E_3^{p+3,q-2}$ moves between even and odd p, we conclude that $E_3^{p,q} = E_\infty^{p,q}$ and therefore we only need to compute one page of the spectral sequence.

To do these calculations, we need to know information about the bundle structure. Recall that the Euler class $\tilde{e} \in H^2(B;\mathbb{Z}^{n-2}) \cong H^2(B;\mathbb{Z}) \otimes \mathbb{Z}^{n-2} \cong \mathbb{Z}^{k-2} \otimes \mathbb{Z}^{n-2}$ completely determines the topology of the T^{n-2}-bundle over B. In particular, since B is simply connected, the fundamental group of M can be calculated as

$$\pi_1(M) = \frac{\mathbb{Z}^{n-2}}{\operatorname{im}(\tilde{e} \colon \mathbb{Z}^{k-2} \to \mathbb{Z}^{n-2})}. \tag{2.189}$$

By assumption $\pi_1(M)$ is trivial which immediately implies $k \geq n$. Furthermore the Smith normal form of $\tilde{e}$ has only 1's along the diagonal. By choosing bases for $H^2(B) = \mathbb{Z}^{k-2}$ and $\pi_1(T^{n-2}) = \mathbb{Z}^{n-2}$ we can splits $\tilde{e}$ up into its column vectors

$$\tilde{e} = [\tilde{e}_1, \ldots, \tilde{e}_{n-2}] \tag{2.190}$$

each of which are Euler classes for an associated $S^1 \subset T^{n-2}$ sub-bundle. If the bases are chosen so that $\tilde{e}$ is in Smith normal form, then

$$\tilde{e}_i = \mathbf{e}_i \in \mathbb{Z}^{k-2} \tag{2.191}$$

form the standard basis for $\mathbb{Z}^{k-2}$.

The Euler class defines the differential

$$d \colon \mathbb{Z}^{n-2} \cong E_2^{0,1} \to E_2^{2,0} \cong \mathbb{Z}^{k-2}$$

in the following way

$$d(\alpha) = d\left(\sum_{i=1}^{n-2} \alpha^i \mathbf{e}_i\right) = \sum_{i=1}^{n-2} \alpha^i \tilde{e}_i \tag{2.192}$$

where $\alpha = \sum_{i=1}^{n-2} \alpha^i \mathbf{e}_i \in \mathbb{Z}^{n-2}$ and $\tilde{e}_i \in \mathbb{Z}^{k-2}$ for each i. We also have a trivial differential

$$d\colon E_2^{p,0} \to E_2^{p+2,-1} = 0. \tag{2.193}$$

Using the graded Leibniz rule and these two differentials, we can generate all of the other differentials. Let $\alpha_i \in \mathbb{Z}^{n-2}$ and $\beta \in H^p(B)$ for $p \in \{0,2,4\}$. The differential $d\colon E_2^{p,q} \to E_2^{p+2,q-1}$ is then

$$d(\beta \otimes \alpha_1 \wedge \cdots \wedge \alpha_q) = d(\beta) \otimes \alpha_1 \wedge \cdots \wedge \alpha_q + \beta \otimes \sum_{i=1}^{q} (-1)^i \alpha_1 \wedge \cdots \wedge d(\alpha_i) \wedge \cdots \wedge \alpha_q \tag{2.194}$$

$$= \beta \otimes \sum_{i=1}^{q} (-1)^i \alpha_1 \wedge \cdots \wedge d(\alpha_i) \wedge \cdots \wedge \alpha_q \tag{2.195}$$

$$= \beta \otimes \sum_{i=1}^{q} (-1)^i d(\alpha_i) \wedge \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_q \tag{2.196}$$

$$= \sum_{i=1}^{q} (-1)^i (\beta \cup d(\alpha_i)) \otimes \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_q \tag{2.197}$$

where the hat is used to indicate an object which has been removed from the product.

We will now continue this calculation for in the $p = 0$ case. Let $\alpha \in \Lambda^q(\mathbb{Z}^{n-2})$ which using multi-index notation can be described as

$$\alpha = \sum_{J \in I_q^{n-2}} \alpha^J \mathbf{e}_J = \sum_{1 \le j_1 < \cdots < j_q \le n-2} \alpha^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_q} \tag{2.198}$$

Observe then the following;

$$d(\alpha) = d\left(\sum_{J \in I_q^{n-2}} \alpha^J \mathbf{e}_J \right) \tag{2.199}$$

$$= d\left(\sum_{1 \le j_1 < \cdots < j_q \le n-2} \alpha^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_q} \right) \tag{2.200}$$

$$= d\left(\sum_{j_i < j_{i+1}} \alpha^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_q} \right) \tag{2.201}$$

Here $j_i < j_{i+1}$ is simply used as a shorthand for $1 \le j_1 < \cdots < j_q \le n-2$.

$$= \sum_{j_i < j_{i+1}} \alpha^{j_1 \dots j_q} \sum_{i=1}^{q} (-1)^{i-1} d(\mathbf{e}_{j_i}) \otimes \mathbf{e}_{j_1} \wedge \cdots \wedge \widehat{\mathbf{e}_{j_i}} \wedge \cdots \wedge \mathbf{e}_{j_q} \tag{2.202}$$

$$= \sum_{j_i < j_{i+1}} \alpha^{j_1 \dots j_q} \sum_{i=1}^{q} (-1)^{i-1} \tilde{e}_{j_i} \otimes \mathbf{e}_{j_1} \wedge \cdots \wedge \widehat{\mathbf{e}_{j_i}} \wedge \cdots \wedge \mathbf{e}_{j_q} \tag{2.203}$$

$$= \sum_{j_i < j_{i+1}} \alpha^{j_1 \dots j_q} \sum_{i=1}^{q} (-1)^{i-1+(i-1)(q-i)} \tilde{e}_{j_i} \otimes \mathbf{e}_{j_{i+1}} \wedge \cdots \wedge \mathbf{e}_{j_q} \wedge \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{i-1}} \tag{2.204}$$

$$= \sum_{j_i < j_{i+1}} \alpha^{j_1 \dots j_q} \sum_{i=1}^{q} (-1)^{(i-1)q} \tilde{e}_{j_i} \otimes \mathbf{e}_{j_{i+1}} \wedge \cdots \wedge \mathbf{e}_{j_q} \wedge \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{i-1}} \tag{2.205}$$

We deduced $i-1+(i-1)(q-i) \equiv (i-1)q \mod 2$ by noting that when q is even, $(i-1)(q-i) \equiv (i-1)i \equiv 0 \mod 2$, and when q is odd $(i-1)(q-i) \equiv (i-1)^2 \equiv i-1 \mod 2$. Now let $j_{q+i} = j_i$ for all i, and we continue or calculations.

$$= \sum_{j_i<j_{i+1}} \alpha^{j_1\cdots j_q} \sum_{i=0}^{q-1} (-1)^{iq} \tilde{e}_{j_{1+i}} \otimes \mathbf{e}_{j_{2+i}} \wedge \cdots \wedge \mathbf{e}_{j_{q+i}} \tag{2.206}$$

$$= \frac{1}{q!} \sum_{j_i\in\{1,\ldots,n-2\}} \alpha^{j_1\cdots j_q} \sum_{i=0}^{q-1} (-1)^{iq} \tilde{e}_{j_{1+i}} \otimes \mathbf{e}_{j_{2+i}} \wedge \cdots \wedge \mathbf{e}_{j_{q+i}} \tag{2.207}$$

The $\frac{1}{q!}$ is placed there to cancel out the redundant terms from no longer requiring a strict ordering of the j_i.

$$= \frac{1}{q!} \sum_{j_1=1}^{n-2} \cdots \sum_{j_q=1}^{n-2} \sum_{i=0}^{q-1} (-1)^{iq} \alpha^{j_1\cdots j_q} \tilde{e}_{j_{1+i}} \otimes \mathbf{e}_{j_{2+i}} \wedge \cdots \wedge \mathbf{e}_{j_{q+i}} \tag{2.208}$$

$$= \frac{1}{q!} \sum_{j_1=1}^{n-2} \cdots \sum_{j_q=1}^{n-2} \sum_{i=0}^{q-1} \alpha^{j_{1+i}\cdots j_{q+i}} \tilde{e}_{j_{1+i}} \otimes \mathbf{e}_{j_{2+i}} \wedge \cdots \wedge \mathbf{e}_{j_{q+i}} \tag{2.209}$$

Recall that cyclic permutations of the indicies in $\alpha^{j_1\cdots j_q}$ also introduces a $(-1)^{iq}$ sign. And since we are summing over all possible combinations of j_i already, we can remove the sum over i.

$$= \frac{1}{(q-1)!} \sum_{j_1=1}^{n-2} \cdots \sum_{j_q=1}^{n-2} \alpha^{j_1\cdots j_q} \tilde{e}_{j_1} \otimes \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_q} \tag{2.210}$$

$$= \sum_{j_1=1}^{n-2} \sum_{1\leq j_2<\cdots<j_q\leq n-2} \alpha^{j_1\cdots j_q} \tilde{e}_{j_1} \otimes \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_q} \tag{2.211}$$

$$= \sum_{\substack{1\leq j_1\leq n-2 \\ 1\leq j_2<\cdots<j_q\leq n-2}} \alpha^{j_1\cdots j_q} \tilde{e}_{j_1} \otimes \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_q} \tag{2.212}$$

$$= \sum_{\substack{1\leq j_1\leq n-2 \\ 1\leq j_2<\cdots<j_q\leq n-2 \\ j_1\neq j_i}} \alpha^{j_1\cdots j_q} \mathbf{e}_{j_1} \otimes \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_q} \tag{2.213}$$

From the above calculations it is clear that $d\colon E_2^{0,q} \to E_2^{2,q-1}$ is injective and

$$E_3^{0,q} = 0 \tag{2.214}$$

for all $q \neq 0$. Note that the image of $d(E_2^{0,q}) \subset E_2^{2,q-1}$ is a primitive submodule. Since $E_2^{2,q-1}$ is torsion free we conclude that the quotient $E_2^{2,q-1}/d(E_2^{0,q})$ is torsion free as well.

We now compute $d\colon \mathbb{Z}^{k-2} \otimes \Lambda^q(\mathbb{Z}^{n-2}) \to \Lambda^{q-1}(\mathbb{Z}^{n-2})$. Let $\mathbf{x} \in \mathbb{Z}^{k-2} \otimes \Lambda^q(\mathbb{Z}^{n-2})$ and observe;

$$
\begin{aligned}
d(\mathbf{x}) &= d\left(\sum_{i=1}^{k-2} \mathbf{e}_i \otimes \alpha^i\right) && (2.215)\\
&= \sum_{i=1}^{k-2} \left(d(\mathbf{e}_i) \otimes \alpha^i + \mathbf{e}_i \cup d(\alpha^i)\right) && (2.216)\\
&= \sum_{i=1}^{k-2} \mathbf{e}_i \cup d(\alpha^i) && (2.217)\\
&= \sum_{i=1}^{k-2} \mathbf{e}_i \cup \left(\sum_{\substack{1\le j_1 \le n-2 \\ 1 \le j_2 < \cdots < j_q \le n-2 \\ j_1 \neq j_i}} \alpha^{ij_1\cdots j_q} \tilde{e}_{j_1} \otimes \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_q} \right) && (2.218)\\
&= \sum_{i=1}^{k-2} \mathbf{e}_i \cup \left(\sum_{\substack{1\le j_1 \le n-2 \\ 1 \le j_2 < \cdots < j_q \le n-2 \\ j_1 \neq j_i}} \alpha^{ij_1\cdots j_q} \mathbf{e}_{j_1} \otimes \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_q} \right) && (2.219)\\
&= \sum_{i=1}^{k-2} \sum_{\substack{1\le j_1 \le n-2 \\ 1 \le j_2 < \cdots < j_q \le n-2 \\ j_1 \neq j_i}} \alpha^{ij_1\cdots j_q} (\mathbf{e}_i \cup \mathbf{e}_{j_1}) \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_q} && (2.220)
\end{aligned}
$$

Recall that the cup product $\mathbb{Z}^{k-2} \otimes \mathbb{Z}^{k-2} \to \mathbb{Z}$ defines a full rank symmetric bi-linear form Q.

$$
\begin{aligned}
&= \sum_{i=1}^{k-2} \sum_{\substack{1\le j_1 \le n-2 \\ 1 \le j_2 < \cdots < j_q \le n-2 \\ j_1 \neq j_i}} Q(\mathbf{e}_i, \mathbf{e}_{j_1}) \alpha^{ij_1\cdots j_q} \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_q} && (2.221)\\
&= \sum_{i=1}^{k-2} \sum_{\substack{j_1=1 \\ j_1 \neq j_i}}^{n-2} \left(\sum_{1 \le j_2 < \cdots < j_q \le n-2} Q(\mathbf{e}_i, \mathbf{e}_{j_1}) \alpha^{ij_1\cdots j_q} \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_q} \right) && (2.222)\\
&= \sum_{i=1}^{k-2} \sum_{\substack{j=1 \\ j \neq j_i}}^{n-2} \left(\sum_{1 \le j_1 < \cdots < j_{q-1} \le n-2} Q(\mathbf{e}_i, \mathbf{e}_{j}) \alpha^{ijj_1\cdots j_{q-1}} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{q-1}} \right) && (2.223)\\
&= \sum_{J \in I_{q-1}^{n-2}} \left(\sum_{i=1}^{k-2} \sum_{\substack{j=1 \\ j \notin J}}^{n-2} Q(\mathbf{e}_i, \mathbf{e}_j) \alpha^{ijJ} \right) \mathbf{e}_J && (2.224)
\end{aligned}
$$

To simplify things we will only look at the coefficients now

$$d(\mathbf{x})^J = \sum_{i=1}^{k-2} \sum_{\substack{j=1 \\ j \notin J}}^{n-2} Q(\mathbf{e}_i, \mathbf{e}_j) \alpha^{ijJ} \tag{2.225}$$

$$= \sum_{i=1}^{k-2} \sum_{\substack{j=1 \\ j \notin J}}^{n-2} Q_{ij} \alpha^{ijJ} \tag{2.226}$$

$$= \sum_{i=1}^{k-2} \sum_{j=1}^{n-2} Q_{ij} \alpha^{ijJ} \tag{2.227}$$

This can be thought of as taking the dot product of Q and α^J in $\mathbb{Z}^{k-2} \otimes \mathbb{Z}^{n-2}$. Of course the j^{th} column of α^J is zero whenever $j \in J$. We can imagine α^J to be in some subspace $V \subset \mathbb{Z}^{k-2} \otimes \mathbb{Z}^{n-2}$ which is isomorphic to $\mathbb{Z}^{k-2} \otimes \mathbb{Z}^{n-q-1}$. The orthogonal complement of this subspace $V^{\perp}$ is the set of matrices where the j^{th} column is zero if $j \notin J$. Since Q is full rank, we know that $Q \notin V^{\perp}$ and thus there exists a $\alpha^J \in V$ such that $Q \cdot \alpha^J \neq 0$. To show that in fact $Q \cdot \alpha^j = 1$ requires additional information.

From the Orlik and Raymond's classification theorem, Theorem 2.34, we know that B is a connected sum of $\mathbb{CP}^2$'s, $\overline{\mathbb{CP}^2}$'s, and $S^2 \times S^2$'s. This means that in the 'standard' basis of $H^2(B)$, the matrix represenation of Q, which we will denote by $Q' \in \mathbb{Z}^{k-2} \otimes \mathbb{Z}^{k-2}$, is a block diagonal matrix with entries $[\pm 1]$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In particular, the columns of Q' are primitive vectors in $\mathbb{Z}^{k-2}$. We are not using the 'standard' basis which means that $Q = P^{-1} Q' P$ for some $P \in GL(k-2, \mathbb{Z})$. Note however that P^{-1} acts on the column vectors and sends primitive vectors to primitive vectors. Similarly P sends the column vectors to linear combinations of columns vectors, and since this is invertible over the integers it sends column vectors to vectors which remain primitive. In particular the column Q_{*l} for $l \notin J$ is primitive. This means there exists a vector $\mathbf{w} \in \mathbb{Z}^{k-2}$ such that $Q_{*l} \cdot \mathbf{w} = Q_{il} w^i = 1$. Now define $\alpha^{ijJ} := w^i$ for $j = l$ and $\alpha^{ijJ} := 0$ for $j \neq l$. By construction $Q \cdot \alpha^J = 1$. Therefore $d \colon E_2^{2,q} \to E_2^{4,q-1}$ is surjective and

$$E_3^{4,q} = 0 \tag{2.228}$$

for all $q \neq n-2$.

We are then left only to compute

$$E_3^{2,q} = \frac{\ker\left(d \colon E_2^{2,q} \to E_2^{4,q-1}\right)}{\operatorname{im}\left(d \colon E_2^{2,q+1} \to E_2^{4,q}\right)} \tag{2.229}$$

Since all three spaces are torsion free, $d \colon E_2^{2,q} \to E_2^{4,q-1}$ is surjective, and $d \colon E_2^{2,q+1} \to E_2^{4,q}$ is injective with a primitive image, we conclude that $E_3^{2,q}$ is torsion free. In particular this means it can be computed simply by counting dimensions;

$$\dim(E_3^{2,q}) = \dim(\ker) - \dim(\operatorname{im}) \tag{2.230}$$

$$= \dim(E_2^{2,q}) - \dim(E_2^{4,q-1}) - \dim(E_2^{0,q+1}) \tag{2.231}$$

$$= \dim\left(\mathbb{Z}^{k-2} \otimes \Lambda^q(\mathbb{Z}^{n-2})\right) - \dim\left(\Lambda^{q-1}(\mathbb{Z}^{n-2})\right) - \dim\left(\Lambda^{q+1}(\mathbb{Z}^{n-2})\right) \tag{2.232}$$

$$= (k-2)\binom{n-2}{q} - \binom{n-2}{q-1} - \binom{n-2}{q+1} \tag{2.233}$$

$$= b_{2+q} \tag{2.234}$$

where b_{2+q} is defined in Equation (2.180). Thus

$$E_3^{2,q} = \mathbb{Z}^{b_{2+q}} \tag{2.235}$$

for all q.

As stated previously, $E_3 = E_\infty$. Therefore when we collect Equations (2.186), (2.187), (2.188), (2.214),(2.228), and (2.235) we see that Equation (2.179) is satisfied and the proof is complete. □

In general the knowing the infinity page of a Serre spectral sequence is useful for computing rational cohomology groups. However the following simple topological lemma, attributed to Jiahao Hu, shows that our specific spectral sequence has additional properties.

Lemma 2.85. *Let B be simply connected and $F \to E \to B$ be a Serre fibration with associated integral cohomology spectral sequence terminating on the r^{th} page, i.e. $E_\infty = E_r$.*

1. *If $E_\infty^{p,q}$ is torsion-free for all $p+q=j$, then $H^j(M;\mathbb{Z})$ is torsion-free.*
2. *If for each j, $E_\infty^{p,q} = 0$ for all but at most one pair (p,q) such that $p+q=j$, then*

$$H^*(E;\mathbb{Z}) \cong Tot^*(E_\infty) \tag{2.236}$$

as graded rings, where

$$Tot^j(E_\infty) = \bigoplus_{p+q=j} E_\infty^{p,q}. \tag{2.237}$$

The first statement of Lemma 2.85 immediately tells us that the cohomology groups (and thus also the homology groups) of M are torsion free. In Lemma 2.86 we explicitly compute the homology groups of the manifolds $M(n,k)$ described in Conjecture A to show that they are identical to the homology groups of M. This proves Part 7 of Theorem D in the special case where M satisfies the hypotheses of Lemma 2.84.

Lemma 2.86. *The Betti numbers of the manifold*

$$M(n,k) = \#_{j=0}^{n-3}\left(j\binom{n-2}{j+1} + (k-n)\binom{n-3}{j}\right) S^{2+j} \times S^{n-j} \tag{2.238}$$

for $k \geq n$ are

$$b_i = \begin{cases} 1 & i = 0, n+2 \\ 0 & i = 1, n-1 \\ (k-2)\binom{n-2}{i-2} - \binom{n-2}{i-3} - \binom{n-2}{i-1} & 2 \leq i \leq n \end{cases} \tag{2.239}$$

Proof. Since $M(n,k)$ is a closed, simply connected $(n+2)$-manifold we know $b_0 = 1 = b_{n+2}$ and $b_1 = 0 = b_{n+1}$. For b_i with $2 \leq i \leq n$ we examine Equation (2.238). For each $0 \leq j \leq n-3$, the coefficient in Equation (2.238) contributes to both b_{2+j} and b_{n-j}. Stated another way, b_i is equal to the $j = i-2$ coefficient in Equation (2.238) plus the $j = n-i$ coefficient. This tells us the Betti numbers of $M(n,k)$ are

$$b_i = \left((i-2)\binom{n-2}{i-1} + (k-n)\binom{n-3}{i-2}\right) + \left((n-i)\binom{n-2}{n-i+1} + (k-n)\binom{n-3}{n-i}\right) \tag{2.240}$$

for $2 \leq i \leq n$. We can now begin to simplify above expression

$$b_i = (i-2)\binom{n-2}{i-1} + (k-n)\binom{n-3}{i-2} + (n-i)\binom{n-2}{i-3} + (k-n)\binom{n-3}{i-3} \tag{2.241}$$

$$= (k-n)\binom{n-2}{i-2} + (n-i)\binom{n-2}{i-3} + (i-2)\binom{n-2}{i-1} \tag{2.242}$$

$$\begin{aligned} &= (k-2)\binom{n-2}{i-2} - \binom{n-2}{i-3} - \binom{n-2}{i-1} \\ &\quad + (n-i+1)\binom{n-2}{i-3} + (i-1)\binom{n-2}{i-1} - (n-2)\binom{n-2}{i-2} \end{aligned} \tag{2.243}$$

The last line is written as the sum of Equation (2.180) and a remainder term. Expanding this remainder in terms of factorials gives

$$\frac{(n-2)!}{(i-3)!(n-1-i)!}\left(\frac{1}{n-i} + \frac{1}{i-2} - \frac{n-2}{(i-2)(n-i)}\right) \tag{2.244}$$

which simplifies to 0. Thus

$$b_i = (k-2)\binom{n-2}{i-2} - \binom{n-2}{i-3} - \binom{n-2}{i-1} \tag{2.245}$$

as desired. ☐

Theorem 2.87. *Let M^{n+2} have k rods and satisfy the hypotheses of Lemma 2.84. Then M satisfies Conjecture A in homology. That is,*

$$H_i(M;\mathbb{Z}) \cong H_i(M(n,k);\mathbb{Z}) \tag{2.246}$$

for all $i = 0, \ldots, n+2$, where $M(n,k)$ is the manifold described in Conjecture A.

Proof. We first use Lemma 2.84 to compute $Tot^i(M) := \bigoplus_{p+q=i} E_\infty^{p,q}$ and find

$$Tot^j(E_\infty) = \mathbb{Z}^{b_i} \tag{2.247}$$

where

$$b_i = \begin{cases} 1 & i = 0, n+2 \\ 0 & i = 1, n-1 \\ (k-2)\binom{n-2}{i-2} - \binom{n-2}{i-3} - \binom{n-2}{i-1} & 2 \leq i \leq n. \end{cases} \tag{2.248}$$

Now using Lemma 2.85 and Poincare duality we see

$$H_i(M;\mathbb{Z}) \cong \mathbb{Z}^{b_i}. \tag{2.249}$$

This agrees the homology groups computed for $M(n,k)$ in Lemma 2.86, thus

$$H_i(M;\mathbb{Z}) \cong H_i(M(n,k);\mathbb{Z}) \tag{2.250}$$

as desired. ☐

Remark 2.88. The second statement from Lemma 2.85 tells us that the cohomology ring of M is equal to the cohomology ring of E_∞, denoted by $Tot^*(E_\infty)$. Having found that $E_\infty = E_3$ in Lemma 2.84, we can say

$$H^*(M;\mathbb{Z}) \cong \left(\frac{\ker(d_2\colon E_2 \to E_2)}{\operatorname{im}(d_2\colon E_2 \to E_2)}, \cup\right) \tag{2.251}$$

where the cup product is then inherited from the cup product on E_2 described in Equation (2.181). Using the isomorphism

$$E_2 \cong \Lambda^*(\mathbb{Z}^{n-2}) \otimes H^*(B) \tag{2.252}$$

and the graded Leibniz rule, the differential d_2 can be described by

$$d_2(\mathbf{e}_i) = \tilde{e}_i \tag{2.253}$$

$$d_2(\mathbf{w}) = 0 \tag{2.254}$$

where $\mathbf{w} \in H^*(B)$, $\mathbf{e}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_{n-2}\} \subset \mathbb{Z}^{n-2}$ is a standard basis element, and $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_{n-2}) \in H^2(B; \mathbb{Z}^{n-2})$ is the Euler class.

Recall that since B is a simply connected 4-manifold with torsion-free cohomology, its cohomology ring $H^*(B)$ is fully described $(H_2(B;\mathbb{Z}), Q)$, where Q is the intersection form. Using the isomorphism $\Psi_*\colon \ker(A) \to H_2(B;\mathbb{Z})$, Theorem B neatly describes $(H_2(B;\mathbb{Z}), Q)$ in terms of the rod structures. Similarly, Theorem 2.81 neatly describes the Euler class in terms of the rod structures. The existence of both of these theorems, along with the fact that Lemma 2.85 gives a way to explicitly compute the cohomology ring in terms of $H^*(B)$ and the Euler class, implies that there ought to exist an elegant formula which computes the cohomology ring of M in terms of the rod structures. Unfortunately as of the writing of this dissertation, such a formula has not been computed.

2.10 Plumbing

Characterizing the domain of outer communication can be done in two main ways. One way is by compactifying the space and classifying it as a simply connected manifold. In the next section we use this method for spatial dimensions 4, 5, and 6 and will explain the obstructions that arise in extending these results into higher dimensions. The other method is by breaking up the domain of outer communication into simpler pieces, then gluing them back together. This is the method of plumbing constructions which will be discussed in this section, whose main purpose is to provide the proof of Theorem E. Since this is a purely topological result, any mention of dimension in this section will refer only to the spatial dimension.

In Theorem E the domain of outer communication is broken up into components based on the number of corners they contain. The pieces which contain no corners are either the asymptotic end M_{end}, or a piece which is homeomorphic to $[0,1] \times D^2 \times T^{n-1}$ which we denote C_k^{n+2}. When a piece contains a single corner it is a neighborhood of that corner, and thus by admissibility it is a tubular neighborhood of a torus $B^4 \times T^{n-2}$. This part of the analysis is identical in the 4-dimensional case and is covered in [25, Theorem 1]. The differences in higher dimensions occur when looking at components which contain at least two corners. A component with exactly two corners will turn out to be the product of a torus T^{n-3} with a disk bundle over a 3-manifold rather than a 2-sphere. For components with more than two corners, we will have to define a generalization of plumbing where the fibers and base space are not the same dimensions.

Theorem 2.89. *Let N be a neighborhood in the orbit space of a portion of the axis Γ with 2 corners and no horizon rods. The total space over N is isomorphic to $T^{n-3} \times \xi$ where the action of $T^n \cong T^{n-3} \times T^3$ acts componentwise. Here ξ is a D^2-bundle over $X \in \{S^3, L(p,q), S^1 \times S^2\}$. The formula for the topologies of X and ξ are computable from the rod structures.*

Proof. The rod diagram of our space has three axis rods separated by two admissible corners. Using Remark 2.21 we can, without changing the topology, transform our rod structures into the form of Equation (2.19), where the last $n-3$ entries of each rod structure are 0. Now the last $n-3$ Killing fields never vanish, hence the total space is a product manifold $T^{n-3} \times \xi$, where the T^n-action splits naturally into T^{n-3} acting on itself and T^3 acting on ξ. Here ξ denotes the manifold given by the rod diagram $\{(1,0,0), (0,1,0), (q,r,p)\}$.

The space ξ can be deformation retracted to the middle axis rod where the second Killing field vanishes. This rod represents a closed manifold $X \in \{S^3, L(p,q), S^1 \times S^2\}$. Fibers over this space correspond to rays extending out from the middle axis rod; see Figure 2.26. Each point in the interior of the middle axis rod corresponds to an entire T^2 while a ray terminating at that point corresponds to $D^2 \times T^2$. Each of our two corners corresponds to an S^1 in our base space X while the adjacent axis rods corresponds to $D^2 \times S^1$. Therefore ξ is a D^2-bundle over X.

To determine the topology of X and ξ, we look at the rod structures. If they are linearly dependent, then by admissibility the rod structures are $\{(1,0,0),(0,1,0),(1,r,0)\}$. Again, there is a free S^1 action, and after factoring out this action, it remains to analyze a disk bundle in 4-dimensions generated by the diagram with rod structures $\{(1,0),(0,1),(1,r)\}$. The base space of this latter bundle is S^2, and its self-intersection number, or equivalently the characteristic number of its Euler class is r, see [25]. In particular, we have $X = S^1 \times S^2$.

If the rod structures $\{(1,0,0),(0,1,0),(q,r,p)\}$ are linearly independent, the base space $X = L(p,q)$. Recall that $L(1,q) = S^3$ for all q. The number of distinct disk bundles, or equivalently $SO(2)$-bundles, over X is determined by the number of homotopy classes of maps from X to $\mathbb{CP}^\infty$, the universal classifying space for $SO(2)$. Since $\mathbb{CP}^\infty$ is an Eilenberg-Maclane space of type $K(\mathbb{Z},2)$, homotopy classes of maps from X to $K(\mathbb{Z},2)$ are classified by $H^2(X;\mathbb{Z}) \cong \mathbb{Z}_p$. The element in a cohomology group which corresponds to a specific bundle ξ is called the *Euler class* $e(\xi)$. This is a total invariant of the bundle ξ.

By the uniqueness of the Hermite normal form, the $r \in \mathbb{Z}_p \cong H^2(L(p,q);\mathbb{Z})$ in the rod structure is uniquely determined for each equivariant homeomorphism class of ξ. Conversely, for each class in $H^2(L(p,q);\mathbb{Z}) \cong \mathbb{Z}_p$ there is a unique disk bundle over $L(p,q)$. Each of these disk bundles admits an effective T^3 action, with T^1 acting on the fibers, and a T^2 acting on the base $L(p,q)$. Thus to each of these disk bundle corresponds a rod diagram with 3 axis rods and two admissible corners. This gives us a one-to-one correspondence between $0 \le r < p$ and $e(\xi) \in H^2(L(p,q),\mathbb{Z})$. Furthermore, for the trivial disk bundle $D^2 \times L(p,q)$ both $e(\xi) = 0$ and $r = 0$. This is because the quotient of $L(p,q)$ by its T^2-action can be represented as an interval where the $(1,0)$ and the (q,p) circles degenerate at the end points. Similarly, the quotient of D^2 by S^1 can be represented by a half open interval where the circle degenerates at the one end point. Taking the product of these two spaces gives us a rod diagram of $\{(1,0,0),(0,1,0),(q,0,p)\}$, from which we deduce that $r = 0$. $\square$

The above theorem shows that the total space over a neighborhood of three consecutive axis rods $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is $T^{n-3} \times \xi$. There is a subtorus T^3 which leaves the slices $\{\varphi\} \times \xi \hookrightarrow T^{n-3} \times \xi$ invariant, and is spanned by the rod structures $\{\mathbf{u},\mathbf{v},\mathbf{w}\} \subset \mathbb{Z}^n$ as follows:

$$T^3 \cong \frac{\mathrm{span}_{\mathbb{R}}\{\mathbf{u},\mathbf{v},\mathbf{w}\}}{\mathbb{Z}^n} \subset \frac{\mathbb{R}^n}{\mathbb{Z}^n} \cong T^n. \tag{2.255}$$

However, $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is not necessarily a primitive set. We may therefore need to perform an integral version of the Gram-Schmidt process to produce a primitive set. Note that in the following lemma, if ξ is a disk bundle over $S^1 \times S^2$, then $p = 0$, $q = 1$, and Equation (2.256) is trivially satisfied.

Lemma 2.90. *Let $\{\mathbf{u},\mathbf{v},\mathbf{w}\} \subset \mathbb{Z}^n$ be consecutive rod structures whose neighborhood lifts to $T^{n-3} \times \xi$ in the total space, where ξ is a disk bundle over $L(p,q)$ with Euler class determined by r. Then there exists a unique vector $\mathfrak{p} \in \mathbb{Z}^n$ satisfying*

$$\mathbf{w} = q\mathbf{u} + r\mathbf{v} + p\mathfrak{p}. \tag{2.256}$$

Furthermore $\{\mathbf{u},\mathbf{v},\mathfrak{p}\} \subset \mathbb{Z}^n$ forms a primitive set.

Proof. The first step is to show that $\mathfrak{p}$ is well defined by (2.256), and is indeed a primitive vector. We put $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ into its Hermite normal form and let A be a coordinate transformation which satisfies $A\mathbf{u} = \mathbf{e}_1$, $A\mathbf{v} = \mathbf{e}_2$, and $A\mathbf{w} = q\mathbf{e}_1 + r\mathbf{e}_2 + p\mathbf{e}_3$. Let $\mathfrak{p} = A^{-1}\mathbf{e}_3$, then clearly $\mathbf{w} - q\mathbf{u} - r\mathbf{v} = p\mathfrak{p}$ is divisible by p. Furthermore, since $\mathbf{e}_3$ is a primitive vector we obtain that $\mathfrak{p} = A^{-1}\mathbf{e}_3$ is primitive as well.

Note that $\{\mathbf{u},\mathbf{v},\mathfrak{p}\}$ is a primitive set if and only if $\mathrm{Det}_3(\mathbf{u},\mathbf{v},\mathfrak{p}) = 1$. By multi-linearity of determinants and Equation (2.256), we compute

$$\mathrm{Det}_3(\mathbf{u}, \mathbf{v}, \mathfrak{p}) = p^{-1}\,\mathrm{Det}_3(\mathbf{u}, \mathbf{v}, \mathbf{w}) = p^{-1}\,\mathrm{Det}_3(\mathbf{e}_1, \mathbf{e}_2, (q, r, p, 0, \ldots, 0)) = 1, \tag{2.257}$$

where the next to last equality follows from the coordinate invariance of Det_3. It follows that $\{\mathbf{u}, \mathbf{v}, \mathfrak{p}\}$ forms a primitive set. □

We now turn to handle portions of the axis with more than two consecutive corners. Portions of the axis with $l+2$ corners will be shown to be $l+1$ disk bundles glued together. This gluing will be a generalization of plumbing. This higher dimensional plumbing, which we will refer to as *toric plumbing* is not a straightforward generalization of other linear plumbings due to the extra circles. For each pair of disk bundles we will define a *plumbing vector* which distiguishes the different ways two disk bundles can be plumbed together. Figure 2.25 shows some examples of the same two disk bundles being plumbed together in different ways to form non-homeomorphic total spaces.

Consider a section of the axis rod with rod structures $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{l+2}\}$. From Theorem 2.89, a neighborhood of each consecutive triple of rod structures $\{\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}\}$ lifts to the total space M as a product $\boldsymbol{\xi}_i \cong T^{n-3} \times \xi_i \subset M$ where ξ_i is a disk bundle with Euler class determined by r_i over either $L(p_i, q_i)$ or $S^1 \times S^2$ if $p_i = 0$. We can arrange the rod structures into Hermite normal form $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{l+2}\}$ so that $A\mathbf{v}_i = \mathbf{w}_i$. Recall that A may not be unique, but the $\mathbf{w}_i$'s are. By remark 2.21, the first three elements are $\mathbf{w}_0 = \mathbf{e}_1$, $\mathbf{w}_1 = \mathbf{e}_2$, and $\mathbf{w}_2 = (q_0, r_0, p_0, 0, \ldots, 0)$. For each i such that $p_i \neq 0$, there exist by Lemma 2.90, vectors $\mathfrak{p}_i$ satisfying

$$\mathbf{w}_{i+2} = q_i\mathbf{w}_i + r_i\mathbf{w}_{i+1} + p_i\mathfrak{p}_i. \tag{2.258}$$

When $p_i = 0$, we define $\mathfrak{p}_i = \mathbf{0}$, and (2.258) is trivially satisfied.

Definition 2.91. The vectors $\mathfrak{p}_i$ satisfying (2.258) are referred to as *plumbing vectors.*

Remark 2.92. If B is a change of coordinates, i.e. a unimodular matrix, then $\{\mathbf{v}_0, \ldots, \mathbf{v}_{l+2}\}$ and $\{B\mathbf{v}_0, \ldots, B\mathbf{v}_{l+2}\}$ have the same Hermite normal form and thus the same plumbing vectors. Therefore the plumbing vectors do not depend on the choice of coordinates, but rather depend only on the topology and toric structure of the total space.

While the set of plumbing vectors is uniquely determined by a set of rod structures, they are not uniquely determined by a set of topologies ξ_i. In Figure 2.25, we present two pairs of examples of the same disk bundles being plumbed using different plumbing vectors. From Remark 2.92 we know that the total spaces will actually have different toric structures, and not just differ by a change of coordinates. Furthermore, in these examples the boundaries of the total spaces have different fundamental groups. Thus plumbing vectors can affect the actual topology of the total space, not just its toric structure.

Plumbing vectors satisfy a number of relations, the first of which is the collection of recursion equations

$$\begin{gathered} \mathbf{w}_0 = \mathbf{e}_1,\ \mathbf{w}_1 = \mathbf{e}_2, \\ \mathbf{w}_{i+2} = q_i\mathbf{w}_i + r_i\mathbf{w}_{i+1} + p_i\mathfrak{p}_i \text{ if } p_i \neq 0, \text{ and} \\ \mathfrak{p}_i = 0 \text{ if } p_i = 0, \end{gathered} \tag{2.259a}$$

for $i = 0, 1, \ldots, l$. These relations are also used to define the plumbing vectors. The next two conditions are admissibility and primitivity. Adjacent rods $\{\mathbf{w}_{i+1}, \mathbf{w}_{i+2}\}$ need to be admissible, i.e. $\mathrm{Det}_2(\mathbf{w}_{i+1}, \mathbf{w}_{i+2}) = 1$. By using the recursion relations and the multilinearity of determinants, this can be rewritten as

$$\mathrm{Det}_2(\mathbf{w}_{i+1}, q_i\mathbf{w}_i + p_i\mathfrak{p}_i) = 1. \tag{2.259b}$$

The primitivity condition

$$\mathrm{Det}_3\{\mathbf{w}_i, \mathbf{w}_{i+1}, \mathfrak{p}_i\} = 1, \tag{2.259c}$$

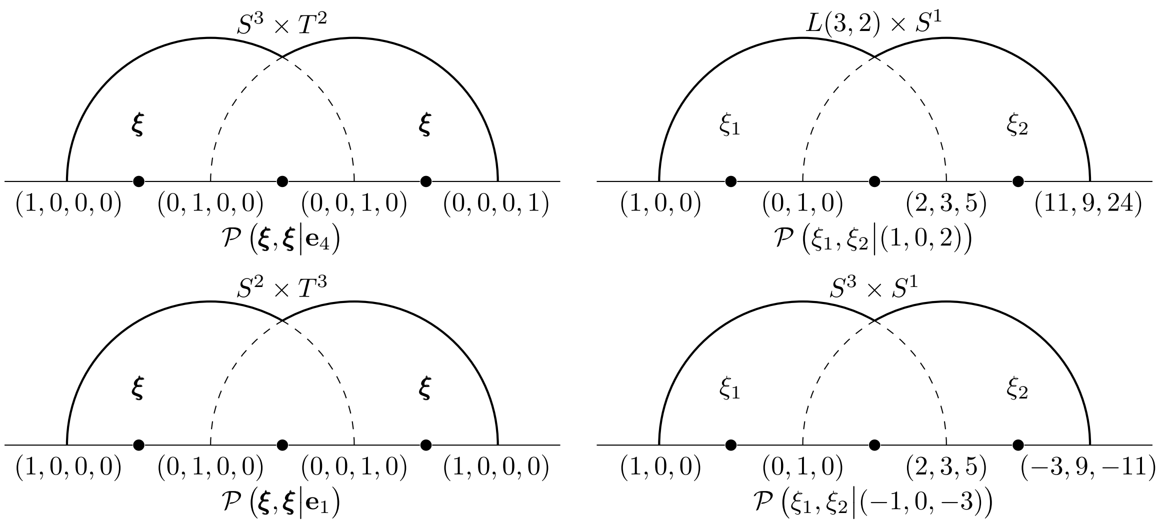


Figure 2.25: The left two examples are different plumbings of the trivial bundle $\boldsymbol{\xi} = S^1 \times D^2 \times S^3$ with itself. In the top left example the plumbing vector is $\mathfrak{p}_1 = \mathbf{e}_4$ while in the bottom left example the plumbing vector is $\mathfrak{p}_1 = \mathbf{e}_1$. The right two examples are different plumbings of ξ_1 over $L(5,2)$ with Euler class determined by 3 and ξ_2 over $L(7,3)$ with Euler class determined by 2. The plumbing vector for the top right example is $\mathfrak{p}_1 = (1,0,2)$ while the plumbing vector for the bottom right example is $\mathfrak{p}_1 = (-1,0,-3)$. We can see that for each pair the topology and toric structure of the total space is different because the plumbing vectors are different. The notation $\mathcal{P}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{u})$ refers to the toric plumbing of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ with plumbing vector $\mathbf{u}$ as defined in Definition 2.95.

when $\mathfrak{p}_i \neq 0$, is guaranteed by Lemma 2.90. If $\mathfrak{p}_i = 0$ then this condition does not apply. Finally, we get two conditions from the fact that $\{\mathbf{w}_0, \ldots, \mathbf{w}_{l+2}\}$ is in Hermite normal form. The first tells us what the last possible nonzero entry is for each plumbing vector. Let $\mathfrak{p}_i = (\mathfrak{p}_{i1}, \ldots, \mathfrak{p}_{in})$, and for convenience define $\mathfrak{p}_{-1}$ to be $\mathbf{e}_2$. If $\mathfrak{p}_{ij} = 0$ for all $j \geq r$ and $-1 \leq i < k$, then

$$\mathfrak{p}_{kj} = 0 \tag{2.259d}$$

for all $j > r$. The second restricts the size of the other entries. If $\mathfrak{p}_{k,r_k}$ is the last non-zero entry of $\mathfrak{p}_k$ and if $\mathfrak{p}_{i,r_k} = 0$ for all $0 \leq i < k$, then $w_{k+2,k}$ is a pivot in the Hermite normal form $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{l+2}\}$ and

$$0 \leq w_{k+2,j} < w_{k+2,r_k}, \tag{2.259e}$$

for $j < r_k$. These relations are collectively referred to as the *plumbing relations*.

