Renormalization and the Teichmüller theory

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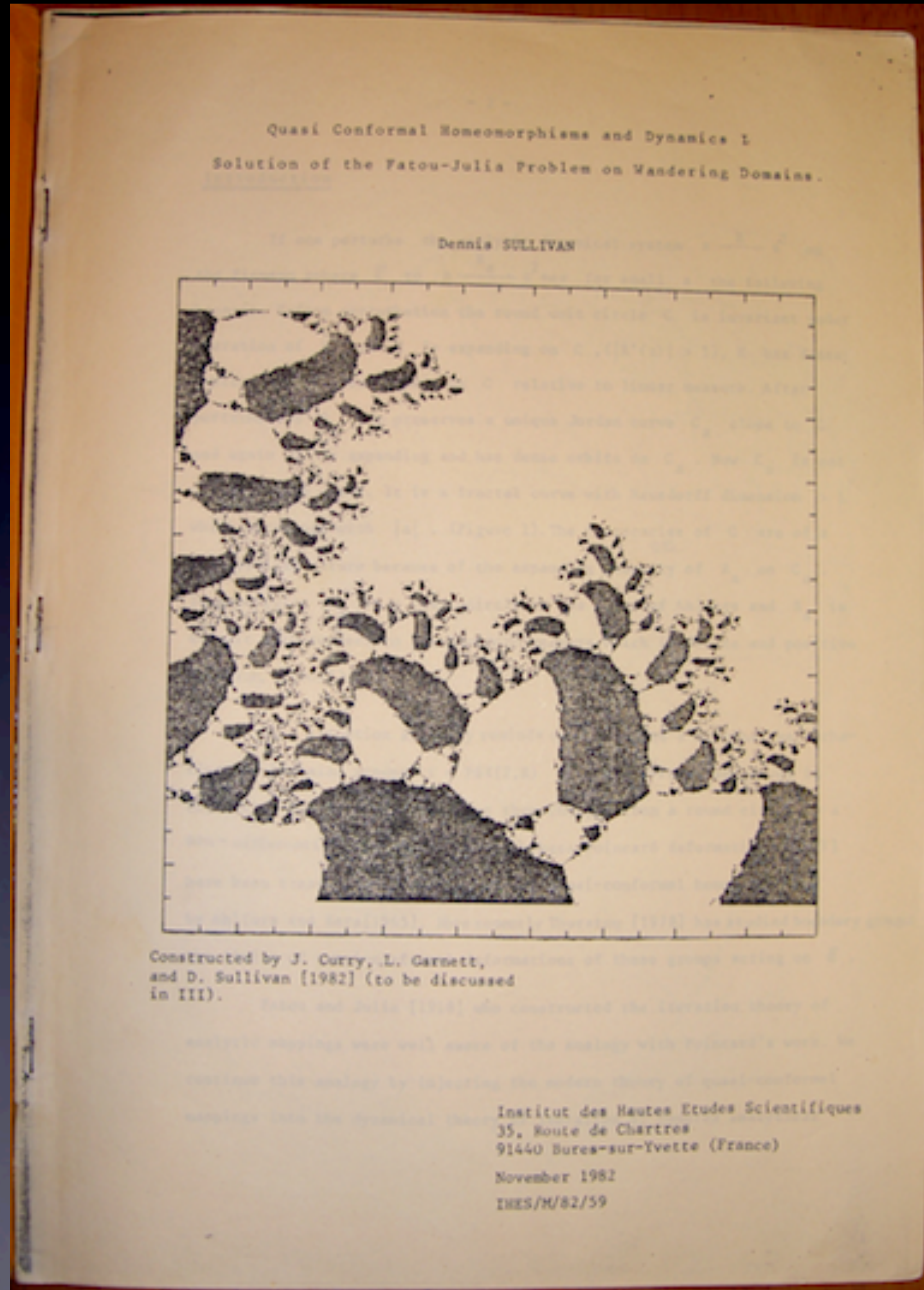
Geometric and Algebraic Structures in Mathematics

A Conference to celebrate Dennis Sullivan's 70th birthday

Simons Center for Geometry and Physics, Stony Brook University

June 4, 2011



Quasi Conformal Homeomorphisms and Dynamics I

Solution of the Fatou-Julia Problem on Wandering Domains.

Dennis SULLIVAN

Constructed by J. Curry, L. Garnett, and D. Sullivan [1982] (to be discussed in III).

Institut des Hautes Etudes Scientifiques
35, Route de Chartres
91440 Bures-sur-Yvette (France)

November 1982

IHES/M/82/59

IHES Preprint 1982

Koebe Distortion Theorem

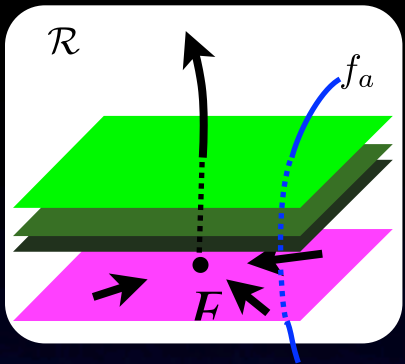
Official: Let $f(z)$ be a univalent (holomorphic injective) function on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. If $f(0) = 0$ and $f'(0) = 1$, then for $|z| < 1$,

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$$

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Sullivan:

Sullivan ICM 1986: proposed the use of Teichmüller space for the convergence of renormalization.



Want contraction on $W^s(F)$

$$d(\mathcal{R}^n f, \mathcal{R}^n g) \to 0$$

Schwarz-Pick Theorem. If $f : \mathbb{D} \to \mathbb{D}$ holomorphic, then

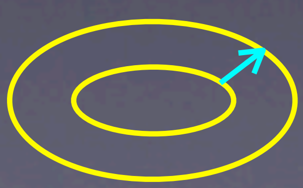
$$d_{\mathbb{D}}(f(x), f(y)) \leq d_{\mathbb{D}}(x, y).$$ non-expanding

Royden-Gardiner Theorem. If $f : Teich(S) \to Teich(S')$ is a holomorphic mapping between Teichmüller spaces, then

$$d_{Teich(S')}(f(x), f(y)) \leq d_{Teich(S)}(x, y).$$ non-expanding

Kobayashi metric

Real bounds $\longrightarrow$ Complex bounds $\longrightarrow$ Contraction in $Teich(\mathcal{L})$



$$mod(U_n' \setminus U_n) \geq m > 0$$ $\mathcal{L} =$ Riemann surface lamination

This talk:

Parabolic/near-parabolic renormalization

study bifurcation of parabolic fixed point
linearization, Siegel disks, Cremer points

(satellite renormalization for MLC???)

a priori bounds and renormalization horseshoe

difference from polynomial-like renormalization:

unbounded geometry, no complex bounds as poly-like maps

Use Teichmüller theory to get contraction

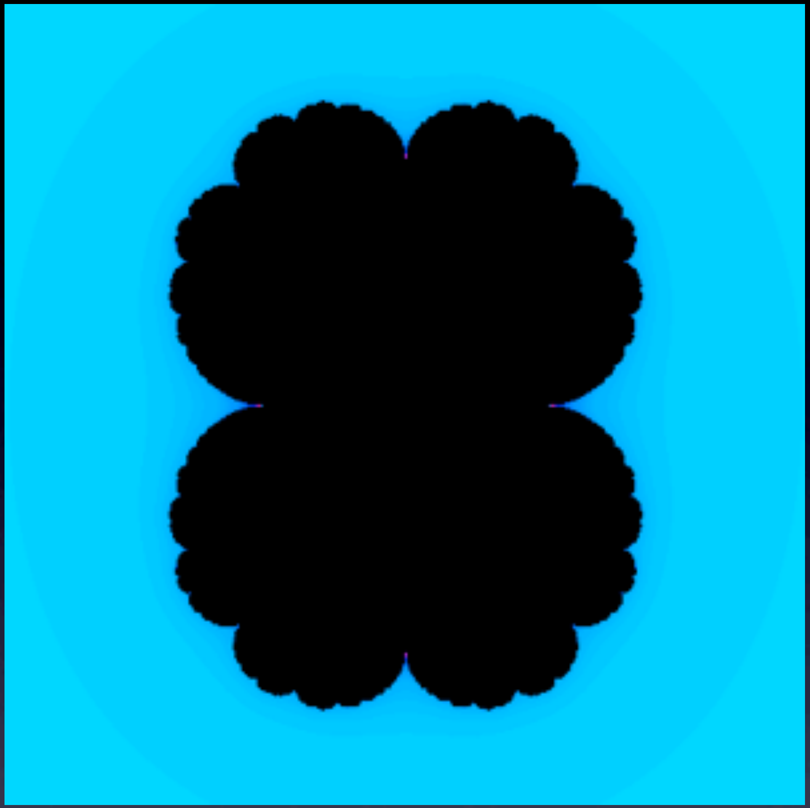
Dictionary: parabolic per. pt $\leftrightarrow$ cusp   Lavaurs map $\leftrightarrow$ geometric limit

large coeff. in continued fraction $\leftrightarrow$ short closed geodesic

Another proof of Lyubich, Graczyk-Swiatek rigidity for infinitely renormalizable real quadratic polynomial using universal Teichmüller space

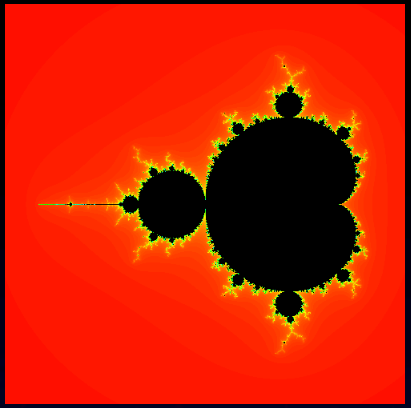
Bifurcation of parabolic fixed point

$(f(z_0) = z_0,\ f'(z_0) = 1)$



$$f_{\frac{1}{4}}(z) = z^2 + \frac{1}{4}$$

$$\xrightarrow{\text{perturb}} f_c(z) = z^2 + c$$



OR

$$f_0(z) = z + z^2$$

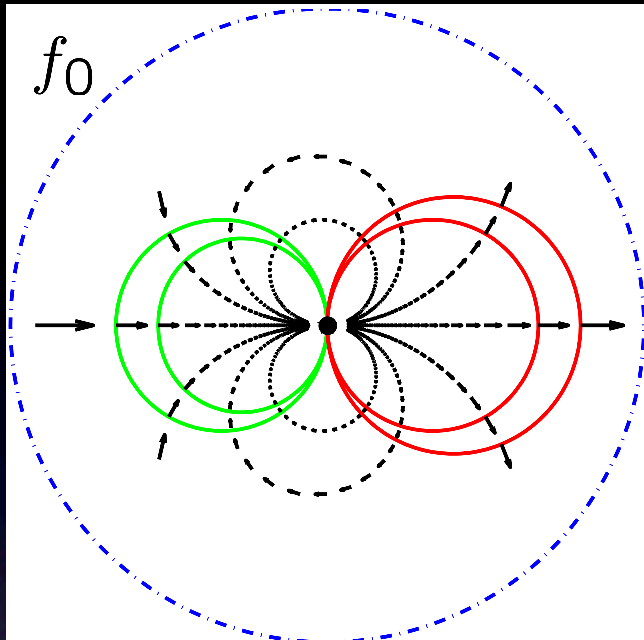
$$\xrightarrow{\text{perturb}} f(z) = e^{2\pi i\alpha} z + z^2$$