Using the plumbing relations one can find all possible values for $\mathfrak{p}_0$. If the base space of ξ_0 is $S^1 \times S^2$ then $p_0 = 0$ so from Equation (2.259a) $\mathfrak{p}_0 = 0$. If $p_0 \neq 0$ then we use Equation (2.259d) to see that $\mathfrak{p}_0 = (a, b, c, 0, \ldots, 0)$ for some integers a, b, and c. This makes $\mathbf{w}_{2,3}$ a pivot, and using the recursive definition (2.259a) in Equation (2.259e) gives us $0 \leq a < c$ and $0 \leq b < c$. Finally the primitivity condition (2.259c) forces $c = 1$ and $\mathfrak{p}_0 = \mathbf{e}_3$. Since $\mathfrak{p}_0$ is determined only by the topology of ξ_0 and not by any plumbing information we do not include it when describing the plumbing of ξ_0 and ξ_1.

Remark 2.93. The plumbing relations (2.259) together with the topologies of the disk bundles completely determine $\mathfrak{p}_1$ in the special case that ξ_0 is a bundle over $S^1 \times S^2$. In the first case when ξ_1 is also a bundle over $S^1 \times S^2$ we have $\mathfrak{p}_1 = 0$ by Equation (2.259a). The second case is where ξ_1 is a disk bundle over a lens space. We now know that $\mathfrak{p}_0 = 0$ so from Theorem 2.89 and Equation (2.259a) $\mathbf{w}_2 = (1, r_0, 0, \ldots, 0)$ and $\mathbf{w}_3 = (r_1, r_0 r_1 + q_1, 0, \ldots, 0) + p_1 \mathfrak{p}_1$. Since $\{\mathbf{w}_1, \mathbf{w}_2, \mathfrak{p}_1\}$ forms a primitive set we know that $\mathfrak{p}_1 = (a, b, c, 0, \ldots, 0)$ for some integers a, b, c. Equation (2.259c) gives $1 = \mathrm{Det}_3\{\mathbf{e}_2, \mathbf{w}_2, \mathfrak{p}_1\} = c$. Now applying

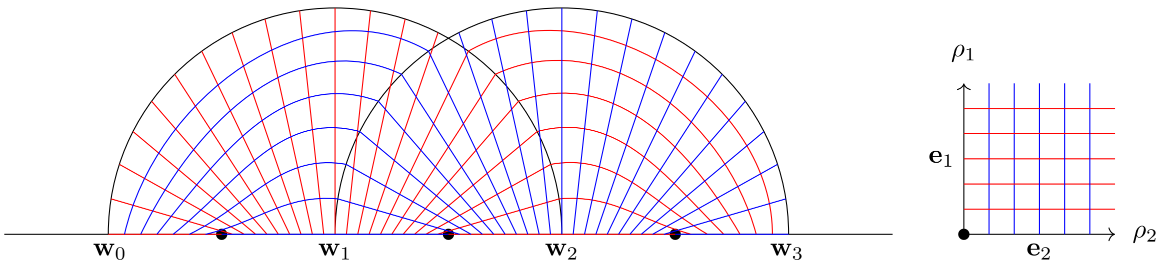


Figure 2.26: In the figure above we have $\mathbf{w}_0 = \mathbf{e}_1$, $\mathbf{w}_1 = \mathbf{e}_2$, $\mathbf{w}_2 = (q_0, r_0, p_0)$, and $\mathbf{w}_3 = q_1\mathbf{w}_1 + r_1\mathbf{w}_2 + p_1\mathfrak{p}_1$, in accordance with Equation (2.259a). The diagram on the left shows a toric plumbing of two disk bundles, $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}_1$, over lens spaces $L(p_0, q_0)$ and $L(p_1, q_1)$, along plumbing vector $\mathfrak{p}_1$. The fibers of $\boldsymbol{\xi}_0$ are given by rays emanating from $\mathbf{w}_1$, while the fibers of $\boldsymbol{\xi}_1$ are given by rays emanating from $\mathbf{w}_2$. One can see that in the overlap that the fibers and sections of the base switch roles between $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}_1$. The overlap is shown again on the right, but this time a transformation matrix has been applied which sends $\mathbf{w}_1$ to $\mathbf{e}_1$ and $\mathbf{w}_2$ to $\mathbf{e}_2$. This allows us to view the overlap as homeomorphic to $\mathbb{C}^2 \times S^1$ with coordinates $(\rho_1 e^{i\phi_1}, \rho_2 e^{i\phi_2}, \phi_3)$.

the pivot condition, Equation (2.259e), we see $0 \leq ar_1p_1 < p_1$. But from Theorem 2.89, $0 \leq r_1 < p_1$ so $a = 0$. Similarly, we get the condition that $0 \leq p_1 b + r_0 r_1 + q_1 < p_1$ which uniquely specifies b. Therefore when $\mathfrak{p}_0 = 0$ and $p_1 \neq 0$ there is only one vector $\mathfrak{p}_1 = (0, b, 1, 0, \ldots, 0)$ which satisfies all the plumbing relations from Equations (2.259).

Proposition 2.94. *There is a one-to-one correspondence between collections of admissible rod structures* $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{l+2}\} \subset \mathbb{Z}^n$ *in Hermite normal form and collections of bundles* $\{\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_l\}$ *paired with a set of primitive vectors* $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\} \subset \mathbb{Z}^n$ *satisfying Equations* (2.259)*.*

Proof. Let $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{l+2}\} \subset \mathbb{Z}^n$ be a collection of admissible rod structures in Hermite normal. Theorem 2.89 shows that from each successive triple $\{\mathbf{w}_i, \mathbf{w}_{i+1}, \mathbf{w}_{i+2}\}$, there is a unique bundle $\boldsymbol{\xi}_i$ which is the lift of a neighborhood of these three rods to the total space M. The rod structures also give the integers q_i, r_i, and p_i used in Definition 2.91 to define the plumbing vectors. Since these $\mathfrak{p}_i$ are indeed plumbing vectors, they satisfy the full plumbing relations in Equations (2.259).

Conversely, let $\{\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_l\}$ be a collection of bundles and let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\} \subset \mathbb{Z}^n$ be a collection of vectors satisfying Equations (2.259). These equations do not make sense without defining rod structures $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{l+2}\}$, which is done in Equation (2.259a). Note that the $\mathbf{w}_i$ are unique since the integers q_i, r_i, and p_i are uniquely defined by each $\boldsymbol{\xi}_i$ in Theorem 2.89. By hypothesis, $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{l+2}\}$ satisfies Equation (2.259b) which can be rewritten as $\mathrm{Det}_2(\mathbf{w}_{i+1}, \mathbf{w}_{i+2}) = 1$, thus proving admissibility. To show that the rod structures are in Hermite normal form, we must verify that the matrix composed of column vectors $\mathbf{w}_i$ satisfies the conditions of Lemma 2.20. To do so, let w_{k+2,h_k} be a pivot. That is, w_{k+2,h_k} is the last nonzero component of $\mathbf{w}_{k+2}$ and $w_{i,h_k} = 0$ for all $0 \leq i < k+2$. Since $\mathbf{w}_{i+2}$ is a linear combination of $\mathbf{w}_i$, $\mathbf{w}_{i+1}$ and $\mathfrak{p}_i$, this is equivalent to the conditions of Equation (2.259e). Therefore whenever w_{k+2,h_k} is a pivot, we have $0 \leq w_{k+2,j} < w_{k+2,h_k}$ for all $j < h_k$. This proves that $\{\mathbf{w}_0, \ldots, \mathbf{w}_{l+2}\}$ is indeed in its unique Hermite normal form. □

Definition 2.95. Let $\boldsymbol{\xi}_i \cong T^{n-3} \times \xi_i$, $i = 0, \ldots, l$, where each ξ_i is a D^2-bundle over either a lens space or $S^1 \times S^2$. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\} \subset \mathbb{Z}^n$ be a collection of primitive vectors satisfying the plumbing relations from Equations (2.259). We define the *toric plumbing* of $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_l$ *along the plumbing vectors* $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ to be the $(n+2)$-dimensional simple T^n-manifold given by rod structures $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_l\}$ where the $\mathbf{w}_i$ are determined by Equations (2.259a). This simple T^n-manifold is denoted by $\mathcal{P}\left(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_l \middle| \mathfrak{p}_1, \ldots, \mathfrak{p}_l\right)$.

Toric plumbing is a generalization of standard equivariant plumbing. In the latter the base and the fiber have the same dimensions, while in the former they do not. To see that this is related to standard plumbing, we restrict ourselves, for the sake of clarity, to $n = 3$ and consider the simple T^3-manifold $\mathcal{P}\left(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1 \middle| \mathfrak{p}_1\right)$. First observe that this is a gluing of $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}_1$. We can see the first inclusion $\boldsymbol{\xi}_0 \hookrightarrow \mathcal{P}\left(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1 \middle| \mathfrak{p}_1\right)$ easily since $\{\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2\}$ is the standard rod diagram for $\boldsymbol{\xi}_0$. To find the inclusion of $\boldsymbol{\xi}_1$ simply apply the coordinate transformation A which sends $\mathbf{w}_1$ to $\mathbf{e}_1$, $\mathbf{w}_2$ to $\mathbf{e}_2$, and sends $\mathfrak{p}_1$ to $\mathbf{e}_3$ if $\mathfrak{p}_1 \neq 0$. Observe that $\{A\mathbf{w}_1, A\mathbf{w}_2, A\mathbf{w}_3\}$ is the standard rod diagram for $\boldsymbol{\xi}_1$. The matrix A exists because of the primitivity condition from Equation (2.259c).

We will now show that the gluing map has a form similar to the gluing map from standard plumbing. The map will be from a subset of $\boldsymbol{\xi}_0$ to a subset of $\boldsymbol{\xi}_1$, as depicted by the overlap in Figure 2.26. This region is an open neighborhood of a single corner, thus is homeomorphic to $S^1 \times B^4$. In both $\boldsymbol{\xi}_0$, and $\boldsymbol{\xi}_1$ this corner represents a single circle, called a *polar circle*, in the base 3-manifold where one of the Killing fields degenerates. It is apparent that the overlap region can be thought of as a trivialization $S^1 \times B^2 \times D^2$ of the D^2-bundle $\boldsymbol{\xi}_i$ over a neighborhood of a polar circle. Here we use B^2 for a disk in the base, as opposed to D^2 for a disk fiber. Just as in standard plumbing we can see from Figure 2.26 that the D^2 fibers, in say $\boldsymbol{\xi}_0$ which are represented by rays emanating from $\mathbf{w}_1$, switch roles in the overlap with the B^2 sections in the base of $\boldsymbol{\xi}_2$, represented by rays emanating from $\mathbf{w}_2$.

In order to fully define the gluing of $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}_1$ we need some automorphism on the overlap $S^1 \times B^2 \times D^2$. The discussion above shows that B^2 and D^2 get switched which leaves the circle S^1 unaccounted for. Since the automorphism must respect the action of T^3 on $S^1 \times B^2 \times D^2$, the image of this S^1 can be represented uniquely by a homotopy class, i.e. an element of $\pi_1(T^3; \mathbb{Z}) \cong \mathbb{Z}^3$. Note however that the S^1 being sent to a vector in $\mathbb{Z}^3$ is not the polar circle $S^1 \subset S^1 \times B^2 \times D^2$ but rather an $S^1 \subset T^3$ which acts upon it. These circle actions are not unique because there are two Killing fields, the ones associated to B^2 and D^2 respectively, which vanish on the polar circle. The only criteria we have for these circle actions is that the Lie group homomorphism from T^3 to T^3 be an isomorphism. This is equivalent to saying that our circle, together with the circle actions on B^2 and D^2, forms an integral basis for $\mathbb{Z}^3$. Letting this circle action be represented by the plumbing vector $\mathfrak{p}_1 \in \mathbb{Z}^3$, we see that this is exactly the statement of Equation (2.259c) for $i = 1$. Therefore we can think of the plumbing vector, defined in Equation (2.259a), as representing the image of our circle action.

Writing a simple T^n-manifold M as a toric plumbing of disk bundles $\mathcal{P}\left(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_l \middle| \mathfrak{p}_1, \dots, \mathfrak{p}_l\right)$ facilitates the analysis of rod diagrams. Indeed $\mathcal{P}\left(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_l \middle| \mathfrak{p}_1, \dots, \mathfrak{p}_l\right)$ and $\mathcal{P}\left(\boldsymbol{\xi}'_0, \boldsymbol{\xi}'_1, \dots, \boldsymbol{\xi}'_l \middle| \mathfrak{p}'_1, \dots, \mathfrak{p}'_l\right)$ can be distinguished easily, as they are isomorphic if and only if $\boldsymbol{\xi}_j \cong \boldsymbol{\xi}'_j$ and $\mathfrak{p}_k = \mathfrak{p}'_k$ for all j and k. To see this, use Proposition 2.94 to get rod structures $\{\mathbf{w}_0, \dots, \mathbf{w}_{l+2}\}$ and $\{\mathbf{w}'_0, \dots, \mathbf{w}'_{l+2}\}$ from the disk bundles and plumbing vectors. These rod structures are automatically in their unique Hermite normal form, and therefore the two simple T^n-manifolds are isomorphic if and only if the rod structures are identical.

Remark 2.96. Given a set of bundles $\{\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_l\}$ it is difficult to determine all possible sets of vectors $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_l\}$ for which the plumbing relations (2.259) are satisfied. However it is fairly easy to check if a given set of vectors $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_l\}$ satisfies the plumbing relations for the bundles $\{\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_l\}$. First check that each $\mathfrak{p}_i$ is a primitive vector. Then simply follow the recursion equations (2.259a) to find all the $\mathbf{w}_i$. If each successive pair $\{\mathbf{w}_i, \mathbf{w}_{i+1}\}$ is admissible, i.e. if their second determinant divisor is 1, then $\{\mathbf{w}_0, \dots, \mathbf{w}_{l+2}\}$ does indeed give a well defined rod diagram for a manifold. Lastly check if $\{\mathbf{w}_0, \dots, \mathbf{w}_{l+2}\}$ is in Hermite normal form. If they are in Hermite normal form then $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_l\}$ are valid plumbing vectors, and in fact are the plumbing vectors for the manifold given by $\{\mathbf{w}_0, \dots, \mathbf{w}_{l+2}\}$.

Figure 2.27 exhibits an example of the decomposition stated in the theorem with three black holes, one toric plumbing, a piece with one corner, and another piece with no corners. Note that the horizons are deformation retracts of the gray areas, hence removing them has no effect on the domain of outer communication. Given any rod diagram, we can decompose the orbit space, minus neighborhoods of the horizon rods, into these components. This completes the proof of Theorem E.

M^5_{end}

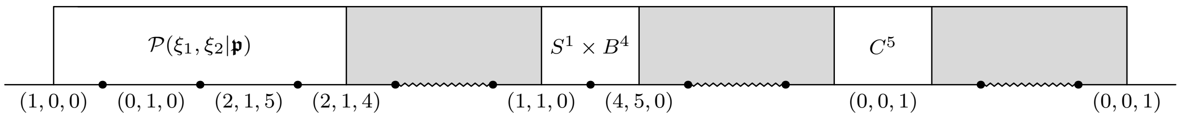


Figure 2.27: This is an example of the decomposition of the domain of outer communication described in Theorem E. The black hole horizons are deformation retracts of the gray areas. There are four remaining pieces. Since this is a topological description, we can deform the black hole horizons to the boundary of the empty regions. In the leftmost piece, ξ_1 is a D^2-bundle over $L(5,2)$ with Euler class determined by 1 and ξ_2 is a bundle over $L(2,1)$ with Euler class 0. The plumbing vector is $\mathfrak{p} = (1,0,2)$. The asymptotic end M_{end} is homeomorphic to $\mathbb{R}_+ \times S^1 \times S^3$. The rightmost piece C^5 is the product $[0,1] \times D^2 \times T^2$.

2.11 Equivariant Cohomology

In this section we will define a few concepts from the world of toric topology and relate them to the tools introduced in Theorems B and C. The majority of the focus will be on equivariant cohomology and the equivariant cohomological rigidity problem, which we will define later. Interestingly, all of the definitions and results from toric topology presented below work equally well for simple T^n-manifolds of any cohomogeneity, not just cohomogeneity two. Despite this fact, we will present the results only for the cohomogeneity two case as the tools we wish to compare them to are defined only for simple T^n-manifolds of dimension $n+2$. Unless otherwise stated, all manifolds in this section are assumed to be closed.

We begin first with the construction of a Stanley-Reisner ring of a simple k-gon (see [4, §3.1] for a discussion on Stanley-Reisner rings of a general simplicial complex).

Definition 2.97. Suppose $\mathcal{P}_k$ is a k-gon with edges $\Gamma_1, \dots, \Gamma_k$. The *face ring* or *Stanley-Reisner ring* of $\mathcal{P}_k$ is a polynomial ring on k generators of degree two

$$\mathbb{Z}[\mathcal{P}_k] := \frac{\mathbb{Z}[\Gamma_1, \dots, \Gamma_k]}{\langle \Gamma_i \Gamma_j | \Gamma_i \cap \Gamma_j = \emptyset \rangle}. \tag{2.260}$$

Next we will define the equivariant cohomology first as a ring, and then as an algebra (see [33] and [4, §B.3] for more details).

Definition 2.98. Given a topological space X and an action of G on X, the *equivariant cohomology ring* of the pair is denoted by $H^*_G(X)$ and defined as the integral cohomology ring

$$H^*_G(X) := H^*(EG \times_G X) \tag{2.261}$$

where EG is the total space of the universal classifying bundle $EG \to BG$ and

$$EG \times_G X := \frac{EG \times X}{G} \tag{2.262}$$

is the quotient of G via the diagonal action.

In our case the group G is the torus T^n and our topological space X is a simple T^n-manifold M^{n+2}. The following lemma [33, Prop. 2.1][49, Theorem 3.5] relates the equivariant cohomology ring of a simple T^n-manifold M^{n+2} with the face ring of its quotient space as a polygon.

Lemma 2.99. *Let M^{n+2} be a closed simply connected T^n-manifold with k rods $\{\Gamma_1, \dots, \Gamma_k\}$. The equivariant cohomology ring of M is isomorphic to the Stanley-Reisner ring of a k-gon, $\mathbb{Z}[\mathcal{P}_k]$. Specifically*

$$H^*_{T^n}(M) \cong \frac{\mathbb{Z}[\tau_1, \dots, \tau_k]}{\langle \tau_i \tau_j | \Gamma_i \cap \Gamma_j = \emptyset \rangle} \tag{2.263}$$

where $\tau_i \in H^2_{T^n}(M)$ is the Poincaré dual of the equivariant cycle corresponding to the co-dimension 2 submanifold $\pi^{-1}(\Gamma_i) \subset M$.

Notice that as a ring, the equivariant cohomology carries very little information about M. It can only be used to distinguish manifolds which have a different number of rods. For instance as rings $H^*_{T^2}(\mathbb{CP}^2) \cong \mathbb{Z}[\mathcal{P}_3] \cong H^*_{T^3}(S^5)$, and thus cannot be distinguished by their equivariant cohomology rings alone. However, the introduction of an algebra structure will be useful in distinguishing them.

Recall that G acts on the universal classifying bundle EG with quotient equal to the universal classifying space BG. This produces a projection map

$$P\colon EG \times_G X \to EG/G =: BG \tag{2.264}$$

The map acts on cohomology

$$P^*\colon H^*(BG) \to H^*(EG \times_G X) =: H^*_G(X) \tag{2.265}$$

to give $H^*_G(X)$ additional structure as algebra over $H^*(BG)$.

Definition 2.100. Given a topological space X and a group action of G on X, the *equivariant cohomology algebra* of the pair is the ring $H^*_G(X)$ equipped with the additional structure of an algebra over $H^*(BG)$ defined by

$$u \cdot w := P^*(u) \cup w \tag{2.266}$$

where $u \in H^*(BG)$, $w \in H^*_G(X)$, and $P^*(u) \cup w$ is the cup product of $P^*(u)$ and w as elements in $H^*(EG \times_G X)$.

The torus T^n is fortunate enough to have a universal classifying space which is a manifold, specifically

$$BT^n \cong (\mathbb{CP}^\infty)^n. \tag{2.267}$$

Additionally the cohomology ring $H^*(BT^n)$ is easy to express. It is the free polynomial algebra on n generators of degree 2, that is

$$H^*(BT^n) \cong \mathbb{Z}[\mathbf{e}^1, \dots, \mathbf{e}^n]. \tag{2.268}$$

Therefore the $H^*(BT^n)$-algebra structure on the equivariant cohomology is completely defined by where $P^*\colon H^*(BT^n) \to H^*_{T^n}(M)$ sends the n generators of $H^*(BT^n)$. The following lemma [49, Lemma 3.6] [33, Prop 2.2] does exactly this.

Lemma 2.101. *Let M^{n+2} be a closed simply connected T^n-manifold with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. The $H^*(BT^n)$-algebra structure on $H^*_{T^n}(M)$ is defined by*

$$P^*(\mathbf{e}^i) := \sum_{j=1}^{k} v^i_j \tau_j \tag{2.269}$$

where $\mathbf{v}_j = (v^1_j, \dots, v^n_j)$.

The above lemma makes clear that the algebra structure is defined entirely by the rod structures. Thus if two simple T^n-manifolds M^{n+2} and N^{n+2} are strongly equivariantly homeomorphic they must have identical equivariant cohomology algebras. If M and N are only weakly equivariantly homeomorphic, that is if there exists a homeomorphism $F\colon M \to N$ and an automorphism $\varphi \in aut(T^n)$ such that

$$F(\mathbf{t} \cdot \mathbf{p}) = \varphi(\mathbf{t}) \cdot F(\mathbf{p}) \tag{2.270}$$

for all $\mathbf{t} \in T^n$ and $\mathbf{p} \in M$, then the equivariant cohomology algebras of M and N must be related in some way. This brings us to the notion of weakly isomorphic equivariant cohomology algebras.

Definition 2.102. Two equivariant cohomology algebras $H^*_{T^n}(M)$ and $H^*_{T^n}(N)$ are *weakly isomorphic* if there exists a ring isomorphism $\Phi\colon H^*_{T^n}(M) \to H^*_{T^n}(N)$ and an automorphism $\varphi \in aut(T^n)$ such that

$$\Phi(u \cdot w) = \varphi^*(u) \cdot \Phi(w) \tag{2.271}$$

for all $u \in H^*(BT^n)$ and $w \in H^*_{T^n}(N)$, where φ^* denotes the automorphism on $H^*(BT^n)$ induced by φ.

We can now finally state and the equivariant cohomology rigidity question and its resolution. The question is: *Is the weak equivariant homeotype of a simple T^n-manifold completely determined by its equivariant cohomology algebra?* This is answered affirmatively by the following theorem.

Theorem 2.103. [48, Theorem 4.2] *Two simple T^n-manifolds M^{n+2} and N^{n+2} are weakly equivariantly homeomorphic if and only if their equivariant cohomology algebras are weakly isomorphic.*

It is interesting that equivariant cohomology algebra turns out to be equivalent to the equivariant homeotype of a simple T^n-manifold. Unfortunately right now this cannot be directly used towards proving Conjecture A which is an attempt to classify the homeotypes of simple T^n-manifolds. Indeed the classification of weakly equivariant homeotypes is trivialized by putting the rod structures in Hermite normal form (see Lemma 2.20). Though clearly the rod structures can be used to reconstruct the manifold (see Corollary 2.15) which means they must contain the information of the homeotype of the manifold. Simply put, the equivariant cohomology algebra is too information rich to be used to classify homeotypes directly.

The equivariant cohomology algebra, which again is equivalent to the rod structures, becomes useful in one of two ways. The first way is to develop a partially forgetful functor which transforms the equivariant cohomology algebra into, or otherwise uses the rod structures to compute, an existing topological invariant. This is the method most commonly employed. See for instance Theorem B where the rod structures are used to compute the intersection form, or Section 3 where they are used to compute the homology groups. The second way the rod structures become useful is in defining auxiliary structure to extend the usefulness of existing topological invariants. This is the rational for defining the 'equivariant intersection form' in Theorem C, which is also proved in Section 3. We end this section with the following construction of the cohomology ring of the manifolds $M(n, n)$ described in Conjecture A. This too is unfortunately not directed useful in proving Conjecture A since the manifolds $M(n, n)$ are already classified by Theorem 2.50. However the construction below partially inspired the CW complex construction in Section 3, and thus we feel it is important to include.

Remark 2.104. The manifolds $M(n, n)$ are entirely determined by their number of rods, and thus the Stanley-Reisner ring can be used to directly compute the cohomology ring. In [4, §4.5] a detailed construction is carried out, which goes as follows. The Stanley-Reisner ring is first defined as

$$\mathbb{Z}[\mathcal{P}_n] := \frac{\mathbb{Z}[\Gamma_1, \ldots, \Gamma_n]}{\langle \Gamma_i\Gamma_j | \Gamma_i \cap \Gamma_j = \emptyset \rangle}. \tag{2.272}$$

Using the exterior algebra $\Lambda^*(\mathbb{Z}^n)$ they then define the quotient algebra

$$R^*(\mathcal{P}_n) := \frac{\Lambda^*(\mathbb{Z}^n) \otimes \mathbb{Z}[\mathcal{P}_n]}{\langle \mathbf{e}_i\Gamma_i \mid i = 1, \ldots, n\rangle}. \tag{2.273}$$

At this point a bi-degree map and a type-$(-1, 0)$ differential operator are introduced to $R^*(\mathcal{P}_n)$ making it a bi-graded co-chain complex;

$$\mathrm{bideg}(\mathbf{e}_i) = (-1, 2) \qquad\qquad d(\mathbf{e}_i) = \Gamma_i \tag{2.274}$$

$$\mathrm{bideg}(\Gamma_i) = (0, 2) \qquad\qquad d(\Gamma_i) = 0. \tag{2.275}$$

Loosing the information of the bi-grading gives a degree map of $\deg(\mathbf{e}_i) = 1$ and $\deg(\Gamma_i) = 2$. The cohomology ring generated by this co-chain complex is isomorphic to the cohomology ring of $M(n, n)$;

$$H^*(M(n, n); \mathbb{Z}) \cong H[R^*(\mathcal{P}_n)]. \tag{2.276}$$

3 Cellular Homology

The main purpose of this section is to prove Theorem C and Part 7 of Theorem D. We in fact end up proving slightly stronger versions of both of these statements. The methods used in these proofs involve constructing a CW complex from the rod structures of M and explicitly computing its cellular homology. The calculations are highly technical. For this reason we present all of the main results here. Complete proofs are also provided here, but they offer little more than a reference to the relevant lemma. Most lemmas will be given their own subsection, with the smaller ones sharing a subsection.

Theorem 3.1 below discusses two main tools and how they interact. The first tool is the well known action of $H_*(T^n)$ on M

$$H_*(T^n) \otimes H_*(M) \to H_*(M)$$

which comes from the group action $T^n \times M \to M$. The second tool is a stratification of the homology groups $H_*(M)$. Indeed, in the construction of the CW complex for M we find that the torus action produces a natural stratification which ends up passing to homology. This gives Equation (3.1) and will be discussed in length in Section 3.2.

Theorem 3.1. *Let M^n be a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ forming the matrix $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$. The homology groups $H_*(M;\mathbb{Z})$ form a stratified $H_*(T^n;\mathbb{Z})$-module*

$$H_i(M;\mathbb{Z}) \cong H_{i,0}(M;\mathbb{Z}) \oplus H_{i-2,2}(M;\mathbb{Z}) \oplus H_{i-4,4}(M;\mathbb{Z}) \tag{3.1}$$

where $[\boldsymbol{\alpha}] \cdot [\mathbf{x}] \in H_{a+c,b}(M;\mathbb{Z})$ for all $[\boldsymbol{\alpha}] \in H_c(T^n;\mathbb{Z})$ and $[\mathbf{x}] \in H_{a,b}(M;\mathbb{Z})$. Any weakly equivariant homeomorphism $(F,\varphi)\colon (M,T^n) \to (M,T^n)$ respects this module structure in the sense that

$$F_*([\boldsymbol{\alpha}] \cdot [\mathbf{x}]) = \varphi_*[\boldsymbol{\alpha}] \cdot F_*[\mathbf{x}] \tag{3.2}$$

where $F_*[\mathbf{x}] \in H_{a,b}(M;\mathbb{Z})$ and $\varphi_*[\boldsymbol{\alpha}] \in H_c(T^n;\mathbb{Z})$.

Proof. The homology groups $H_{a,b}(M;\mathbb{Z})$ are defined in Section 3.2 and are shown in Lemma 3.7 to have the property that $[\boldsymbol{\alpha}] \cdot [\mathbf{x}] \in H_{a+c,b}(M;\mathbb{Z})$ for all $[\boldsymbol{\alpha}] \in H_c(T^n;\mathbb{Z})$ and $[\mathbf{x}] \in H_{a,b}(M;\mathbb{Z})$. The second property, Equation (3.2), is proven in Lemma 3.8. □

The following Theorem 3.2 shows how to compute the homology groups. Notice that $H_{i,0}$ and $H_{i-4,4}$ have simple formulas while the formula $H_{i-2,2}$ is quite complicated. Intuitively $H_{i,0}$ is the portion of the homology which comes from the fundamental group. If the fundamental group has a non-trivial free part then $M^{n+2} \cong T^{n-l} \times N^{l+2}$ for some simple T^l-manifold N. The portion of the homology which comes T^{n-l} is $H_{i-4,4}$. The remaining portion is $H_{i-2,2}$ which comes from N. This is computed using a function $\Lambda^{i-2}(\mathrm{id} \otimes A)$ which maps $\Lambda^{i-2}(\mathbb{Z}^n) \otimes \mathbb{Z}^k$ to $\Lambda^{i-1}(\mathbb{Z}^n)$ by

$$\Lambda^{i-2}(\mathrm{id} \otimes A)(\omega \otimes \mathbf{w}) := \omega \wedge A(\mathbf{w}). \tag{3.3}$$

The group $H_{i-2,2}$ contains the bulk of the homological information of M and is simplified in Theorem 3.3.

Theorem 3.2. *Let M^n be a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ forming the rank-l matrix $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$. Define $E \subset \{1, \ldots, k\}$ so that there is a corner at $\Gamma_c \cap \Gamma_{c+1}$ for every $c \in E$. The bi-graded* *homology groups of M are*

$$H_{i,0}(M;\mathbb{Z}) \cong \frac{\Lambda^i(\mathbb{Z}^n)}{\Lambda^{i-1}(\mathbb{Z}^n)\cdot\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}} \cong \Lambda^i(\mathbb{Z}^{n-l}) \oplus \bigoplus_{j=1}^{l} (\mathbb{Z}/s_j\mathbb{Z})^{\binom{n-j}{i-1}} \tag{3.4}$$

$$H_{i-2,2}(M;\mathbb{Z}) \cong \frac{\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))}{\Lambda^{i-3}(\mathbb{Z}^n)\cdot\{\mathbf{v}_a\otimes\mathbf{e}_a, \mathbf{v}_{c+1}\otimes\mathbf{e}_c + \mathbf{v}_c\otimes\mathbf{e}_{c+1}|, a\in\{1,\ldots,k\}, c\in E\}} \tag{3.5}$$

$$H_{i-4,4}(M;\mathbb{Z}) \cong \begin{cases} \Lambda^{n+2-i}(\mathbb{Z}^{n-l}) & M \text{ is closed} \\ \{0\} & M \text{ is not closed.} \end{cases} \tag{3.6}$$

Proof. The $H_{i,0}(M;\mathbb{Z})$ homology group is calculated in Lemma 3.10. Similarly the $H_{i-4,4}(M;\mathbb{Z})$ homology group is calculated in Lemma 3.11. Equation (3.5) is technically proven in Lemma 3.13 however is not discussed until Lemma 3.22. □

Theorem 3.3. *Let M^{n+2} be a simple T^n-manifold with k rods and m corners. If $H_1(M;\mathbb{Q}) = \{0\}$ then $H_{i-2,2}(M;\mathbb{Q})$ can be computed from*

$$\dim(H_{i-2,2}(M;\mathbb{Q})) = k\binom{n}{i-2} - \binom{n}{i-1} - k\binom{n-1}{i-3} - m\binom{n-2}{i-3}. \tag{3.7}$$

If instead $H_1(M;\mathbb{Q}) = \mathbb{Q}^{n-l}$, then $M \cong T^{n-l}\times N^{l+2}$ for some simple T^l-manifold N, and $H_{i-2,2}(M;\mathbb{Q})$ can be computed using the universal coefficient theorem and the following Künneth-like formula

$$H_{a,b}(M;\mathbb{Z}) \cong \bigoplus_{c+d=a} H_c(T^{n-l};\mathbb{Z})\otimes H_{d,b}(N;\mathbb{Z}). \tag{3.8}$$

Proof. The Künneth-like formula is found in the proof of Lemma 3.11 as Equation (3.111). Equation (3.7) is derived in Lemma 3.22. □

In the following theorem we show that $H_*(M)$ is torsion free when M is simply connected. More importantly we confrim that Conjecture A holds in homology.

Theorem 3.4 (Theorem D, Part 7)**.** *Let M^{n+2} be a closed simply connected T^n-manifold with k rods. The integral homology of M is torsion-free and has Betti numbers*

$$b_i = \begin{cases} 1 & i = 0, n+2 \\ 0 & i = 1, n+1 \\ k\binom{n-2}{i-2} - \binom{n}{i-1} & 2 \le i \le n. \end{cases} \tag{3.9}$$

In particular M has the same integral homology as $M(n,k)$ in Conjecture A;

$$H_i(M;\mathbb{Z}) \cong H_i(M(n,k);\mathbb{Z}). \tag{3.10}$$

Proof. When $i \in \{0,1,n+1,n+2\}$ the formula is obvious so let $2 \le i \le n$. In Lemma 3.22 we see that $H_{i-2,2}(M;\mathbb{Z})$ is free with dimension

$$\dim(H_{i-2,2}(M;\mathbb{Z})) = k\binom{n}{i-2} - \binom{n}{i-1} - k\binom{n-1}{i-3} - k\binom{n-2}{i-3}. \tag{3.11}$$

Using the formulas from Theorem 3.2 we see that when M is simply connected $H_{i-2,2}(M;\mathbb{Z}) \cong H_i(M;\mathbb{Z})$.

In Lemma 2.86 we see that $H_i(M(n,k);\mathbb{Z})$ is free with dimension

$$\dim(H_i(M(n,k);\mathbb{Z})) = (k-2)\binom{n-2}{i-2} - \binom{n-2}{i-3} - \binom{n-2}{i-1}. \tag{3.12}$$

A simple application of the following binomial coefficient identities will show that all three of these expressions are identical;

$$\binom{n}{i-2} = \binom{n-1}{i-3} + \binom{n-2}{i-3} + \binom{n-2}{i-2} \tag{3.13}$$

$$\binom{n}{i-1} = \binom{n-1}{i-1} + 2\binom{n-2}{i-2} + \binom{n-2}{i-3}. \tag{3.14}$$

□

The final main result of this section is the proof of a slight generalization of Theorem C to work for non-simply connected manifolds as well. We write the theorem in its entirety here, where all the homology groups $H_i(M;\mathbb{Z})$ and $H_j(M;\mathbb{Z})$ in the statement of Theorem C are replaces by the subgroups $H_{i-2,2}(M;\mathbb{Z}) \subset H_i(M;\mathbb{Z})$ and $H_{j-2,2}(M;\mathbb{Z}) \subset H_j(M;\mathbb{Z})$.

Theorem 3.5 (Theorem C$^+$). *Let M^{n+2} be a simple T^n-manifold with rod structures $\{\mathbf{v}_1,\dots,\mathbf{v}_k\} \subset \mathbb{Z}^n$ defining the linear map*

$$\Lambda^{i-2}(\mathrm{id}\otimes A)\colon \Lambda^{i-2}(\mathbb{Z}^n)\otimes\mathbb{Z}^k \to \Lambda^{i-1}(\mathbb{Z}^n)$$

by sending $\alpha\otimes\mathbf{e}_a$ to $\alpha\wedge\mathbf{v}_a$ for each basis element $\mathbf{e}_a\in\mathbb{Z}^k$ and each $\alpha\in\Lambda^{i-2}(\mathbb{Z}^n)$.

a. For each $2\le i\le n$ there exists a surjective homomorphism

$$\Psi_{i*}\colon \ker(\Lambda^{i-2}(\mathrm{id}\otimes A)) \to H_{i-2,2}(M;\mathbb{Z}) \tag{3.15}$$

$$(\alpha_1,\dots,\alpha_k)\mapsto[\boldsymbol{\alpha}] \tag{3.16}$$

which is described explicitly in terms of the rod structures.

b. The map Ψ_{i*} well-defines a bilinear form, which we refer to as an equivariant intersection form*,*

$$Q\colon H_{i-2,2}(M;\mathbb{Z})\otimes H_{j-2,2}(M;\mathbb{Z})\to H_{i+j-2}(T^n;\mathbb{Z}) \tag{3.17}$$

by

$$Q([\boldsymbol{\alpha}],[\boldsymbol{\beta}]) := \sum_{1\le a<b\le k-1} \alpha_a\wedge\mathbf{v}_a\wedge\beta_b\wedge\mathbf{v}_b \in \Lambda^{i+j-2}(\mathbb{Z}^n)\cong H_{i+j-2}(T^n;\mathbb{Z}) \tag{3.18}$$

where $[\boldsymbol{\alpha}]$ and $[\boldsymbol{\beta}]$ are homology classes in $H_{i-2,2}(M;\mathbb{Z})$ and $H_{j-2,2}(M;\mathbb{Z})$ respectively.

c. Assume M is simply connected so that $H_2(M;\mathbb{Z})\cong H_{i-2,2}(M;\mathbb{Z})$. When $i=2$ the map $\Psi_{2*}\colon \ker(A)\cong\mathbb{Z}^{k-n}\to H_2(M;\mathbb{Z})$ is an isomorphism. When $i+j=n+2$ the equivariant intersection form $Q\colon H_i(M;\mathbb{Z})\otimes H_j(M;\mathbb{Z})\to H_n(T^n;\mathbb{Z})\cong\mathbb{Z}$ agrees with the intersection pairing on $H_*(M;\mathbb{Z})$.

d. Let $(F,\varphi)\colon (M^{m+2},T^m)\to(N^{n+2},T^n)$ be a weakly equivariant submersion between simply connected T-manifolds with equivariant intersection forms Q_M and Q_N respectively. Then

$$Q_N(F_*[\boldsymbol{\alpha}],F_*[\boldsymbol{\beta}]) = \varphi_*Q_M([\boldsymbol{\alpha}],[\boldsymbol{\beta}]) \tag{3.19}$$

where $[\boldsymbol{\alpha}]$ and $[\boldsymbol{\beta}]$ are homology classes in $H_i(M;\mathbb{Z})$ and $H_j(M;\mathbb{Z})$ respectively.

Proof. For Part a, the homomorphism $\Psi_{i*}\colon \ker(\Lambda^{i-2}(\mathrm{id}\otimes A))\to H_{i-2,2}(M;\mathbb{Z})$ is shown to be surjective in Lemma 3.13, and defined explicitly in terms of cell structures in Equations (3.137), (3.138), and (3.145).

Lemma 3.17 shows that Ψ_{i*} defines a bilinear form Q exactly as described in Part b. The first statement of Part c is proven in Corollary 3.14 while the second statement is proven in Lemma 3.18. Part d is proven in Lemma 3.21. □

Many of the techniques used here work even when M is not a manifold. That is, the rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ need not form admissible corners and M may in fact be a simple T^n-orbifold (see Definition 2.56). However the main focus of this dissertation is on manifolds, so if needed admissibility of the rod structures will be assumed without statement.