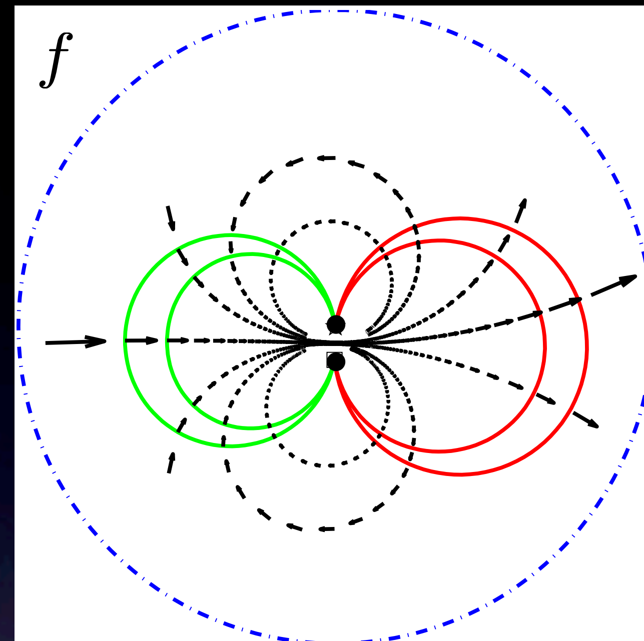
Discontinuous change of Julia sets

Creates complicated/rich Julia sets

$$\text{Thm. } \exists c \in \partial M \ \ HD(J(f_c)) = 2. \qquad HD(\partial M) = 2.$$

Parabolic Implosion (Douady-Hubbard-Lavaurs)





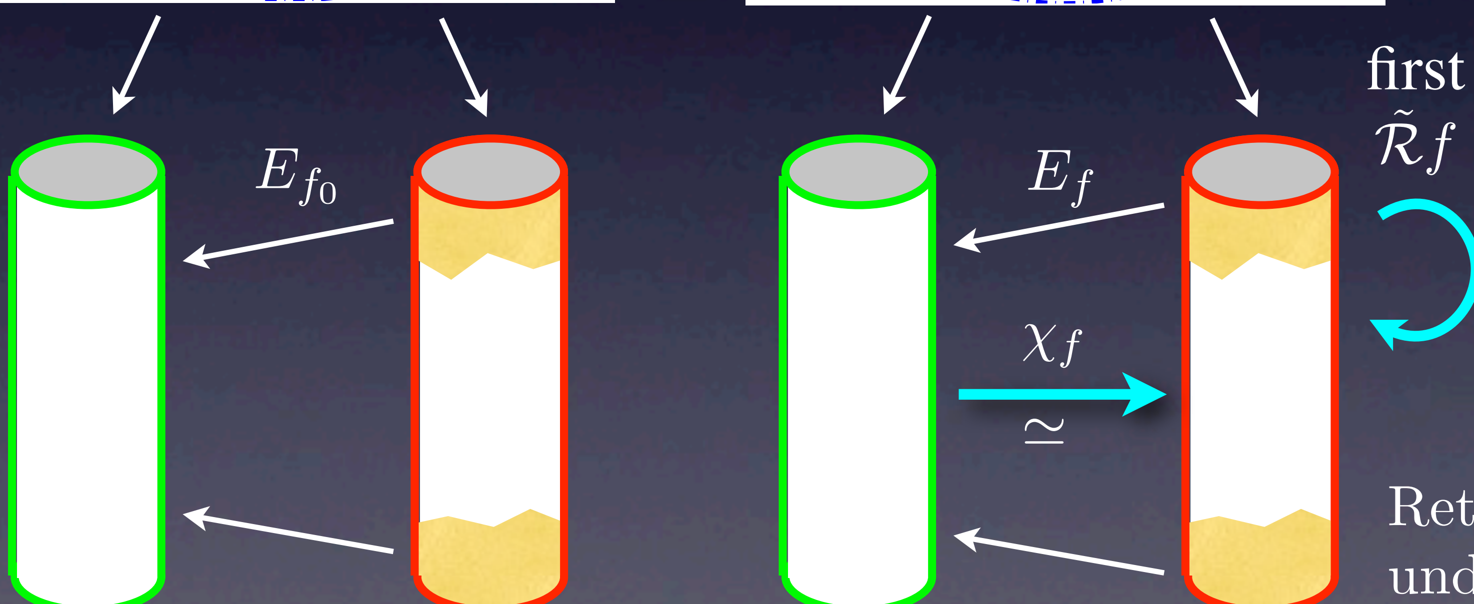
$f'(0) = e^{2\pi i \alpha}$

α small

$|\arg \alpha| < \frac{\pi}{4}$

first return map

$\tilde{\mathcal{R}} f = \chi_f \circ E_f$



Return map can be understood via the horn map E_{f_0} and rotation number α

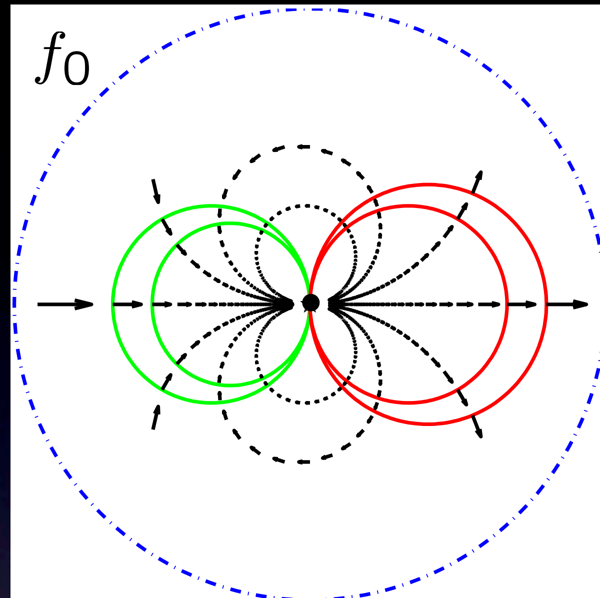
E_f depends continuously on f

(after a suitable normalization)