3.1 Chain Complex

Lemma 3.6. *Let M^{n+2} be a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{Z}^n$ and corners at $\Gamma_c \cap \Gamma_{c+1}$ if and only if $c \in E \subset \{1, \ldots, k\}$. There exists a CW complex X which is homeomorphic to M. The cellular chain complex of X has additional structure as a $\Lambda^*(\mathbb{Z}^n)$-module*

$$C_*(X) = \frac{\Lambda^*(\mathbb{Z}^n) \cdot G}{\Lambda^*(\mathbb{Z}^n) \cdot R} \tag{3.20a}$$

with generators

$$G = \{\mathbf{x}_a, \xi_a \mathbf{x}_a, I_{2a}, B, \mathfrak{c}_c | c \in E, a \in \{1, \ldots, k\}, \mathbf{x}_a \in \{p_{2a-1}, p_{2a}, I_{2a-1}\}\} \tag{3.20b}$$

and relations

$$R = \{\mathbf{v}_a \cdot \xi_a \mathbf{x}_a, \mathbf{v}_c \cdot \mathfrak{c}_c, \mathbf{v}_{c+1} \cdot \mathfrak{c}_c | c \in E, a \in \{1, \ldots, k\}, \mathbf{x}_a \in \{p_{2a-1}, p_{2a}, I_{2a-1}\}\}. \tag{3.20c}$$

The degree map and boundary operators are defined as

$$\deg(p_b) = 0 \qquad \partial(p_b) = 0 \tag{3.21a}$$
$$\deg(I_b) = 1 \qquad \partial(I_b) = p_{b+1} - p_b \tag{3.21b}$$
$$\deg(\xi_a \mathbf{x}_a) = \deg(\mathbf{x}_a) + 2 \qquad \partial(\xi_a \mathbf{x}_a) = \mathbf{v}_a \cdot \mathbf{x}_a + \xi_a \partial(\mathbf{x}_a) \tag{3.21c}$$
$$\deg(B) = 2 \qquad \partial(B) = I_1 + \cdots + I_{2k} \tag{3.21d}$$
$$\deg(\mathfrak{c}_c) = 4 \qquad \partial(\mathfrak{c}_c) = \mathbf{v}_c \cdot \xi_{c+1} p_{2c+1} + \mathbf{v}_{c+1} \cdot \xi_c p_{2c} + \mathbf{v}_c \wedge \mathbf{v}_{c+1} \cdot I_{2c} \tag{3.21e}$$
$$\deg(\alpha \cdot \mathbf{y}) = \deg(\alpha) + \deg(\mathbf{y}) \qquad \partial(\alpha \cdot \mathbf{y}) = (-1)^{\deg(\alpha)} \alpha \cdot \partial(\mathbf{y}) \tag{3.21f}$$

for $c \in E$, $b \in \{1, \ldots, 2k\}$, $a \in \{1, \ldots, k\}$, $\mathbf{x}_a \in \{p_{2a-1}, p_{2a}, I_{2a-1}\}$, $\mathbf{y} \in G$, and $\alpha \in \Lambda^i(\mathbb{Z}^n)$ for $i = \deg(\alpha)$.

Proof. Before we begin constructing X we need to check that the object defined in Lemma 3.6 is indeed a chain complex. This means we need to check that equations

$$\deg(\partial(\mathbf{z})) = \deg(\mathbf{z}) - 1 \tag{3.22}$$
$$\partial^2(\mathbf{z}) = 0 \tag{3.23}$$

hold for all $\mathbf{z} \in C_*(X)$. Since $\partial \colon C_*(X) \to C_*(X)$ is a linear operator, we need only show these statements hold for simple elements of $C_*(X)$. For both of these equations let $c \in E$, $b \in \{1, \ldots, 2k\}$, $a \in \{1, \ldots, k\}$, $\mathbf{x}_a \in \{p_{2a-1}, p_{2a}, I_{2a-1}\}$, $\mathbf{y} \in G$, and $\alpha \in \Lambda^i(\mathbb{Z}^n)$ for $i = \deg(\alpha)$. Equation (3.22) is shown by the following simple computations:

$$\deg(\partial(I_b)) = \deg(p_{b+1} - p_b) = 0 \tag{3.24}$$

$$\begin{aligned}\deg(\partial(\xi_a \mathbf{x}_a)) &= \deg(\mathbf{v}_a \cdot \mathbf{x}_a + \xi_a \partial(\mathbf{x}_a)) = \max\{\deg(\mathbf{v}_a \cdot \mathbf{x}_a), \deg(\xi_a \partial(\mathbf{x}_a))\} \\ &= \max\{1 + \deg(\mathbf{x}_a), 2 + \deg(\partial(\mathbf{x}_a))\} = \deg(\mathbf{x}_a) + 1\end{aligned} \tag{3.25}$$

$$\deg(\partial(B)) = \deg(I_1 + \cdots + I_{2k}) = 1 \tag{3.26}$$

$$\begin{aligned}\deg(\partial(\mathfrak{c}_c)) &= \deg(\mathbf{v}_c \cdot \xi_{c+1} p_{2c+1} + \mathbf{v}_{c+1} \cdot \xi_c p_{2c} + \mathbf{v}_c \wedge \mathbf{v}_{c+1} \cdot I_{2c}) \\ &= \max\{\deg(\mathbf{v}_c \cdot \xi_{c+1} p_{2c+1}), \deg(\mathbf{v}_c \wedge \mathbf{v}_{c+1} \cdot I_{2c})\} \\ &= \max\{1 + \deg(p_{2c+1}) + 2, 2 + \deg(I_{2c})\} = 3\end{aligned} \tag{3.27}$$

$$\deg(\partial(\alpha \cdot \mathbf{y})) = \deg(\alpha \cdot \partial(\mathbf{y})) = \deg(\alpha) + \deg(\partial(\mathbf{y})) = \deg(\alpha \cdot \mathbf{y}) - 1 \tag{3.28}$$

Similar computations below prove Equation (3.23).

$$\partial^2(p_b) = \partial(0) = 0 \tag{3.29}$$

$$\partial^2(I_b) = \partial(p_{b+1} - p_b) = 0 \tag{3.30}$$

$$\partial^2(\xi_a p_{2a-1}) = \partial(\mathbf{v}_a \cdot p_{2a-1}) + \partial(\xi_a \partial(p_{2a-1})) = -\mathbf{v}_a \cdot \partial(p_{2a-1}) + \partial(\xi_a 0) = 0 \tag{3.31}$$

By the same argument $\partial^2(\xi_a p_{2a}) = 0$.

$$\begin{aligned}\partial^2(\xi_a I_{2a-1}) &= \partial(\mathbf{v}_a \cdot I_{2a-1}) + \partial(\xi_a \partial(I_{2a-1})) = -\mathbf{v}_a \cdot \partial(I_{2a-1}) + \partial(\xi_a (p_{2a} - p_{2a-1})) \\ &= -\mathbf{v}_a \cdot (p_{2a} - p_{2a-1}) + \mathbf{v}_a \cdot (p_{2a} - p_{2a-1}) = 0\end{aligned} \tag{3.32}$$

$$\partial^2(B) = \partial(I_1 + \cdots + I_{2k}) = (p_2 - p_1) + \cdots + (p_{2k+1} - p_{2k}) = p_{2k+1} - p_1 = 0 \tag{3.33}$$

The points are labeled cyclically so p_{2k+1} and p_1 are the exact same 0-cell.

$$\begin{aligned}\partial^2(\mathfrak{c}_c) &= \partial\left(\mathbf{v}_c \cdot \xi_{c+1} p_{2c+1} + \mathbf{v}_{c+1} \cdot \xi_c p_{2c} + \mathbf{v}_c \wedge \mathbf{v}_{c+1} \cdot I_{2c}\right) \\ &= -\mathbf{v}_c \wedge \mathbf{v}_{c+1} \cdot p_{2c+1} - \mathbf{v}_{c+1} \wedge \mathbf{v}_c \cdot p_{2c} + \mathbf{v}_c \wedge \mathbf{v}_{c+1} \cdot (p_{2c+1} - p_{2c}) = 0\end{aligned} \tag{3.34}$$

$$\partial^2(\alpha \cdot \mathbf{y}) = (-1)^i \alpha \cdot \partial^2(\mathbf{y}) = (-1)^i \alpha \cdot 0 = 0 \tag{3.35}$$

Therefor the object defined as $C_*(X)$ in Lemma 3.6 is indeed a chain complex. The remainer of this proof is dedicated to constructing X.

To construct a CW complex X, we will decompose M into pieces of three different "types". Doing this will allow us to build the CW complex in three stages $X_0 \subset X_1 \subset X_2 = X \cong M$. At the 0^{th} stage X_0 consists only of pieces of Type-0, at the 1^{st} stage X_1 is created by attaching all the pieces of Type-1 to X_0, and at the 2^{nd} and final stage $X_2 = X$ is created by attaching all the Type-2 pieces to X_1. The pieces are defined as follows. Consider the projection map $\pi \colon M \to M/T^n$. There is a single Type-0 piece defined to be the total space over the interior of the base $\pi^{-1}((M/T^n) \setminus \partial(M/T^n))$. Type-1 pieces are defined to be tubular neighborhoods of the total space over the interior of each of the k axis rods, $\pi^{-1}(\Gamma_i \setminus \partial\Gamma_i)$. The Type-2 pieces are similarly defined to be tubular neighborhoods of the total space over each of the corners, $\pi^{-1}(\Gamma_i \cap \Gamma_{i+1})$. This decomposition can be seen in Figure 3.1 for a simple T^n-space with three rods and two corners.

We begin at the 0^{th} stage. Notice that the single Type-0 piece, $\pi^{-1}(B \setminus \partial B)$, is homeomorphic to the product of the torus T^n and a 2-ball $\mathbb{B}$. We will construct the CW complex X_0 by taking the product of CW complexes for these two spaces. Consider a cell structure on the closed 2-ball $\mathbb{B}$ with $2k$ 0-cells, $2k$ 1-cells,

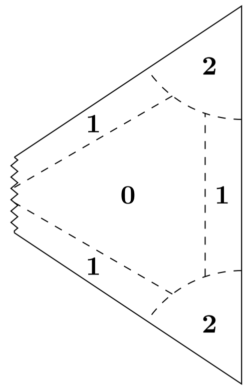
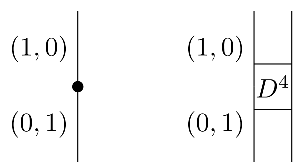


Figure 3.1: Decomposition

Figure 3.2: $\mathbb{R}^4 = T^2 \times (1, \infty) \times \mathbb{R} \cup \mathbb{D}^2 \times S^1 \times$



and 1 2-cell. This is defined by

$$\mathbb{B}^0 := \bigcup_{a=1}^{2k} p_a \tag{3.36}$$

$$\mathbb{B}^1 := \mathbb{B}^0 \cup_{f_a} \bigcup_{i=a}^{2k} I_a \tag{3.37}$$

$$\mathbb{B}^2 := \mathbb{B}^1 \cup_g B \tag{3.38}$$

where the attaching maps $f_a \colon \partial I_a \to \mathbb{B}^0$ and $g \colon \partial B \to \mathbb{B}^1$ send ∂I_a to $\{p_{a+1}\} \cup \{p_a\}$ and $S^1 = \partial B$ to $I_1 \cup \cdots \cup I_{2k}$ in a way that $\mathbb{B}$ is homeomorphic to a 2-ball and the associated chain complex for $\mathbb{B}$ is

$$C_*(\mathbb{B}) = C_0(\mathbb{B}) \oplus C_1(\mathbb{B}) \oplus C_2(\mathbb{B}) \tag{3.39}$$

$$C_0(\mathbb{B}) = \mathrm{span}_{\mathbb{Z}}\{p_1, \ldots, p_{2k}\} \qquad \partial p_i = 0 \tag{3.40}$$

$$C_1(\mathbb{B}) = \mathrm{span}_{\mathbb{Z}}\{I_1, \ldots, I_{2k}\} \qquad \partial I_i = p_{i+1} - p_i \tag{3.41}$$

$$C_2(\mathbb{B}) = \mathrm{span}_{\mathbb{Z}}\{B\} \qquad \partial B = I_1 + \cdots + I_{2k}. \tag{3.42}$$

Next, recall that there exists a CW-complex for T^n with exactly $\binom{n}{i}$ i-cells. We denote the 0-cell by $\{p\}$, the 1-cells by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, the 2-cells by $\{\mathbf{e}_i \wedge \mathbf{e}_j | 1 \leq i < j \leq n\}$, and so on. The associated chain complex is therefore

$$C_*(T^n) = \bigoplus_{i=0}^{n} C_i(T^n) \tag{3.43}$$

$$C_0(T^n) = \mathrm{span}_{\mathbb{Z}}\{p\} \tag{3.44}$$

$$C_i(T^n) = \mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_i} \mid 1 \leq j_1 < \cdots < j_i \leq n\}. \tag{3.45}$$

The boundary operator on $C_*(T^n)$ must be trivial since $C_i(T^n) \cong \mathbb{Z}^{\binom{n}{i}} \cong H_i(T^n;\mathbb{Z})$. Furthermore $H_*(T^n;\mathbb{Z})$ has a natural ring structure on it coming from the ring structure on $H^*(T^n;\mathbb{Z})$ via Poincaré duality. This coincides with the ring structure on $C_*(T^n)$ implied by the wedge product notation. Therefore $C_*(T^n)$ has a natural ring structure isomorphic to the exterior algebra $\Lambda^*(\mathbb{Z}^n)$, or equivalently to the free $\Lambda^*(\mathbb{Z}^n)$-module on a single element which we denote by

$$C_*(T^n) = \Lambda^*(\mathbb{Z}^n) \cdot \{1\}. \tag{3.46}$$

Finally, we define X_0 to be the Cartesian product of $\mathbb{B}$ and T^n with the associated cell structures described above. This multiplies the associated chain complexes so that

$$C_*(X_0) = C_*(T^n) \otimes C_*(\mathbb{B}) \tag{3.47}$$

and

$$C_i(X_0) = C_i(T^n) \otimes C_0(\mathbb{B}) \oplus C_{i-1}(T^n) \otimes C_1(\mathbb{B}) \oplus C_{i-2}(T^n) \otimes C_2(\mathbb{B}). \tag{3.48}$$

Rearranging the terms and using the fact that $C_*(T^n) \cong \Lambda^*(\mathbb{Z}^n)$ we can see

$$C_*(X_0) = \Lambda^*(\mathbb{Z}^n) \otimes (C_0(\mathbb{B}) \oplus C_1(\mathbb{B}) \oplus C_2(\mathbb{B})) \tag{3.49}$$

and therefore $C_*(X_0)$ is a free $\Lambda^*(\mathbb{Z}^n)$-module over the generators of $C_0(\mathbb{B}) \oplus C_1(\mathbb{B}) \oplus C_2(\mathbb{B})$. In particular

$$C_*(X_0) = \Lambda^*(\mathbb{Z}^n) \cdot \{p_1, \dots, p_{2k}, I_1, \dots, I_{2k}, B\} \tag{3.50}$$

where the degree map and boundary operator are defined by the graded Leibniz rule;

$$\deg(p_b) = 0 \qquad \partial(p_b) = 0 \tag{3.51a}$$
$$\deg(I_b) = 1 \qquad \partial(I_b) = p_{b+1} - p_b \tag{3.51b}$$
$$\deg(B) = 2 \qquad \partial(B) = I_1 + \cdots + I_{2k} \tag{3.51c}$$
$$\deg(\alpha \cdot \mathbf{y}) = \deg(\alpha) + \deg(\mathbf{y}) \qquad \partial(\alpha \cdot \mathbf{y}) = (-1)^{\deg(\alpha)} \alpha \cdot \partial(\mathbf{y}) \tag{3.51d}$$

for $b \in \{1, \dots, 2k\}$, $\mathbf{y} \in \{p_1, \dots, p_{2k}, I_1, \dots, I_{2k}, B\}$, and $\alpha \in \Lambda^i(\mathbb{Z}^n)$ for $i = \deg(\alpha)$.

Now that the 0^{th} stage is complete we have a CW complex for $X_0 \subset X$, which can be thought of as the single Type-0 piece of X. In the 1^{st} stage we will add on the Type-1 pieces. There are k such pieces, each associated to an axis rod Γ_a and are homeomorphic to the product of an interval and $([0,1] \times T^n)/\sim$, where $(0, \boldsymbol{\theta}) \sim (0, \boldsymbol{\theta} + \lambda \mathbf{v}_a)$ for $\boldsymbol{\theta} \in T^n$ and $\lambda \in \mathbb{R}$. Written another way, each piece is homeomorphic to $I \times \mathbb{D}^2 \times T^{n-1}$ where $T^{n-1} \cong \frac{\mathbb{R}^n/\mathbf{v}_a\mathbb{R}}{\mathbb{Z}^n}$ and $\mathbb{D}^2 \cong ([0,1] \times \mathbf{v}_a\mathbb{R}/\mathbb{Z})/\sim$. These pieces will be created by attaching several smaller cells.

Each rod structure $\mathbf{v}_a \in \mathbb{Z}^n$ naturally defines an element $\mathbf{v}_a \in \Lambda^1(\mathbb{Z}^n)$, which intern naturally defines the linear combination of 1-cells $\mathbf{v}_a \cdot p_{2a-1} = v_a^1 \mathbf{e}_1 \cdot p_{2a-1} + \cdots + v_a^n \mathbf{e}_n \cdot p_{2a-1}$. Define a new 2-cell by attaching $\partial\mathbb{D}^2$ to $\mathbf{e}_1 \cdot p_{2a-1} \cup \cdots \cup \mathbf{e}_n \cdot p_{2a-1}$ so that at the chain level the boundary operator sends this 2-cell to $\mathbf{v}_a \cdot p_{2a-1} \in C_1(X_0)$. Similarly, define another 2-cell by attaching $\partial\mathbb{D}^2$ to $\mathbf{v}_a \cdot p_{2a}$. Denote these 2-cells by $\xi_a p_{2a-1}$ and $\xi_a p_{2a}$ respectively. Next we define the 3-cell $\xi_a I_{2a-1}$ by attaching $\partial(\mathbb{D}^2 \times [0,1])$ so that $\mathbb{D}^2 \times \{1\}$ is sent to $\xi_a p_{2a}$, $\mathbb{D}^2 \times \{0\}$ is sent to $\xi_a p_{2a-1}$ with the opposite orientation, and $S^1 \times [0,1]$ is sent to $\mathbf{v}_a \cdot I_{2a-1}$. On the level of chains this is described as

$$\partial(\xi_a p_{2a-1}) = \mathbf{v}_a \cdot p_{2a-1} \tag{3.52}$$
$$\partial(\xi_a p_{2a}) = \mathbf{v}_a \cdot p_{2a} \tag{3.53}$$
$$\partial(\xi_a I_{2a-1}) = \mathbf{v}_a \cdot I_{2a-1} + \xi_a p_{2a} - \xi_a p_{2a-1}. \tag{3.54}$$

We now need to attach the torus, $T^{n-1} \cong \frac{\mathbb{R}^n/\mathbf{v}_a\mathbb{R}}{\mathbb{Z}^n}$. Since $\mathbf{v}_a$ is primitive, we have $\mathbb{Z}^n/\mathbf{v}_a\mathbb{Z} \cong \mathbb{Z}^{n-1}$ and therefore can choose a basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}\}$ for $\mathbb{Z}^n/\mathbf{v}_a\mathbb{Z}$ so that $\mathbf{f}_j \wedge \mathbf{v}_a \in \Lambda^2(\mathbb{Z}^n)$. These basis vectors will define the higher dimensional cells. Let $1 \leq j_1 < \cdots < j_i \leq n-1$ define the $(i+2)$-cell $\mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \cdot \xi_a p_{2a-1}$ by attaching its boundary to $\mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \wedge \mathbf{v}_a \cdot p_{2a-1}$. In the same manner as above we define the $(i+2)$ and $(i+3)$-cells $\mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \cdot \xi_a p_{2a}$ and $\mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \cdot \xi_a I_{2a-1}$ so that on the chain level

$$\partial\left(\mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \cdot \xi_a p_{2a-1}\right) = (-1)^i \mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \wedge \mathbf{v}_a \cdot p_{2a-1} \tag{3.55}$$

$$\partial\left(\mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \cdot \xi_a p_{2a}\right) = (-1)^i \mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \wedge \mathbf{v}_a \cdot p_{2a} \tag{3.56}$$

$$\partial\left(\mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \cdot \xi_a I_{2a-1}\right) = (-1)^i \mathbf{f}_{j_1} \wedge \cdots \wedge \mathbf{f}_{j_i} \wedge \mathbf{v}_a \cdot \left(I_{2a-1} + \xi_a p_{2a} - \xi_a p_{2a-1}\right). \tag{3.57}$$

This gives $C_*(X_1)$ a natural structure of a direct sum of $C_*(X_0)$ and several free $\Lambda^*(\mathbb{Z}^{n-1})$-modules,

$$\bigoplus_{\substack{a \in \{1,\ldots,k\} \\ \mathbf{x}_a \in \{p_{2a-1}, p_{2a}, I_{2a-1}\}}} \Lambda^*(\mathbb{Z}^{n-1}) \cdot \{\xi_a \mathbf{x}_a\}. \tag{3.58}$$

To see $C_*(X_1)$ as a $\Lambda^*(\mathbb{Z}^n)$-module, we use the fact that $\Lambda^*(\mathbb{Z}^{n-1}) \cdot \{\xi_a \mathbf{x}_a\}$ is itself a $\Lambda^*(\mathbb{Z}^n)$-module;

$$\Lambda^*(\mathbb{Z}^{n-1}) \cdot \{\xi_a \mathbf{x}_a\} \cong \Lambda^*(\mathbb{Z}^n/\mathbf{v}_a\mathbb{Z}) \cdot \{\xi_a \mathbf{x}_a\} \cong \frac{\Lambda^*(\mathbb{Z}^n) \cdot \{\xi_a \mathbf{x}_a\}}{\Lambda^*(\mathbb{Z}^n) \cdot \{\mathbf{v}_a \cdot \xi_a \mathbf{x}_a\}}. \tag{3.59}$$

This shows

$$C_*(X_1) = \frac{\Lambda^*(\mathbb{Z}^n) \cdot G_1}{\Lambda^*(\mathbb{Z}^n) \cdot R_1} \tag{3.60}$$

with

$$G_1 = \{\mathbf{x}_a, \xi_a \mathbf{x}_a, I_{2a}, B, | a \in \{1, \ldots, k\}, \mathbf{x}_a \in \{p_{2a-1}, p_{2a}, I_{2a-1}\}\} \tag{3.61}$$

and

$$R_1 = \{\mathbf{v}_a \cdot \xi_a \mathbf{x}_a, | a \in \{1, \ldots, k\}, \mathbf{x}_a \in \{p_{2a-1}, p_{2a}, I_{2a-1}\}\}. \tag{3.62}$$

The degree map and boundary operator are defined by Equations (3.51) and

$$\deg(\xi_a \mathbf{x}_a) = \deg(\mathbf{x}_a) + 2 \qquad \partial(\xi_a \mathbf{x}_a) = \mathbf{v}_a \cdot \mathbf{x}_a + \partial(\mathbf{x}_a) \tag{3.63}$$

for $a \in \{1, \ldots, k\}$ and $\mathbf{x}_a \in \{p_{2a-1}, p_{2a}, I_{2a-1}\}$.

In the 2^{nd} and last stage we construct $X = X_2$ by adding the pieces of Type-2 to X_1. These pieces are associated to the corners $\Gamma_a \cap \Gamma_{a+1}$ and are homeomorphic to $([0,1]^2 \times T^n)/\sim$ where $(0, y, \boldsymbol{\theta}) \sim (0, y, \boldsymbol{\theta} + \lambda \mathbf{v}_a)$ and $(x, 0, \boldsymbol{\theta}) \sim (x, 0, \boldsymbol{\theta} + \mu \mathbf{v}_{a+1})$ for $x, y \in [0,1]$, $\boldsymbol{\theta} \in T^n$, and $\lambda, \mu \in \mathbb{R}$. Of course since M is a manifold, a tubular neighborhood of a corner must be homeomorphic to $\mathbb{D}^4 \times T^{n-2}$ [reference something] where $T^{n-2} \cong \frac{\mathbb{R}^n/\operatorname{span}_{\mathbb{R}}\{\mathbf{v}_a, \mathbf{v}_{a+1}\}}{\mathbb{Z}^n}$, $T^2 \cong \frac{\operatorname{span}_{\mathbb{R}}\{\mathbf{v}_a, \mathbf{v}_{a+1}\}}{\mathbb{Z}^n \cap \operatorname{span}_{\mathbb{R}}\{\mathbf{v}_a, \mathbf{v}_{a+1}\}}$, and $\mathbb{D}^4 \cong ([0,1]^2 \times T^2)/\sim$.

The $\mathbb{D}^4$ piece defines a new 4-cell $\mathfrak{c}_a$. From the construction of $([0,1]^2 \times T^2)/\sim$ and by looking at Figure 3.3 for reference, we know how $\partial\left(([0,1]^2 \times T^2)/\sim\right)$ is attached to X_1. The $(\{0\} \times [0,1] \times T^2)/\sim$ piece is sent to $\mathbf{v}_a \cdot \xi_{a+1} p_{2a+1}$, the $([0,1] \times \{0\} \times T^2)/\sim$ piece is sent to $\mathbf{v}_{a+1} \cdot \xi_a p_{2a}$, and the remaining $((\{1\} \times [0,1] \cup [0,1] \times \{1\}) \times T^2)/\sim$ piece is sent to $\mathbf{v}_a \wedge \mathbf{v}_{a+1} \cdot I_{2a}$. On the chain level this is described by

$$\partial(\mathfrak{c}_a) = \mathbf{v}_a \cdot \xi_{a+1} p_{2a+1} + \mathbf{v}_{a+1} \cdot \xi_a p_{2a} + \mathbf{v}_a \wedge \mathbf{v}_{a+1} \cdot I_{2a}. \tag{3.64}$$

To describe the remaining $(i+4)$-cells we just repeat the construction done in the 1^{st} stage. At each corner $\{\mathbf{v}_a, \mathbf{v}_{a+1}\}$ is a primitive pair, so $\mathbb{Z}^n/\operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_a, \mathbf{v}_{a+1}\} \cong \mathbb{Z}^{n-2}$. Thus it is possible to choose a basis $\{\mathbf{g}_1, \ldots, \mathbf{g}_{n-2}\}$ for $\mathbb{Z}^n/\operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_a, \mathbf{v}_{a+1}\}$ so that $\mathbf{g}_j \wedge \mathbf{v}_a \in \Lambda^2(\mathbb{Z}^n/\mathbf{v}_{a+1}\mathbb{Z}) \cong \frac{\Lambda^2(\mathbb{Z}^n)\cdot\{1\}}{\Lambda^1(\mathbb{Z}^n)\cdot\{\mathbf{v}_{a+1}\}}$, $\mathbf{g}_j \wedge \mathbf{v}_{a+1} \in$

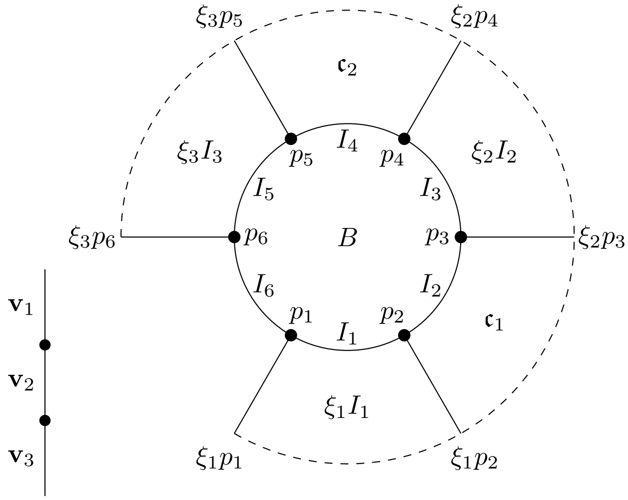


Figure 3.3: This figure is a little deceptive because it makes it seem that the outermost circle is a boundary. That ξI and $\mathfrak{c}$ have 4 edges, when infact they both have 3. The outermost circle corresponds to the center of a disk.

$\Lambda^2(\mathbb{Z}^n/\mathbf{v}_a\mathbb{Z}) \cong \frac{\Lambda^2(\mathbb{Z}^n)\cdot\{1\}}{\Lambda^1(\mathbb{Z}^n)\cdot\{\mathbf{v}_a\}}$, and $\mathbf{g}_j \wedge \mathbf{v}_a \wedge \mathbf{v}_{a+1} \in \Lambda^3(\mathbb{Z}^n)$. We can therefore define the $(i+4)$-cell $\mathbf{g}_{j_1} \wedge \cdots \wedge \mathbf{g}_{j_i} \cdot \mathfrak{c}_a$ so that

$$\partial\left(\mathbf{g}_{j_1} \wedge \cdots \wedge \mathbf{g}_{j_i} \cdot \mathfrak{c}_a\right) = (-1)^i \mathbf{g}_{j_1} \wedge \cdots \wedge \mathbf{g}_{j_i} \cdot \partial(\mathfrak{c}_a) \in C_{i+3}(X_1). \tag{3.65}$$

This makes $C_*(X_2)$ the direct sum of $C_*(X_1)$ and several free $\Lambda^*(\mathbb{Z}^{n-2})$-modules,

$$\bigoplus_{c \in E} \Lambda^*(\mathbb{Z}^{n-2}) \cdot \{\mathfrak{c}_c\}. \tag{3.66}$$

Of course this is also a $\Lambda^*(\mathbb{Z}^n)$-module since

$$\Lambda^*(\mathbb{Z}^{n-2}) \cdot \{\mathfrak{c}_c\} \cong \Lambda^*(\mathbb{Z}^n/\operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_c, \mathbf{v}_{c+1}\}) \cdot \{\mathfrak{c}_c\} \cong \frac{\Lambda^*(\mathbb{Z}^n) \cdot \{\mathfrak{c}_c\}}{\Lambda^*(\mathbb{Z}^n) \cdot \{\mathbf{v}_c \cdot \mathfrak{c}_c, \mathbf{v}_{c+1} \cdot \mathfrak{c}_c\}}. \tag{3.67}$$

Therefore $X = X_2$ has a chain complex of

$$C_*(X) = \frac{\Lambda^*(\mathbb{Z}^n) \cdot G}{\Lambda^*(\mathbb{Z}^n) \cdot R} \tag{3.68}$$

where

$$\deg(\alpha \cdot \mathbf{y}) = \deg(\alpha) + \deg(\mathbf{y}) \qquad\qquad \partial(\alpha \cdot \mathbf{y}) = (-1)^{\deg(\alpha)} \alpha \cdot \partial(\mathbf{y}) \tag{3.69}$$

for $\mathbf{y} \in G$ and $\alpha \in \Lambda^i(\mathbb{Z}^n)$ for $i = \deg(\alpha)$. The generators are

$$G = G_1 \cup \{\mathfrak{c}_c | c \in E\} \tag{3.70}$$

and relations

$$R = R_1 \cup \{\mathbf{v}_c \cdot \mathfrak{c}_c, \mathbf{v}_{c+1} \cdot \mathfrak{c}_c | c \in E\}. \tag{3.71}$$

Equations (3.51), (3.63), and (3.69) agree with Equations (3.21) as desired and the proof is complete.

□

3.2 Bi-Graded Homology Groups

Lemma 3.7. *Let M^{n+2} be a simple T^n-manifold. The torus action on M make the homology groups of M into a stratified $H_*(T^n;\mathbb{Z})$-module;*

$$H_i(M;\mathbb{Z}) \cong H_{i,0}(M;\mathbb{Z}) \oplus H_{i-2,2}(M;\mathbb{Z}) \oplus H_{i-4,4}(M;\mathbb{Z}), \tag{3.72}$$

where $[\alpha]\cdot[\mathbf{x}] \in H_{a+c,b}(M;\mathbb{Z})$ for all $[\alpha] \in H_c(T^n;\mathbb{Z})$ and $[\mathbf{x}] \in H_{a,b}(M;\mathbb{Z})$.

Proof. Let X be the CW-complex for M defined in Lemma 3.6 and $C_*(X)$ be its cellular chain complex. Let ∂_i denote the restriction of ∂ to $C_i(X)$. The homology groups of X are defined as $H_i(X;\mathbb{Z}) := \frac{\ker(\partial_i)}{\mathrm{im}(\partial_{i+1})}$. To stratify $H_i(X;\mathbb{Z})$ we need to first stratify $C_i(X)$ by introducing a bi-degree map for simple elements in $C_*(X)$. This is defined in the following way

$$\mathrm{bideg}(p_b) = (0,0) \tag{3.73a}$$
$$\mathrm{bideg}(I_b) = (-1,2) \tag{3.73b}$$
$$\mathrm{bideg}(\xi_a\mathbf{x}_a) = (0,2) + \mathrm{bideg}(\mathbf{x}_a) \tag{3.73c}$$
$$\mathrm{bideg}(B) = (-2,4) \tag{3.73d}$$
$$\mathrm{bideg}(\mathfrak{c}_c) = (0,4) \tag{3.73e}$$
$$\mathrm{bideg}(\alpha\cdot\mathbf{y}) = (\deg(\alpha),0) + \mathrm{bideg}(\mathbf{y}), \tag{3.73f}$$

for $c \in E$, $b \in \{1,\dots,2k\}$, $a \in \{1,\dots,k\}$, $\mathbf{x}_a \in \{p_{2a-1}, p_{2a}, I_{2a-1}\}$, $\mathbf{y} \in G$, and $\alpha \in \Lambda^i(\mathbb{Z}^n)$ for $i = \deg(\alpha)$. Notice that for all $\mathbf{z} \in C_*(X)$ where bideg is defined, if $\mathrm{bideg}(\mathbf{z}) = (a,b)$ then $\deg(\mathbf{z}) = a+b$.

Next, extend the definition of bideg to linear combinations $\lambda\mathbf{z} + \mathbf{z}' \in C_{a+b}(X)$ of simple elements with the same bi-degree in following way

$$\mathrm{bideg}(\lambda\mathbf{z}+\mathbf{z}') := \begin{cases} \mathrm{bideg}(\mathbf{z}) & \mathrm{bideg}(\mathbf{z}) = \mathrm{bideg}(\mathbf{z}') \\ \text{not defined} & \mathrm{bideg}(\mathbf{z}) \neq \mathrm{bideg}(\mathbf{z}') \end{cases} \tag{3.74}$$

where $\lambda \in \mathbb{Z}$ and $\mathbf{z}, \mathbf{z}' \in C_{a+b}(X)$. By construction,

$$C_{a,b}(X) := \{\mathbf{z} \in C_*(X) \mid \mathrm{bideg}(\mathbf{z}) = (a,b)\} \tag{3.75}$$

is now a well defined subspace of $C_{a+b}(X)$. By using Equations (3.73) and (3.20) we see

$$C_{i,0}(M) := \Lambda^i(\mathbb{Z}^n)\cdot\{p_1,\dots,p_{2k}\} \tag{3.76}$$

$$C_{i-2,2}(M) := \Lambda^{i-1}(\mathbb{Z}^n)\cdot\{I_1,\dots,I_{2k}\} \oplus \bigoplus_{a=1}^{k} \frac{\Lambda^{i-2}(\mathbb{Z}^n)\cdot\{\xi_a p_{2a-1}, \xi_a p_{2a}\}}{\Lambda^{i-3}(\mathbb{Z}^n)\cdot\{\mathbf{v}_a\cdot\xi_a p_{2a-1}, \mathbf{v}_a\cdot\xi_a p_{2a}\}} \tag{3.77}$$

$$\begin{aligned} C_{i-4,4}(M) := {} & \Lambda^{i-2}(\mathbb{Z}^n)\cdot\{B\} \oplus \bigoplus_{a=1}^{k} \frac{\Lambda^{i-3}(\mathbb{Z}^n)\cdot\{\xi_a I_{2a-1}\}}{\Lambda^{i-4}(\mathbb{Z}^n)\cdot\{\mathbf{v}_a\cdot\xi_a I_{2a-1}\}} \\ & \oplus \bigoplus_{c\in E} \frac{\Lambda^{i-4}(\mathbb{Z}^n)\cdot\{\mathfrak{c}_c\}}{\Lambda^{i-5}(\mathbb{Z}^n)\cdot\{\mathbf{v}_c\cdot\mathfrak{c}_c, \mathbf{v}_{c+1}\cdot\mathfrak{c}_c\}}. \end{aligned} \tag{3.78}$$

In particular, $C_i(X)$ is broken up into three three distinct subspaces which direct sum so that

$$C_i(X) = C_{i,0}(X) \oplus C_{i-2,2}(X) \oplus C_{i-4,4}(X). \tag{3.79}$$

To show that this stratification passes to homology, let $\partial_{a,b}$ denote the restriction of ∂ to $C_{a,b}(X) \subset$

$$
\begin{tikzcd}
0 \arrow[d] & 0 & 0 & 0 \arrow[dl] & 0 \arrow[dl] & 0 \arrow[dl] \\
\vdots \arrow[d] & & \cdots \arrow[dl] & \cdots \arrow[dl] & \cdots \arrow[dl] & \arrow[dl] \\
C_{i+2} \arrow[d] & 0 & C_{i+2,0} \arrow[dl] & C_{i,2} \arrow[dl] & C_{i-2,4} \arrow[dl] & 0 \arrow[dl] \\
C_{i+1} \arrow[d] & 0 & C_{i+1,0} \arrow[dl] & C_{i-1,2} \arrow[dl] & C_{i-3,4} \arrow[dl] & 0 \arrow[dl] \\
C_{i} \arrow[d] & 0 & C_{i,0} \arrow[dl] & C_{i-2,2} \arrow[dl] & C_{i-4,4} \arrow[dl] & 0 \arrow[dl] \\
C_{i-1} \arrow[d] & 0 & C_{i-1,0} \arrow[dl] & C_{i-3,2} \arrow[dl] & C_{i-5,4} \arrow[dl] & 0 \arrow[dl] \\
C_{i-2} \arrow[d] & 0 & C_{i-2,0} \arrow[dl] & C_{i-4,2} \arrow[dl] & C_{i-6,4} \arrow[dl] & 0 \arrow[dl] \\
\vdots \arrow[d] & & \cdots \arrow[dl] & \cdots \arrow[dl] & \cdots \arrow[dl] & \\
0 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

Figure 3.4: Caption

$C_{a+b}(X)$. Using Equations (3.21) and (3.73) it is easy to check that

$$\operatorname{bideg}(\partial(\mathbf{z})) = \operatorname{bideg}(\mathbf{z}) + (1, -2) \tag{3.80}$$

for all $\mathbf{z} \in C_{a,b}(X)$. This induces the chain complex diagram seen in Figure 3.4 and defines the homology groups

$$H_{a,b}(X;\mathbb{Z}) := \frac{\ker(\partial_{a,b})}{\operatorname{im}(\partial_{a-1,b+2})}. \tag{3.81}$$

Observe that $\frac{\ker(\partial_{a,b})}{\operatorname{im}(\partial_{a-1,b+2})} \subset \frac{\ker(\partial_{a+b})}{\operatorname{im}(\partial_{a+b+1})}$ and therefore $H_{a,b}(X;\mathbb{Z}) \subset H_{a+b}(X;\mathbb{Z})$. Equation (3.79) then show the homology groups split as

$$H_i(X;\mathbb{Z}) \cong H_{i,0}(X;\mathbb{Z}) \oplus H_{i-2,2}(X;\mathbb{Z}) \oplus H_{i-4,4}(X;\mathbb{Z}) \tag{3.82}$$

as required.