$$\chi_f(z) = z - \frac{1}{\alpha}$$

Parabolic Renormalization

$$f_0(z) = z + a_2 z^2 + \ldots$$
$$a_2 \neq 0$$



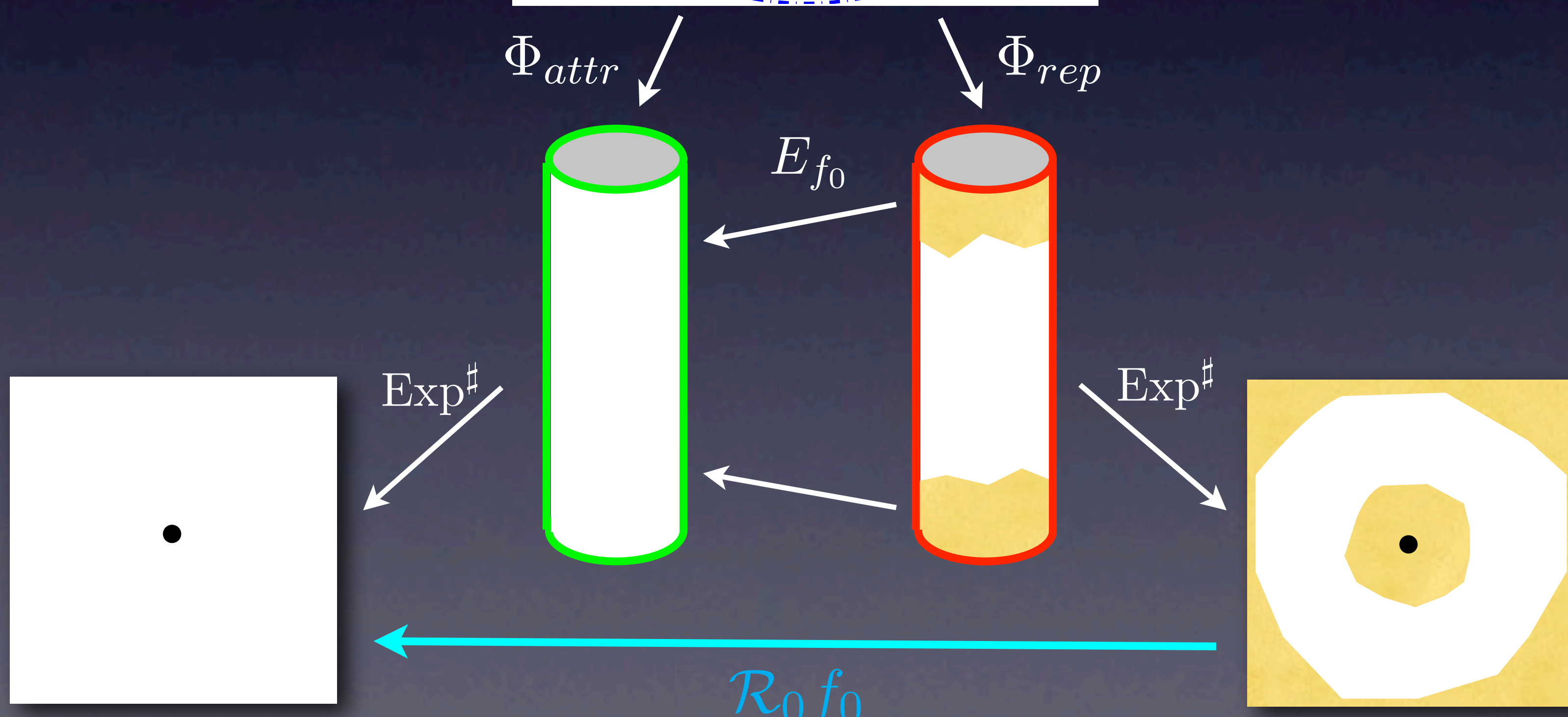
$$\mathrm{Exp}^\sharp(z) = e^{2\pi i z} : \mathbb{C}/\mathbb{Z} \xrightarrow{\simeq} \mathbb{C}^*$$

Parabolic Renormalization

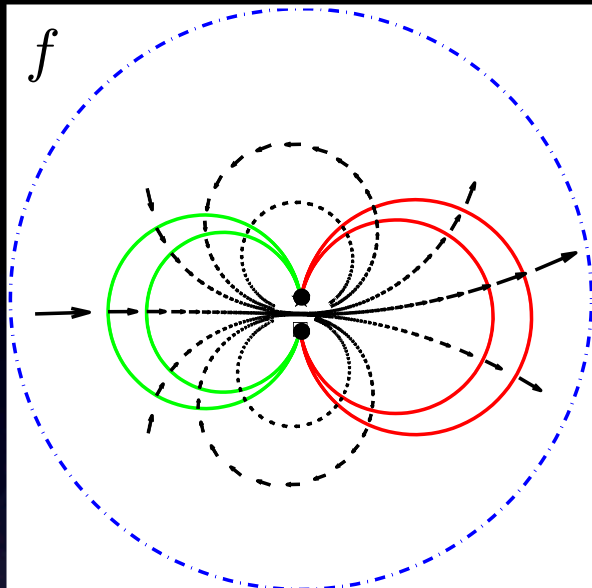
$$\mathcal{R}_0 f_0 = \mathrm{Exp}^\sharp \circ E_{f_0} \circ (\mathrm{Exp}^\sharp)^{-1}$$

Normalization

$$\mathcal{R}_0 f_0(z) = z + \ldots$$
$$E_{f_0}(z) = z + o(1) \quad (\mathrm{Im}\, z \to +\infty)$$



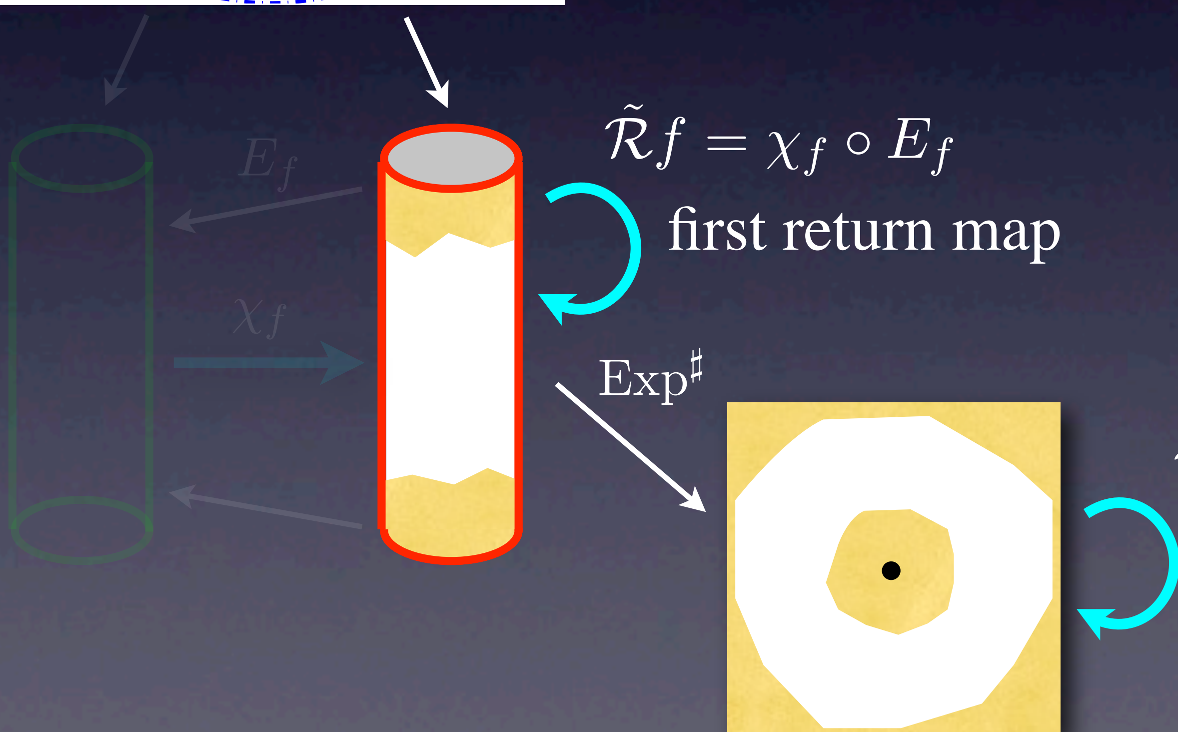
Near-parabolic Renormalization



$$\mathcal{R}f = \mathrm{Exp}^\sharp \circ \tilde{\mathcal{R}} f \circ (\mathrm{Exp}^\sharp)^{-1}$$
$$= \mathrm{Exp}^\sharp \circ \chi_f \circ E_f \circ (\mathrm{Exp}^\sharp)^{-1}$$
$$= e^{2\pi i \beta} z + O(z^2)$$

where $\beta = -\dfrac{1}{\alpha} \pmod{\mathbb{Z}}$

or $\alpha = \dfrac{1}{m - \beta} \quad (m \in \mathbb{N})$

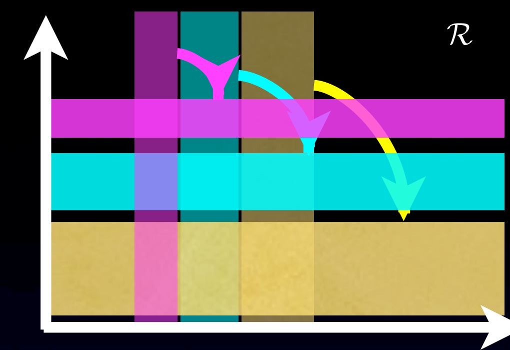


$\mathcal{R}f$

Return map $\mathcal{R}f$ of perturbed map $f(z) = e^{2\pi i \alpha} z + \ldots$ is almost $e^{-2\pi i/\alpha} \mathcal{R}_0 f_0$

For $N \in \mathbb{N}$, let $Irrat_N$ be the set of irrational number of high type:

$$Irrat_N \ni \alpha = \pm \cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{\ddots}}} \qquad \text{where } a_i \in \mathbb{N} \text{ and } a_i \geq N,$$



For a neighborhood V of 0, define $P(z) = z(1+z)^2$ and

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} \,\middle|\, \begin{array}{c} \varphi : V \to \mathbb{C} \text{ is univalent (with qc extension)} \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}$$

Theorem (Inou & S.): For some V and N, the near-parabolic renormalization $\mathcal{R}$ from

$$\{e^{2\pi i \alpha} f : \alpha \in Irrat_N, f \in \mathcal{F}_1\} = Irrat_N \times \mathcal{F}_1$$

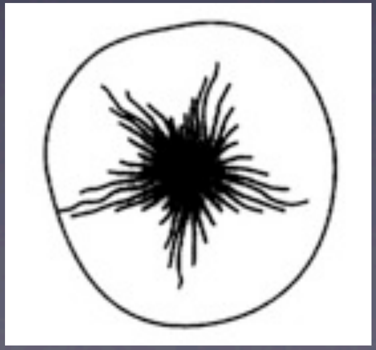
to itself is well defined. Moreover $\mathcal{R}(e^{2\pi i \alpha} z + z^2)$ belong to the above set for $\alpha \in Irrat_N$.
It is hyperbolic; expanding along α direction and uniformly contracting along $\mathcal{F}_1$ direction.