Finally choose homology classes $[\mathbf{x}] \in H_{a,b}(M;\mathbb{Z})$ and $[\alpha] \in H_c(T^n;\mathbb{Z})$ with representatives $\mathbf{x}, \mathbf{x}' \in [\mathbf{x}]$ and $\alpha, \alpha' \in [\alpha]$. Using the cell structure for T^n described in the proof of Lemma 3.6 we see that the boundary operator is trivial on $C_*(T^n)$ and there is only one represenative of each homology class, thus $\alpha = \alpha'$. We also know that since $\mathbf{x}, \mathbf{x}' \in C_{a,b}(M)$ are both representatives of the same homology class, there exists a $\mathbf{y} \in C_{a-1,b+2}$ such that $\partial(\mathbf{y}) = \mathbf{x} - \mathbf{x}'$. This shows $\alpha \cdot \mathbf{x} - \alpha' \cdot \mathbf{x}' = \alpha \cdot (\mathbf{x} - \mathbf{x}') = \alpha \cdot \partial(\mathbf{y}) = \partial((-1)^c \alpha \cdot \mathbf{y})$ and therefore

$$[\alpha] \cdot [\mathbf{x}] := [\alpha \cdot \mathbf{x}]$$

is a well defined product of homology classes. Thus $H_{*,*}(X;\mathbb{Z})$ is a stratified $H_*(T^n;\mathbb{Z})$-module as desired and the proof is complete. $\square$

3.3 Weakly Equivariant Maps I

Lemma 3.8. *Let M^{m+2} be a simple T^m-manifold and let N^{n+2} be a simple T^n-manifold. Let $\{\Gamma_1^M, \ldots, \Gamma_k^M\}$ be rods for M with rod structures $\{\mathbf{v}_1^M, \ldots, \mathbf{v}_k^M\}$, defining rod structures and rods for N similarly. Suppose there exists a weakly equivariant map $(F, \varphi)\colon (M, T^m) \to (N, T^n)$ which induces a homeomorphism between the quotient spaces M/T^m and N/T^n with the property that*

$$\pi_N(F(\pi_M^{-1}(\Gamma_i^M))) = \Gamma_i^N \tag{3.83}$$

$$\varphi(\mathbf{v}_i^M) = \mathbf{v}_i^N \tag{3.84}$$

for all $i = 1, \ldots, k$. Then F induces a homomorphism on the bi-graded homology groups

$$F_*\colon H_{a,b}(M;\mathbb{Z}) \to H_{a,b}(N;\mathbb{Z})$$

with the property that

$$F_*([\alpha]\cdot[\mathbf{x}]) = \varphi_*([\alpha])\cdot F_*([\mathbf{x}]) \tag{3.85}$$

for all $[\alpha] \in H_*(T^m;\mathbb{Z})$ and $[\mathbf{x}] \in H_{*,*}(M;\mathbb{Z})$.

Proof. Using Lemma 3.6 we can construct a CW complex X for M and Y for N. We will distinguish the cells in X from Y by labeling them with the superscript M or N. For instance, ${}^M_a$ is a $(-,2)$-cell in X. Using the map F we will assign a cell structure to N which agrees with the cell structure on Y. This is a common procedure for CW complexes, however we will alter the process slightly by using a bi-grading of the cells and a boundary operator which sends (a,b)-cells to $(a+1, b-2)$-cells. This will allow us to assign a cell structure to N in an order which is more natural to the construction in Lemma 3.6.

Let's begin by first defining for each $a \in \{1, \ldots, 2k\}$ the $(0,0)$-cell map

$$g_a^{(0,0)} := F|_{\{p_a^M\}} \tag{3.86a}$$

which sends the $(0,0)$-cell $\{p_a^M\}$ to $\{F(p_a^M)\} \subset N$. This defines the $(0,0)$-skeleton in N. Note that each point $F(p_a^M)$ is distinct since by hypothesis F induces an homeomorphism on the quotient spaces. We now define the $(-1,2)$-cell maps

$$g_b^{(-1,2)} := F|_{I_b^M} \tag{3.86b}$$

for each $b \in \{1, \ldots, k\}$ in the same way. Note that boundary of each interval I_b is sent to the $(0,0)$-cells $F(p_{b+1}^M) - F(p_b^M)$ which is part of the $(0,0)$-skeleton of N. Thus the maps $\{g_b^{(-1,2)}\}$ define the $(-1,2)$-skeleton on N. We similarly construct the map

$$g^{(-2,4)} := F|_{B^M} \tag{3.86c}$$

to define the $(-2,4)$-skeleton.

We now define the $(1,0)$-cell maps $g_{a,j}^{(1,0)}\colon [0,1] \to N$ by $g_{a,j}^{(1,0)}(t) := t\mathbf{e}_j \cdot g_a^{(0,0)}(p_a^M)$ where we are using the action of $\mathbb{R}^n/\mathbb{Z}^n$ on N, and $\mathbf{e}_j \in \mathbb{Z}^n$ is a standard basis element. Note that when $t = 1$ the action is trivial, giving $g_{a,i}^{(1,0)}$ empty boundary as expected. Though a similar procedure we define the higher dimensional cell maps as maps from $[0,1]^i$ in the following way

$$g_{a,J}^{(i,0)}(t_1, \ldots, t_i) := (t_1\mathbf{e}_{j_1} + \cdots + t_i\mathbf{e}_{j_i}) \cdot g_a^{(0,0)}(p_a^M) \tag{3.86d}$$

$$g_{b,J}^{(i-1,2)}(t_1, \ldots, t_i) := (t_1\mathbf{e}_{j_1} + \cdots + t_i\mathbf{e}_{j_i}) \cdot g_b^{(-1,2)}(I_b^M) \tag{3.86e}$$

$$g_J^{(i-2,4)}(t_1, \ldots, t_i) := (t_1\mathbf{e}_{j_1} + \cdots + t_i\mathbf{e}_{j_i}) \cdot g^{(-2,4)}(B^M) \tag{3.86f}$$

for every $a \in \{1, \ldots, 2k\}$, $b \in \{1, \ldots, k\}$ and $J \in I_i^n$.

Note that Equation (3.86k) defines n distinct $(0,2)$-maps. We now define the remain $2k$ $(0,2)$-cell maps by

$$g_a^{(0,2)} := F|_{\xi_a^M p_a^M} \tag{3.86g}$$

for $a \in \{1,\dots,2k\}$. Note that $g_a^{(0,2)}$ is a map from a disk $\mathbb{D}^2$ sitting inside of M to N. The boundary of $g_a^{(0,2)}$ is by definition the restriction of $g_a^{(0,2)}$ to the boundary of the disk. The boundary of $\xi_a^M p_a^M$ is $\mathbf{v}_j^M \cdot p_a^M = \{t\mathbf{v}_j^M \cdot p_a^M | t \in [0,1]\} \subset M$ where $a = 2j$ or $a = 2j-1$. Since F is weakly equivariant we know its image is $\{\varphi(t\mathbf{v}_j^M) \cdot F(p_a^M) | t \in [0,1]\} \subset N$. By hypothesis $\varphi(\mathbf{v}_j^M) = \mathbf{v}_j^N$ and thus the image is $\{t\mathbf{v}_j^N \cdot F(p_a^M) | t \in [0,1]\}$ which is equal to the image of $\sum_{i=1}^n (\mathbf{v}_j^N)^i g_{a,i}^{(1,0)}$. This makes the boundary of $g_a^{(0,2)}$ lie in the $(1,0)$-skeleton, as expected. Through a similar procedure we define the cell maps

$$g_b^{(-1,4)} := F|_{\xi_b^M I_b^M} \tag{3.86h}$$

$$g_c^{(0,4)} := F|_{\mathfrak{c}_c^M} \tag{3.86i}$$

for all $b \in \{1,\dots,k\}$ and $c \in E \subset \{1,\dots,k\}$. The final maps we must define are the ones coming from the torus action on the images of $\xi_a^M p_a^M$, $\xi_b^M I_b^M$, and $\mathfrak{c}_c^M$. For that we define

$$g_{a,J}^{(i,2)}(t_1,\dots,t_i,x) := (t_1\mathbf{e}_{j_1} + \cdots + t_i\mathbf{e}_{j_i}) \cdot g_a^{(0,2)}(x) \tag{3.86j}$$

$$g_{b,J}^{(i-1,4)}(t_1,\dots,t_i,x) := (t_1\mathbf{e}_{j_1} + \cdots + t_i\mathbf{e}_{j_i}) \cdot g_b^{(-1,4)}(x) \tag{3.86k}$$

$$g_J^{(i,4)}(t_1,\dots,t_i,x) := (t_1\mathbf{e}_{j_1} + \cdots + t_i\mathbf{e}_{j_i}) \cdot g^{(0,4)}(x). \tag{3.86l}$$

The maps defined in Equations (3.86) cover N, any two maps of the same dimension do not overlap except on the boundary, and image of the boundary of any map is contained in the image of maps of lower dimensions. Thus Equations (3.86) give a well-defined cell structure on N. In particular this cell structure was constructed to make it identical to the one constructed for N using Lemma 3.6 with the identification of

$$g_a^{(0,0)} \mapsto p_a^N \tag{3.87a}$$

$$g_a^{(-1,2)} \mapsto I_a^N \tag{3.87b}$$

$$g^{(-2,4)} \mapsto B^N \tag{3.87c}$$

$$g_a^{(0,2)} \mapsto \xi_a^N p_a^N \tag{3.87d}$$

$$g_b^{(-1,4)} \mapsto \xi_b^N I_b^N \tag{3.87e}$$

$$g_c^{(0,4)} \mapsto \mathfrak{c}_c^N \tag{3.87f}$$

for $a \in \{1,\dots,2k\}$, $b \in \{1,\dots,k\}$, and $c \in E$. Moreover, by construction F induces a map on the chain complex of

$$F_*(\alpha \cdot \mathbf{y}^M) = \varphi_*(\alpha) \cdot \mathbf{y}^N := \varphi_*(\alpha) \cdot F_*(\mathbf{y}^M) \tag{3.88}$$

where $\mathbf{y} \in \{p_a, \xi_a p_a, I_a, \xi_b I_b, \mathfrak{c}_c, B | a \in \{1,\dots,2k\}, b \in \{1,\dots,k\}, c \in E\}$, $\alpha \in \Lambda^i(\mathbb{Z}^m)$, and $\varphi_*\colon \Lambda^i(\mathbb{Z}^m) \to \Lambda^i(\mathbb{Z}^n)$ is the homomorphism induced by $\varphi\colon \mathbb{Z}^m \to \mathbb{Z}^n$. This homomorphism passes to the bi-graded homology groups and proved Equation (3.85). □

Remark 3.9. Many common simple T^n-manifolds admit non-trivial weakly equivariant automorphisms which can be used to infer symmetries of the homology groups. For example consider $M(5,5)$ as defined in Conjecture A and Table 2.18. This 7-manifold has rod structures $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ and admits a weakly equivariant automorphism of order 5 defined by rotating the quotient space $1/5^{th}$ of a rotation and sending the basis vectors $\mathbf{e}_i$ to $\mathbf{e}_{i+1}$. This shows that $\mathbb{Z}_5$ acts on $H_{*,*}$, but does not determine if it acts trivially or not. Later during the discussion of $H_{i-2,2}$, we will see that almost any such automorphism acts non-trivially on $H_{i-2,2}$. In particular, there is a non-trivial $\mathbb{Z}_5$ action on the non-trivial groups $H_{i-2,2}(M(5,5);\mathbb{Z})$ for $i = 3, 4$.

3.4 Computing $H_{i,0}(X;\mathbb{Z})$

Lemma 3.10. *Let M^{n+2} be a simple T^n-manifold with rod structures $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\} \subset \mathbb{Z}^n$. Using the fundamental group of M*

$$\pi_1(M) \cong \frac{\mathbb{Z}^n}{\operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}} \cong \mathbb{Z}^{n-l} \oplus \bigoplus_{j=1}^{l} (\mathbb{Z}/s_j\mathbb{Z}) \tag{3.89}$$

the $(i,0)$-homology group can be calculated as

$$H_{i,0}(X;\mathbb{Z}) \cong \frac{\Lambda^i(\mathbb{Z}^n)}{\Lambda^{i-1}(\mathbb{Z}^n)\cdot\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}} \cong \Lambda^i(\mathbb{Z}^{n-l}) \oplus \bigoplus_{j=1}^{l} (\mathbb{Z}/s_j\mathbb{Z})^{\binom{n-j}{i-1}}. \tag{3.90}$$

Proof. Let X be the CW-complex of M. By definition $H_{i,0}(X;\mathbb{Z})$ is the quotient of $\ker(\partial_{i,0}\colon C_{i,0} \to C_{i+1,-2})$ and $\operatorname{im}(\partial_{i-1,2}\colon C_{i-1,2} \to C_{i,0})$. However $C_{i+1,-2} \cong \{0\}$ is trivial so $\ker(\partial_{i,0}) = C_{i,0}$ and we see that

$$H_{i,0}(X;\mathbb{Z}) \cong C_{i,0}(X)/\operatorname{im}(\partial_{i-1,2}).$$

Next, Equation (3.77) shows us that the homology classes $[\alpha\cdot p_a], [\alpha\cdot p_b] \in C_{i,0}(X)/\operatorname{im}(\partial_{i-1,2})$ are homologous since

$$\begin{aligned} \alpha\cdot p_b - \alpha\cdot p_a &= \alpha\cdot(p_b - p_a) && (3.91)\\ &= \alpha\cdot\partial(I_a + I_{a+1} + \cdots + I_{b-1} + I_b) && (3.92)\\ &= \pm\partial\left(\alpha\cdot(I_a + \cdots + I_b)\right). && (3.93) \end{aligned}$$

Furthermore, Equation (3.76) shows that every element of $C_{i,0}(X)$ is just a linear combination $\sum_{a=1}^{2k} \alpha^a \cdot p_a$. Therefore references to specific points p_a can be dropped when computing this homology group $H_{i,0}(X;\mathbb{Z})$. This leads to the simplification

$$\frac{C_{i,0}(X)}{\operatorname{im}(\partial_{i-1,2})} \cong \frac{\Lambda^i(\mathbb{Z}^n)}{\partial\left(\bigoplus_{a=1}^{k} \frac{\Lambda^{i-1}(\mathbb{Z}^n)\cdot\{\xi_a\}}{\Lambda^{i-2}(\mathbb{Z}^n)\cdot\{\mathbf{v}_a\cdot\xi_a\}}\right)}. \tag{3.94}$$

Lastly observe that in the denominator $\partial(\mathbf{v}_b\cdot\xi_b) = -\mathbf{v}_b\wedge\mathbf{v}_b = 0 \in \Lambda^2(\mathbb{Z}^n)$. This shows that $\partial\left(\Lambda^{i-2}(\mathbb{Z}^n)\cdot\{\mathbf{v}_a\cdot\xi_a\}\right) \cong \{0\}$ for all $a = 1,\ldots,k$. Therefore

$$\operatorname{im}(\partial_{i-2,2}) \cong \partial\left(\bigoplus_{a=1}^{k} \Lambda^{i-1}(\mathbb{Z}^n)\cdot\{\xi_a\}\right) \cong \bigoplus_{a=1}^{k} \Lambda^{i-1}(\mathbb{Z}^n)\cdot\{\mathbf{v}_a\} \tag{3.95}$$

and we recover the first half of Equation (3.90).

To prove the second half of Equation (3.90) define $Z_{i,0} := \Lambda^i(\mathbb{Z}^n)$ and $B_{i,0} := \Lambda^{i-1}(\mathbb{Z}^n)\cdot\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ so that $H_{i,0}(X;\mathbb{Z}) \cong Z_{i,0}/B_{i,0}$. Recall from Lemma 2.22 that the Smith normal form of a set of rod structures $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\} \subset \mathbb{Z}^k$ is a unique collection of vectors $\{s_1\mathbf{e}_1,\ldots,s_l\mathbf{e}_l\} \subset \mathbb{Z}^k$ such that $\operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_1,\ldots,\mathbf{v}_k\} = \operatorname{span}_{\mathbb{Z}}\{s_1\mathbf{e}_1,\ldots,s_l\mathbf{e}_l\}$ and $s_j|s_{j+1}$ for all $1 \le j < l \le \min\{n,k\}$. The first property shows us

$$\Lambda^{i-1}(\mathbb{Z}^n)\cdot\{\mathbf{v}_1,\ldots,\mathbf{v}_k\} \cong \Lambda^{i-1}(\mathbb{Z}^n)\cdot\{s_1\mathbf{e}_1,\ldots,s_l\mathbf{e}_l\} \tag{3.96}$$

while the second one shows that

$$s_{j'}\mathbf{e}_{j'} \wedge \mathbf{e}_j \in \operatorname{span}_{\mathbb{Z}}\{s_j\mathbf{e}_j \wedge \mathbf{e}_{j'}\} \tag{3.97}$$

will always hold for any $1 \leq j < j' \leq l$. Using these two equations we see

$$\begin{aligned} B_{i,0} &= \Lambda^{i-1}(\mathbb{Z}^n) \cdot \{s_1\mathbf{e}_1, \dots, s_l\mathbf{e}_l\} && (3.98)\\ &= \operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{i-1}} \wedge s_j\mathbf{e}_j | 1 \leq j_1 < \cdots < j_{i-1} \leq n, 1 \leq j \leq l\} && (3.99)\\ &= \operatorname{span}_{\mathbb{Z}}\{s_{j_1}\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_i} \mid 1 \leq j_1 < \cdots < j_i \leq n, j_1 \leq l\}. && (3.100) \end{aligned}$$

We can similarly express $Z_{i,0}$ as

$$\begin{aligned} Z_{i,0} &= \Lambda^i(\mathbb{Z}^n) && (3.101)\\ &= \Lambda^i(\operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_1, \dots, \mathbf{e}_n\}) && (3.102)\\ &= \Lambda^i(\operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_{l+1}, \dots, \mathbf{e}_n\}) \oplus \Lambda^{i-1}(\mathbb{Z}^n) \cdot \{\mathbf{e}_1, \dots, \mathbf{e}_l\} && (3.103)\\ &= \Lambda^i(\mathbb{Z}^{n-l}) \oplus \operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_i} \mid 1 \leq j_1 < \cdots < j_i \leq n, j_1 \leq l\}. && (3.104) \end{aligned}$$

Notice that in both $B_{i,0}$ and $Z_{i,0}$, for each $j = 1, \dots, l$, there are $\binom{n-j}{i-1}$ distinct basis elements that are wedge products which start with $\mathbf{e}_j$. Therefore the quotient space is equal to

$$Z_{i,0}/B_{i,0} \cong \Lambda^i(\mathbb{Z}^{n-l}) \oplus \bigoplus_{j=1}^{l} (\mathbb{Z}/s_j\mathbb{Z})^{\binom{n-j}{i-1}} \tag{3.105}$$

as desired. $\square$

3.5 Computing $H_{i-4,4}(X;\mathbb{Z})$

Lemma 3.11. *Suppose M^{n+2} is a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ forming matrix $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$ of rank l. There exists a simple T^l-manifold N^{l+2} such that M is homeomorphic to $T^{n-l} \times N$ and*

$$H_{i-4,4}(M;\mathbb{Z}) \cong H_{i-2-l}(T^{n-l};\mathbb{Z}) \otimes H_{l+2}(N;\mathbb{Z}) \cong \begin{cases} \Lambda^{i-2-l}(\mathbb{Z}^{n-l}) & M \text{ is closed} \\ \{0\} & M \text{ is not closed.} \end{cases} \tag{3.106}$$

Proof. Without loss of generality assume that the rod structures $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ are in Hermite normal form so that $\mathbf{v}_i \in \operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_1, \dots, \mathbf{e}_l\} \subset \mathbb{Z}^n$ for all i. Consider M as the quotient space $(M/T^n \times T^n)/\sim$ and define $N := (M/T^n \times T^l)/\sim$ by collapsing the last $n-l$ circles in $T^n \cong T^l \times T^{n-l}$. Clearly $M \cong T^{n-l} \times N$ and we can apply Kuneth formula to get

$$H_*(M;\mathbb{Z}) \cong H_*(T^{n-l};\mathbb{Z}) \otimes H_*(N;\mathbb{Z}). \tag{3.107}$$

Now construct the CW complex X for M as usual. Consider the chain complex $C_*(X) = \frac{\Lambda^*(\mathbb{Z}^n)\cdot G}{\Lambda^*(\mathbb{Z}^n)\cdot R}$ from Lemma 3.6 as an Abelian group. Define the subgroup $C_*(Y) := \frac{\Lambda^*(\operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_1,\dots,\mathbf{e}_l\})\cdot G}{\Lambda^*(\operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_1,\dots,\mathbf{e}_l\})\cdot R} \subset C_*(X)$ and construct a subcomplex $Y \subset X$ to be the collection of cells whose associated chains are in $C_*(Y)$. Clearly $C_*(Y)$ is the associated chain complex to the CW complex Y, and Y is a CW complex associated to N. Next observe that $\Lambda^*(\mathbb{Z}^n) \cong \Lambda^*(\operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_{l+1}, \dots, \mathbf{e}_n\}) \otimes \Lambda^*(\operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_1, \dots, \mathbf{e}_l\})$ which means in particular $C_*(X) \cong C_*(T^{n-l}) \otimes C_*(Y)$ since $C_*(T^{n-l}) \cong \Lambda^*(\operatorname{span}_{\mathbb{Z}}\{\mathbf{e}_{l+1}, \dots, \mathbf{e}_n\})$. The exact isomorphism $C_*(T^{n-l}) \otimes C_*(Y) \to C_*(X)$ is described by $(\alpha, \mathbf{y}) \mapsto \alpha \cdot \mathbf{y}$ for all $\mathbf{y} \in C_*(Y)$ and $\alpha \in C_*(T^{n-l})$.

Note that $C_*(Y)$ inherits the bigrading from $C_*(X)$ so $\operatorname{bideg}_Y(\mathbf{y}) = \operatorname{bideg}_X(\mathbf{y}) = \operatorname{bideg}(\mathbf{y})$. Using this, combined with the fact that $\mathrm{bideg}(\alpha \cdot \mathbf{y}) = \deg(\alpha) + \mathrm{bideg}(\mathbf{y})$, we can see the following isomorphism

$$C_{*,*}(X) \cong C_*(T^{n-l}) \otimes C_{*,*}(Y) \tag{3.108}$$

$$C_{a,b}(X) \cong \bigoplus_{c+d=a} C_c(T^{n-l}) \otimes C_{d,b}(Y). \tag{3.109}$$

Because Y inherits the boundary operator from X, this is indeed an isomorphism of chain complexes and not just an isomorphism of Abelian groups. Explicitly $\partial_Y(\mathbf{y}) = \partial_X(\mathbf{y}) = \partial(\mathbf{y})$ implies that $\partial_{T^{n-l}\times Y}(\alpha, \mathbf{y}) \mapsto \partial_X(\alpha \cdot \mathbf{y})$ as seen in the following calculation:

$$\begin{aligned} \partial_{T^{n-l}\times Y}(\alpha, \mathbf{y}) &= (\partial_{T^{n-1}}(\alpha), \mathbf{y}) + (-1)^{\deg(\alpha)}(\alpha, \partial_Y(\mathbf{y})) \\ &= (-1)^{\deg(\alpha)}(\alpha, \partial_Y(\mathbf{y})) \\ &= (-1)^{\deg(\alpha)}(\alpha, \partial_X(\mathbf{y})) \\ (-1)^{\deg(\alpha)}(\alpha, \partial_X(\mathbf{y})) &\mapsto (-1)^{\deg(\alpha)}\alpha \cdot \partial_X(\mathbf{y}) \\ &= \partial_X(\alpha \cdot \mathbf{y}). \end{aligned}$$

Since $C_{*,*}(X)$ and $C_*(T^{n-l}) \otimes C_{*,*}(Y)$ are isomorphic as stratified chain complex, the isomorphism passes to homology.

$$H_{*,*}(M;\mathbb{Z}) \cong H_*(T^{n-l};\mathbb{Z}) \otimes H_{*,*}(N;\mathbb{Z}) \tag{3.110}$$

$$H_{a,b}(M;\mathbb{Z}) \cong \bigoplus_{c+d=a} H_c(T^{n-l};\mathbb{Z}) \otimes H_{d,b}(N;\mathbb{Z}) \tag{3.111}$$

Next we will show that $H_{j-4,4}(N;\mathbb{Z}) \cong \{0\}$ for all $j < l+2$ and thus

$$H_{i-4,4}(M;\mathbb{Z}) \cong H_{i-2-l}(T^{n-l};\mathbb{Z}) \otimes H_{l-2,4}(N;\mathbb{Z}). \tag{3.112}$$

By definition the homology group $H_{j-4,4}(N;\mathbb{Z})$ is equal to $\frac{\ker(\partial\colon C_{j-4,4}(Y)\to C_{j-3,2}(Y))}{\mathrm{im}(\partial\colon C_{j-5,6}(Y)\to C_{j-4,4}(Y))}$. However $C_{j-5,6}(Y) = \{0\}$ which means $H_{j-4,4}(N;\mathbb{Z}) = \ker(\partial_{j-4,4}) = \{\mathbf{y} \in C_{j-4,4}(Y) | \partial(\mathbf{y}) = 0\}$. For the following calculations let $\mathbb{Z}^l$ denote $\mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_1, \ldots, \mathbf{e}_l\} \subset \mathbb{Z}^n$ and let $\mathbf{v}_a \in \mathbb{Z}^l$ denote the vector consisting of only the first l entries of $\mathbf{v}_a \in \mathbb{Z}^n$ (i.e., the only possibly non-zero entries of $\mathbf{v}_a$). Now consider $C_{j-4,4}(Y)$ expressed using Equation (3.78) as

$$\Lambda^{j-2}(\mathbb{Z}^l)\cdot\{B\} \oplus \frac{\Lambda^{j-3}(\mathbb{Z}^l)\cdot\{\xi_a I_{2a-1} | a \in \{1,\ldots,k\}\}}{\Lambda^{j-4}(\mathbb{Z}^l)\cdot\{\mathbf{v}_a\cdot\xi_a I_{2a-1} | a \in \{1,\ldots,k\}\}} \oplus \frac{\Lambda^{j-4}(\mathbb{Z}^l)\cdot\{\mathfrak{c}_c | c \in E\}}{\Lambda^{j-5}(\mathbb{Z}^l)\cdot\{\mathbf{v}_c\cdot\mathfrak{c}_c, \mathbf{v}_{c+1}\cdot\mathfrak{c}_c | c \in E\}} \tag{3.113}$$

and let $\mathbf{y} \in C_{j-4,4}(Y)$. This means there exists $\alpha \in \Lambda^{j-2}(\mathbb{Z}^n)$, $\beta^b \in \Lambda^{j-3}(\mathbb{Z}^n)$ for $b \in \{1,\ldots,k\}$, and $\gamma^c \in \Lambda^{j-4}(\mathbb{Z}^n)$ for $c \in E$ such that

$$\mathbf{y} = \alpha \cdot B + \sum_{b=1}^{k} \beta^b \cdot \xi_b I_{2b-1} + \sum_{c\in E} \gamma^c \cdot \mathfrak{c}_c. \tag{3.114}$$

Examining $\partial(\mathbf{y})$ we see

$$\partial(\alpha\cdot B) = (-1)^{j-2}\alpha\cdot(I_1 + \cdots + I_{2k}) \tag{3.115}$$

$$\partial(\beta^b\cdot\xi_b I_{2b-1}) = (-1)^{j-3}\left(\beta^b\wedge\mathbf{v}_b\cdot I_{2b-1} + \beta^b\cdot\xi_b(p_{2b} - p_{2b-1})\right) \tag{3.116}$$

$$\partial(\gamma^c\cdot\mathfrak{c}_c) = (-1)^{j-4}\gamma^c\wedge\left(\mathbf{v}_c\cdot\xi_{c+1}p_{2c+1} + \mathbf{v}_{c+1}\cdot\xi_c p_{2c} + \mathbf{v}_c\wedge\mathbf{v}_{c+1}\cdot I_{2c}\right). \tag{3.117}$$

From the above equations, if $\partial(\mathbf{y}) = 0$, we can deduce

$$\alpha \cdot I_{2b-1} = \beta^b \wedge \mathbf{v}_b \cdot I_{2b-1} \tag{3.118}$$

$$-\alpha \cdot I_{2b} = \gamma^b \wedge \mathbf{v}_b \wedge \mathbf{v}_{b+1} \cdot I_{2b} \tag{3.119}$$

$$\beta^b \cdot \xi_b p_{2b} = \gamma^b \wedge \mathbf{v}_{b+1} \cdot \xi_b p_{2b} \tag{3.120}$$

$$-\beta^b \cdot \xi_b p_{2b-1} = \gamma^{b-1} \wedge \mathbf{v}_{b-1} \cdot \xi_b p_{2b-1} \tag{3.121}$$

for all $b = 1, \ldots, k$.

Assuming $\partial(\mathbf{y}) = 0$, we will show that $\mathbf{y} \neq 0$ if and only if $\alpha \in \Lambda^{j-2}(\mathbb{Z}^l)$ is non-trivial. The if direction is obvious since $\alpha \neq 0$ automatically means $\mathbf{y} \neq 0$. For the only if direction assume that $\mathbf{y} \neq 0$ so at least one of the terms in Equation (3.114) must be non-zero. Assume there exists a $b \in \{1, \ldots, k\}$ such that $\beta^b \cdot \xi_b I_{2b-1} \in \frac{\Lambda^{j-3}(\mathbb{Z}^l)\cdot\{\xi_b I_{2b-1}\}}{\Lambda^{j-3}(\mathbb{Z}^l)\cdot\{\mathbf{v}_b\cdot\xi_b I_{2b-1}\}} \subset C_{j-4,4}(Y)$ is non-trivial. By Equation (3.59) this means $\beta^b \cdot \xi_b I_{2b-1}$ is a non-trivial element in $\Lambda^{j-3}(\mathbb{Z}^l/\mathbf{v}_b\mathbb{Z}) \cdot \{\xi_b I_{2b-1}\}$ and therefore $\beta^b \wedge \mathbf{v}_b \neq 0$. Using equation (3.118) we see $\alpha \neq 0$. Similarly assume there exists a $c \in E$ such that $\gamma^c \cdot \mathfrak{c}_c \in \frac{\Lambda^{j-4}(\mathbb{Z}^l)\cdot\{\mathfrak{c}_c\}}{\Lambda^{j-4}(\mathbb{Z}^l)\cdot\{\mathbf{v}_c\cdot\mathfrak{c}_c,\mathbf{v}_{c+1}\cdot\mathfrak{c}_c\}} \subset C_{j-4,4}(Y)$ is non-trivial. Equation (3.67) show $\gamma^c \cdot \mathfrak{c}_c$ is non-trivial in $\Lambda^{j-4}(\mathbb{Z}^l/\operatorname{span}_{\mathbb{Z}}\{\mathbf{v}_c, \mathbf{v}_{c+1}\}) \cdot \{\mathfrak{c}_c\}$ and therefore $\gamma^c \wedge \mathbf{v}_c \wedge \mathbf{v}_{c+1} \neq 0$. By Equation (3.119) we again see $\alpha \neq 0$ as desired.

Observe that M (or equivalently N) being not closed means that it is missing a corner, or in other words there exists an $a \in \{1, \ldots, k\} \setminus E$. Since $a \notin E$ the term γ^a is automatically 0 and by Equation (3.119) $\alpha = 0$ and thus $\mathbf{y} = 0$ as well. This shows $H_{j-4,4}(N; \mathbb{Z}) \cong \{0\}$ for all j whenever M is not closed. For the remainder of the proof we will assume that M is closed, that $j < l + 2$, and assume by contradiction that $\mathbf{y} \neq 0$.

Note that for any $\eta \in \Lambda^{l+1-j}(\mathbb{Z}^n)$ we have $\partial(\eta \cdot \mathbf{y}) = (-1)^{l+1-j}\eta \cdot \partial(\mathbf{x}) = 0$ and thus $\eta \cdot \mathbf{y} \in \ker(\partial_{l-3,4}) \cong H_{l-3,4}(N; \mathbb{Z})$. In particular, if $\mathbf{y} \neq 0$ then it is possible to choose η so that $\mathbf{z} := \eta \cdot \mathbf{y} \in H_{l-3,4}(N; \mathbb{Z})$ is non-trivial. To see this first we express $\alpha \in \Lambda^{j-2}(\mathbb{Z}^l)$ as

$$\alpha = \sum_{J \in I^l_{j-2}} \alpha^J \mathbf{e}_{a_1} \wedge \cdots \wedge \mathbf{e}_{a_{j-2}} \tag{3.122}$$

where $\alpha^J \in \mathbb{Z}$ and $I^l_{j-2} := \{J = \{a_1, a_2, \ldots, a_{j-2}\} | 1 \leq a_1 < \cdots < a_{j-2} \leq l\}$. Since $\alpha \neq 0$ there exists a $J \in I^l_{j-2}$ such that $\alpha^J \neq 0$. Without loss of generality assume that $\alpha^{J_l} \neq 0$ where $J_l := \{1, 2, \ldots, j-2\} \in I^l_{j-2}$. For $q \in \{1, \ldots, j-2\}$ let $J_q := \{1, 2, \ldots, q-1, q+1, \ldots, j-2, l\} \in I^l_{j-2}$. Now observe that

$$\alpha \wedge \mathbf{e}_{j-1} \wedge \cdots \wedge \mathbf{e}_{l-1} = \alpha^{J_l} \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{l-1} + \sum_{q=1}^{j-2} \alpha^{J_q} \mathbf{e}_1 \wedge \cdots \wedge \widehat{\mathbf{e}_q} \wedge \cdots \wedge \mathbf{e}_l. \tag{3.123}$$

This is a sum of $j - 1$ linearly independent vectors in $\Lambda^{l-1}(\mathbb{Z}^l)$ and is therefore equal to 0 if and only if each vector is 0. By construction $J_l \neq 0$ which means $\alpha \wedge \mathbf{e}_{j-1} \wedge \cdots \wedge \mathbf{e}_{l-1} \neq 0$. Letting $\eta := \mathbf{e}_{j-1} \wedge \cdots \wedge \mathbf{e}_{l-1}$ we see that $\mathbf{z}$ is a non-trivial element in $H_{l-3,4}(N; \mathbb{Z}) \subset H_{l+1}(N; \mathbb{Z})$.

Recall that the matrix of rod structures A has rank l, and thus by Lemma 3.10 $H_1(N; \mathbb{Q}) \cong H_{1,0}(N; \mathbb{Z}) \otimes \mathbb{Q} \cong \{0\}$, or $b_1(N) = 0$. This leads to a contradiction. Observe that $\partial(m\mathbf{z}) = 0$ and that $m\mathbf{z} \neq 0$ for all non-zero integers $m \in \mathbb{Z}$. This means $\mathbf{z}$ generates a free, infinite cyclic subgroup $\mathbb{Z} \cong \langle \mathbf{z} \rangle \subset H_{l-3,4}(N; \mathbb{Z})$. Since $H_{l-3,4}(N; \mathbb{Z}) \subset H_{l+1}(N; \mathbb{Z})$ and N is a closed, oriented $(l + 2)$-manifold, Poincaré duality shows that $H_1(N; \mathbb{Z})$ also contains a free, infinite cyclic subgroup. In particular $b_1(N) \neq 0$.

The last step of the proof is to show that $H_{l-2,4}(N; \mathbb{Z}) \cong H_{l+2}(N; \mathbb{Z})$. If N is not closed then this is trivial as both groups are $\{0\}$. Assuming N is closed let $[N] \in H_{l+2}(N; \mathbb{Z}) \cong \ker(\partial_{l+2})$ denote the fundamental class and let $N \in C_{l+2}(Y) \cong C_{l+2,0}(Y) \oplus C_{l,2}(Y) \oplus C_{l-2,4}(Y)$ be a chain which represents $[N]$. However using Equations (3.76) we see $C_{l+2,0}(Y) \cong \{0\}$. Equations (3.77) shows

$$C_{l,2}(Y) \cong \bigoplus_{a=1}^{k} \frac{\Lambda^l(\mathbb{Z}^l) \cdot \{\xi_a p_{2a-1}, \xi_a p_{2a}\}}{\Lambda^{l-1}(\mathbb{Z}^l) \cdot \{\mathbf{v}_a \cdot \xi_a p_{2a-1}, \mathbf{v}_a \cdot \xi_a p_{2a}\}}. \tag{3.124}$$

Every l-form in $\Lambda^l(\mathbb{Z}^l)$ is a multiple of the "volume form" $\omega \in \Lambda^l(\mathbb{Z}^l)$, and since $\mathbf{v}_a$ is primitive there exists a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_l\}$ for $\mathbb{Z}^l$ with $\mathbf{u}_1 = \mathbf{v}_a$ such that $\omega = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_l = \pm(\mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_l) \wedge \mathbf{v}_a$. Therefore any l-form in the numerator of Equation (3.124) also appears in the denominator which means $C_{l,2}(Y) \cong \{0\}$. As a result $N \in C_{l-2,4}(Y) \cong C_{l+2}(Y)$ and thus $H_{l-2,4}(N;\mathbb{Z}) \cong H_{l+2}(N;\mathbb{Z})$. □

Remark 3.12. Recall that when M^{n+2} is closed, Poincaré duality is equivalent to the perfect pairings on the free and torsion parts of homology

$$fH_i(M;\mathbb{Z}) \otimes fH_{n+2-i}(M;\mathbb{Z}) \to \mathbb{Z} \tag{3.125}$$
$$\tau H_i(M;\mathbb{Z}) \otimes \tau H_{n+1-i}(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}. \tag{3.126}$$

We can combine this with the stratification of homology

$$fH_i(M;\mathbb{Z}) \cong fH_{i,0}(M;\mathbb{Z}) \oplus fH_{i-2,0}(M;\mathbb{Z}) \oplus fH_{i-4,0}(M;\mathbb{Z}) \tag{3.127}$$
$$\tau H_i(M;\mathbb{Z}) \cong \tau H_{i,0}(M;\mathbb{Z}) \oplus \tau H_{i-2,0}(M;\mathbb{Z}). \tag{3.128}$$

This leads to six perfect pairings on the free part and three perfect pairings on the torsion parts. As we know, the Poincaré dual of $[N]$ in $H_*(T^{n-l} \times N;\mathbb{Z})$ is $[T^{n-l}]$ which is a homology class in $H_{n-l,0}(T^{n-l} \times N;\mathbb{Z})$. Since $[N]$ generates the homology $H_{*-4,4}(M;\mathbb{Z})$ we see that $fH_{i,0}$ is "Poincaré dual" to $fH_{n-2-i,4}$. This reduces the six perfect pairings to two

$$fH_{i,0}(M;\mathbb{Z}) \otimes fH_{n-2-i,4}(M;\mathbb{Z}) \to \mathbb{Z} \tag{3.129}$$
$$fH_{i-2,2}(M;\mathbb{Z}) \otimes fH_{n-i,2}(M;\mathbb{Z}) \to \mathbb{Z}. \tag{3.130}$$

In addition, we can plainly see that $\tau H_{*,0}$ cannot be self-dual. This reduces the number of torsion perfect pairings to two.

$$\tau H_{i,0}(M;\mathbb{Z}) \otimes \tau H_{n-i-1,2}(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \tag{3.131}$$
$$\tau H_{i-2,2}(M;\mathbb{Z}) \otimes \tau H_{n-i-1,2}(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \tag{3.132}$$

3.6 Algebraic Representatives of $H_{i-2,2}(X;\mathbb{Z})$

Lemma 3.13. *Let M^{n+2} be a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{Z}^n$ defining the matrix $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$. Using A construct the linear map*

$$\Lambda^{i-2}(\mathrm{id} \otimes A)\colon \Lambda^{i-2}(\mathbb{Z}^n) \otimes \mathbb{Z}^k \to \Lambda^{i-1}(\mathbb{Z}^n) \tag{3.133}$$

defined using the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\} \subset \mathbb{Z}^k$ by

$$\Lambda^{i-2}(\mathrm{id} \otimes A)(\beta \otimes \mathbf{e}_a) := \beta \wedge A(\mathbf{e}_a) = \beta \wedge \mathbf{v}_a. \tag{3.134}$$

for all $a = 1, \ldots, k$ and $\beta \in \Lambda^{i-2}(\mathbb{Z}^n)$. There exists an explicit surjective homomorphism

$$\Psi_{i*}\colon \ker(\Lambda^{i-2}(\mathrm{id} \otimes A)) \subset \Lambda^{i-2}(\mathbb{Z}^n) \otimes \mathbb{Z}^k \to H_{i-2,2}(M;\mathbb{Z}) \tag{3.135}$$

with a kernel of

$$\ker(\Psi_{i*}) = \Lambda^{i-3}(\mathbb{Z}^n) \cdot \{\mathbf{v}_b \otimes \mathbf{e}_b, \mathbf{v}_{c+1} \otimes \mathbf{e}_c + \mathbf{v}_c \otimes \mathbf{e}_{c+1} | b \in \{1, \ldots, k\}, c \in E\}. \tag{3.136}$$

Proof. Define the homomorphism

$$\Psi_i \colon \Lambda^{i-2}(\mathbb{Z}^n) \otimes \mathbb{Z}^k \to C_{i-2,2} \tag{3.137}$$

by

$$\Psi_i(\beta \otimes \mathbf{e}_j) = \beta \cdot \xi_j p_{2j-1} + \beta \wedge \mathbf{v}_j \cdot (I_1 + \cdots + I_{2j-2}) \tag{3.138}$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ are the standard basis elements of $\mathbb{Z}^k$ and $\beta \in \Lambda^{i-2}(\mathbb{Z}^n)$. The map Ψ_i extends linearly so that when

$$\mathbf{x} = \sum_{j=1}^{k} \beta^j \otimes \mathbf{e}_j \tag{3.139}$$

we have

$$\begin{aligned} \partial(\Psi_i(\mathbf{x})) &= \partial\left(\Psi_i\left(\sum_{j=1}^{k} \beta^j \otimes \mathbf{e}_j\right)\right) && (3.140)\\ &= \sum_{j=1}^{k} \partial\left(\beta^j \cdot \xi_j p_{2j-1} + \beta^j \wedge \mathbf{v}_j \cdot (I_1 + \cdots + I_{2j-2})\right) && (3.141)\\ &= (-1)^{i-2} \sum_{j=1}^{k} \left(\beta^j \wedge \mathbf{v}_j \cdot p_{2j-1} - \beta^j \wedge \mathbf{v}_j \cdot (p_{2j-1} - p_1)\right) && (3.142)\\ &= (-1)^{i-2} \left(\sum_{j=1}^{k} \beta^j \wedge \mathbf{v}_j\right) \cdot p_1. && (3.143) \end{aligned}$$

By construction $\Psi_i(\mathbf{x}) \in \ker(\partial_{i-2,2})$ if and only if $0 = \sum_{j=1}^{k} \beta^j \wedge \mathbf{v}_j$.