Applications

Theorem (Buff-Chéritat): $\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}$ $\;Area(J(e^{2\pi i\alpha}z + z^2)) > 0$.

Theorem (S.): If $f = e^{2\pi i\alpha}h$, $h \in \mathcal{F}_1$, $\alpha \in Irrat_N$ and f is linearizable at 0 (Brjuno condition), then the boundary of its Siegel disk is a Jordan curve.

Theorem (S.): If $f = e^{2\pi i\alpha}h$, $h \in \mathcal{F}_1$, $\alpha \in Irrat_N$ and f is not linearizable at 0, then there exists an invariant set Λ_f (maximal hedgehog) such that f is homeomorphic on Λ_f; Λ_f contains 0 and a critical point; $\Lambda_f \setminus \{0\}$ consists disjoint arcs ending at 0.
Moreover for two such maps with the same α, there exists a quasiconformal map conjugating on Λ_f, which is asymptotically conformal at the critical orbit.



A Priori Bounds

Claim: The parabolic renormalization $\mathcal{R}_0$ is well-defined in $\mathcal{F}_1$ and the image is contained in $\mathcal{F}_1$.

By the continuity of the horn map, the near-parabolic renormalization $\mathcal{R}_0$ is well-defined as a self map of $\{e^{2\pi i\alpha}h : \alpha \in Irrat_N, h \in \mathcal{F}_1\}$ for large N.

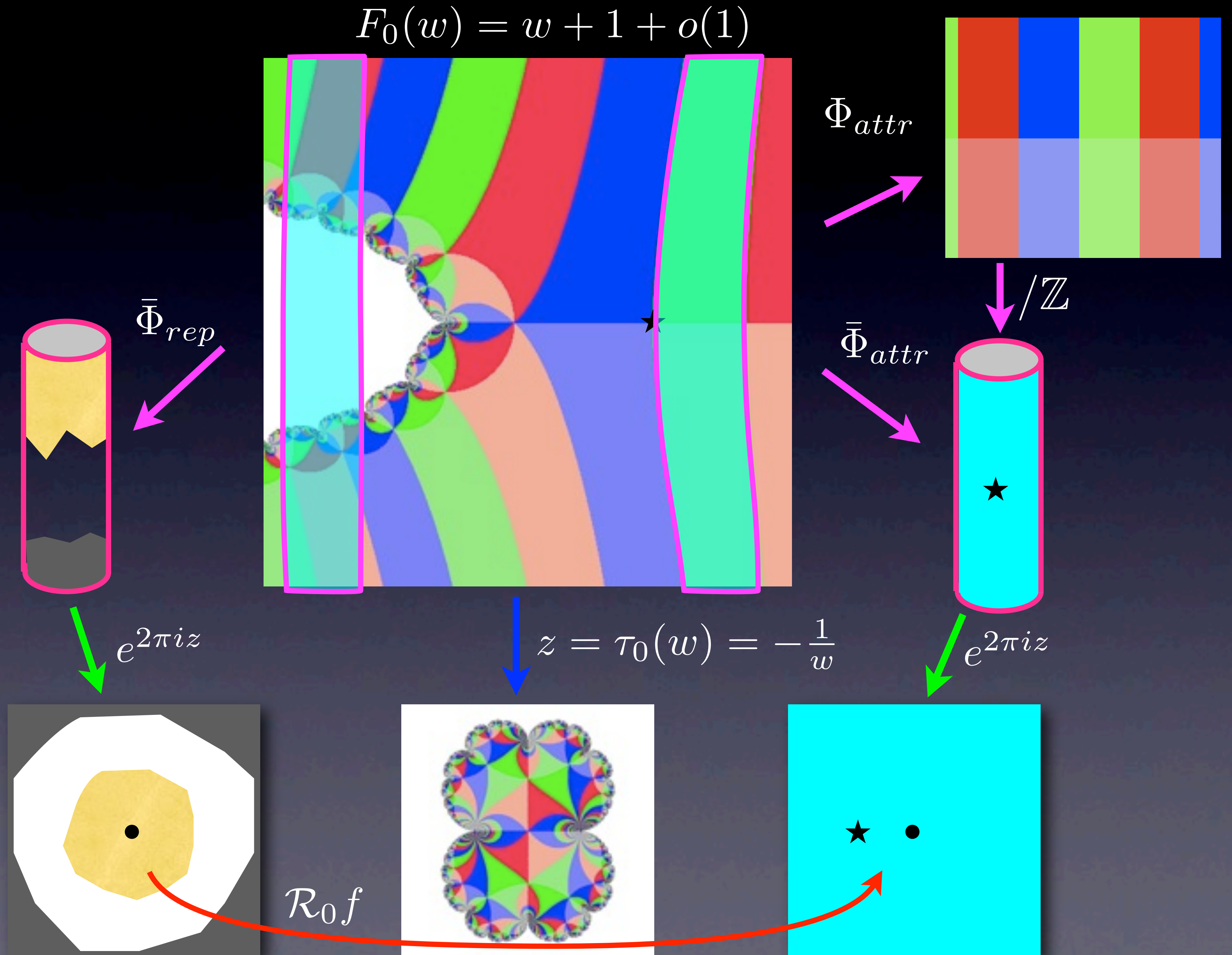
Idea of proof:

Why $P \circ \varphi^{-1}$?

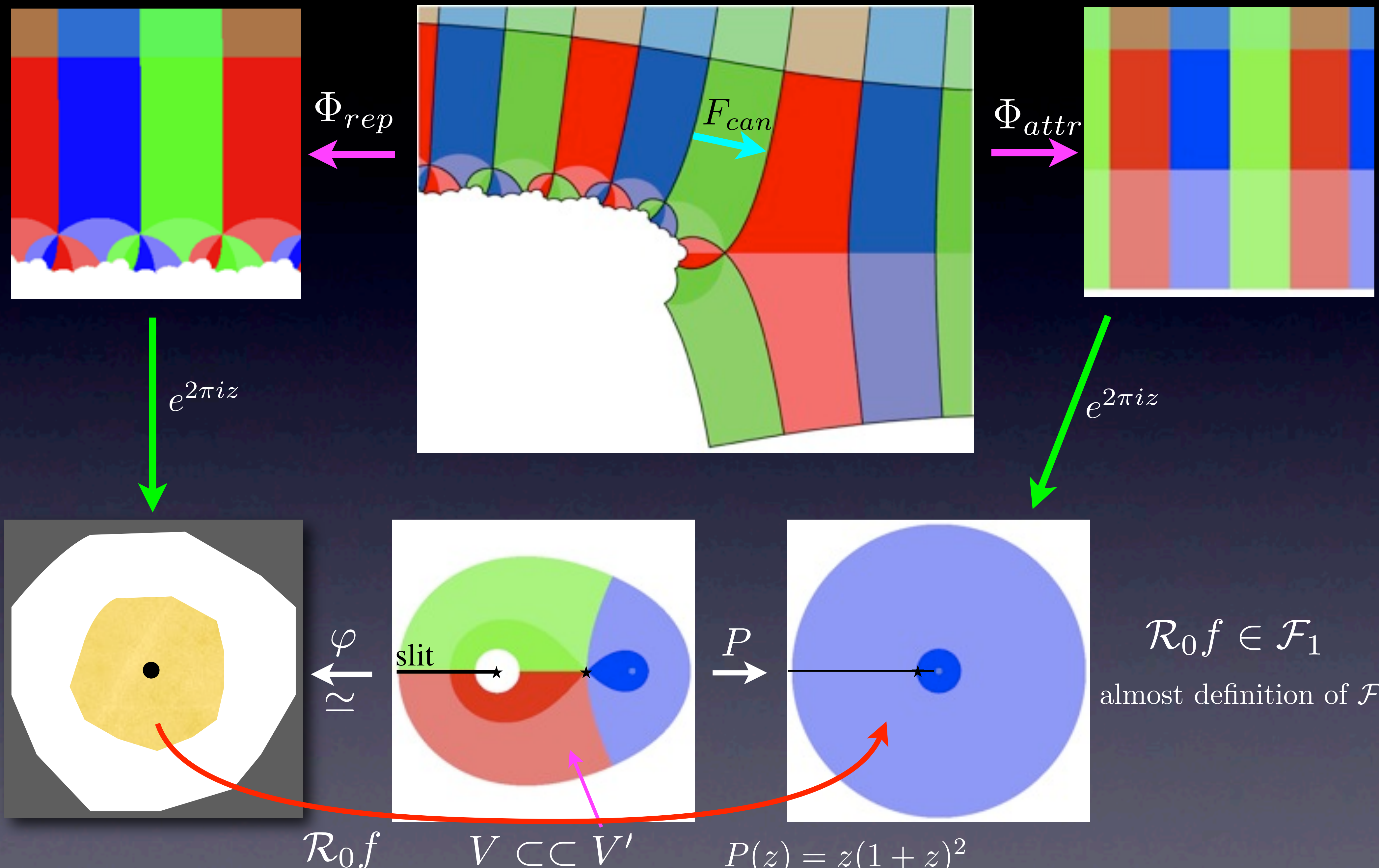
Since the construction of $\mathcal{R}_0 f$ involves the uniformization of the cylinder (a transcendental operation), we can't compute the derivatives of $\mathcal{R}_0 f$, etc. Try to characterize it by a (partial) covering property.

In order to justify the following arguments, one has to check many inequalities using Koebe distortion theorems and variants.

Basic checkerboard pattern for $f_0(z) = z + z^2$



Truncated pattern induces a cubic-like covering



$P \leftrightarrow$ pattern (universal) $\quad \varphi \leftrightarrow$ shape of domain (depends on f)

Proof of Contraction

How to prove that $\mathcal{R}_0 : \mathcal{F}_1 \to \mathcal{F}_1$ is a contraction. (The proof for $\mathcal{R}_\alpha$ is similar.)

Don't Recall the definition of $\mathcal{R}_0$

$f(z) = z + a_2 z^2 + \dots \quad (a_2 \neq 0)$

$\longrightarrow$ black box operation $\longrightarrow$ $\mathcal{R}_0 f(z) = z + \dots$

But we can't even compute $(\mathcal{R}_0 f)'(0)$, $(\mathcal{R}_0 f)''(0)$ etc.