Notice that $\Lambda^{i-2}(\mathrm{id} \otimes A)(\mathbf{x}) = \sum_{j=1}^{k} \beta^j \wedge \mathbf{v}_j$ which means $\Psi_i(\ker(\Lambda^{i-2}(\mathrm{id} \otimes A))) \subset \ker(\partial_{i-2,2})$. This allows us to define the homomorphism

$$\Psi_{i*} \colon \ker(\Lambda^{i-2}(\mathrm{id} \otimes A)) \to \frac{\ker(\partial_{i-2,2})}{\mathrm{im}(\partial_{i-3,4})} \cong H_{i-2,2}(X; \mathbb{Z}) \tag{3.144}$$

simply by taking the homology class of the image of Ψ;

$$\Psi_{i*}(\mathbf{x}) := [\Psi_i(\mathbf{x})]. \tag{3.145}$$

To show that Ψ_{i*} is surjective, choose a homology class $[\mathbf{y}] \in H_{i-2,2}(X; \mathbb{Z})$. We will construct a representative $\mathbf{y}_4 \in [\mathbf{y}] \subset \ker(\partial_{i-2,2})$ that is in the image of Ψ_i.

A generic representative $\mathbf{y}_1 \in [\mathbf{y}]$ will be of the form

$$\mathbf{x}_1 = \sum_{j=1}^{k} \left(\alpha^j \cdot \xi_j p_{2j-1} + \beta^j \cdot \xi_j p_{2j} + \gamma^j \cdot I_{2j-1} + \delta^j \cdot I_{2j}\right) \in \ker(\partial_{i-2,2}) \tag{3.146}$$

for some where $\alpha^j, \beta^j \in \Lambda^{i-2}(\mathbb{Z}^n)$ and $\gamma^j, \delta^j \in \Lambda^{i-1}(\mathbb{Z}^n)$. This can be simplified in the following way

$$\begin{aligned}
\mathbf{y}_1 &= \alpha^j \cdot \xi_j p_{2j-1} + \beta^j \cdot \xi_j p_{2j} + \gamma^j \cdot I_{2j-1} + \delta^j \cdot I_{2j} && (3.147)\\
&= (\alpha^j + \beta^j) \cdot \xi_j p_{2j-1} + \beta^j \cdot \xi_j (p_{2j} - p_{2j-1}) + \gamma^j \cdot I_{2j-1} + \delta^j \cdot I_{2j} && (3.148)\\
&= (\alpha^j + \beta^j) \cdot \xi_j p_{2j-1} + (-1)^{i-2} \left(\partial(\beta^j \cdot \xi_j I_{2j-1}) - \beta^j \wedge \mathbf{v}_j \cdot I_{2j-1}\right) + \gamma^j \cdot I_{2j-1} + \delta^j \cdot I_{2j} && (3.149)\\
&= (\alpha^j + \beta^j) \cdot \xi_j p_{2j-1} + \left(\gamma^j + (-1)^{i-2}\beta^j \wedge \mathbf{v}_j\right) \cdot I_{2j-1} + \delta^j \cdot I_{2j} + \partial(\dots). && (3.150)\\
& && (3.151)
\end{aligned}$$

Therefore there exists a representative $\mathbf{y}_2 \in [\mathbf{y}]$ of the form

$$\mathbf{y}_2 = \alpha^j \cdot \xi_j p_{2j-1} + \beta^j \cdot I_{2j-1} + \gamma^j \cdot I_{2j} \tag{3.152}$$

with $\alpha^j \in \Lambda^{i-2}(\mathbb{Z}^n)$ and $\beta^j, \gamma^j \in \Lambda^{i-1}(\mathbb{Z}^n)$.

Now we use the condition that $\partial(\mathbf{y}_2) = 0$ to simplify the representative further.

$$\begin{aligned}
0 &= \partial(\mathbf{y}_2) && (3.153)\\
&= \partial\left(\alpha^j \cdot \xi_j p_{2j-1} + \beta^j \cdot I_{2j-1} + \gamma^j \cdot I_{2j}\right) && (3.154)\\
&= (-1)^{i-2}\alpha^j \wedge \mathbf{v}_j \cdot p_{2j-1} + (-1)^{i-1}\beta^j \cdot (p_{2j} - p_{2j-1}) + (-1)^{i-1}\gamma^j \cdot (p_{2j+1} - p_{2j}) && (3.155)\\
&= (-1)^{i-2}\left((\alpha^j \wedge \mathbf{v}_j + \beta^j - \gamma^{j-1}) \cdot p_{2j-1} + (\gamma^j - \beta^j) \cdot p_{2j}\right) && (3.156)
\end{aligned}$$

Hence $\beta^j = \gamma^j$ in Equation (3.152) which means there exists an even simpler representative, $\mathbf{y}_3 \in [\mathbf{y}]$, of the form

$$\mathbf{y}_3 = \alpha^j \cdot \xi_j p_{2j-1} + \beta^j \cdot (I_{2j-1} + I_{2j}). \tag{3.157}$$

We can preform one final simplification. Consider

$$\begin{aligned}
0 &= \partial(\mathbf{y}_3) && (3.158)\\
&= \partial\left(\alpha^j \cdot \xi_j p_{2j-1} + \beta^j \cdot (I_{2j-1} + I_{2j})\right) && (3.159)\\
&= (-1)^{i-2}(\alpha^j \wedge \mathbf{v}_j) \cdot p_{2j-1} + (-1)^{i-1}\beta^j \cdot (p_{2j+1} - p_{2j-1}) && (3.160)\\
&= (-1)^{i-2}\left((\alpha^j \wedge \mathbf{v}_j + \beta^j - \beta^{j-1}) \cdot p_{2j-1}\right). && (3.161)
\end{aligned}$$

Therefore

$$\beta^{j-1} = \beta^j + \alpha^j \wedge \mathbf{v}_j \tag{3.162}$$

for all $j = 1, \dots, k$ (where $\beta^0 = \beta^k$). By fixing β^k, this gives the formula

$$\beta^j = \beta^k + \alpha^{j+1} \wedge \mathbf{v}_{j+1} + \dots + \alpha^k \wedge \mathbf{v}_k \tag{3.163}$$

for $j = 1, \dots, k-1$. Finally note that $\sum_{j=1}^{k} (\beta^j - \beta^k) \cdot (I_{2j-1} + I_{2j})$ is homologous to $\sum_{j=1}^{k} \beta^j \cdot (I_{2j-1} + I_{2j})$ since $\partial(\beta^k \cdot B) = (-1)^{i-1}\beta^k(I_1 + \dots + I_{2k})$. So there exists representative $\mathbf{y}_4 \in [\mathbf{y}]$ for any choice of β^k. Setting $\beta^k = 0$ we have the following formula

$$\mathbf{y}_4 = \sum_{j=1}^{k} \alpha^j \cdot \xi_j p_{2j-1} + \sum_{j=1}^{k-1} (\alpha^{j+1} \wedge \mathbf{v}_{j+1} + \dots + \alpha^k \wedge \mathbf{v}_k) \cdot (I_{2j-1} + I_{2j}). \tag{3.164}$$

Lastly, one need to rearrange terms in Equation (3.164) to express it in terms of Ψ_i.

$$\mathbf{y}_4 = \sum_{j=1}^{k} \alpha^j \cdot \xi_j p_{2j-1} + \sum_{j=1}^{k-1} (\alpha^{j+1} \wedge \mathbf{v}_{j+1} + \cdots + \alpha^k \wedge \mathbf{v}_k) \cdot (I_{2j-1} + I_{2j}) \tag{3.165}$$

$$= \sum_{j=1}^{k} \alpha^j \cdot \xi_j p_{2j-1} + \sum_{j=1}^{k-1} \sum_{l=j+1}^{k} (\alpha^l \wedge \mathbf{v}_l) \cdot (I_{2j-1} + I_{2j}) \tag{3.166}$$

$$= \sum_{j=1}^{k} \alpha^j \cdot \xi_j p_{2j-1} + \sum_{1 \leq j < l \leq k} (\alpha^l \wedge \mathbf{v}_l) \cdot (I_{2j-1} + I_{2j}) \tag{3.167}$$

$$= \sum_{j=1}^{k} \alpha^j \cdot \xi_j p_{2j-1} + \sum_{l=2}^{k} \sum_{j=1}^{l-1} (\alpha^l \wedge \mathbf{v}_l) \cdot (I_{2j-1} + I_{2j}) \tag{3.168}$$

$$= \sum_{j=1}^{k} \alpha^j \cdot \xi_j p_{2j-1} + \sum_{j=2}^{k} \sum_{l=1}^{j-1} (\alpha^j \wedge \mathbf{v}_j) \cdot (I_{2l-1} + I_{2l}) \tag{3.169}$$

$$= \sum_{j=1}^{k} \alpha^j \cdot \xi_j p_{2j-1} + \sum_{j=2}^{k} \alpha^j \wedge \mathbf{v}_j \cdot (I_1 + \cdots + I_{2j-2}) \tag{3.170}$$

$$= \Psi_i \left(\sum_{j=1}^{k} \alpha^j \otimes \mathbf{e}_j \right). \tag{3.171}$$

Thus Ψ_{i*} is indeed surjective.

We now need to calculate $\ker(\Psi_{i*})$. First observe that $\Psi_i(\beta \otimes \mathbf{e}_j) = 0$ if and only if $\beta \in \Lambda^{i-3}(\mathbb{Z}^n) \cdot \{\mathbf{v}_j\}$. Therefore $\ker(\Psi_i) = \Lambda^{i-3}(\mathbb{Z}^n) \cdot \{\mathbf{v}_b \otimes \mathbf{e}_b | b \in \{1, \ldots, k\}\}$, which happens to be contained in $\ker(\Lambda^{i-2}(\mathrm{id} \otimes A))$. This means the kernel of Ψ_{i*} can be calculated as

$$\ker(\Psi_{i*}) = \Psi_i^{-1}(\{0\}) \cup \Psi_i^{-1}(\mathrm{im}(\partial_{i-3,4}) \setminus \{0\}). \tag{3.172}$$

Having already computed $\Psi_i^{-1}(\{0\})$ we are left with computing $\Psi_i^{-1}(\mathrm{im}(\partial_{i-3,4}) \setminus \{0\})$. To do this we will choose an arbitrary non-trivial element $\mathbf{y} \in C_{i-3,4}(X)$ and compute its boundary. We will then derive relations for its coefficients based on the assumption that $\partial(\mathbf{y}) = \Psi_i(\mathbf{x})$ for some $\mathbf{x} \in \ker(\Lambda^{i-2}(\mathrm{id} \otimes A))$.

Let

$$\mathbf{y} = (-1)^{i-1} \left(\alpha \cdot B + \sum_{j=1}^{k} \beta_j \xi_j I_{2j-1} + \varepsilon_j \gamma_j \mathfrak{c}_j \right) \in C_{i-3,4}(X) \tag{3.173}$$

where

$$\varepsilon_j := \begin{cases} 1 & j \in E \\ 0 & j \notin E \end{cases} \tag{3.174}$$

and $\alpha \in \Lambda^{i-1}(\mathbb{Z}^n)$, $\beta_j \in \Lambda^{i-2}(\mathbb{Z}^n)$, and $\gamma_j \in \Lambda^{i-3}(\mathbb{Z}^n)$. The $(-1)^{i-1}$ in front is put in place to cancel the sign coming from the graded-Leibniz rule.

$$\partial(\mathbf{y}) = (-1)^{i-1}\partial\left(\alpha \cdot B + \sum_{j=1}^{k} \beta_j \xi_j I_{2j-1} + \varepsilon_j \gamma_j \mathfrak{c}_j\right) \tag{3.175}$$

$$\begin{aligned} &= \alpha \cdot (I_1 + \cdots + I_{2k}) \\ &\quad - \sum_{j=1}^{k} (\beta_j \wedge \mathbf{v}_j \cdot I_{2j-1} + \beta_j \cdot \xi_j (p_{2j} - p_{2j-1})) \\ &\quad + \sum_{j=1}^{k} \varepsilon_j \gamma_j \wedge (\mathbf{v}_j \cdot \xi_{j+1} p_{2j+1} + \mathbf{v}_{j+1} \cdot \xi_j p_{2j} + \mathbf{v}_j \wedge \mathbf{v}_{j+1} \cdot I_{2j}) \end{aligned} \tag{3.176}$$

$$\begin{aligned} &= \sum_{j=1}^{k} \Big((\alpha - \beta_j \wedge \mathbf{v}_j) \cdot I_{2j-1} + (\alpha + \varepsilon_j \gamma_j \wedge \mathbf{v}_j \wedge \mathbf{v}_{j+1}) \cdot I_{2j} \\ &\quad + (\varepsilon_j \gamma_j \wedge \mathbf{v}_{j+1} - \beta_j) \cdot \xi_j p_{2j} + \beta_j \cdot \xi_j p_{2j-1} + \varepsilon_j \gamma_j \wedge \mathbf{v}_j \cdot \xi_{j+1} p_{2j+1} \Big) \end{aligned} \tag{3.177}$$

We can stop here and determine three relations that α, β_j and γ_j have to satisfy. First, nothing in the image of Ψ_i contains any p_{2j} terms. Thus the coefficient infront of the p_{2j} terms must be 0.

$$\beta_j = \varepsilon_j \gamma_j \wedge \mathbf{v}_{j+1} = \begin{cases} \gamma_j \wedge \mathbf{v}_{j+1} & \text{corner at } (p_{2j}, p_{2j+1}) \\ 0 & \text{horizon at } (p_{2j}, p_{2j+1}) \end{cases} \tag{3.178}$$

Second, the coefficients in front of the I_{2j-1} and I_{2j} term must agree.

$$-\beta_j \wedge \mathbf{v}_j = \varepsilon_j \gamma_j \wedge \mathbf{v}_j \wedge \mathbf{v}_{j+1} = \begin{cases} \gamma_j \wedge \mathbf{v}_j \wedge \mathbf{v}_{j+1} & \text{corner at } (p_{2j}, p_{2j+1}) \\ 0 & \text{horizon at } (p_{2j}, p_{2j+1}) \end{cases} \tag{3.179}$$

Third, the $(I_{2k-1} + I_{2k})$ term never appears in the image of Ψ_i.

$$\alpha + \varepsilon_k \gamma_k \wedge \mathbf{v}_k \wedge \mathbf{v}_1 = 0 = \alpha - \beta_k \wedge \mathbf{v}_k \tag{3.180}$$

These three relations tell us that $\mathbf{y}$ is entirely determined by γ_j. In particular

$$(-1)^{i-1}\mathbf{y} = -\varepsilon_k \gamma_k \wedge \mathbf{v}_k \wedge \mathbf{v}_1 \cdot B + \sum_{j=1}^{k} \varepsilon_j \left(\gamma_j \wedge \mathbf{v}_{j+1} \cdot \xi_j I_{2j-1} + \gamma_j \cdot \mathfrak{c}_j\right). \tag{3.181}$$

This simplified form allows us to continue our calculations

$$\begin{aligned} \partial(\mathbf{y}) &= \sum_{j=1}^{k} \Big((\alpha + \varepsilon_j \gamma_j \wedge \mathbf{v}_j \wedge \mathbf{v}_{j+1}) \cdot (I_{2j-1} + I_{2j}) \\ &\quad + \varepsilon_j \left(\gamma_j \wedge \mathbf{v}_{j+1} \cdot \xi_j p_{2j-1} + \gamma_j \wedge \mathbf{v}_j \cdot \xi_{j+1} p_{2j+1}\right) \Big) \end{aligned} \tag{3.182}$$

$$\begin{aligned} &= \sum_{j=1}^{k} \Big((\alpha + \varepsilon_j \gamma_j \wedge \mathbf{v}_j \wedge \mathbf{v}_{j+1}) \cdot (I_{2j-1} + I_{2j}) \\ &\quad + (\varepsilon_j \gamma_j \wedge \mathbf{v}_{j+1} + \varepsilon_{j-1} \gamma_{j-1} \wedge \mathbf{v}_{j-1}) \cdot \xi_j p_{2j-1} \Big) \end{aligned} \tag{3.183}$$

The last step is to rearrange terms so that $\partial(\mathbf{y})$ in the standard form of the image of Ψ_i. For ease of notation, let $a_j := (\alpha + \varepsilon_j \gamma_j \wedge \mathbf{v}_j \wedge \mathbf{v}_{j+1})$ for $1 \leq j \leq k-1$ and $a_0 := a_k := 0$.

$$\sum_{j=1}^{k} a_j \cdot (I_{2j-1} + I_{2j}) = \sum_{j=1}^{k} \sum_{l=1}^{j} a_j \cdot (I_{2l-1} + I_{2l}) - \sum_{j=1}^{k} \sum_{l=1}^{j-1} a_j \cdot (I_{2l-1} + I_{2l}) \tag{3.184}$$

$$= \sum_{j=1}^{k-1} \sum_{l=1}^{j} a_j \cdot (I_{2l-1} + I_{2l}) - \sum_{j=2}^{k} \sum_{l=1}^{j-1} a_j \cdot (I_{2l-1} + I_{2l}) \tag{3.185}$$

$$= \sum_{j=1}^{k-1} \sum_{l=1}^{j} a_j \cdot (I_{2l-1} + I_{2l}) - \sum_{j=1}^{k-1} \sum_{l=1}^{j} a_{j+1} \cdot (I_{2l-1} + I_{2l}) \tag{3.186}$$

$$= \sum_{j=1}^{k-1} (a_j - a_{j+1}) \cdot \sum_{l=1}^{j} (I_{2l-1} + I_{2l}) \tag{3.187}$$

$$= \sum_{j=1}^{k-1} (a_j - a_{j+1}) \cdot (I_1 + \cdots + I_{2j}) \tag{3.188}$$

$$= \sum_{j=2}^{k} (a_{j-1} - a_j) \cdot (I_1 + \cdots + I_{2j-2}) \tag{3.189}$$

$$= \sum_{j=1}^{k} (a_{j-1} - a_j) \cdot (I_1 + \cdots + I_{2j-2}) \tag{3.190}$$

$$\tag{3.191}$$

We conclude with the following

$$
\begin{aligned}
\partial(\mathbf{y}) &= \sum_{j=1}^{k}(a_{j-1}-a_j)\cdot(I_1+\cdots+I_{2j-2}) + (\varepsilon_j\gamma_j\wedge\mathbf{v}_{j+1}+\varepsilon_{j-1}\gamma_{j-1}\wedge\mathbf{v}_{j-1})\cdot\xi_j p_{2j-1} && (3.192)\\
&= \sum_{j=1}^{k}\Big(((\alpha+\varepsilon_{j-1}\gamma_{j-1}\wedge\mathbf{v}_{j-1}\wedge\mathbf{v}_j)-(\alpha+\varepsilon_j\gamma_j\wedge\mathbf{v}_j\wedge\mathbf{v}_{j+1}))\cdot(I_1+\cdots+I_{2j-2}) && (3.193)\\
&\quad + (\varepsilon_j\gamma_j\wedge\mathbf{v}_{j+1}+\varepsilon_{j-1}\gamma_{j-1}\wedge\mathbf{v}_{j-1})\cdot\xi_j p_{2j-1}\Big)\\
&= \sum_{j=1}^{k}\Big((\varepsilon_{j-1}\gamma_{j-1}\wedge\mathbf{v}_{j-1}\wedge\mathbf{v}_j-\varepsilon_j\gamma_j\wedge\mathbf{v}_j\wedge\mathbf{v}_{j+1})\cdot(I_1+\cdots+I_{2j-2}) && (3.194)\\
&\quad + (\varepsilon_j\gamma_j\wedge\mathbf{v}_{j+1}+\varepsilon_{j-1}\gamma_{j-1}\wedge\mathbf{v}_{j-1})\cdot\xi_j p_{2j-1}\Big)\\
&= \sum_{j=1}^{k}\Big((\varepsilon_{j-1}\gamma_{j-1}\wedge\mathbf{v}_{j-1}+\varepsilon_j\gamma_j\wedge\mathbf{v}_{j+1})\wedge\mathbf{v}_j\cdot(I_1+\cdots+I_{2j-2}) && (3.195)\\
&\quad + (\varepsilon_j\gamma_j\wedge\mathbf{v}_{j+1}+\varepsilon_{j-1}\gamma_{j-1}\wedge\mathbf{v}_{j-1})\cdot\xi_j p_{2j-1}\Big)\\
&= \sum_{j=1}^{k}\Big((\varepsilon_j\gamma_j\wedge\mathbf{v}_{j+1})\cdot\xi_j p_{2j-1}+(\varepsilon_j\gamma_j\wedge\mathbf{v}_{j+1})\wedge\mathbf{v}_j\cdot(I_1+\cdots+I_{2j-2})\Big) && (3.196)\\
&\quad + \sum_{j=1}^{k}\Big((\varepsilon_j\gamma_j\wedge\mathbf{v}_j)\cdot\xi_{j+1}p_{2j+1}+(\varepsilon_j\gamma_j\wedge\mathbf{v}_j)\wedge\mathbf{v}_{j+1}\cdot(I_1+\cdots+I_{2j+1})\Big)\\
&= \sum_{c\in E}\Big((\gamma_c\wedge\mathbf{v}_{c+1})\cdot\xi_c p_{2c-1}+(\gamma_c\wedge\mathbf{v}_{c+1})\wedge\mathbf{v}_c\cdot(I_1+\cdots+I_{2c-2})\Big) && (3.197)\\
&\quad + \sum_{c\in E}\Big((\gamma_c\wedge\mathbf{v}_c)\cdot\xi_{c+1}p_{2c+1}+(\gamma_c\wedge\mathbf{v}_c)\wedge\mathbf{v}_{c+1}\cdot(I_1+\cdots+I_{2c})\Big)\\
&= \sum_{c\in E}\Psi_i(\gamma_c\wedge\mathbf{v}_{c+1}\otimes\mathbf{e}_c)+\sum_{c\in E}\Psi_i(\gamma_c\wedge\mathbf{v}_c\otimes\mathbf{e}_{c+1}) && (3.198)\\
&= \Psi_i\left(\sum_{c\in E}\gamma_c\cdot(\mathbf{v}_{c+1}\otimes\mathbf{e}_c+\mathbf{v}_c\otimes\mathbf{e}_{c+1})\right) && (3.199)\\
&= \Psi_i(\mathbf{x}) && (3.200)
\end{aligned}
$$

where $\mathbf{x}\in\Lambda^{i-3}(\mathbb{Z}^n)\cdot\{\mathbf{v}_{c+1}\otimes\mathbf{e}_c+\mathbf{v}_c\otimes\mathbf{e}_{c+1}|c\in E\}$ as desired. This shows $\Psi_i^{-1}(\mathrm{im}(\partial_{i-3,4})\setminus\{0\}) = \Lambda^{i-3}(\mathbb{Z}^n)\cdot\{\mathbf{v}_{c+1}\otimes\mathbf{e}_c+\mathbf{v}_c\otimes\mathbf{e}_{c+1}|c\in E\}$ and proves Equation (3.136).

□

The above lemma is especially simple when $i=2$ since the map $\Lambda^{i-2}(\mathrm{id}\otimes A)$ becomes simply A. Additionally the kernel vanishes since $i<3$, and thus $\Psi_{2*}\colon \ker(A)\to H_{0,2}(M;\mathbb{Z})$ is an isomorphism. Finally when M is simply connected, from Lemmas 3.10 and 3.11 we see that $H_i = H_{i-2,2}$, in particular $H_2(M;\mathbb{Z}) = H_{0,2}(M;\mathbb{Z})$. This special case is recorded in the following corollary.

Corollary 3.14. *For any simply connected T^n-manifold M^{n+2} with rod structures forming the matrix $A\colon\mathbb{Z}^k\to\mathbb{Z}^n$, there exists an isomorphism*

$$\Psi_*\colon \ker(A)\to H_2(M;\mathbb{Z}) \tag{3.201}$$

defined explicitly in terms of the rod structures.

3.7 Geometric Representatives of $H_{i-2,2}(X;\mathbb{Z})$

Lemma 3.15. *Let M^{n+2} be a simple T^n-manifold with projection map $\pi\colon M \to M/T^n$. Choose an interior point $R_0 \in M/T^n \setminus \partial(M/T^n)$ and a collection of line segments $R_a \subset M/T^n$ that connect R_0 to each of the k rods $\Gamma_a \subset \partial(M/T^n)$, as described in Figure 2.16 and in the proof of Lemma 2.38. For every homology class $[\mathbf{y}] \in H_{i-2,2}(M;\mathbb{Z})$ there exists an i-dimensional CW-complex Y and a continuous function $f\colon Y \to M$ such that $[\mathbf{y}]$ is realized by the pushforward of the fundamental class $[Y] \in H_i(Y;\mathbb{Z})$;*

$$f_*[Y] = [\mathbf{y}]. \tag{3.202}$$

Furthermore, the projection of the image of Y is contained in the curves R_a;

$$f(Y) \subset \bigcup_{a=1}^{k} \pi^{-1}(R_a). \tag{3.203}$$

Proof. This proof has two steps to it. The first step is to show that there exists a CW complex Y and a continuous function $f\colon Y \to M$ which satisfies Equation (3.202) and where the projection of the image is distinct from the corners, i.e. $\pi(f(Y)) \cap \Gamma_a \cap \Gamma_{a+1} = \emptyset$ for all $a = 1, \ldots, k$. The second step is to deform f so that the projection of the image is contained in the curves R_a, i.e. Equation (3.203) is satisfied. This second step is a trivial task and a diagram of the procedure can be seen in Figure 2.14. The remainder of this proof is dedicated to proving the first step by explicitly constructing the CW complex Y and a the map $f\colon Y \to M$.

Let X be the CW complex for M. As observed in the previous lemma, every homology class $[\mathbf{y}] \in H_{i-2,2}(X;\mathbb{Z})$ can be represented by $\Psi_i(\mathbf{x}) = \mathbf{y} \in [\mathbf{y}]$ for some $\mathbf{x} \in \ker(\Lambda^{i-2}(\mathrm{id} \otimes A))$. This means there is a representative $\mathbf{y} \in [\mathbf{y}]$ is of the form

$$\begin{aligned}
\mathbf{y} &= \Psi_i(\mathbf{x}) \\
&= \Psi_i\left(\sum_{a=1}^{k} \beta^a \otimes \mathbf{e}_a\right) \\
&= \sum_{a=1}^{k} \beta^a \cdot \xi_a p_{2a-1} + \beta^a \wedge \mathbf{v}_a \cdot (I_1 + \cdots + I_{2a-2}).
\end{aligned} \tag{3.204}$$

Consider the cell structure of $\mathbb{D}^2 = p \cup S^1 \cup D^2$ with a single 0, 1, and 2-cell so that $\partial\mathbb{D}^2 = S^1$. For each $a \in \{1, \ldots, k\}$ define the cell map $f_{a^-}\colon \mathbb{D}^2 \to X$ by sending p to p_{2a-1}, D^2 to $\xi_a p_{2a-1}$, and S^1 to $\mathbf{v}_a \cdot p_{2a-1}$. Using the usual cell structure on T^{i-2} with exactly $\binom{i-2}{j}$ j-cells, define $Y_{a^-} := T^{i-2} \times \mathbb{D}^2$. Extend the map to $f_{a^-}\colon Y_{a^-} \to X$ by sending $T^{i-2} \times \{p\}$ to $\beta^a \cdot p_{2a-1}$, $T^{i-2} \times B$ to $\beta^a \cdot \xi_a p_{2a-1}$, and $\partial(T^{i-2} \times \mathbb{D}^2) = T^{i-2} \times S^1$ to $\beta^a \wedge \mathbf{v}_a \cdot p_{2a-1}$. This makes $f_{a^-}\colon Y_{a^-} \to X$ a cellular map between CW complexes. Next let $Y_{a^+} := T^{i-1} \times [0,1]$ be a CW complex and define the cell map $f_{a^+}\colon Y_{a^+} \to X$ by sending $T^{i-1} \times (0,1)$ to $\beta^a \wedge \mathbf{v}_a \cdot (I_1 + \cdots + I_{2a-2})$, sending $T^{i-1} \times \{0\}$ to $-\beta^a \wedge \mathbf{v}_a \cdot p_{2a-1}$, and sending $T^{i-1} \times \{1\}$ to $\beta^a \wedge \mathbf{v}_a \cdot p_1$.

Now defined the CW complex Y_a by gluing Y_{a^-} and Y_{a^+} together in the obvious way, by attaching $\partial(Y_{a^-})$ to $T^{i-1} \times \{0\} \subset \partial(Y_{a^+})$. Define the cell map $f_a\colon Y_a \to X$ by $f_a|_{Y_{a^\pm}} = f_{a^\pm}$. Note that the image of any cell map is a CW sub-complex, and that the union of any sum-complexes is also a sub-complex. In particular $\bigcup_{a=1}^{k} f_a(\partial Y_a)$ is a CW complex which we can use to define Y. Let $Y_0 := \bigcup_{a=1}^{k} f_a(\partial Y_a)$ and define $Y := Y_0 \cup_{f_1} Y_1 \cup_{f_2} \cdots \cup_{f_k} Y_k$ by attaching ∂Y_a to Y_0 via $f_a\colon \partial Y_a \to Y_0$.

If the CW complex Y has a fundamental class $[Y]$, then it is a primitive element in the top cellular homology group $H_i(Y;\mathbb{Z})$. Since $C_{i+1}(Y) = \{0\}$, we know that every element in $H_i(Y;\mathbb{Z})$ is uniquely described by a closed chain in $C_i(Y)$. Each Y_a has two i-cells which we denote by Y_{a^-} and Y_{a^+}. These are the only i-cells in Y and thus $C_i(Y) = \mathrm{span}_{\mathbb{Z}}\{Y_{1^-}, Y_{1^+}, \ldots, Y_{k^-}, Y_{k^+}\}$. By construction $\partial Y_{a^+} = \beta^a \wedge \mathbf{v}_a \cdot p_1 - \partial Y_{a^-}$ where $\beta^a \wedge \mathbf{v}_a \cdot p_1 \in C_{i-1}(Y_0) \subset C_{i-1}(Y)$. Recall that $\mathbf{x} \in \ker(\Lambda^{i-2}(\mathrm{id} \otimes A))$ and observe $\sum_{a=1}^{k} \beta^a \wedge \mathbf{v}_a = \Lambda^{i-2}(\mathrm{id} \otimes A)(\sum_{a=1}^{k} \beta^a \otimes \mathbf{e}_a) = \Lambda^{i-2}(\mathrm{id} \otimes A)(\mathbf{x})$. Therefore an i-chain given by an element in $\mathrm{span}_{\mathbb{Z}}\{Y_{1^-}, Y_{1^+}, \ldots, Y_{k^-}, Y_{k^+}\}$ is closed if and only if all the coefficients are identical. This means $H_i(Y;\mathbb{Z}) \cong \mathbb{Z}$ and we can define the fundamental class $[Y]$ as being represented by the closed chain $Y_{1^-} + Y_{1^+} + \cdots + Y_{k^-} + Y_{k^+} \in C_i(Y)$.

The map $f\colon Y \to X$ was defined by $f|_{Y_{a^\pm}} = f_{a^\pm}$. By construction f_{a^-} sends the i-cell Y_{a^-} to $\beta^a \cdot \xi_a p_{2a-1}$ and f_{a^+} sends Y_{a^+} to $\beta^a \wedge \mathbf{v}_a \cdot (I_1 + \cdots + I_{2a-2})$. Therefore the induced map $f_*\colon C_i(Y) \to C_i(X)$ sends $Y_{1^-} + Y_{1^+} + \cdots + Y_{k^-} + Y_{k^+}$ to $\mathbf{y}$ and $f_*\colon H_i(Y;\mathbb{Z}) \to H_i(X;\mathbb{Z})$ sends $[Y]$ to $[\mathbf{y}]$, satisfying Equation (3.202).

Finally we must show that $f(Y)$ is disjoint from each corner $\pi^{-1}(\Gamma_c \cap \Gamma_{c+1})$. For each $c \in E$, the set $\pi^{-1}(\Gamma_c \cap \Gamma_{c+1})$ is entirely contained in the $(n+2)$-cell $\gamma^c \cdot \mathfrak{c}_c$ for some $\gamma^c \in \Lambda^{n-2}(\mathbb{Z}^n)$. The cell map $f\colon Y \to X$ misses these cells and thus $f(Y)$ misses the corners.

$\square$

Remark 3.16. Under certain circumstances Lemma 3.15 can be upgraded to guarantee that Y is a smooth closed manifold, rather than merely a CW complex. Assume that M is simply connected and $[\mathbf{y}] = [\boldsymbol{\alpha}] \cdot [\mathbf{z}] \in H_{i-2,2}(M;\mathbb{Z})$ where $[\mathbf{z}] \in H_j(M;\mathbb{Z})$ and $[\boldsymbol{\alpha}] \in H_{i-j}(T^n;\mathbb{Z})$ for $j \leq 3$. The Hurewicz Isomorphism Theorem guarantees that $[\mathbf{z}] = g_*[S^j]$ for some continuous map $g\colon S^j \to M$. Then since $\dim(S^j) + \dim(\pi^{-1}(\Gamma_a \cap \Gamma_{a+1})) < \dim(M^{n+2})$ we can use transversality to force $\pi(g(S^j))$ to be disjoint from every corner. Now let $h\colon T^{i-j} \to T^n$ be a continuous map so that so that $h_*[T^{i-j}] = [\boldsymbol{\alpha}] \in H_{i-j}(T^n;\mathbb{Z})$. Define the map $f\colon T^{i-j} \times S^j \to M$ by $f(\phi, p) = h(\phi) \cdot g(p)$. Letting $Y := T^{i-j} \times S^j$ we see that $f\colon Y \to M$ satisfies the conclusion of Lemma 3.15.

3.8 Equivariant Intersection Form

Lemma 3.17. *Let M^{n+2} be a simple T^n-manifold with rod structures $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ forming the matrix $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$. There is a bilinear form*

$$Q\colon H_{i-2,2}(M;\mathbb{Z}) \otimes H_{j-2,2}(M;\mathbb{Z}) \to H_{i+j-2}(T^n;\mathbb{Z}) \tag{3.205}$$

which for $\Psi_{i*}\left(\sum_{a=1}^{k} \alpha^a \otimes \mathbf{e}_a\right) = [\mathbf{x}] \in H_{i-2,2}(M;\mathbb{Z})$ and $\Psi_{j*}\left(\sum_{b=1}^{k} \beta^b \otimes \mathbf{e}_b\right) = [\mathbf{y}] \in H_{j-2,2}(M;\mathbb{Z})$ is defined as

$$Q([\mathbf{x}], [\mathbf{y}]) := \sum_{1 \leq a < b \leq k-1} \alpha^a \wedge \mathbf{v}_a \wedge \beta^b \wedge \mathbf{v}_b \in \Lambda^{i+j-2}(\mathbb{Z}^n) \cong H_{i+j-2}(T^n;\mathbb{Z}). \tag{3.206}$$

Proof. First note that $\sum_{a=1}^{k} \alpha^a \otimes \mathbf{e}_a \in \ker(\Lambda^{i-2}(\mathrm{id} \otimes A)) \subset \Lambda^{i-2}(\mathbb{Z}^n) \otimes \mathbb{Z}^k$ so $\alpha^a \in \Lambda^{i-2}(\mathbb{Z}^n)$ is an $(i-2)$-form on $\mathbb{Z}^n$. Similarly β^b is a $(j-2)$-form and $\alpha^a \wedge \mathbf{v}_a \wedge \beta^b \wedge \mathbf{v}_b$ is an $(i+j-2)$-form. Using the cell structure for T^n described in the proof of Lemma 3.6, we see that $\alpha^a \wedge \mathbf{v}_a \wedge \beta^b \wedge \mathbf{v}_b$ is naturally described as an element of $H_{i+j-2}(T^n;\mathbb{Z})$. The only thing to prove is that Q is well-defined.

To that end, we construct a new bilinear form (analogous to the bilinear form defined in Remark 2.43)

$$D\colon \ker(\Lambda^{i-2}(\mathrm{id} \otimes A)) \otimes \ker(\Lambda^{j-2}(\mathrm{id} \otimes A)) \to \Lambda^{i+j-2}(\mathbb{Z}^n) \tag{3.207}$$

which for $\boldsymbol{\alpha} := \sum_{a=1}^{k} \alpha^a \otimes \mathbf{e}_a$ and $\boldsymbol{\beta} := \sum_{b=1}^{k} \beta^b \otimes \mathbf{e}_b$ is defined by

$$D(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \sum_{1 \leq a < b < k} \alpha^a \wedge \mathbf{v}_a \wedge \beta^b \wedge \mathbf{v}_b. \tag{3.208}$$

Next choose an element $\boldsymbol{\gamma} \in \ker(\Psi_{i*}) \subset \ker(\Lambda^{i-2}(\mathrm{id} \otimes A))$. Observe that $\Psi_{i*}(\boldsymbol{\alpha}) = \Psi_{i*}(\boldsymbol{\alpha} + \boldsymbol{\gamma})$. Thus if $D(\boldsymbol{\alpha} + \boldsymbol{\gamma}, \boldsymbol{\beta}) = D(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for all $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$, then Q is well defined.

From Lemma 3.13 we know $\boldsymbol{\gamma} \in \Lambda^{i-3}(\mathbb{Z}^n) \cdot \{\mathbf{v}_a \otimes \mathbf{e}_a, \mathbf{v}_c \otimes \mathbf{e}_{c+1} + \mathbf{v}_{c+1} \otimes \mathbf{e}_c | a \in \{1, \ldots, k\}, c \in E\}$. To show that $D(\boldsymbol{\alpha} + \boldsymbol{\gamma}, \boldsymbol{\beta}) = D(\boldsymbol{\alpha}, \boldsymbol{\beta})$, first let $\gamma \in \Lambda^{i-3}(\mathbb{Z}^n)$ and fix $a \in \{1, \ldots, k\}$. Observe that

$$D\left(\gamma \cdot \mathbf{v}_a \otimes \mathbf{e}_a, \boldsymbol{\beta}\right) = \sum_{a<b<k} \gamma \wedge \mathbf{v}_a \wedge \mathbf{v}_a \wedge \beta^b \wedge \mathbf{v}_b = 0.$$

The same result happens for any fixed $c \in E$, namely if $\boldsymbol{\gamma} = \gamma \cdot (\mathbf{v}_c \otimes \mathbf{e}_{c+1} + \mathbf{v}_{c+1} \otimes \mathbf{e}_c)$ then

$$\begin{aligned} D\left(\boldsymbol{\gamma}, \boldsymbol{\beta}\right) &= \sum_{c+1<b<k} \gamma \wedge \mathbf{v}_c \wedge \mathbf{v}_{c+1} \wedge \beta^b \wedge \mathbf{v}_b + \sum_{c<b<k} \gamma \wedge \mathbf{v}_{c+1} \wedge \mathbf{v}_c \wedge \beta^b \wedge \mathbf{v}_b \\ &= \gamma \wedge \mathbf{v}_c \wedge \mathbf{v}_{c+1} \wedge \beta^{c+1} \wedge \mathbf{v}_{c+1} + \sum_{c+1<b<k} \gamma \wedge \left(\mathbf{v}_c \wedge \mathbf{v}_{c+1} + \mathbf{v}_{c+1} \wedge \mathbf{v}_c\right) \wedge \beta^b \wedge \mathbf{v}_b \\ &= 0 \end{aligned}$$

Since D is bilinear we see $D(\boldsymbol{\gamma}, \boldsymbol{\beta}) = 0$ for any $\boldsymbol{\gamma} \in \ker(\Psi_{i*}) \subset \ker(\Lambda^{i-2}(\mathrm{id} \otimes A))$ and $\boldsymbol{\beta} \in \ker(\Lambda^{j-2}(\mathrm{id} \otimes A))$. Therefore $D(\boldsymbol{\alpha} + \boldsymbol{\gamma}, \boldsymbol{\beta}) = D(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for all $\boldsymbol{\alpha} \in \ker(\Lambda^{i-2}(\mathrm{id} \otimes A))$ and Q is well defined. ☐

The Lemma 3.18 below relates the equivariant intersection form with the intersection pairing from singular homology. The proof is a straightforward generalization of the proof of Theorem 2.40 which, amongst other things, shows that shows that the equivariant intersection form and the intersection form coincide in dimension 4. In the statement of the lemma we also use the fact that when M is simply connected, the homology groups $H_{i-2,2}(M; \mathbb{Z})$ and $H_i(M; \mathbb{Z})$ are isomorphic. This can be confirmed by Lemmas 3.10 and 3.11.

Lemma 3.18. *When M is simply connected and $i + j = n + 2$, the equivariant intersection form*

$$Q \colon H_i(M; \mathbb{Z}) \otimes H_j(M; \mathbb{Z}) \to H_n(T^n; \mathbb{Z}) \cong \mathbb{Z} \tag{3.209}$$

agrees with the intersection pairing from singular homology,

$$\cap \colon H_i(M; \mathbb{Z}) \otimes H_j(M; \mathbb{Z}) \to \mathbb{Z}.$$

Proof. Let us briefly review the intersection pairing from singular homology. A singular i-chain is a formal sum of oriented continuous maps from an i-simplex to M. Given an i-chain and a j-chain, each consisting of a single oriented map which happen to be transverse to each other, and with $i + j = \dim(M)$, it is possible compute an integer value known as the signed intersection number. This pairing is extended to all pairs of i-chains and j-chains by linearity. If the j-chain is the boundary of a $(j + 1)$-chain, then the signed intersection number will always be zero. This pairing then passes to singular homology to produce the intersection pairing.

In Lemma 3.15 it was shown that every homology class $[\mathbf{y}] \in H_{i-2,2}(M; \mathbb{Z}) \cong H_i(M; \mathbb{Z})$ can be represented by a continuous map from a CW complex in the space, $f \colon Y \to M$. Note that Y is not an i-simplex but, by restricting f to the i-cells of Y, the map $f \colon Y \to M$ can be thought of as a formal sum of maps from i-simplexes into M and is thus a singular i-chain. This i-chain is in particular a representative of $[\mathbf{y}]$ in singular homology. In a similar manor we use Lemma 3.15 to produce a representative $g \colon Z \to M$ for $[\mathbf{z}] \in H_j(M; \mathbb{Z})$. Assume without loss of generality that f and g are transverse to each other. We can now compute the intersection pairing $[\mathbf{y}] \cap [\mathbf{z}] \in \mathbb{Z}$ by computing the signed intersection number of f and g, which we will denote by $\#(f \cap g)$.

The maps f and g can be homotoped so that the images of $\pi \circ f$ and $\pi \circ g$ lie in k line segments. Denote the portion of Y corresponding to the a^{th} line segment as Y_a, and denote the restriction of f to Y_a as

$f_a := f|_{Y_a}$. Similarly define Z_b and g_b. By breaking up f and g into pieces, we can compute $\#(f \cap g)$ by

$$\#(f \cap g) = \sum_{a,b=1}^{k} \#(f_a \cap g_b).$$

A similar computation was done in the proof of Theorem 2.40, where it was found that $\#(f_a \cap g_b) = 0$ unless $1 \leq a < b \leq k-1$.

To compute $\#(f_a \cap g_b)$ we must go back to the construction of f and g in Lemma 3.15. Suppose that $\sum_{a=1}^{k} \alpha^a \otimes \mathbf{e}_a = \mathbf{x} \in \ker(\Lambda^{i-2}(\mathrm{id} \otimes A))$ and $\sum_{b=1}^{k} \beta^b \otimes \mathbf{e}_b = \mathbf{w} \in \ker(\Lambda^{j-2}(\mathrm{id} \otimes A))$ so that $\Psi_i(\mathbf{x}) = \mathbf{y} \in [\mathbf{y}]$ and $\Psi_j(\mathbf{w}) = [\mathbf{z}] \in H_j(M;\mathbb{Z})$. For any interior point in the curve $p \in \pi \circ f_a(Y_a)$ we see that $f_a(Y_a) \cap \pi^{-1}(p) \subset \pi^{-1}(p) \cong T^n$ is an oriented $(i-1)$-dimensional subtorus, representing the homology class $\alpha^a \wedge \mathbf{v}_a \in \Lambda^{i-1}(\mathbb{Z}^n) \cong H_{i-1}(T^n;\mathbb{Z})$. Similarly $g_b(Z_B) \cap \pi^{-1}(p) \subset T^n$ is a $(j-1)$-dimensional subtorus, representing the homology class $\beta^b \wedge \mathbf{v}_b \in \Lambda^{j-1}(\mathbb{Z}^n) \cong H_{j-1}(T^n;\mathbb{Z})$. Because $(i-1)+(j-1) = n$, the intersection pairing of these two forms is a single integer which is equivalent to the signed intersection number of f_a and g_b. Representing the integers as $\Lambda^n(\mathbb{Z}^n) \cong \mathbb{Z}$, we see that the signed intersection number of f_a and g_b is $\alpha^a \wedge \mathbf{v}_a \wedge \beta^b \wedge \mathbf{v}_b$. This proves that the intersection pairing of $[\mathbf{y}]$ and $[\mathbf{z}]$ is

$$[\mathbf{y}] \cap [\mathbf{z}] = \sum_{1 \leq a < b \leq k-1} \alpha^a \wedge \mathbf{v}_a \wedge \beta^b \wedge \mathbf{v}_b \in \Lambda^n(\mathbb{Z}^n) \cong \mathbb{Z}. \tag{3.210}$$

This is equivalent to the equivariant intersection form $Q([\mathbf{y}], [\mathbf{z}])$. □

3.9 Weakly Equivariant Maps II

Lemma 3.19. *Let $(F, \varphi)\colon (M^{m+2}, T^m) \to (N^{n+2}, T^n)$ be a weakly equivariant map satisfying the hypotheses of Lemma 3.8. Then for all $2 \leq i \leq m$*

$$F_*(\Psi_{i*}^M(\mathbf{w})) = \Psi_{i*}^N(\varphi_*(\mathbf{w})) \tag{3.211}$$

for all $\mathbf{w} \in \ker(\Lambda^{i-2}(\mathrm{id} \otimes A^M)) \subset \Lambda^{i-2}(\mathbb{Z}^m) \otimes \mathbb{Z}^k$ where $\varphi_*\colon \Lambda^{i-2}(\mathbb{Z}^m) \otimes \mathbb{Z}^k \to \Lambda^{i-2}(\mathbb{Z}^n) \otimes \mathbb{Z}^k$ is the homomorphism induced by $\varphi\colon \mathbb{Z}^m \to \mathbb{Z}^n$ via $\varphi_*(\mathbf{v} \otimes \mathbf{e}_a) := \varphi(\mathbf{v}) \otimes \mathbf{e}_a$.

Proof. The proof is a simple computation. Let $\mathbf{w} = \sum\limits_{a=1}^{k} \alpha^a \otimes \mathbf{e}_a \in \ker(\Lambda^{i-2}(\mathrm{id} \otimes A^M))$. Observe the following chain of equalities which uses notation defined in the proof of Lemma 3.8.

$$\begin{aligned}
F_*(\Psi_{i*}^M(\mathbf{w})) &= F_*\left(\Psi_{i*}^M\left(\sum_{a=1}^{k} \alpha^a \otimes \mathbf{e}_a\right)\right) &(3.212)\\
&= \sum_{a=1}^{k} F_*\left(\Psi_{i*}^M\left(\alpha^a \otimes \mathbf{e}_a\right)\right) &(3.213)\\
&= \sum_{a=1}^{k} F_*\left(\alpha^a \cdot \xi_a^M p_{2a-1}^M + \alpha^a \wedge \mathbf{v}_a^M \cdot (I_1^M + \cdots + I_{2a-2}^M)\right) &(3.214)\\
&= \sum_{a=1}^{k} \left(\varphi(\alpha^a) \cdot F_*(\xi_a^M p_{2a-1}^M) + \varphi(\alpha^a) \wedge \varphi(\mathbf{v}_a^M) \cdot (F_*(I_1^M) + \cdots + F_*(I_{2a-2}^M))\right) &(3.215)
\end{aligned}$$

We now use Equation (3.88) from the proof of Lemma 3.8 to express image of the image of F_* in terms of N.

$$F_*(\Psi^M_{i*}(\mathbf{w})) = \sum_{a=1}^{k} \left(\varphi(\alpha^a) \cdot \xi^N_a p^N_{2a-1} + \varphi(\alpha^a) \wedge \mathbf{v}^N_a \cdot (I^N_1 + \cdots + I^N_{2a-2})\right) \tag{3.216}$$

$$= \sum_{a=1}^{k} \Psi^N_{i*}\left(\varphi(\alpha^a) \otimes \mathbf{e}_a\right) \tag{3.217}$$

$$= \Psi^N_{i*}\left(\sum_{a=1}^{k} \varphi(\alpha^a) \otimes \mathbf{e}_a\right) \tag{3.218}$$

$$= \Psi^N_{i*}(\varphi_*(\mathbf{w})) \tag{3.219}$$

□

The above lemma has a special case when $i = 2$ and the manifolds are simply connected. First Corollary 3.14 shows that Ψ induces an isomorphism on the second integral homology group. Second, $\varphi_*\colon \Lambda^{i-2}(\mathbb{Z}^m)\otimes\mathbb{Z}^k\Lambda^{i-2}(\mathbb{Z}^n)\otimes\mathbb{Z}^k$ in Lemma 3.19 reduces to the identity map on $\mathbb{Z}^k$ when $i = 2$. This special case is recorded in the following corollary.

Corollary 3.20. *Let M^{m+2} be a simply connected T^m-manifold with projection map $\pi_M\colon M \to M/T^m$, rods $\{\Gamma^M_1, \ldots, \Gamma^M_k\}$, rod structures $\{\mathbf{v}^M_1, \ldots, \mathbf{v}^M_k\}$ forming the matrix $A^M\colon \mathbb{Z}^k \to \mathbb{Z}^m$ and isomorphism $\Psi^M_*\colon \ker(A^M) \to H_2(M;\mathbb{Z})$. Similarly define N^{n+2} to be a simply connected T^n-manifold. Suppose there exists a weakly equivariant map $(F,\varphi)\colon (M,T^m) \to (N,T^n)$ which induces a homeomorphism between the quotient spaces M/T^m and N/T^n with the property that*

$$\pi_N(F(\pi^{-1}_M(\Gamma^M_i))) = \Gamma^N_i \tag{3.220}$$

$$\varphi(\mathbf{v}^M_i) = \mathbf{v}^N_i \tag{3.221}$$

for all $i = 1,\ldots,k$. Then $\ker(A^M) \subset \ker(A^N) \subset \mathbb{Z}^k$ and

$$F_*(\Psi^M_*(\mathbf{w})) = \Psi^N_*(\mathbf{w}) \tag{3.222}$$

for all $\mathbf{w} \in \ker(A^M)$.

Lemma 3.21. *Let $(F,\varphi)\colon (M^{m+2},T^m) \to (N^{n+2},T^n)$ be a weakly equivariant map satisfying the hypotheses of Lemma 3.8. Then for any $i,j \in \{2,\ldots,m\}$*

$$Q_N(F_*[\mathbf{x}], F_*[\mathbf{y}]) = \varphi_*\left(Q_M([\mathbf{x}],[\mathbf{y}])\right) \tag{3.223}$$

for all $[\mathbf{x}] \in H_{i-2,2}(M;\mathbb{Z})$ and $[\mathbf{y}] \in H_{j-2,2}(M;\mathbb{Z})$ where $\varphi_*\colon H_{i+j-2}(T^m;\mathbb{Z}) \to H_{i+j-2}(T^n;\mathbb{Z})$ is the homomorphism induced from $\varphi\colon T^m \to T^n$.

Proof. Using Lemma 3.13 let $\mathbf{w}_x := \sum_{a=1}^{k} \alpha^a\otimes\mathbf{e}_a \in \ker(\Lambda^{i-2}(\mathrm{id}\otimes A^M))$ and $\mathbf{w}_y := \sum_{b=1}^{k} \beta^b\otimes\mathbf{e}_b \in \ker(\Lambda^{j-2}(\mathrm{id}\otimes A^M))$ so that $\Psi^M_{i*}(\mathbf{w}_x) = [\mathbf{x}] \in H_{i-2,2}(M;\mathbb{Z})$ and $\Psi^M_{j*}(\mathbf{w}_y) = [\mathbf{y}] \in H_{j-2,2}(M;\mathbb{Z})$. Now observe the following chain of equalities.

$$\varphi_*\left(Q_M([\mathbf{x}],[\mathbf{y}])\right) = \varphi_*\left(Q_M(\Psi^M_{i*}(\mathbf{w}_x), \Psi^M_{j*}(\mathbf{w}_y))\right) \tag{3.224}$$

$$= \varphi_*\left(\sum_{1\le a<b<k} \alpha^a \wedge \mathbf{v}^M_a \wedge \beta^b \wedge \mathbf{v}^M_b\right) \tag{3.225}$$

The above line is simply the definition of Q_M from Lemma 3.17.

$$\begin{aligned}
\varphi_*\left(Q_M([\mathbf{x}],[\mathbf{y}])\right) &= \sum_{1\leq a<b<k} \varphi_*\left(\alpha^a \wedge \mathbf{v}_a^M \wedge \beta^b \wedge \mathbf{v}_b^M\right) && (3.226)\\
&= \sum_{1\leq a<b<k} \varphi(\alpha^a) \wedge \varphi(\mathbf{v}_a^M) \wedge \varphi(\beta^b) \wedge \varphi(\mathbf{v}_b^M) && (3.227)\\
&= \sum_{1\leq a<b<k} \varphi(\alpha^a) \wedge \mathbf{v}_a^N \wedge \varphi(\beta^b) \wedge \mathbf{v}_b^N && (3.228)\\
&= Q_N\left(\Psi_{i*}^N\left(\sum_{a=1}^{k} \varphi(\alpha^a)\otimes \mathbf{e}_a\right), \Psi_{i*}^N\left(\sum_{b=1}^{k} \varphi(\beta^b)\otimes \mathbf{e}_b\right)\right) && (3.229)\\
&= Q_N\left(\Psi_{i*}^N(\varphi_*(\mathbf{w}_x)), \Psi_{j*}^N(\varphi_*(\mathbf{w}_y))\right) && (3.230)
\end{aligned}$$

We now use Lemma 3.19 and conclude

$$\begin{aligned}
\varphi_*\left(Q_M([\mathbf{x}],[\mathbf{y}])\right) &= Q_N\left(F_*(\Psi_{i*}^M(\mathbf{w}_x)), F_*(\Psi_{j*}^M(\mathbf{w}_y))\right) && (3.231)\\
&= Q_N(F_*[\mathbf{x}], F_*[\mathbf{y}]). && (3.232)
\end{aligned}$$

□

3.10 Computing $H_{i-2,2}(X;\mathbb{Z})$

Lemma 3.22. *Let M be a simple T^n-manifold of dimension $(n+2)>4$ with k rods and $m\leq k$ corners. If the first rational homology group of M vanishes, then for $2\leq i\leq n$*

$$H_{i-2,2}(M;\mathbb{Q}) \cong \mathbb{Q}^{b_i} \tag{3.233}$$

where

$$b_i = k\binom{n}{i-2} - \binom{n}{i-1} - k\binom{n-1}{i-3} - m\binom{n-2}{i-3}. \tag{3.234}$$

If in addition M is simply connected, then

$$H_{i-2,2}(M;\mathbb{Z}) \cong \mathbb{Z}^{b_i}. \tag{3.235}$$

Proof. In Lemma 3.13 a surjective homomorphism

$$\Psi_{i*}\colon \ \ker(\Lambda^{i-2}(\mathrm{id}\otimes A)) \to H_{i-2,2}(M;\mathbb{Z}) \tag{3.236}$$

was constructed. This gives a representation of the integral homology group

$$H_{i-2,2}(M;\mathbb{Z}) \cong \frac{\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))}{\ker(\Psi_{i*})} \tag{3.237}$$

where the kernel of Ψ_{i*} was computed to be

$$\ker(\Psi_{i*}) = \Lambda^{i-3}(\mathbb{Z}^n)\cdot\{\mathbf{v}_a\otimes\mathbf{e}_a, \mathbf{v}_{c+1}\otimes\mathbf{e}_c + \mathbf{v}_c\otimes\mathbf{e}_{c+1} | a\in\{1,\ldots,k\}, c\in E\} \tag{3.238}$$

where $E\subset\{1,\ldots,k\}$ is the set of corners of M. Recall that any rational homology group is a vector space and as such its only invariant is its dimension. The dimension can be computed by taking the difference of the dimensions in the numerator and denominator in Equation (3.237);

$$b_i = \dim(\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))\otimes\mathbb{Q}) - \dim(\ker(\Psi_{i*})\otimes\mathbb{Q}). \tag{3.239}$$

To prove that the integral homology is also only defined by b_i amounts to showing that $H_{i-2,2}(M;\mathbb{Z})$ is torsion-free. To do this we will first show that $\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))$ is a free $\mathbb{Z}$-module so that no torsion is inherited from it. Then we will show that $\ker(\Psi_{i*}) \subset \ker(\Lambda^{i-2}(\mathrm{id}\otimes A))$ is a primitive sub-latice or sub-module, so no new torsion is introduce by taking a quotient.

Our first goal is to compute the dimension of $\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))$. To that end consider the domain and range of $\Lambda^{i-2}(\mathrm{id}\otimes A)$ thought of as a linear map between $\mathbb{Z}$-modules

$$\begin{aligned}
\mathrm{dom}(\Lambda^{i-2}(\mathrm{id}\otimes A)) &= \Lambda^{i-2}(\mathbb{Z}^n)\otimes\mathbb{Z}^k && (3.240)\\
&= \mathrm{span}_{\mathbb{Z}}\{\alpha\otimes\mathbf{w}|\alpha\in\Lambda^{i-2}(\mathbb{Z}^n),\mathbf{w}\in\mathbb{Z}^k\} && (3.241)\\
\mathrm{range}(\Lambda^{i-2}(\mathrm{id}\otimes A)) &= \Lambda^{i-2}(\mathbb{Z}^n)\cdot A(\mathbb{Z}^k) && (3.242)\\
&= \mathrm{span}_{\mathbb{Z}}\{\alpha\wedge A(\mathbf{w})|\alpha\in\Lambda^{i-2}(\mathbb{Z}^n),\mathbf{w}\in\mathbb{Z}^k\}. && (3.243)
\end{aligned}$$

If we tensor these spaces with $\mathbb{Q}$ then they become vector spaces and the rank-nullity theorem will allow us to compute the dimension of $\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))\otimes\mathbb{Q}$ from the dimensions of $\mathrm{dom}(\Lambda^{i-2}(\mathrm{id}\otimes A))\otimes\mathbb{Q}$ and $\mathrm{range}(\Lambda^{i-2}(\mathrm{id}\otimes A))\otimes\mathbb{Q}$. Note that since $\mathrm{range}(\Lambda^{i-2}(\mathrm{id}\otimes A))\subset\Lambda^{i-1}(\mathbb{Z}^n)$ and $\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))\subset\Lambda^{i-2}(\mathbb{Z}^n)\otimes\mathbb{Z}^k$ are all clearly finitely generated free $\mathbb{Z}$-modules, they too are described by a single invariant known as the dimension. Thus for ease of notation we will suppress the $\mathbb{Q}$ when computing dimension.

Computing the dimension of the domain is trivial since the dimension of a tensor product is the product of the dimension, thus

$$\dim(\Lambda^{i-2}(\mathbb{Z}^n)\otimes\mathbb{Z}^k) = k\binom{n}{i-2}. \tag{3.244}$$

Computing $\dim(\mathrm{range}(\Lambda^{i-2}(\mathrm{id}\otimes A)))$ is more difficult but will be made easy by introducing new bases for $\mathbb{Z}^n$ and $\mathbb{Z}^k$ coming from the Smith normal form of A (see Lemma 2.22). As an abuse of notation we will denote the elements of both new bases as $\mathbf{f}_a$, so that $\{\mathbf{f}_1,\dots,\mathbf{f}_k\}\subset\mathbb{Z}^k$ is a basis for $\mathbb{Z}^k$ and $\{\mathbf{f}_1,\dots,\mathbf{f}_n\}\subset\mathbb{Z}^n$ is a basis for $\mathbb{Z}^n$. The fact that $\mathbb{Z}^k$ and $\mathbb{Z}^n$ are formally distinct vector spaces (i.e., a single vector cannot be in both vector spaces) will hopefully mitigate any confusion that this abuse of notation causes. Recall the that Smith normal form of a rank-l linear map $A\colon\mathbb{Z}^k\to\mathbb{Z}^n$ is given by $UAV=S$ where $U\in GL(n,\mathbb{Z})$, $V\in GL(k,\mathbb{Z})$, and S is the diagonal matrix $\mathrm{diag}(s_1,\dots,s_l,0,\dots,0)$. Using the standard bases $\{\mathbf{e}_1,\dots,\mathbf{e}_k\}\subset\mathbb{Z}^k$ and $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}\subset\mathbb{Z}^n$ we define the new bases as

$$\mathbf{f}_a := V(\mathbf{e}_a) = V_a^b\mathbf{e}_b\in\mathbb{Z}^k \tag{3.245}$$
$$\mathbf{f}_a := U^{-1}(\mathbf{e}_a) = (U^{-1})_a^b\mathbf{e}_b\in\mathbb{Z}^n. \tag{3.246}$$

Observe that the matrix representation of A in these new bases is the Smith normal form of the matrix representation of A in the standard bases;

$$\begin{aligned}
A(\mathbf{f}_a) &= A(V_a^b\mathbf{e}_b)\\
&= A_a^cV_c^b\mathbf{e}_b\\
&= (AV)_a^b\mathbf{e}_b\\
&= (U^{-1}S)_a^b\mathbf{e}_b\\
&= (U^{-1})_a^cS_c^b\mathbf{e}_b\\
&= (U^{-1})_a^cs_c\mathbf{e}_c\\
&= U^{-1}(s_a\mathbf{e}_a)\\
&= s_a\mathbf{f}_a.
\end{aligned}$$

In this basis the kernel and range of A become easy to represent;

$$\ker(A) = \mathrm{span}_{\mathbb{Z}}\{\mathbf{f}_{l+1}, \ldots, \mathbf{f}_k\} \cong \mathbb{Z}^{k-l} \tag{3.247}$$

$$\mathrm{range}(A) = \mathrm{span}_{\mathbb{Z}}\{s_1\mathbf{f}_1, \ldots, s_l\mathbf{f}_l\} \cong \mathbb{Z}^l. \tag{3.248}$$

We can also define an "orthogonal complement" $\ker(A)^\perp \subset \mathbb{Z}^k$ as

$$\ker(A)^\perp := \mathrm{span}_{\mathbb{Z}}\{\mathbf{f}_1, \ldots, \mathbf{f}_l\} \cong \mathbb{Z}^l \tag{3.249}$$

so that $\mathbb{Z}^k = \ker(A) \oplus \ker(A)^\perp$. Note that $\ker(A)^\perp$ is only "orthogonal" to $\ker(A)$ with respect to the dot-product using the $\{\mathbf{f}_a\} \subset \mathbb{Z}^k$ basis, and not some underlying inner-product on $\mathbb{Z}^k$.

With this new notation we can find a new set of generators for the range;

$$\begin{aligned}
\mathrm{range}(\Lambda^{i-2}(\mathrm{id}\otimes A)) &= \mathrm{span}_{\mathbb{Z}}\{\alpha \wedge A(\mathbf{w}) | \alpha \in \Lambda^{i-2}(\mathbb{Z}^n), \mathbf{w} \in \mathbb{Z}^k\} && (3.250)\\
&= \mathrm{span}_{\mathbb{Z}}\{\alpha \wedge A(\mathbf{w}) | \alpha \in \Lambda^{i-2}(\mathbb{Z}^n), \mathbf{w} \in \ker(A)^\perp\} && (3.251)\\
&= \mathrm{span}_{\mathbb{Z}}\{\mathbf{f}_J \wedge A(\mathbf{f}_a) | J \in I^n_{i-2}, a \in \{1, \ldots, l\}\} && (3.252)\\
&= \mathrm{span}_{\mathbb{Z}}\{\mathbf{f}_J \wedge s_a\mathbf{f}_a | J \in I^n_{i-2}, a \in \{1, \ldots, l\}\} && (3.253)\\
&= \mathrm{span}_{\mathbb{Z}}\{\mathbf{f}_J \wedge s_a\mathbf{f}_a | J \in I^n_{i-2}, a \in \{1, \ldots, l\}, a \notin J\} && (3.254)\\
&= \mathrm{span}_{\mathbb{Z}}\{\mathbf{f}_J \wedge s_a\mathbf{f}_a | J \in I^n_{i-2}, a \in \{1, \ldots, l\}, a < J\}. && (3.255)
\end{aligned}$$

Where we are using the now standard multi-index notation with J and I^n_{i-2} and $a < J$ means $a < j_1$. Observe that by construction the generators in Equation (3.255) are all linearly independent. Computing the dimension is now a simple combinatorics problem. For each $a \in \{1, \ldots, l\}$ we must choose the distinct values for the $i-2$ numbers $\{j_1, \ldots, j_{i-2}\}$ from our set of $n-a$ numbers $\{a+1, \ldots, n\}$;

$$\dim(\mathrm{range}(\Lambda^{i-2}(\mathrm{id}\otimes A))) = \sum_{a=1}^{l} \binom{n-a}{i-2} = \binom{n}{i-1} - \binom{n-l}{i-1}. \tag{3.256}$$

The rank-nullity theorem then shows

$$\dim(\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))) = k\binom{n}{i-2} + \binom{n-l}{i-1} - \binom{n}{i-1}. \tag{3.257}$$

The above equation is true for any linear map $A\colon \mathbb{Z}^k \to \mathbb{Z}^n$ of rank l. However by hypothosis the $H_1(M;\mathbb{Q}) = \{0\}$ which from Lemma 3.10 means $l = n$. Therefore when the first rational homology group vanishes

$$\dim(\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))) = k\binom{n}{i-2} - \binom{n}{i-1}. \tag{3.258}$$

Our next goal in this proof is to compute the dimension of $\ker(\Psi_{i*})$.

$$\ker(\Psi_{i*}) = \Lambda^{i-3}(\mathbb{Z}^n) \cdot \{\mathbf{v}_a \otimes \mathbf{e}_a, \mathbf{v}_{a+1} \otimes \mathbf{e}_a + \mathbf{v}_a \otimes \mathbf{e}_{a+1} | a \in E\}. \tag{3.259}$$

We will denote this space by N and split it up into two sub-spaces,

$$N_1 := \Lambda^{i-3}(\mathbb{Z}^n) \cdot \{\mathbf{v}_a \otimes \mathbf{e}_a | a \in \{1, \ldots, k\}\} \tag{3.260}$$

$$N_2' := \Lambda^{i-3}(\mathbb{Z}^n) \cdot \{\mathbf{v}_{a+1} \otimes \mathbf{e}_a + \mathbf{v}_a \otimes \mathbf{e}_{a+1} | a \in \{1, \ldots, k\}\}, \tag{3.261}$$

noting that $N = N_1 + N_2'$.

To compute the dimension of N_1 we must first express it as a $\mathbb{Z}$-module

$$\begin{aligned}
N_1 &= \Lambda^{i-3}(\mathbb{Z}^n)\cdot\{\mathbf{v}_a\otimes\mathbf{e}_a | a\in\{1,\dots,k\}\} && (3.262)\\
&= \operatorname{span}_{\mathbb{Z}}\{\beta\wedge\mathbf{v}_a\otimes\mathbf{e}_a | \beta\in\Lambda^{i-3}(\mathbb{Z}^n), a\in\{1,\dots,k\}\} && (3.263)\\
&= \bigoplus_{a=1}^{k}\operatorname{span}_{\mathbb{Z}}\{\beta\wedge\mathbf{v}_a\otimes\mathbf{e}_a | \beta\in\Lambda^{i-3}(\mathbb{Z}^n)\} && (3.264)\\
&= \bigoplus_{a=1}^{k}\operatorname{span}_{\mathbb{Z}}\{\beta\wedge\mathbf{v}_a\otimes\mathbf{e}_a | \beta\in\Lambda^{i-3}(\mathbb{Z}^n), \beta\wedge\mathbf{v}_a\neq 0\}. && (3.265)
\end{aligned}$$

Now for each $a\in\{1,\dots,k\}$ we will define a basis $\{\mathbf{u}_a^1,\dots,\mathbf{u}_a^n\}$ of $\mathbb{Z}^n$ where $\mathbf{u}_a^n := \mathbf{v}_a$. Recall that this is always possible because $\mathbf{v}_a\in\mathbb{Z}^n$ is a primitive vector. By expressing β in this basis we observe

$$\begin{aligned}
N_1 &= \bigoplus_{a=1}^{k}\operatorname{span}_{\mathbb{Z}}\{\mathbf{u}_a^{j_1}\wedge\cdots\wedge\mathbf{u}_a^{j_3}\wedge\mathbf{v}_a\otimes\mathbf{e}_a | J\in I_{i-3}^n, n\notin J\} && (3.266)\\
&= \bigoplus_{a=1}^{k}\operatorname{span}_{\mathbb{Z}}\{\mathbf{u}_a^{j_1}\wedge\cdots\wedge\mathbf{u}_a^{j_3}\wedge\mathbf{u}_a^{n}\otimes\mathbf{e}_a | J\in I_{i-3}^{n-1}\}. && (3.267)
\end{aligned}$$

By construction the set of generators in the above equation are linearly independent which means

$$\dim(N_1) = k\binom{n-1}{i-3}. \tag{3.268}$$

At this point we are going to pause and define two new manifolds. Let both M' and $\overline{M}$ be simple T^n-manifolds defined by the rod structures $\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$, but with M' having 0 corners and $\overline{M}$ having k corners. Note that if $m=0$ then $M=M'$, $E=\emptyset$, $N_2'=\{0\}$, and

$$H_{i-2,2}(M';\mathbb{Z}) \cong \frac{\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))}{N_1}. \tag{3.269}$$

However, notice in the definition of N_1 that since $\mathbf{v}_i$ are primitive vectors, the sublatice (submodule) $N_1\subset\Lambda^{i-2}(\mathbb{Z}^n)\otimes\mathbb{Z}^k$ must be primitive. In particular, N_1 is a primitive sublatice of $\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))$ as well hence $H_{i-2,2}(M';\mathbb{Z})$ is a free $\mathbb{Z}$-module. This means $H_{i-2,2}(M';\mathbb{Z})\cong\mathbb{Z}^{b_i'}$ where

$$b_i' := k\binom{n}{i-2} - \binom{n}{i-1} - k\binom{n-1}{i-3}. \tag{3.270}$$

Observe that since each corner in $\overline{M}$ is equivalent to T^{n-2}, the space M' can be thought of as $\overline{M}$ with k copies of T^{n-2} removed, $M'\cong\overline{M}\setminus k\cdot T^{n-2}$. Assume for now that M is simply connected (which is equivalent to saying that M' and $\overline{M}$ are simply connected). This means each of the tori coming from each of the corners is null-homotopic. This gives us the remarkable identity

$$M'\cong\overline{M}\#(S^{n+2}\setminus k\cdot T^{n-2}). \tag{3.271}$$

We can now use Meyer-Viatoris to compute the integral homology of $\overline{M}$.

$$\cdots\to H_j(S^{n+1})\to H_j(\overline{M})\oplus H_j(S^{n+2}\setminus k\cdot T^{n-2})\to H_j(M')\to H_{j-1}(S^{n+1})\to\dots \tag{3.272}$$

Since $H_i(S^{n+2}\setminus k\cdot T^{n-2})\cong\mathbb{Z}^{k\binom{n-2}{i-3}}$ for $0<i<n+2$, we can solve for the homology of $\overline{M}$ and see

$$H_i(\overline{M})\cong\mathbb{Z}^{b_i'-k\binom{n-2}{i-3}} \tag{3.273}$$

for $2 \leq i \leq n$. With a similar exercise we observe that

$$M \cong \overline{M}\#(S^{n+2} \setminus (k-m) \cdot T^{n-2}) \tag{3.274}$$

and therefore

$$H_i(M) \cong \mathbb{Z}^{b_i' - m\binom{n-2}{i-3}} \cong \mathbb{Z}^{b_i}, \tag{3.275}$$

establishing Equation (3.235) as desired. (Note that since M is simply connected, $H_{i,0}(M) = H_{i-4,4}(M) = \{0\}$ and thus $H_{i-2,2}(M) = H_i(M)$.)

Now that the statement has been proven for the the simply connected case, we return to the case where $H_1(M;\mathbb{Z}) \neq \{0\}$ but $H_1(M;\mathbb{Q}) = \{0\}$. Recall Corollary 2.58 proves that M has a torsion free cover $\widetilde{M}$ which is also a simple T^n-manifold with k rods. Since $H_1(M;\mathbb{Q}) = \{0\}$ we know that $\widetilde{M}$ is in fact simply connected and thus $H_i(\widetilde{M};\mathbb{Z}) \cong \mathbb{Z}^{b_i}$. Now since $\widetilde{M}$ is a cover of M there exists a surjection $H_i(\widetilde{M};\mathbb{Q}) \to H_i(M;\mathbb{Q})$ between their rational homologies [16, Proposition 3G.1]. This provides an upper bound to the dimension

$$\dim(H_{i-2,2};\mathbb{Q}) \leq \dim(H_i(M;\mathbb{Q})) \leq b_i. \tag{3.276}$$

To prove that b_i is also a lower bound we return to the $\mathbb{Z}$-module N_2'.

$$\begin{aligned} N_2' &:= \Lambda^{i-3}(\mathbb{Z}^n) \cdot \{\mathbf{v}_{a+1} \otimes \mathbf{e}_a + \mathbf{v}_a \otimes \mathbf{e}_{a+1} | a \in E\} && (3.277)\\ &= \operatorname{span}_{\mathbb{Z}}\{\beta \wedge (\mathbf{v}_{a+1} \otimes \mathbf{e}_a + \mathbf{v}_a \otimes \mathbf{e}_{a+1}) | \beta \in \Lambda^{i-2}(\mathbb{Z}^n), a \in E\}. && (3.278) \end{aligned}$$

Now for each $a \in \{1, \ldots, k\}$ we define the basis $\{\mathbf{u}_a^1, \ldots, \mathbf{u}_a^n\}$ of $\mathbb{Z}^n$ with the property that both $\mathbf{u}_a^n := \mathbf{v}_a$ and $\mathbf{u}_a^{n-1} := \mathbf{v}_{a+1}$. This is always possible because whenever there is a corner at $\Gamma_a \cap \Gamma_{a+1}$, the rod structures $\{\mathbf{v}_a, \mathbf{v}_{a+1}\}$ are admissible and thus form a primitive set. In this basis we can express N_2' as

$$N_2' = \operatorname{span}_{\mathbb{Z}}\{\mathbf{u}_a^{j_1} \wedge \cdots \wedge \mathbf{u}_a^{j_{i-3}} \wedge (\mathbf{u}_a^{n-1} \otimes \mathbf{e}_a + \mathbf{u}_a^n \otimes \mathbf{e}_{a+1}) | J \in I_{i-3}^n, a \in E\} \tag{3.279}$$

Now let $\beta = \mathbf{u}_a^{j_1} \wedge \cdots \wedge \mathbf{u}_a^{j_{i-3}}$ and supposed $n-1 \in J$. Then

$$\begin{aligned} \beta \wedge (\mathbf{v}_{a+1} \otimes \mathbf{e}_a + \mathbf{v}_a \otimes \mathbf{e}_{a+1}) &= \mathbf{u}_a^{j_1} \wedge \cdots \wedge \mathbf{u}_a^{j_{i-4}} \wedge \mathbf{u}_a^{n-1} \wedge (\mathbf{v}_{a+1} \otimes \mathbf{e}_a + \mathbf{v}_a \otimes \mathbf{e}_{a+1}) && (3.280)\\ &= \mathbf{u}_a^{j_1} \wedge \cdots \wedge \mathbf{u}_a^{j_{i-4}} \wedge \mathbf{u}_a^{n-1} \wedge (\mathbf{u}_a^{n-1} \otimes \mathbf{e}_a + \mathbf{u}_a^n \otimes \mathbf{e}_{a+1}) && (3.281)\\ &= \mathbf{u}_a^{j_1} \wedge \cdots \wedge \mathbf{u}_a^{j_{i-4}} \wedge \mathbf{u}_a^{n-1} \wedge \mathbf{u}_a^n \otimes \mathbf{e}_{a+1} && (3.282)\\ &= \mathbf{u}_a^{j_1} \wedge \cdots \wedge \mathbf{u}_a^{j_{i-4}} \wedge \mathbf{u}_a^n \wedge \mathbf{u}_a^{n-1} \otimes \mathbf{e}_{a+1} && (3.283)\\ &= \mathbf{u}_a^{j_1} \wedge \cdots \wedge \mathbf{u}_a^{j_{i-4}} \wedge \mathbf{u}_a^n \wedge \mathbf{v}_{a+1} \otimes \mathbf{e}_{a+1} && (3.284)\\ &\in N_1 && (3.285) \end{aligned}$$

A similar argument shows that if instead $n \in J$ then $\beta \wedge (\mathbf{v}_{a+1} \otimes \mathbf{e}_a + \mathbf{v}_a \otimes \mathbf{e}_{a+1})$ is also in N_1. This gives N_1 and N_2' a non-trivial intersection and makes computing the dimension of N more difficult. To avoid this problem we define a new subspace $N_2 \subset N_2'$ as

$$N_2 := \operatorname{span}_{\mathbb{Z}}\{\mathbf{u}_a^{j_1} \wedge \cdots \wedge \mathbf{u}_a^{j_{i-3}} \wedge (\mathbf{u}_a^{n-1} \otimes \mathbf{e}_a + \mathbf{u}_a^n \otimes \mathbf{e}_{a+1}) | J \in I_{i-3}^{n-2}, a \in E\} \tag{3.286}$$

noting that $N = N_1 + N_2$. Without even determining whether the generators in Equation (3.286) are basis vectors or not, we can see that an upper bound for the dimension of N_2 is

$$\dim(N_2) \leq m\binom{n-2}{i-3}. \tag{3.287}$$

This gives an upper bound for the dimension of N and thus a lower bound for $\dim(H_{i-2,2}(M;\mathbb{Q})) =$

$\dim(\ker(\Lambda^{i-2}(\mathrm{id}\otimes A))) - \dim(N)$. In particular we see

$$\dim(H_{i-2,2}(M;\mathbb{Q})) \geq k\binom{n}{i-2} - \binom{n}{i-1} - k\binom{n-1}{i-3} - m\binom{n-2}{i-3} = b_i. \tag{3.288}$$

Equation (3.233) is now satisfied and the proof is complete.

□

4 Einstein Equations

In Section 4 we examine the Einstein Equations and prove Theorems F, G, and H. The first of these, Theorem F, is the generalization of the harmonic map method [27, 28] into higher dimensions. The other two, Theorems G and H, are more geometric and topological in nature, but rely on Theorem F. It is therefore important that we begin with an overview of the harmonic map method and its assumptions.

The vacuum Einstein equations are notoriously difficult to solve in general. However when sufficient symmetry is assumed the equations sometimes reduce to a more manageable form. In our case the vacuum Einstein equations on a Lorentzian manifold $(\mathcal{M}^{n+3}, g)$ admitting symmetry group $\mathbb{R} \times U(1)^n$ reduce to a harmonic map equation for $\varphi\colon \mathbb{R}^3 \to SL(n+1,\mathbb{R})/SO(n+1)$. The domain for the harmonic map is obtained from the orbit space $\mathcal{M}^{n+3}/[\mathbb{R} \times U(1)^n]$, which is homeomorphic to the right half plane $\{(\rho, z) : \rho > 0\}$ [15, 19, 20], by adding an ignorable angular coordinate $\phi \in [0, 2\pi)$, yielding $\mathbb{R}^3$ parametrized by the cylindrical coordinates (ρ, z, ϕ). The harmonic map itself is axisymmetric, as it does not depend on ϕ. Uniqueness theorems for higher dimensional stationary n-axisymmetric black holes ultimately reduce to the uniqueness question for such harmonic maps with prescribed axis behavior [20].

Geometrically, we must assume that $(\mathcal{M}^{n+3}, g)$ is a connected asymptotically locally Kaluza-Klein stationary vacuum spacetime, with 2, 3, or 4 'large' spatial asymptotically (locally) flat dimensions. By *asymptotically locally Kaluza-Klein* we refer to a spacetime which asymptotes to the ideal geometry $\left(\mathbb{R}^{4-s,1}/G\right) \times T^{n+s-2}$, where T^{n+s-2} is any flat torus, $G \subset O(4-s)$ is a discrete subgroup of spatial rotations, and $s \in \{0, 1, 2\}$. If G is trivial, then the moniker 'locally' is removed from the terminology. Note that we cannot have more than 4 'large' dimensions since $\mathbb{R}^5$ does not admit an effective T^3 action, and fewer than 2 'large' dimensions is just not interesting. We will refer to such spacetimes as *well-behaved* if the orbits of the stationary Killing field are complete, the domain of outer communication (DOC) is globally hyperbolic, and the DOC contains an acausal spacelike connected hypersurface which is asymptotic to the canonical slice in the asymptotic end and whose boundary is a compact cross section of the horizon. These assumptions are used for the reduction of the stationary vacuum equations (preformed in Section 4.1), and are consistent with [18].

The prescribed axis behaviour of our harmonic comes in the form of a *rod data set* $\{(\mathbf{v}_1, \Gamma_1, \mathbf{c}_1), \ldots, (\mathbf{v}_k, \Gamma_k, \mathbf{c}_k)\}$ consisting of *rod structures* $\mathbf{v}_i \in \mathbb{Z}^n$ (see Section 2.2), *axis rods* $\Gamma_i \subset \Gamma \subset \partial\mathbb{R}^2_+$ (see Section 2.1), and *potential constants* $\mathbf{c}_i \in \mathbb{R}^n$ (see Section 4.2). This information encodes an approximate solution to the harmonic map equations, referred to as a *model map*. We then say that the model map *corresponds* to the rod data set. The connected spaces in-between axis rods, which are intervals in the real line, are referred to as *horizon rods*. These represent the horizon cross-sections of black holes. If all horizon rods have nonzero length, then the rod data is associated with nondegenerate black hole solutions [20, Lemma 7]. Note that rod data with no horizon rods is still considered nondegenerate. The prescribed harmonic map problem is solved by finding a solution which is *asymptotic* to the model map. A precise description of the properties required for the model map is given in Definition 4.3, and the notion of asymptotic maps is reviewed in Definition 4.7. Theorem F is a generalization of Theorem 1 in [28]. In particular, it extends the previous result to higher dimensions, and removes the assumption of a compatibility condition for the rod data. However the notion of *admissibility*, which was explained in Section 2.2, is still retained since this is required to ensure that the total space arising from the rod structures is a manifold.

4.1 Harmonic Map Equations

Let $(\mathcal{M}^{n+3}, g)$, $n \geq 1$ be a well-behaved asymptotically Kaluza-Klein stationary n-axisymmetric vacuum spacetime, that is, any time slice $(M^{n+2}, g) \subset (\mathcal{M}^{n+3}, g)$ admits it admits $U(1)^n$ as a subgroup of its isometry group, and thus admits an effective T^n-action. As a consequence of topological censorship [8], the fundamental group of the asymptotic end of M is equal to the fundamental group of M. This forces the orbit space M/T^n to be simply connected and precludes the existence of any orbifold points in the quotient map $M \to M/T^n$, thus satisfying the conclusion of Theorem 2.3 and making M a simple T^n-manifold (see Definition 2.4). In particular, the spacetime metric g may be written in Weyl-Papapetrou coordinates [18, Theorem 8] as

$$g = f^{-1}e^{2\sigma}(d\rho^2 + dz^2) - f^{-1}\rho^2 dt^2 + \sum_{i,j=1}^{n} f_{ij}(d\phi^i + w^i dt)(d\phi^j + w^j dt), \tag{4.1}$$

where (f_{ij}) is an $n\times n$ symmetric positive definite matrix with determinant f, and f_{ij}, w^j, σ are all functions of ρ and z. To avoid writing sums we will use Einstein notation with the following notation convention.

Notation 4.1. For the remainder of Section 4 unless stated otherwise:
the indices i, j, k, etc..., are assumed to range over $\{1,\dots,n\}$, and
the indices μ,ν,λ, etc..., are assumed to range over $\{t,\rho,z\}$.
Furthermore, ∂_i denotes ∂_{ϕ^i}, and dx^μ denotes $d\rho$, dz, or dt depending on the value of μ.

Remark 4.2. The relation between rod structures and axis rods can be seen in the metric. On each axis rod, a circle corresponding to the rod structure collapses. This means the associated Killing field vanishes. If $\mathbf{v} = (v^1,\dots,v^n)$ is the rod structure for the axis rod Γ_q, then its Killing field is $X_q := v^i\partial_i$. Thus $z\in\Gamma_q$ if and only if

$$\lim_{\rho\to 0} g(X_q, X_q) = \lim_{\rho\to 0} f_{ij}v^i v^j = 0. \tag{4.2}$$

To see the harmonic map equations let

$$g_3 := e^{2\sigma}(d\rho^2 + dz^2) - \rho^2 dt^2, \qquad \mathcal{A}^{(j)} := w^j dt, \tag{4.3}$$

then the vacuum equations imply

$$d(ff_{ij}\star_3 d\mathcal{A}^{(j)}) = 0, \tag{4.4}$$

where $\star_3$ represents the Hodge dual operator with respect to g_3. Thus, there exist globally defined *twist potentials* ω_i such that

$$d\omega_i = 2ff_{ij}\star_3 d\mathcal{A}^{(j)}. \tag{4.5}$$

The value of the twist potentials on axes adjacent to the horizons determines the angular momenta of the black holes. Next, note that we can write the 3-dimensional reduced Einstein-Hilbert action [32] as

$$S = \int_{\mathbb{R}\times(\mathcal{M}^{n+3}/[\mathbb{R}\times U(1)^n])} R^{(3)}\star_3 1 + \frac{1}{4}\operatorname{Tr}(\Phi^{-1}d\Phi\wedge\star_3\Phi^{-1}d\Phi), \tag{4.6}$$

where

$$\Phi = \begin{pmatrix} f^{-1} & -f^{-1}\omega_i \\ -f^{-1}\omega_i & f_{ij} + f^{-1}\omega_i\omega_j \end{pmatrix}, \quad i,j = 1,..,n, \tag{4.7}$$

is symmetric, positive definite, and satisfies $\det(\Phi) = 1$. By varying the action with respect to Φ and applying $\mathbb{R}$-symmetry, a majority of the reduced Einstein vacuum equations becomes equivalent to setting a quantity τ known as *tension* to zero:

$$\begin{aligned} \tau^{f_{lj}} &:= \Delta f_{lj} - f^{km}\nabla^\mu f_{lm}\nabla_\mu f_{kj} + f^{-1}\nabla^\mu\omega_l\nabla_\mu\omega_j = 0, \\ \tau^{\omega_j} &:= \Delta\omega_j - f^{kl}\nabla^\mu f_{jl}\nabla_\mu\omega_k - f^{lm}\nabla^\mu f_{lm}\nabla_\mu\omega_j = 0. \end{aligned} \tag{4.8}$$

The above are the equations for a harmonic map $\varphi : \mathbb{R}^3\setminus\Gamma \to SL(n+1,\mathbb{R})/SO(n+1)$, where φ is defined in terms of (f_{ij},ω_i) coordinates (see Sections 4.2 and 4.3). Given a solution to this system, the remaining metric components w^i and σ may be found by solving the quadrature equations [21, Pg. 34]

$$\begin{aligned}
\rho^{-1}\sigma_{,\rho} &= \frac{1}{8}f^{-2}\left(f_{,\rho}^2 + f_{,z}^2\right) + \frac{1}{8}f^{ij}f^{kl}\left(f_{ik,\rho}f_{jl,\rho} - f_{ik,z}f_{jl,z}\right) \\
&\quad + \frac{1}{4}f^{-1}f^{ij}\left(\omega_{i,\rho}\omega_{j,\rho} - \omega_{i,z}\omega_{j,z}\right) \\
\rho^{-1}\sigma_{,z} &= \frac{1}{4}f^{-2}f_{,\rho}f_{,z} + \frac{1}{4}f^{ij}f^{kl}f_{ik,\rho}f_{jl,z} + \frac{1}{2}f^{-1}f^{ij}\omega_{i,\rho}\omega_{j,z} \\
w^i_{,\rho} &= \rho f^{-1}f^{ij}\omega_{j,z} \\
w^i_{,z} &= -\rho f^{-1}f^{ij}\omega_{j,\rho}.
\end{aligned} \tag{4.9}$$

Therefore, the stationary vacuum equations in the n-axially symmetric setting are equivalent to a harmonic map problem with prescribed singularities on Γ, a subset of the z-axis which represents the axes of the $U(1)^n$-action or rather those points associated with a nontrivial isotropy group.

4.2 Model Map

In this section we construct the model map $\varphi_0 \colon \mathbb{R}^3 \setminus \Gamma \to SL(n+1,\mathbb{R})/SO(n+1)$ seen in Theorem F. This map describes the singular behavior of the desired harmonic map near the axis Γ, as well as the asymptotics at infinity. The model map can be viewed as an approximate solution to the singular harmonic map problem near the axes and at infinity [28, 54]. We define a model map as follows.

Definition 4.3. A map $\varphi_0 \colon \mathbb{R}^3 \setminus \Gamma \to SL(n+1,\mathbb{R})/SO(n+1)$ is a *model map* if

1. $|\tau(\varphi_0)|$ is bounded, where τ denotes the tension of φ_0, and
2. there is a positive function function $u \in C^2(\mathbb{R}^3)$ with $\Delta u \le -|\tau(\varphi_0)|$ and $u \to 0$ at infinity.

It should be noted that if $|\tau(\varphi_0)| = O(r^{-\alpha})$ as $r \to \infty$, for some $\alpha > 2$, then this is sufficient to satisfy condition (2). In order to facilitate the construction of the model map, we will utilize the following parameterization of the target space. Namely, the target space is parameterized by (F, ω), where $F = (f_{ij})$ is a symmetric positive definite $n \times n$ matrix and $\omega = (\omega_i)$ is an n-tuple corresponding to the twist potentials. On each axis rod, the Dirichlet boundary data for ω_i is constant. These so called *potential constants* $\mathbf{c}_i$, defined by

$$\mathbf{c}_i := (\omega_1, \ldots, \omega_n)|_{\Gamma_i}, \tag{4.10}$$

determine the angular momenta of the horizons and do not vary between adjacent axis rods which are separated by a corner. In (F, ω) coordinates, the metric on the target space $SL(n+1,\mathbb{R})/SO(n+1)$ may be expressed as (see [32])

$$\frac{1}{4}\frac{df^2}{f^2} + \frac{1}{4}f^{ij}f^{kl}df_{ik}df_{jl} + \frac{1}{2}\frac{f^{ij}d\omega_i d\omega_j}{f} = \frac{1}{4}[\mathrm{Tr}(F^{-1}dF)]^2 + \frac{1}{4}\,\mathrm{Tr}(F^{-1}dFF^{-1}dF) + \frac{1}{2}\frac{d\omega^t F^{-1}d\omega}{f}, \tag{4.11}$$

where $f = \det F$ and $F^{-1} = (f^{ij})$ is the inverse matrix. By setting

$$H = F^{-1}\nabla F, \qquad G = f^{-1}F^{-1}(\nabla\omega)^2, \qquad K = f^{-1}F^{-1}\nabla\omega, \tag{4.12}$$

it follow from (4.8) that the squared norm of the tension becomes

$$|\tau|^2 = \frac{1}{4}[\mathrm{Tr}(\operatorname{div} H + G)]^2 + \frac{1}{4}\,\mathrm{Tr}[(\operatorname{div} H + G)(\operatorname{div} H + G)] + \frac{1}{2}f(\operatorname{div} K)^t F(\operatorname{div} K). \tag{4.13}$$

It is clear from (4.13) that the tension norm is invariant under the transformation

$$F \mapsto hFh^t \quad \text{and} \quad \omega \mapsto h\omega, \tag{4.14}$$

for any $h \in SL(n,\mathbb{R})$. Note that $\det h = 1$ is not required for this to hold when ω is constant, since G and K are then zero. The next result generalizes the model map construction from lower dimensions that was presented in [27, 28].

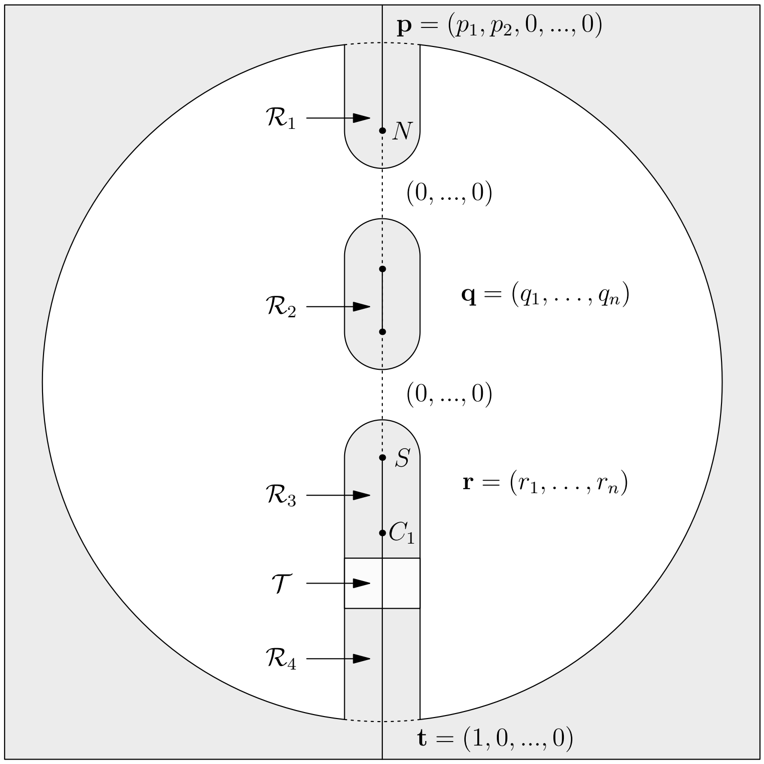


Figure 4.1: This diagram depicts the various regions used in the construction of the model map. Axis rod structures are represented by $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{t}$, while horizon rods are indicated by dashed lines.

Lemma 4.4. *For any admissible rod data set, with nondegenerate horizons, there exists a corresponding model map* $\varphi_0 \colon \mathbb{R}^3 \setminus \Gamma \to SL(n+1,\mathbb{R})/SO(n+1)$, *for* $n \geq 2$, *having tension decay at infinity given by* $|\tau| = O(r^{-5/2})$.

Proof. We first present a proof for the rod data set corresponding to two horizons and a single corner, as shown in Figure 4.1. At the end of the proof, we will indicate the necessary adjustments for the general case. Observe that in the diagram there are four neighborhoods $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$ associated with certain axis rods, having rod structures $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{t}$ respectively. The model map will be constructed separately in each of these regions. The following two harmonic functions on $\mathbb{R}^3 \setminus \Gamma$ will play an important role in the construction

$$u_a = \log(r_a - (z-a)) = \log(2r_a \sin^2(\theta_a/2)), \qquad v_a = \log(r_a + (z-a)) = \log(2r_a \cos^2(\theta_a/2)), \tag{4.15}$$

where $r_a = \sqrt{\rho^2 + (z-a)^2}$ is the Euclidean distance from the point $z = a$ on the z-axis, and θ_a is the polar angle.

Consider first the case in which the asymptotic end is modeled on $L(p,q) \times T^{n-2}$, where $0 \leq q < p$. By applying Lemma 2.20 if necessary, it may be assumed without loss of generality that the rod structures on the semi-infinite rods are $\mathbf{p} = (p_1, p_2, 0, \ldots, 0)$ with $p_2 > 0$, and $\mathbf{t} = (1, 0, \ldots, 0)$. The model map outside of a large ball (corresponding to the shaded region outside of the circle in Figure 4.1) and in the regions $\mathcal{R}_1$ and $\mathcal{R}_4$, may then be given by

$$F_1 = h\tilde{F}_1 h^t, \qquad \omega = h\tilde{\omega}(\theta), \tag{4.16}$$

where $\tilde{\omega}$ is a function of $\theta = \theta_0$ alone described below and

$$\tilde{F}_1 = \mathrm{diag}\,(e^{u_0 - \log 2}, e^{v_0 - \log 2}, 1, ..., 1), \qquad h = \begin{pmatrix} 0 & \sqrt{p_2} & 0 \\ 1/\sqrt{p_2} & -p_1/\sqrt{p_2} & 0 \\ 0 & 0 & \mathbf{I}_{n-2} \end{pmatrix}, \tag{4.17}$$

with $\mathbf{I}_{n-2}$ representing the identity matrix. Notice that, up to multiplication by constants, h^t sends $\mathbf{t} \mapsto \mathbf{e}_2$ and $\mathbf{p} \mapsto \mathbf{e}_1$. Thus, the matrix F_1 possesses the appropriate kernel at the semi-infinite rods to encode the given rod structures. Moreover, since $\varphi_0 = (F_1, \omega)$ is obtained from the map $(\tilde{F}_1, \tilde{\omega})$ by applying an isometry to the target space, and $\tilde{F}_1$ arises from the canonical flat metric on $\mathbb{R}^4 \times T^{n-2}$, it follows that $\operatorname{div} H = \operatorname{div} F_1^{-1} \nabla F_1 = 0$. We may further choose $\tilde{\omega}(\theta)$ to be constant for $\theta \in [0, \epsilon] \cup [\pi - \epsilon, \pi]$, thus showing that (F_1, ω) is harmonic in $\mathcal{R}_1$ and $\mathcal{R}_4$. The constants are chosen to coincide with the prescribed potential constants on the axis rods. Within the remaining angular interval, $\tilde{\omega}(\theta)$ may be prescribed arbitrarily as long as it is smooth. In order to verify the decay of the tension for this map in the range $\theta \in [\epsilon, \pi - \epsilon]$, observe that since $F_1 = O(r)$, $f = O(r^2)$, $|\nabla \omega| = O(r^{-1})$, and $\operatorname{div} K = O(r^{-4})$ we have

$$f(\operatorname{div} K)^t F_1 (\operatorname{div} K) = O(r^{-5}), \qquad G = O(r^{-4}). \tag{4.18}$$

Hence $|\tau|$ decays like $r^{-5/2}$, which is sufficient. Similarly, in the case where the asymptotic end is modeled on $S^2 \times T^{n-1}$, we can without loss of generality assume that the rod structures on both the semi-infinite rods are $(1, 0, \ldots, 0)$. The model map outside of the large ball and in the regions $\mathcal{R}_1$ and $\mathcal{R}_4$ is now given by

$$F_1 = \mathrm{diag}\,(e^u, 1, ..., 1), \qquad \omega = \omega(\theta), \tag{4.19}$$

where $u = 2 \log \rho$ and ω is constant on $\theta \in [0, \epsilon] \cup [\pi - \epsilon, \pi]$. As before, the tension decays as $|\tau| = O(r^{-5/2})$ when $r \to \infty$.

Next consider the compact region $\mathcal{R}_2$ below the first horizon. The poles in this region are located at $z = a$ and $z = b$, $a < b$, and the rod structure is $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. The model map in this region is defined by

$$F_2 = h_2 \tilde{F}_2 h_2^t, \qquad \omega = c_2, \tag{4.20}$$

where $\tilde{F}_2 = \mathrm{diag}\,(e^u, 1, ..., 1)$, $u = u_a - u_b$, and

$$h_2 = \left(\left[\mathbf{q}, \mathbf{e}_2, \ldots, \mathbf{e}_n \right]^t \right)^{-1}. \tag{4.21}$$

The constant vector c_2 is chosen to agree with the prescribed potential constants on the rod. As pointed out in the remark preceding the lemma, $\det h_2 = 1$ is not required here since ω is constant. It follows that the map $\varphi_0 = (F_2, \omega)$ is harmonic in region $\mathcal{R}_2$.

Now we will deal with the regions $\mathcal{R}_3$, $\mathcal{R}_4$ and the transition region $\mathcal{T}$ between them. Let the pole S be at $z = s > 0$ and the corner C_1 be at $z = 0$. The rod structure above the corner C_1 is $\mathbf{r} = (r_1, \ldots, r_n)$, and below the corner is $\mathbf{t} = (1, 0, \ldots, 0)$. Because of admissibility, we can without loss of generality assume that $r_2 > 0$. As above we set ω to be a constant c_3, agreeing with the prescribed potential constant on the rods, in the entire southern tubular neighborhoods $\mathcal{R}_3$ and $\mathcal{R}_4$. Let

$$\tilde{F}_3 = \mathrm{diag}\,(e^u, e^v, 1, ..., 1), \qquad u = (u_0 - \log 2) - \lambda(z)(u_s - \log 2), \qquad v = v_0 - \log 2, \tag{4.22}$$

where $\lambda = \lambda(z)$ is a smooth cut-off function which is 1 near $\mathcal{R}_3$ and 0 near $\mathcal{R}_4$. Define the map in region $\mathcal{R}_3$ by

$$F_3 = h_3 \tilde{F}_3 h_3^t, \qquad \omega = c_3, \tag{4.23}$$

where

$$h_3 = \sqrt{p_2}\left(\left[\mathbf{r}, \mathbf{e}_1, \mathbf{e}_3, \ldots, \mathbf{e}_n\right]^t\right)^{-1}. \tag{4.24}$$

We have already given the map in $\mathcal{R}_4$. In order to define the map in $\mathcal{T}$, set $h_3(z)$ to be a smooth curve of invertible $n \times n$ matrices which connects h_3 in (4.24) to h in (4.17). Note that this is possible since both endpoint matrices have negative determinant, and that the curve may be chosen so that the second column of $(h_3(z)^t)^{-1}$ remains the constant vector $1/\sqrt{p_2}\mathbf{e}_1$. The map $F_3(z) = h_3(z)\tilde{F}_3(z)h_3^t(z)$ then identifies the correct rod structures, and agrees with the previously defined map on $\mathcal{R}_4$. Since $\omega = c_3$, we have $G = K = 0$ in $\mathcal{R}_3 \cup \mathcal{R}_4$. It remains to show that $\operatorname{div} F_3^{-1}\nabla F_3$ is bounded on the transition region $\mathcal{T}$, since it vanishes on the complement. To see this, compute

$$\begin{aligned}\operatorname{div} F_3^{-1}\nabla F_3 =& [\nabla(\tilde{F}_3 h_3^t)^{-1}] \cdot (h_3^{-1}\nabla h_3)\tilde{F}_3 h_3^t + (\tilde{F}_3 h_3^t)^{-1}\operatorname{div}(h_3^{-1}\nabla h_3)\tilde{F}_3 h_3^t \\ &+ (\tilde{F}_3 h_3^t)^{-1}(h_3^{-1}\nabla h_3) \cdot \nabla(\tilde{F}_3 h_3^t) + (\nabla h_3^{-t}) \cdot (\tilde{F}_3^{-1}\nabla\tilde{F}_3)h_3^t \\ &+ h_3^{-t}\operatorname{div}(\tilde{F}_3^{-1}\nabla\tilde{F}_3)h_3^t + h_3^{-t}(\tilde{F}_3^{-1}\nabla\tilde{F}_3) \cdot \nabla h_3^t + \operatorname{div}(h_3^{-t}\nabla h_3).\end{aligned} \tag{4.25}$$

Note that $|\nabla u|$ and $\partial_z v = 1/r$ are clearly bounded in $\mathcal{T}$. Moreover, the second row of $h_3^{-1}\nabla h_3$ vanishes, and this leads to the desired boundedness of $\operatorname{div} F_3^{-1}\nabla F_3$. Indeed, consider the first term on the right-hand side of (4.25), namely

$$[\nabla(\tilde{F}_3 h_3^t)^{-1}] \cdot (h_3^{-1}\nabla h_3)\tilde{F}_3 h_3^t = \left[\left(h_3^t\right)^{-1}\partial_z \tilde{F}_3^{-1} + \partial_z\left(h_3^t\right)^{-1} \cdot \tilde{F}_3^{-1}\right](h_3^{-1}\partial_z h_3)\tilde{F}_3 h_3^t. \tag{4.26}$$

The only potential difficulty in bounding this expression on $\mathcal{T}$ arises from the function e^{-v}, in $\tilde{F}_3^{-1}$ and $\partial_z\tilde{F}_3^{-1}$. However, since $h_3^{-1}\partial_z h_3$ has a vanishing second row, the products

$$\tilde{F}_3^{-1} \cdot (h_3^{-1}\partial_z h_3), \qquad \partial_z\tilde{F}_3^{-1} \cdot (h_3^{-1}\partial_z h_3), \tag{4.27}$$

no longer contain e^{-v} and the first term of (4.25) is controlled. The remaining terms may be handled analogously. It follows that (4.25) is bounded, and hence the model map $\varphi_0 = (F_3, \omega)$ has bounded tension in a tubular neighborhood of the two southern most rods. This treats the case in which the asymptotic end is modeled on $L(p,q) \times T^{n-2}$, and a similar procedure may be used in the case that the asymptotic end is modeled on $S^2 \times T^{n-1}$.

We will now address the multiple corner case. Any connected component of the axis consists of a consecutive sequence of axis rods. To construct the model map in a tubular neighborhood of such a component, first divide this region into neighborhoods centered at corners and transition regions between corners. The basic block consists of two such neighborhoods around adjacent corners C_n and C_s, and the transition region $\mathcal{T}$ between them. It suffices to illustrate the map construction in such blocks, as the full map may then be obtained by combining the individual pieces to handle any rod structure configuration.

Consider a basic block with rod structures $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ on axis rods Γ_1, Γ_2, and Γ_3 respectively, moving from north to south. Note that $\mathbf{p}$ and $\mathbf{q}$, as well as $\mathbf{q}$ and $\mathbf{r}$, must be linearly independent since the corners C_n and C_s are admissible. It follows that there is a collection of standard basis vectors $\{\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-2}}\}$ that complete $\{\mathbf{p}, \mathbf{q}\}$ to a basis, and similarly for $\{\mathbf{q}, \mathbf{r}\}$. We may then form the matrices

$$h_{\mathbf{p},\mathbf{q}} = \left(\left[\mathbf{p}, \mathbf{q}, \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-2}}\right]^t\right)^{-1}, \qquad h_{\mathbf{r},\mathbf{q}} = \left(\left[\mathbf{r}, \mathbf{q}, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-2}}\right]^t\right)^{-1}. \tag{4.28}$$

Next define $F_0 = \operatorname{diag}(e^u, e^v, 1, ..., 1)$ where u and v are harmonic, with e^u vanishing on Γ_1 and Γ_3, and e^v vanishing on Γ_2. These functions may be given as the sum of logarithms of the form (4.15). Then F_0 corresponds to the rod structures $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_1$ on Γ_1, Γ_2, and Γ_3 respectively. Consider a smooth curve of invertible $n \times n$ matrices $h_{\mathbf{p}|\mathbf{r},\mathbf{q}}(z)$ which agrees with $h_{\mathbf{p},\mathbf{q}}$ on Γ_1 and in a neighborhood of C_n, and transitions over $\mathcal{T} \subset \Gamma_2$ so that it agrees with $h_{\mathbf{r},\mathbf{q}}$ on Γ_3 and in a neighborhood of C_s. The existence of such a curve is possible since we may assume that the determinants of $h_{\mathbf{p},\mathbf{q}}$ and $h_{\mathbf{r},\mathbf{q}}$ have the same sign by replacing $\mathbf{r}$ with $-\mathbf{r}$ if necessary. Moreover, the curve may be designed such that the second column of $\left(h_{\mathbf{p}|\mathbf{r},\mathbf{q}}(z)^t\right)^{-1}$ is the constant vector $\mathbf{q}$. This implies that the second row of $h^{-1}_{\mathbf{p}|\mathbf{r},\mathbf{q}}\nabla h_{\mathbf{p}|\mathbf{r},\mathbf{q}}$ vanishes, so that with the help of (4.25) we find that $\operatorname{div} F^{-1}\nabla F$ remains bounded along $\mathcal{T}$, where $F = h_{\mathbf{p}|\mathbf{r},\mathbf{q}} F_0 h^t_{\mathbf{p}|\mathbf{r},\mathbf{q}}$. The model map $\varphi_0 = (F, \omega)$ on the basic block, with ω constant, then has bounded tension.

Lastly, it remains to treat the case of multiple blocks within an axis component. To accomplish this, take u and v harmonic so that e^u and e^v vanish in an alternating fashion on the string of axis rods. The diagonal matrix F_0 is then defined along the entire string. We will inductively construct the model map on basic block assemblies. As a demonstration of this, consider adding an additional rod Γ_4, with rod structure $\mathbf{w}$, to the sequence of three rods discussed above which we call basic block $\mathcal{B}_1$. We may view the Γ_2, Γ_3, Γ_4 string, with rod structures $\mathbf{q}$, $\mathbf{r}$, $\mathbf{w}$, as a basic block $\mathcal{B}_2$; the corner between the third and fourth rod will be denoted by C_w. The map has already been defined into a neighborhood of Γ_3, and may be extended into a neighborhood of Γ_4 as follows. Recall that the maps

$$F_1 = h_{\mathbf{p}|\mathbf{r},\mathbf{q}} F_0 h^t_{\mathbf{p}|\mathbf{r},\mathbf{q}}, \qquad F_2 = h_{\mathbf{r},\mathbf{q}|\mathbf{w}} F_0 h^t_{\mathbf{r},\mathbf{q}|\mathbf{w}}, \tag{4.29}$$

are defined on the basic blocks $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, and identify the desired rod structures. However, they do not necessarily coincide on the overlap regions. In order to remedy this situation, let $h_4(z)$ be a smooth curve of invertible $n \times n$ matrices connecting $h_{\mathbf{r},\mathbf{q}}$ to $h_{\mathbf{r},\mathbf{w}}$ with a transition over $\tilde{\mathcal{T}} \subset \Gamma_3$. This is possible since by replacing $\mathbf{w}$ with $-\mathbf{w}$ if necessary, we may assume that both endpoint matrices have determinants of the same sign. Moreover, this curve may be chosen such that the first column of $\left(h_4(z)^t\right)^{-1}$ remains the constant vector $\mathbf{r}$. Set $F = h_4(z) F_0 h_4(z)^t$ on Γ_3, and observe that this agrees with F_1 and F_2 near the corners C_s and C_w, respectively, so that F is naturally defined on all of $\mathcal{B}_1 \cup \mathcal{B}_2$. Since the first row of $h_4^{-1}\nabla h_4$ vanishes, we find with the aid of (4.25) that $\operatorname{div} F^{-1}\nabla F$ remains bounded along Γ_3. The model map $\varphi_0 = (F, \omega)$ on the two basic blocks, with ω constant, then has bounded tension. We may continue this process inductively to treat any number of consecutive axis rods. $\square$

Remark 4.5. In [27, 28] an additional technical assumption on the rod structures, known as the compatibility condition, was used for the construction of the model map. The condition, which is not required for Lemma 4.4, states that given three adjacent rod structures with admissible corners, say (m, n), (p, q), and (r, s), the following inequality must hold

$$mr(mq - np)(ps - rq) \leq 0 \ . \tag{4.30}$$

This turns out not to be a geometric condition, as it can always be achieved by a change of coordinates. To see this, first assume without loss of generality that the determinants $(mq - np)$ and $(ps - rq)$ are 1, by possibly replacing (p, q) or (r, s) or both with the vector of the same length and opposite direction. Note that this operation does not alter the isotropy subgroup prescribed by the rod structure. Next apply the unimodular matrix

$$U = \begin{pmatrix} q & -p \\ -n & m \end{pmatrix} \tag{4.31}$$

to obtain the rod structures $U \cdot \{(m, n), (p, q), (r, s)\} = \{(1, 0), (0, 1), (r', s')\}$, for some $r', s' \in \mathbb{Z}$. Then Equation (4.30) is clearly satisfied for the new set of rod structures.

Remark 4.6. Lemma 4.4 and Remark 4.5 provide the proof of Part a from Theorem F.

4.3 Energy Estimates

In this section we show how the energy estimates based on horocyclic coordinates can be generalized from the lower rank target space setting that was treated in [28, Section 6]. The target space is now $SL(n+1, \mathbb{R})/SO(n+1)$, which is a noncompact symmetric space of dimension $n(n+3)/2$ and rank n. For convenience we denote $G = SL(n+1, \mathbb{R})$, $K = SO(n+1)$, and $\mathbf{X} = G/K$. The Iwasawa decomposition is given by

$G = NAK$, where A is the abelian group

$$A = \{\text{diag}(e^{\lambda_1}, ..., e^{\lambda_{n+1}}) \mid \prod_{i=1}^{n+1} e^{\lambda_i} = 1\}, \tag{4.32}$$

and N is the nilpotent subgroup of upper triangular matrices with diagonal entries set to 1. Thus, given $g \in G$ there are unique elements $m \in N$, $a \in A$, and $k \in K$ with $g = mak$, and the symmetric space $\mathbf{X}$ may be identified with the subgroup NA. Denote $x_0 = [Id] \in \mathbf{X}$ and note that the orbits $A \cdot x_0 =: \mathfrak{F}_{x_0}$ and $N \cdot x_0$ are respectively a maximal flat and a horocycle. The former is an n-dimensional totally geodesic submanifold with vanishing sectional curvature, and the latter is an $n(n+1)/2$-dimensional submanifold with the property that each flat which is asymptotic to the same Weyl chamber at infinity has an orthogonal intersection with the horocycle in a single point. Furthermore, since each point $x \in \mathbf{X}$ may be uniquely expressed as $ma \cdot x_0$, the assignment $x \mapsto \mathfrak{F}_x = ma \cdot \mathfrak{F}_{x_0}$ yields a smooth foliation whose leaves are the flats $\{m \cdot \mathfrak{F}_{x_0}\}_{m \in N}$; the flat $\mathfrak{F}_x$ orthogonally interects the horocycle $N \cdot x$ only at x. In this manner, the pair (a, m) gives rise to a horocyclic orthogonal coordinate system for $\mathbf{X}$.

A Euclidean coordinate system $r = (r_1, \ldots, r_n)$ may be introduced on $\mathfrak{F}_{x_0}$, and can then be pushed forward to each flat $m \cdot \mathcal{F}_{x_0}$ so that the horocyclic coordinates (a, m) may be represented by (r, m). Furthermore, each r' defines a diffeomorphism (translation) $(r, m) \mapsto (r + r', m)$ that preserves the m-coordinates, and for each $m' \in N$ there is an isometry that preserves the r-coordinates $(r, m) \mapsto (r, m'm)$. These r-translations map horocycles to horocylces, and therefore may be used to push forward a system of global coordinates $\theta = (\theta^1, \ldots, \theta^{n(n+1)/2})$ on $N \cdot x_0 \cong \mathbb{R}^{n(n+1)/2}$ to all horocycles. It follows that (r, θ) form a set of global coordinates on $\mathbf{X}$ in which the coordinate fields ∂_{r_i} and ∂_{θ^j} are orthogonal, and such that the G-invariant Riemannian metric on $\mathbf{X}$ is expressed as

$$\mathbf{g} = dr^2 + Q(d\theta, d\theta) = \sum_{i=1}^{n} dr_i^2 + \sum_{j,l=1}^{n(n+1)/2} Q_{jl} d\theta^j d\theta^l, \tag{4.33}$$

where the coefficients $Q_{jl}(r, \theta)$ are smooth functions. Moreover, the proof of [28, Lemma 8] generalizes in a direct manner to the current setting to yield the uniform bounds

$$bQ(\xi, \xi) \leq \partial_{r_i} Q(\xi, \xi) \leq cQ(\xi, \xi), \tag{4.34}$$

for all $i = 1, \ldots, n$ and $\xi \in \mathbb{R}^{n(n+1)/2}$ where $0 < b < c$. With the help of (4.34), by expressing the harmonic map equations in the horocyclic parameterization we may establish energy bounds on compact subsets away from the axis. In particular, if $\varphi : \mathbb{R}^3 \setminus \Gamma \to \mathbf{X}$ is a harmonic map and $\Omega \subset \mathbb{R}^3 \setminus \Gamma$ is a bounded domain then the harmonic energy restricted to Ω satisfies

$$E_\Omega(\varphi) \leq \mathcal{C}, \tag{4.35}$$

where the constant $\mathcal{C}$ depends only on the maximum distance $\sup_{y \in \Omega} d_{\mathbf{X}}(\varphi(y), x_0)$.

Definition 4.7. Two maps $\varphi_1,\ \varphi_2 \colon \mathbb{R}^3 \setminus \Gamma \to \mathbf{X}$ are *asymptotic* if there exists a constant C such that $d_{\mathbf{X}}(\varphi_1, \varphi_2) \leq C$, and $d_{\mathbf{X}}(\varphi_1(y), \varphi_2(y)) \to 0$ as $|y| \to \infty$.

The distance between the model map and solutions to the harmonic map Dirichlet problem on an exhausting sequence of domains may be estimated via a maximum principle argument [54], which is based on convexity of the distance function in the nonpositively curved target. This supremum bound together with the energy bound, allow for an application of standard elliptic theory to control all higher order derivatives. The sequence of harmonic maps on exhausting domains will then subconverge to the desired solution, for details see [28, §6 & 7]. We record this conclusion as the following result.

Lemma 4.8. *Let φ_0 be a model map. Then there exists a unique harmonic map $\varphi : \mathbb{R}^3 \setminus \Gamma \to \mathbf{X}$ such that φ is asymptotic to φ_0.*

We are now ready to prove Theorem F.

Theorem 4.9 (Theorem F)**.** *Suppose that $\{(\mathbf{v}_1, \Gamma_1, \mathbf{c}_1), \ldots, (\mathbf{v}_k, \Gamma_k, \mathbf{c}_k)\}$ is an n-dimensional admissible rod data set with non-degenerate horizon rods.*

(a) There exists a model map $\varphi_0 : \mathbb{R}^3 \setminus \Gamma \to SL(n+1, \mathbb{R})/SO(n+1)$ which corresponds to the rod data set.

(b) There exists a unique harmonic map $\varphi : \mathbb{R}^3 \setminus \Gamma \to SL(n+1, \mathbb{R})/SO(n+1)$ which is asymptotic to the model map φ_0.

(c) A well-behaved asymptotically (locally) Kaluza-Klein solution of the $(n+3)$-dimensional vacuum Einstein equations admitting the isometry group $\mathbb{R} \times U(1)^n$ can be constructed from φ. Such a metric is smooth except possibly along the finite axis rods where conical singularities may be present.

(d) Any time slice of the spacetime produced is a simple T^n-manifold which agrees with the rod data.

Proof. As stated in Remark 4.6, Lemma 4.4 and Remark 4.5 provide the proof of Part a. Lemma 4.8 establishes Part b. Since φ is asymptotic to φ_0, it can be shown in the same way as [28, Theorem 11], that the two maps respect the same rod data set. Furthermore, Part c may be established analogously to [28, §8]. Finally Part d is established by applying Remark 4.2 to any time slice of $(\mathcal{M}, g)$. This completes the proof. □

4.4 Conical Singularities

In this section we will discuss the presence of conical singularities on axis rods and prove Theorem G. In order to do this we will need to introduce two new definitions.

Definition 4.10. We say that (M, g) is a *simple T^n-manifold*, or *simple T^n Riemannian manifold*, if (M, g) is a Riemannian manifold equipped with an effective isometric T^n-action so that M is a simple T^n-manifold in the sense of Definition 2.4.

If (M^{n+2}, g) is a simple T^n-manifold with an asymptotic end, then the interior of the image of $\pi \colon M \to M/T^n$ is usually described as the open half plane $(M/T^n) \setminus \partial(M/T^n) \cong \mathbb{R}^2_+$. The standard (ρ, z)-coordinates on the closed half plan, $(\rho, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, gives a global coordinate system for the quotient space $M/T^n \cong \mathbb{R}^2_{\geq 0} \setminus \Gamma$. Recall that $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$ and that as an abuse of notation we use Γ_i to both denote a portion of the boundary of the quotient space which we call an axis rod $\Gamma_i \subset \partial(M/T^n)$ and to denote the subset $\pi^{-1}(\Gamma_i) \subset M$. In addition we can use the standard coordinates on the torus, $(\phi^1, \ldots, \phi^n) \in \mathbb{R}/\mathbb{Z} \times \cdots \times \mathbb{R}/\mathbb{Z}$ to create a global coordinate system $(\rho, z, \phi^1, \ldots, \phi^n)$ on $M \setminus \Gamma$. We call this coordinate system (ρ, z)-*coordinates* or *cylindrical coordinates*. Note that in the same way polar coordinates on $\mathbb{R}^2$ are not defined at the origin, this (ρ, z)-coordinate system is not defined on Γ and thus cannot be extended to a global coordinate system on all of M. However $M \setminus \Gamma$ is a dense open subset of M which means that the (ρ, z)-coordinate system is an *almost global* coordinate system. More importantly any C^0 metric on M can be fully described by its restriction to $M \setminus \Gamma$ and thus can be fully described by the (ρ, z)-coordinate system. In this coordinate system, any metric g on $M \setminus \Gamma$ appears as

$$g = \sum_{i,j=1}^{n} \sum_{\mu,\nu \in \{\rho, z\}} \left(g_{\mu\nu} dx^\mu dx^\nu + g_{\mu j} dx^\mu d\phi^j + g_{i\nu} d\phi^i dx^\nu + g_{ij} d\phi^i d\phi^j \right) \tag{4.36}$$

where dx^μ and dx^ν represent $d\rho$ and dz, depending on the value of μ and ν. For the remainder of Section 4.4 we will be using the following notation convention.

Notation 4.11. For the remainder of Section 4.4,
when using Einstein summation notation Latin indices will be assumed to range over $\{1,\ldots,n\}$,
while Greek indices will be assumed to range over $\{\rho,z\}$.

Definition 4.12. Let (M^{n+2},g) be a simple T^n-manifold with rod data $\{(\mathbf{v}_1,\Gamma_1),\ldots,(\mathbf{v}_k,\Gamma_k)\}$ and Riemannian metric $g\in C^0(M)\cap C^2(M\setminus\Gamma)$. Using (ρ,z)-coordinates choose a point $(0,z_0)\in\Gamma_j\subset\partial\mathbb{R}^2_{\geq 0}$ and define the Killing field $X_j:=v_j^i\partial_{\phi^j}$ where $\mathbf{v}_j=(v_j^1,\ldots,v_j^n)$. The *cone angle* at $(0,z_0)$ is

$$\alpha(z_0):=\lim_{\rho\to 0}\frac{\int_0^{2\pi}\sqrt{g(X_j,X_j)}}{\int_0^{\rho}\sqrt{g(\partial_\rho,\partial_\rho)}}=\lim_{\rho\to 0}\frac{\text{circumference}}{\text{radius}}. \tag{4.37}$$

If the cone angle is constant for all $(0,z)\in\Gamma_j$ then we denote its value as α_j.

Remark 4.13. When the spacetime $(\mathcal{M},g)$ is written in Weyl-Papapetrou coordinates

$$g=f^{-1}e^{2\sigma}(d\rho^2+dz^2)-f^{-1}\rho^2dt^2+f_{ij}(d\phi^i+w^idt)(d\phi^j+w^jdt),$$

the equation for the cone angles of the time slice (M,g) reduces to

$$\alpha(z_0)=2\pi\lim_{\rho\to 0}\frac{\sqrt{f_{ij}v_q^iv_q^j}}{\sqrt{\rho^2f^{-1}e^{2\sigma}}} \tag{4.38}$$

where $z_0\in\Gamma_q$ and $\mathbf{v}_q=(v_q^1,\ldots,v_q^n)$ is the rod structure for Γ_q.

Lemma 4.14 below is a simple yet powerful observation about cone angles and isometric torus actions. In the statement of the lemma we use the phrase locally isometric, which may be defined differently by different source. For our purposes, two Riemannian manifolds (X,g) and (Y,h) are *locally isometric* if for all $p\in X$ and $q\in Y$ there exists open sets $U,W\subset X$ and $V,Z\subset Y$ with $p\in U$ and $q\in Z$ such that there exists isometries $F\colon(U,g)\to(V,h)$ and $G\colon(Z,h)\to(W,g)$. Note that this definition does not depend on the existence of any globally defined map $(X,g)\to(Y,h)$ or $(Y,h)\to(X,g)$. In particular any two open subsets of Euclidean space are locally isometric, regardless of their topologies.

Lemma 4.14. *Let (M^{n+2},g) be a simple T^n-manifold with rod data $\{(\mathbf{e}_1,\Gamma_1),\ldots,(\mathbf{e}_k,\Gamma_k)\}$ for $k\leq n$, possibly with conical singularities along Γ. If each cone angle α_i is constant along Γ_i, then there exists a metric g' for M such that:*

1. *(M,g') is a simple T^n-manifold with the same rod data as (M,g),*
2. *$(M\setminus\Gamma,g')$ is locally isometric to $(M\setminus\Gamma,g)$, and*
3. *(M,g') is without conical singularities.*

Proof. For all $j>k$ define $\beta_j:=1$ and for all $j\leq k$ define β_j by the cone angle, $\beta_j:=\frac{1}{2\pi}\alpha_j$. The new metric g' is defined on $M\setminus\Gamma$ by g and expressed in terms of Equation (4.36) as

$$g'_{\mu\nu}:=g_{\mu\nu}\qquad\qquad g'_{\mu j}:=\frac{1}{\beta_j}g_{\mu j}\qquad\qquad g'_{ij}:=\frac{1}{\beta_i\beta_j}g_{ij}. \tag{4.39}$$

We now extend g' by continuity from a metric defined only on $M\setminus\Gamma$ to a metric defined on all of M. Observe that (M,g') is a simple T^n-manifold as T^n still acts effectively and isometrically on (M,g'). It also has the same rod data $\{(\mathbf{e}_1,\Gamma_1),\ldots,(\mathbf{e}_k,\Gamma_k)\}$ as (M,g) whenever the Killing field $X_j=\partial_{\phi^j}$ vanishes on (M,g) it also must vanish on (M,g').

Note that g' is not the pullback of g by some 'change of coordinates map' on $M\setminus\Gamma$. However one could image g' begin the pullback of g via a local isometry $F\colon(U,g')\to(V,g)$ which we will now define. First choose a points on the torus $\boldsymbol{\theta}_0, \boldsymbol{\eta}_0 \in (\mathbb{R}/(2\pi\mathbb{Z}))^n$. Now for sufficiently small ε we define the open sets $U, V \subset M \setminus \Gamma$ in coordinates by

$$U := (\mathbb{R}^2_{\geq 0} \setminus \Gamma) \times (\theta_0^1 - \varepsilon, \theta_0^1 + \varepsilon) \times \cdots \times (\theta_0^n - \varepsilon, \theta_0^n + \varepsilon) \tag{4.40}$$
$$V := (\mathbb{R}^2_{\geq 0} \setminus \Gamma) \times (\beta_1(\eta_0^1 - \varepsilon), \beta_1(\eta_0^1 + \varepsilon)) \times \cdots \times (\beta_n(\eta_0^n - \varepsilon), \beta_n(\eta_0^n + \varepsilon)). \tag{4.41}$$

The map

$$F(\rho, z, \phi^1, \ldots, \phi^n) := (\rho, z, \beta_1(\phi^1 - \theta_0^1 + \eta_0^1), \ldots, \beta_n(\phi^n - \theta_0^n + \eta_0^n)) \tag{4.42}$$

is then a diffeomorphism between U and V. Moreover, $F\colon (U, g') \to (V, g)$ is an isometry since $g(F_*(\partial_\rho), F_*(\partial_{\phi^j})) = g(\partial_\rho, \frac{1}{\beta_j}\partial_{\phi^j}) = g'_{\rho j} = g'(\partial_\rho, \partial_{\phi^j})$. By preforming similar calculations with $g'_{\mu\nu}$, $g'_{\mu j}$, and g'_{ij} we can see that indeed $F^*(g) = g'$, or more specifically $F^*(g|_V) = g'|_U$. Since the points $\boldsymbol{\theta}_0$ and $\boldsymbol{\eta}_0$ were chosen arbitrarily we conclude that $(M \setminus \Gamma, g')$ and $(M \setminus \Gamma, g)$ are indeed locally isometric.

The final thing to check is that (M, g') is without conical singularities. This is a simple calculation. By definition the cone angle (M, g') at $(0, z_0) \in \Gamma_j$ is

$$\alpha'(z_0) := \lim_{\rho \to 0} \frac{\int_0^{2\pi} \sqrt{g'(X_j, X_j)}}{\int_0^\rho \sqrt{g'(\partial_\rho, \partial_\rho)}}. \tag{4.43}$$

Since $X_j = \partial_{\phi^j}$ we see $g'(X_j, X_j) = g'_{jj} = \frac{1}{\beta_j^2} g_{jj}$. Similarly $g'(\partial_\rho, \partial_\rho) = g'_{\rho\rho} = g_{\rho\rho}$. This simplifies to

$$\alpha'(z_0) = \frac{1}{\beta_j} \lim_{\rho \to 0} \frac{\int_0^{2\pi} \sqrt{g(X_j, X_j)}}{\int_0^\rho \sqrt{g(\partial_\rho, \partial_\rho)}} = \frac{\alpha_j}{\beta_j} \tag{4.44}$$

where α_j is the cone angle for (M, g) at any point $(0, z) \in \Gamma_j$. By construction $\beta_j = \frac{1}{2\pi}\alpha_j$ and thus $\alpha'(z_0) = 2\pi$ for all $(0, z_0) \in \Gamma_j$ and for all j. Since the cone angle for (M, g') is 2π along all of Γ we conclude that (M, g') is without conical singularities. The proof is now complete. $\square$

Suppose the simple T^n-manifold (M^{n+2}, g) given in Lemma 4.14 is a time slice of a stationary spacetime $\mathcal{M} \cong M \times \mathbb{R}$. Expressing the full Lorentzian metric g in Weyl-Papapetrou coordinates as

$$g = f^{-1}e^{2\sigma}(d\rho^2 + dz^2) - f^{-1}\rho^2 dt^2 + f_{ij}(d\phi^i + w^i dt)(d\phi^j + w^j dt), \tag{4.45}$$

we see that Lemma 4.14 produces a corresponding Lorentzian metric

$$g' = f^{-1}e^{2\sigma}(d\rho^2 + dz^2) - f^{-1}\rho^2 dt^2 + f_{ij}(\frac{1}{\beta_i}d\phi^i + w^i dt)(\frac{1}{\beta_j}d\phi^j + w^j dt). \tag{4.46}$$

The spacetimes $(\mathcal{M} \setminus \Gamma, g)$ and $(\mathcal{M} \setminus \Gamma, g')$ are in fact locally isometric. The collection of local isometries expressed in Equation (4.42) is modified to

$$F(t, \rho, z, \phi^1, \ldots, \phi^n) := (t, \rho, z, \beta_1(\phi^1 - \theta_0^1 + \eta_0^1), \ldots, \beta_n(\phi^n - \theta_0^n + \eta_0^n)) \tag{4.47}$$

so that time is include. As a result if $(\mathcal{M}, g)$ solves the vacuum Einstein equations then so does $(\mathcal{M}, g')$. Of course any stationary vacuum solution of the Einstein equations with symmetry group $U(1)^n$ can be expressed in Weyl-Papapetrou form [18, Theorem 8], and must solve the harmonic map equations (see Section 4.1). This is made explicit in the following lemma.

Lemma 4.15. *The spacetime metric g' can be put in Weyl-Papapetrou form as*

$$g' = f'^{-1}e^{2\sigma'}(d\rho^2 + dz^2) - f'^{-1}\rho^2 dt'^2 + f'_{ij}(d\phi^i + w'^i dt')(d\phi^j + w'^j dt') \tag{4.48}$$

where

$$f'_{ij} := \frac{1}{\beta_i \beta_j} f_{ij} \qquad \sigma' := \sigma - \log(\beta)$$

$$\beta := \beta_1 \cdots \beta_n \qquad t' := \frac{1}{\beta} t$$

$$f' := \det(f'_{ij}) = \beta^{-2} f \qquad w'^i := \beta \beta_i w^i.$$

If the spacetime metric g satisfies the harmonic map equations, then so does g', having twist potentials

$$\omega'_i = \beta^{-1} \beta_i^{-1} \omega_i. \tag{4.49}$$

Proof. The metrics g and g' are defined in Equations (4.45) and (4.46) respectively. As state above in the discussion above, that fact that $(\mathcal{M}, g)$ and $(\mathcal{M}, g')$ are locally isometric means that g' can be written in Weyl-Papapetrou form and that it satisfies the harmonic map equations if and only if g does. The only claims that need to be checked are the exact values of the metric coefficients of g' in terms of g. This is done with a simple computation. Plug in $f'^{-1} = \beta^2 f^{-1}$, $e^{2\sigma'} = \beta^2 e^{2\sigma}$, $w'^i = \beta \beta_i w^i$, and $dt' = \beta^{-1} dt$ into Equation (4.46) and observe that it reduces to Equation (4.48). For the harmonic map equations (see Equations (4.8) and (4.9)) one needs to check that $\omega'_i := \beta^{-1} \beta_i^{-1} \omega_i$ satisfies $d\omega'_i = 2 f' f'_{ij} \star'_3 d\mathcal{A}'^{(j)}$. Once this is established then the rest of the proof reduces to confirming that each term in the harmonic map equations scales properly with β and β_i. □

In order to apply Lemma 4.14 to the metrics produced by Theorem F we must know that the cone angles remain constant along the axis rods. This was shown in the $(4+1)$-dimensional setting in [28, Remark 8.1.2], but the argument works without issue in the higher dimensional setting. We record the results in the following lemma.

Lemma 4.16. *Any solution to the Einstein equations $(\mathcal{M}, g)$ produced by Theorem F has cone angles constant along its axis rods.*

The step last in the proof of Theorem G is showing that the metrics produced by Lemma 4.14 are indeed asymptotically Kaluza-Klein. This is proven in the following lemma.

Lemma 4.17. *Let (M^{n+2}, g) be a simple T^n-manifold satisfying the hypotheses of Lemma 4.14. If (M, g) is asymptotically Kaluza-Klein, then (M, g') is also asymptotically Kaluza-Klein.*

Proof. First observe that the semi-infinite axis rods have rod structures $\mathbf{e}_1$ and $\mathbf{e}_k$, thus the 'horizon at infinity' has topology of $S^3 \times T^{n-2}$. In particular this means there are 4 'large' spatial dimensions and g asymptotes to $\delta = \delta_{\mathbb{R}^4 \times T^{n-2}}$, the product of the Euclidean metric on $\mathbb{R}^4$ and $n-2$ metrics on S^1. Importantly the definition of asymptotically Kaluza-Klein does not depend on the size of the various S^1's that make up the torus. We will shows that g' asymptotes to a metric δ' which identical to δ, except with the circles possibly being of different sizes. Below we prove the statement for $k = n = 3$, however the proof works without issue for all $n \geq 2$ and all $k \leq n$.

By definition g asymptotes to an ideal metric on $\mathbb{R}^4 \times S^1$,

$$\delta = \delta_{\mathbb{R}^4 \times S^1} = \delta_{\mathbb{R}^4} + a^2 d\psi^2 = dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2 + a^2 d\psi^2 \tag{4.50}$$

where $a > 0$ is some constant which represents the 'radius' of the circle S^1. By 'asymptotes' we mean $g = h + \delta$ where h decays to 0 sufficiently quickly as r grows, with r defined by $r^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2$. We wish to show that g' asymptotes δ' where δ' is also defined by Equation (4.50), except with constant $a' > 0$. This will involve transforming the $(\rho, z, \phi^1, \phi^2, \phi^3)$ coordinate system into $(x_1, y_1, x_2, y_2, \psi)$.

First assume that (M, g) has rod data $\{(\mathbf{e}_1, \Gamma_1), (\mathbf{e}_3, \Gamma_2), (\mathbf{e}_2, \Gamma_3)\}$ where Γ_1 and Γ_2 are the two semi-infinite axis rods. With this we can set $\psi = \phi^3$. We now introduce the intermediate step of Hopf coordinates on the (ρ, z)-plane, defined by

$$r\sin(2\eta) = \rho \tag{4.51}$$

$$r\cos(2\eta) = z \tag{4.52}$$

where $0 \leq 2\eta \leq \pi$ is the angle past the positive z-axis. Next using ϕ^1 and ϕ^2 we introduce bi-polar coordinates $(r_1, \phi^1, r_2, \phi^2)$ on $\mathbb{R}^4$. The radial components are expressed in terms of Hopf coordinates by

$$r_1 = r\sin(\eta) \tag{4.53}$$

$$r_2 = r\cos(\eta). \tag{4.54}$$

Finally, these bi-polar coordinates can be transformed into the usual (x_1, y_1, x_2, y_2) Cartesian coordinates with the following

$$x_i = r_i \cos(\phi^i) \tag{4.55}$$

$$y_i = r_i \sin(\phi^i). \tag{4.56}$$

Notice that $\rho^2 + z^2 = r^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2$. In particular this means the metric g becomes arbitrarily close to δ for sufficiently large $z > 0$. The cone angle of Γ_1 can be computed at any $(0, z) \in \Gamma_1$, in particular for large $z > 0$ where the metric is sufficiently close to δ. Since δ is without cone angles we conclude that g is without cone angles on Γ_1. The same argument applies to Γ_3. Therefore the metric components of g', expressed in terms of the (ρ, z)-coordinate system in Equation (4.36), are identical to that of g with the exception that $g'_{\mu 3} = \frac{1}{\beta_3} g_{\mu 3}$, $g'_{i3} = \frac{1}{\beta_3} g_{i3}$, and $g'_{33} = \frac{1}{\beta_3^2} g_{33}$ for $\mu \in \{\rho, z\}$ and $i \in \{1, 2\}$. By construction, $\psi = \phi^3$ and none of coordinates (x_1, y_1, x_2, y_2) depend at all on ϕ^3. Therefore in the $(x_1, y_1, x_2, y_2, \psi)$ coordinate system we can see

$$g'_{p_i q_j} = g_{p_i q_j} \qquad g'_{p_i \psi} = \frac{1}{\beta_3} g_{p_i \psi} \qquad g'_{\psi\psi} = \frac{1}{\beta_3^2} g_{\psi\psi} \tag{4.57}$$

where $p, q \in \{x, y\}$ and $i, j \in \{1, 2\}$. Now using $g = h + \delta$ we define h' and δ' to be the components of g' which come from h and δ respectively. In particular we see that h' decays to 0 just as fast as h does, and

$$\delta' = dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2 + \left(\frac{a}{\beta_3}\right)^2 d\psi^2. \tag{4.58}$$

Since g' asymptotes to δ' and δ' is a flat metric on $\mathbb{R}^4 \times S^1$ with a circle of radius $\frac{a}{\beta_3}$ we conclude that g' is in fact asymptotically Kaluza-Klein. The proof is now complete. $\square$

We are now ready to prove Theorem G.

Theorem 4.18 (Theorem G)**.** *Given n-dimensional rod data $\{(\mathbf{e}_1, \Gamma_1), \dots, (\mathbf{e}_k, \Gamma_k)\}$ with $k \leq n$ and non-degenerate horizon rods, there exists a choice of potential constants $\{\mathbf{c}_1, \dots, \mathbf{c}_k\} \subset \mathbb{R}^n$ such that the spacetime produced by Theorem F is without conical singularities.*

Proof. First choose arbitrary potential constants $\{\mathbf{d}_1, \dots \mathbf{d}_k\} \subset \mathbb{R}^n$ subject to the constraints that $\mathbf{d}_j = \mathbf{d}_{j+1}$ if $\Gamma_j \cap \Gamma_{j+1} \neq \emptyset$, and denote them by $\mathbf{d}_j = (d_j^1, \dots, d_j^n)$. Applying Theorem F to the rod data set $\{(\mathbf{e}_1, \Gamma_1, \mathbf{d}_1), \dots, (\mathbf{e}_k, \Gamma_k, \mathbf{d}_k)\}$ produces a spacetime $(\mathcal{M}, g)$ which agrees with the rod data. Using Lemma 4.16 we know that the cone angles along each axis rod Γ_j are constant, say $2\pi\beta_j$. This allows us to apply Lemma 4.14 and produce a new asymptotically Kaluza-Klein (Lemma 4.17) spacetime metric g' where all cone angles are 2π. The spacetimes $(\mathcal{M}, g)$ and $(\mathcal{M}, g')$ are locally isometric and in particular we see from Lemma 4.15 that $(\mathcal{M}, g')$ satisfies the harmonic map equations. The twist potentials of g and g' are related by Equation (4.49), $\omega'_i = \beta^{-1}\beta_i^{-1}\omega_i$. Since potential constants are defined by the restriction of the twist potentials to the axis rods, we see that

$$\mathbf{c}_j := \left(\frac{d_j^1}{\beta\beta_j}, \dots, \frac{d_j^n}{\beta\beta_j} \right) \tag{4.59}$$

are the potential constants for g'. Therefore $(\mathcal{M}, g')$ is solution to the harmonic map equations without conical singularities coming from rod data $\{(\mathbf{e}_1, \Gamma_1, \mathbf{c}_1), \dots, (\mathbf{e}_k, \Gamma_k, \mathbf{c}_k)\}$.

□

Example 4.19: Consider 3-dimensional rod data of the form $\{(\mathbf{e}_1, \Gamma_1), (\mathbf{e}_2, \Gamma_2), (\mathbf{e}_3, \Gamma_3)\}$ with two non-degenerate horizon rods between the axis rods. Theorem G produces a well-behaved asymptotically Kaluza-Klein stationary vacuum spacetime $(\mathcal{M}^{5+1}, g)$. Geometrically the spacetime has 4 'large' spatial dimensions, yet topologically $\mathcal{M}$ is simply connected (see Theorem 2.25). In fact using Theorem 2.50 we can see that it's Cauchy surface is diffeomorphic to $S^5 \setminus 3 \cdot S^1$. From Remark 2.24 we see the horizon cross section is $2 \cdot (S^3 \times S^1)$, which makes this a pair of balanced 'black rings'. We are tentatively calling these *tri-rings* since it takes 3 dimensional torus symmetry to define, and obviously because it evokes the similar sounding bi-rings [11] and di-rings [13, 22].

Remark 4.20. The arguments employed in the proof of Theorem 4.18 also work for rod data of the form $\{(\mathbf{e}_1, \Gamma_1), \dots, (\mathbf{e}_k, \Gamma_k), (\mathbf{e}_1, \Gamma_{k+1})\}$ with $k \leq n$. The crux of the proof of Theorem 4.18 is in Lemma 4.14, which uses the fact that the rod structures are 'orthogonal' to each other to simplify cone angle calculation. In short, having rod structures of the form $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ means that adjusting the cone angle for Γ_i involves only scaling the i^{th} circle in T^n, and does not affect the cone angle for Γ_j at all. However in [27, §6] (see also proof of Lemma 4.17) it was shown that the semi-infinite axis rods are without conical singularities. This means the circles corresponding to the first and last rod structures do not need to be scaled at all, and thus do not need to be orthogonal to each other. In particular having the first and last rod structures be identical poses no problem at all. This argument was used in a previous work [24, Proposition 7.2] to produce a spacetime devoid of conical singularities having rod structures $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\}$ with black holes at the two corners, which was used as a counterexample to a conjecture by Hollands and Ishibashi [18, Conjecture 1].

4.5 Dimensional Reduction

The purpose of this section is to prove Theorem H. We begin by giving a definition of the k-reduced Kaluza-Klein equations.

Definition 4.21. A spacetime $(\mathcal{M}, g)$ is said to solve the *k-reduced Kaluza-Klein equations* if there exists a principal T^k-bundle $\widetilde{\mathcal{M}}$ over $\mathcal{M}$ with Riemannian submersion $P\colon (\widetilde{\mathcal{M}}, \tilde{g}) \to (\mathcal{M}, g)$ such that $(\widetilde{\mathcal{M}}, \tilde{g})$ solves the vacuum Einstein equations.

The 1-reduced Kaluza-Klein equations are perhaps the most famous. This was Kaluza's and Klein's famously 'almost successful' attempts to unify general relativity with electro-magnetism, resulting in the Einstein-Maxwell-Dilaton field equations (sometimes refered to as *dilaton gravity*[45, pg. 349]). Higher dimensional generalizations, which we refer to as the k-reduced Kaluza-Klein equations (also refered to as *axion-dilaton gravity* [45, pg. 349]), have less immediate physical interpretations but are nevertheless mathematically interesting. A derivation of the action functional for these equations can be found in [45, §11.4], while the field equations themselves can be found in [47]. Interestingly, in the proof of Theorem H we produce regular solutions to these field equations despite not needing to know any information about them beyond Definition 4.21. We now present the proof of Theorem H without delay.

Theorem 4.22 (Theorem H)**.** *Suppose that $\{(\mathbf{v}_1, \Gamma_1), \dots, (\mathbf{v}_k, \Gamma_k)\}$ is n-dimensional rod data with non-degenerate horizon rods. If the solutions produced in Theorem G are analytically regular, then there exists* *a well-behaved regular asymptotically Kaluza-Klein stationary solution of the $(n+3)$-dimensional k-reduced Kaluza-Klein equations with time slice admitting the $U(1)^n$ symmetry group, and in particular agreeing with the rod data.*

Proof. Let M be the simple T^n-manifold defined by the rod data $\{(\mathbf{v}_1,\Gamma_1),\dots,(\mathbf{v}_k,\Gamma_k)\}$. Define a new simple T^{n+k}-manifold $\widetilde{M}$ with rod data $\{(\tilde{\mathbf{v}}_1,\Gamma_1),\dots,(\tilde{\mathbf{v}}_k,\Gamma_k)\}$ where $\tilde{\mathbf{v}}_i$ is defined by

$$\tilde{\mathbf{v}}_i := (\mathbf{v}_i,\mathbf{e}_i)\in\mathbb{Z}^n\oplus\mathbb{Z}^k. \tag{4.60}$$

Corollary 2.63 shows that the map $P\colon \widetilde{M}\to M$ given by

$$P(p,\phi^1,\dots,\phi^{n+k}) := (p,\phi^1,\dots,\phi^n) \tag{4.61}$$

defines a principal T^k-bundle over M with free T^k-action on $\widetilde{M}$ generated by $\mathbf{e}_{n+i}\mathbb{R}/\mathbb{Z}\subset\mathbb{R}^{n+k}/\mathbb{Z}^{n+k}$. Now define the change of basis matrix $U\in SL(n+k,\mathbb{Z})$ by $U(\tilde{\mathbf{v}}_i):=\mathbf{e}_{n+i}$ for $i=1,\dots,k$ and $U(\mathbf{e}_j):=\mathbf{e}_j$ for $j=1,\dots,n$. Applying this matrix to $\widetilde{M}$ we see that it admits rod structures $\{\mathbf{e}_{n+1},\dots,\mathbf{e}_{n+k}\}$.

Applying Theorem G to the rod data $\{(\mathbf{e}_{n+1},\Gamma_1),\dots,(\mathbf{e}_{n+k},\Gamma_k)\}$ produces a stationary spacetime $(\widetilde{\mathcal{M}},\tilde{g})$ with time slice $(\widetilde{M},\tilde{g})$ agreeing with the rod data. Now apply the change of basis U^{-1} so that $(\widetilde{M},\tilde{g})$ has rod data $\{(\tilde{\mathbf{v}}_1,\Gamma_1),\dots,(\tilde{\mathbf{v}}_k,\Gamma_k)\}$. Extend the projection map $P\colon\widetilde{M}\to M$ to include t so that $P\colon\widetilde{\mathcal{M}}\to\mathcal{M}:=M\times\mathbb{R}$ is a submersion in the sense that $dP\colon T\widetilde{\mathcal{M}}\to T\mathcal{M}$ is everywhere surjective. This defines a vertical distribution $\mathcal{V}:=\ker(dP)\subset T\widetilde{\mathcal{M}}$ in the tangent space of $\widetilde{\mathcal{M}}$, and using the metric $\tilde{g}$ gives its orthogonal complement horizontal distribution $\mathcal{H}:=\ker(dP)^\perp$.

The fact that T^k acts by isometries on $\widetilde{\mathcal{M}}$ allows us to 'pushforward' the metric $\tilde{g}$ and define a metric $g:=P_*(\tilde{g})$ on $\mathcal{M}$. To see that such a pushforward is possible, choose a point $\mathbf{p}\in\mathcal{M}$, tangent vectors $X,Y\in T_{\mathbf{p}}\mathcal{M}$, and curves $\gamma,\eta\subset\mathcal{M}$ with $\gamma'(0)=X$ and $\eta'(0)=Y$. Now choose any $\tilde{\mathbf{p}}_0\in P^{-1}(\mathbf{p})$ and define the unique lifts $\tilde{\gamma}_0,\tilde{\eta}_0\subset\widetilde{\mathcal{M}}$ with $\tilde{\gamma}_0(0)=\tilde{\mathbf{p}}_0=\tilde{\eta}_0(0)$ and let $\tilde{X}_0:=\tilde{\gamma}_0'(0)$ and $\tilde{Y}_0:=\tilde{\eta}_0'(0)$. For any other point $\tilde{\mathbf{p}}_1\in P^{-1}(\mathbf{p})$ the vectors $\tilde{X}_1:=\tilde{\gamma}_1'(0)$ and $\tilde{Y}_1:=\tilde{\eta}_1'(0)$ are defined similarly. Since T^k acts transitively on the fibers of P, there exists a $\mathbf{t}\in T^k$ such that $\mathbf{t}\cdot\tilde{\mathbf{p}}_0=\tilde{\mathbf{p}}_1$. Because $\mathbf{t}$ is an isometry we know $\mathbf{t}^*(\tilde{g}_{\tilde{\mathbf{p}}_1})=\tilde{g}_{\tilde{\mathbf{p}}_0}$. By uniqueness of lifts $\mathbf{t}(\tilde{\gamma}_0)=\tilde{\gamma}_1$ thus $\mathbf{t}_*(\tilde{X}_0)=\tilde{X}_1$, and similarly $\mathbf{t}_*(\tilde{Y}_0)=\tilde{Y}_1$. Therefore we see $\tilde{g}_{\tilde{\mathbf{p}}_0}(\tilde{X}_0,\tilde{Y}_0)=\mathbf{t}^*(\tilde{g}_{\tilde{\mathbf{p}}_1})(\tilde{X}_0,\tilde{Y}_0)=\tilde{g}_{\tilde{\mathbf{p}}_1}(\mathbf{t}_*\tilde{X}_0,\mathbf{t}_*\tilde{Y}_0)=\tilde{g}_{\tilde{\mathbf{p}}_1}(\tilde{X}_1,\tilde{Y}_1)$. This means that there exists a well-defined metric g on $\mathcal{M}$ coming from $\tilde{g}$. Specifically

$$g_{\mathbf{p}}(X,Y):=\tilde{g}_{\tilde{\mathbf{p}}}(\tilde{X},\tilde{Y}) \tag{4.62}$$

for any $\tilde{\mathbf{p}}\in P^{-1}(\mathbf{p})$ and $\tilde{X}\in\mathcal{H}\cap dP_{\tilde{\mathbf{p}}}^{-1}(X)$ and $\tilde{Y}\in\mathcal{H}\cap dP_{\tilde{\mathbf{p}}}^{-1}(Y)$. By construction

$$dP_{\tilde{\mathbf{p}}}\colon(\mathcal{H}_{\tilde{\mathbf{p}}},\tilde{g}_{\tilde{\mathbf{p}}}|_{\mathcal{H}})\to(T_{\mathbf{p}}\mathcal{M},g_{\mathbf{p}}) \tag{4.63}$$

is an isometry, and thus $P\colon(\widetilde{\mathcal{M}},\tilde{g})\to(\mathcal{M},g)$ is a Riemannian submersion. Since $(\widetilde{\mathcal{M}},\tilde{g})$ is assumed to be a complete regular (i.e. satisfy both analytic and geometric regularity) smooth Lorentzian manifold, and T^k acts freely and by isometries, we know the quotient space $(\widetilde{\mathcal{M}},\tilde{g})/T^k\cong P_*(\widetilde{\mathcal{M}},\tilde{g})\cong(\mathcal{M},g)$ is also a complete regular smooth Lorentzian manifold. In particular $(\mathcal{M},g)$ is a regular solution to the k-reduced Kaluza-Klein equations. □

In Theorem H the dimension of the reduction is chosen to be equal to the number of rods k. This is in no way optimal. In fact in almost all cases this number can be lowered to to $k-2$.

The metric for g can be explicitly described in terms of $\tilde{g}$. Let $(\widetilde{\mathcal{M}},\tilde{g})$ be the spacetime described above, written in Weyl-Papapetrou coordinates as

$$\tilde{g}=\frac{e^{2\sigma}}{\tilde{f}}(d\rho^2+dz^2)-\frac{\rho^2}{\tilde{f}}dt^2+\sum_{\mathbf{i},\mathbf{j}=1}^{n+k}\tilde{f}_{\mathbf{ij}}(d\phi^{\mathbf{i}}+w^{\mathbf{i}}dt)(d\phi^{\mathbf{j}}+w^{\mathbf{j}}dt). \tag{4.64}$$

Using the submersion $P\colon \widetilde{\mathcal{M}} \to \mathcal{M}$

$$P(t,\rho,z,\phi^1,\ldots,\phi^{n+k}) := (t,\rho,z,\phi^1,\ldots,\phi^n) \tag{4.65}$$

we wish to compute the 'pushforward' metric $g := P_*(\tilde{g})$. It turns out that g will be able to be expressed in a form that is close, but not exactly equivalent, to Weyl-Papapetrou form. This is shown in Lemma 4.24 below. This lemma, its proof, and everything in this section that follows will become much easier once we introduce the following notation convention.

Notation 4.23. For the remainder of Section 4.5:
the indices i, j, k, etc..., are assumed to range over $\{1,\ldots,n\}$,
the indices a, b, c, etc..., are assumed to range over $\{n+1,\ldots,n+k\}$,
the indices $\mathbf{i},\mathbf{j},\mathbf{k}$, etc..., are assumed to range over $\{1,\ldots,n+k\}$,
the indices μ,ν,λ, etc..., are assumed to range over $\{t,\rho,z\}$,
and the indices $\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\lambda}$, etc..., are assumed to range over $\{t,\rho,z,1,\ldots,n\}$.
Furthermore, ∂_i, ∂_a, and $\partial_{\mathbf{i}}$ denote ∂_{ϕ^i}, ∂_{ϕ^a}, and $\partial_{\phi^{\mathbf{i}}}$ respectively, and dx^μ is assumed to be $d\rho$, dz, or dt depending on the value of μ.

Lemma 4.24. *Suppose $(\widetilde{\mathcal{M}},\tilde{g})$ and $(\mathcal{M},g)$ are the spacetimes described in Theorem 4.22. There exists one-forms $\mathcal{B}^a := \mathcal{B}^a_i d\phi^i + \mathcal{B}^a_t dt$ and symmetric positive definite matrices f_{ij} and κ_{ab}, all of which only depend on ρ and z, such that the spacetime metrics $\tilde{g}$ and g can be expressed as*

$$\tilde{g} = g + \kappa_{ab}(d\phi^a + \mathcal{B}^a)(d\phi^b + \mathcal{B}^b) \tag{4.66}$$

and

$$g = \frac{e^{2\sigma}}{f\kappa}(d\rho^2 + dz^2) - \frac{\rho^2}{f\kappa}dt^2 + f_{ij}(d\phi^i + w^i dt)(d\phi^j + w^j dt). \tag{4.67}$$

Proof. We begin by writing $\tilde{g}$ in its Weyl-Papapetrou form;

$$\tilde{g} = \tilde{f}^{-1}e^{2\sigma}(d\rho^2 + dz^2) - \tilde{f}^{-1}\rho^2 dt^2 + \tilde{f}_{\mathbf{ij}}(d\phi^{\mathbf{i}} + w^{\mathbf{i}}dt)(d\phi^{\mathbf{j}} + w^{\mathbf{j}}dt). \tag{4.68}$$

In order to extract g from $\tilde{g}$ we will need to seperate the $d\phi^a$ and $d\phi^i$ parts from each other in Equation (4.68). The first step to doing this is decomposing $\tilde{f}_{\mathbf{ij}}$ into its parts like so

$$\tilde{f}_{\mathbf{ij}} = \left[\begin{array}{c|c} f_{ij} + \mathcal{B}^a_i\kappa_{ab}\mathcal{B}^b_j & \mathcal{B}^a_i\kappa_{ab} \\ \hline \kappa_{ab}\mathcal{B}^b_j & \kappa_{ab} \end{array}\right]. \tag{4.69}$$

Recall that any symmetric invertible matrix can be decomposed in this way with the pieces f_{ij}, $\mathcal{B}^a_i$, and κ_{ab} are all being defined by this decomposition;

$$\kappa_{ab} := \tilde{f}_{ab} \tag{4.70}$$

$$\mathcal{B}^a_i := \kappa^{ac}\tilde{f}_{ci} \tag{4.71}$$

$$f_{ij} := \tilde{f}_{ij} - \mathcal{B}^a_i\kappa_{ab}\mathcal{B}^b_j. \tag{4.72}$$

It should be pointed out that κ_{ab} and f_{ij} are both positive definite symmetric matrices of size $k \times k$ and $n \times n$ respectively. We are also using the convention that f and κ denote the determinants of f_{ij} and κ_{ab} respectively, and f^{ij} and κ^{ab} denote the entries of the inverse matrices $(f_{ij})^{-1}$ and $(\kappa_{ab})^{-1}$. The inverse and determinant of $\tilde{f}_{\mathbf{ij}}$ can be computed from Equation (4.69) using variations on the Woodbury matrix identity. We find

$$\tilde{f}^{\mathbf{ij}} = \left[\begin{array}{c|c} f^{ij} & -f^{ik}\mathcal{B}^b_k \\ \hline -\mathcal{B}^a_k h^{kj} & \kappa^{ab} + \mathcal{B}^a_i f^{ij}\mathcal{B}^b_j \end{array}\right] \tag{4.73}$$

and

$$\tilde{f} = f\kappa. \tag{4.74}$$

If we further define

$$\mathcal{B}^a_t := w^a \tag{4.75}$$

so that

$$\mathcal{B}^a := \mathcal{B}^a_i d\phi^i + \mathcal{B}^a_t dt \tag{4.76}$$

becomes a one-form, we can compactly rewrite Equation (4.68) using these new terms. This results in the following expression;

$$\tilde{g} = \frac{e^{2\sigma}}{f\kappa}(d\rho^2 + dz^2) - \frac{\rho^2}{f\kappa}dt^2 + f_{ij}(d\phi^i + w^i dt)(d\phi^j + w^j dt) + \kappa_{ab}(d\phi^a + \mathcal{B}^a)(d\phi^b + \mathcal{B}^b). \tag{4.77}$$

Equation (4.77) is convenient because with it we can immediately see

$$\tilde{g}(\partial_a, \partial_j - \mathcal{B}^c_j\partial_c) = 0 = \tilde{g}(\partial_a, \partial_t - \mathcal{B}^c_t\partial_c).$$

This combined with $\tilde{g}(\partial_a, \partial_\rho) = \tilde{g}(\partial_a, \partial_z) = 0$ shows $\mathcal{H}$ to be generated by Z_μ and Z_i where

$$\begin{aligned} Z_i &:= \partial_i - \mathcal{B}^c_i\partial_c \\ Z_t &:= \partial_t - \mathcal{B}^c_t\partial_c \\ Z_\rho &:= \partial_\rho \\ Z_z &:= \partial_z. \end{aligned} \tag{4.78}$$

Moreover Equation (4.65) shows that ∂_c generate the kernel of $dP\colon T\widetilde{\mathcal{M}} \to T\mathcal{M}$, thus the pushforwards of Z_i and Z_μ are the standard coordinate vector fields;

$$dP(Z_i) = \partial_i \qquad\qquad dP(Z_\mu) = \partial_\mu. \tag{4.79}$$

Using the fact that $P\colon (\widetilde{\mathcal{M}}, \tilde{g}) \to (\mathcal{M}, g)$ is a Riemannian submersion allows us to now directly compute the metric coefficients of g in terms of how $\tilde{g}$ acts on the horizontal vectors. The results of these computations are:

$$\begin{aligned} g(\partial_i, \partial_j) &= \tilde{g}(Z_i, Z_j) = f_{ij} \\ g(\partial_i, \partial_t) &= \tilde{g}(Z_i, Z_t) = f_{ij}w^j \\ g(\partial_t, \partial_t) &= \tilde{g}(Z_t, Z_t) = f_{ij}w^iw^j - \frac{\rho^2}{f\kappa} \\ g(\partial_\rho, \partial_\rho) &= \tilde{g}(Z_\rho, Z_\rho) = \frac{e^{2\sigma}}{f\kappa}. \end{aligned} \tag{4.80}$$

Therefore the spacetime metric g is shown to be

$$g = \frac{e^{2\sigma}}{f\kappa}(d\rho^2 + dz^2) - \frac{\rho^2}{f\kappa}dt^2 + f_{ij}(d\phi^i + w^i dt)(d\phi^j + w^j dt) \tag{4.81}$$

and the proof is complete. □

Since the simple T^{n+k}-manifold $\widetilde{M}$ was constructed from the simple T^n-manifold M, we already know what the relation between their rod structures is. However we can see this relation explicitly in the metric as well. Recall from Remark 4.2 that the $(n+k)$-dimensional rod data $\{(\mathbf{v}_1, \Gamma_1), \ldots, (\mathbf{v}_k, \Gamma_k)\}$ of $(\widetilde{\mathcal{M}}, \tilde{g})$ can be recovered from the metric by the formula

$$\lim_{\rho\to 0} \tilde{f}_{\mathbf{ij}} v_q^{\mathbf{j}} = 0 \tag{4.82}$$

for all $\mathbf{i}$ and for all points $z \in \Gamma_q$ if and only if $\mathbf{v}_q = (v_q^1, \ldots, v_q^{n+k})$ is the rod structure for Γ_q. By splitting $\tilde{f}_{\mathbf{ij}}$ up into its parts we can recover the rod data for $(\mathcal{M}, g)$ as well. This is easiest to see in matrix form;

$$\tilde{f}_{\mathbf{ij}} v_q^{\mathbf{j}} = \left[\begin{array}{c|c} f_{ij} + \mathcal{B}_i^a \kappa_{ab} \mathcal{B}_j^b & \mathcal{B}_i^a \kappa_{ab} \\ \hline \kappa_{ab} \mathcal{B}_j^b & \kappa_{ab} \end{array}\right] \begin{bmatrix} v_q^j \\ v_q^b \end{bmatrix} = \begin{bmatrix} f_{ij} v_q^j + \mathcal{B}_i^a \left(\kappa_{ab} \mathcal{B}_j^b v_q^j + \kappa_{ab} v_q^b\right) \\ \kappa_{ab} \mathcal{B}_j^b v_q^j + \kappa_{ab} v_q^b \end{bmatrix}. \tag{4.83}$$

Since the left hand side of this expression goes to zero, both components of the right hand side go to zero. By reranging terms we can easy see that $f_{ij} v_q^j$ goes to zero as well, and thus

$$\mathbf{v}_q := (v_q^1, \ldots, v_q^n) \tag{4.84}$$

is the rod structure for the Γ_q rod on $(\mathcal{M}, g)$.

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