Remember

$$\mathcal{F}_1 = \left\{ f = \underset{\text{fixed}}{P} \circ \underset{\text{varies}}{\varphi^{-1}} \;\middle|\; \begin{array}{c} \varphi : V \to \mathbb{C} \text{ is univalent (with qc extension)} \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}$$

$$\mathcal{F}_1 \ni f = P \circ \varphi^{-1} \xrightarrow{\quad \mathcal{R}_0 \quad} \mathcal{R}_0 f = P \circ \psi^{-1} \in \mathcal{F}_1$$

ψ extends conformally to a larger region V'

$\mathcal{R}_0$ is holomorphic: a holomorphic family φ_λ gives a holomorphic family ψ_λ on V'

Proof of Contraction: part 2

$W := \mathbb{C} \setminus V \ \ni \ W' := \mathbb{C} \setminus V'$ (both isomorphic to $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$)

Teichmüller space of *W*:

$Teich(W) := \left\{ \mu(z)\frac{d\bar{z}}{dz} \text{ on } W \right\} / \sim$ ("same boundary value" for qc map)

Teichmüller infinitesimal (Finsler) metric

$$||\mu||_{Teich} = \sup \left\{ \iint_W q(z)\mu(z)\,dx\,dy \;\middle|\; \begin{array}{c} q(z)dz^2 \text{ integrable holomorphic quadratic} \\ \text{differential with } \displaystyle\iint_W |q(z)|\,dx\,dy = 1 \end{array} \right\}$$

Identify $\mathcal{F}_1$ with $Teich(W)$

$$\mathcal{F}_1 \ni f = P \circ \varphi^{-1} \longrightarrow \varphi \longrightarrow \text{qc extension } \tilde{\varphi} \longrightarrow [\mu_{\tilde{\varphi}}|_W] \in Teich(W)$$

$$\begin{tikzcd}
\mathcal{F}_1 \ni f = P \circ \varphi^{-1} \arrow[rr, "\mathcal{R}_0"] \arrow[d, leftrightarrow] & & \mathcal{R}_0 f = P \circ \psi^{-1} \in \mathcal{F}_1 \arrow[d, leftrightarrow] \\
Teich(W) \ni [\mu_{\tilde{\varphi}}|_W] \arrow[dr] & & [\mu_{\tilde{\psi}}|_W] \in Teich(W) \\
& {[\mu_{\tilde{\psi}}|_{W'}] \in Teich(W')} \arrow[ur, "\Xi"'] &
\end{tikzcd}$$

non-expanding
by Royden-Gardiner

Contraction!

Proof of Contraction: part 3

$\Xi : Teich(W') \to Teich(W)$

$$\text{induced by } \mu \mapsto \mu' = \begin{cases} \mu & \text{on } W' \\ 0 & \text{on } W \setminus W' \end{cases}$$

$$||D_\mu \Xi||_{Teich} = \sup \left\{ \frac{\iint_{\varphi_\mu(W')} |q(z)| dx\, dy}{\iint_{\varphi_\mu(W)} |q(z)| dx\, dy} \,\middle|\, \begin{array}{l} q(z)dz^2 \text{ integrable holomorphic} \\ \text{quadratic differential on } \varphi_\mu(W) \end{array} \right\}$$

Claim: $||D_\mu \Xi||_{Teich} \leq \lambda := \exp(-2\pi \, mod(W \setminus W'))$.

follows from modulus-area-inequality and isoperimetric inequality for holomorphic quadratic differential on a punctured disk.

$$\implies d(\mathcal{R}_0 f, \mathcal{R}_0 g) \leq \lambda d(f, g)$$

Another Application of Teichmüller contraction: Rigidity

Theorem (Lyubich, Gracyk-Swiatek): Suppose that $f = f_c$ and $\hat{f} = f_{\hat{c}}$ ($c, \hat{c} \in [-2, \frac{1}{4}]$) are combinatorially equivalent (or topologically conjugate) and that they are infinitely renormalizable. Then f and $\hat{f}$ are quasi-symmetrically conjugate on their postcritical sets.

Consequences:

qs-conj. on their postcrit. set $\implies$ quasiconf-conj. on $\mathbb{C}$ $\implies$ conformally-conj on $\mathbb{C}$ $\implies$ $c = \hat{c}$

Hyperbolic parameters are dense among real quadratic polynomials.

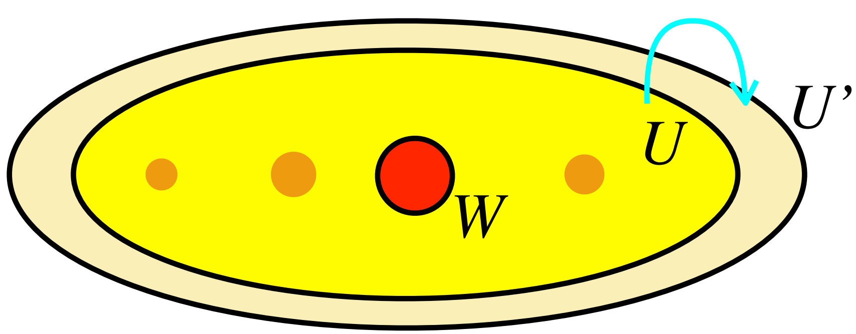
1st reduction to one step renormalization

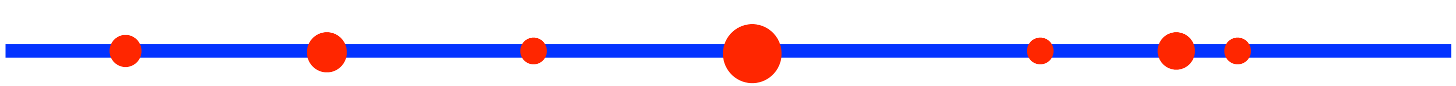
By the Complex Bounds (Levin-van Strien, Lyubich-Yampolsky, Graczk-Swiatek, Sands) the rigidity theorem reduces to:

Theorem: For any $m > 0$, there exists $K \geq 1$ such that if $f : U \to U'$ and $\hat{f} : \hat{U} \to \hat{U}'$ are real (symmetric) quadratic-like mapping with $mod(U' \setminus U), mod(\hat{U}' \setminus \hat{U}) \geq m$ and if they are (once) renormalizable with the same type and period > 2, then there exists a K-quasiconformal partial conjugacy (defined later).

Here K depends only on m and is independent of the combinatorics (e.g. period).

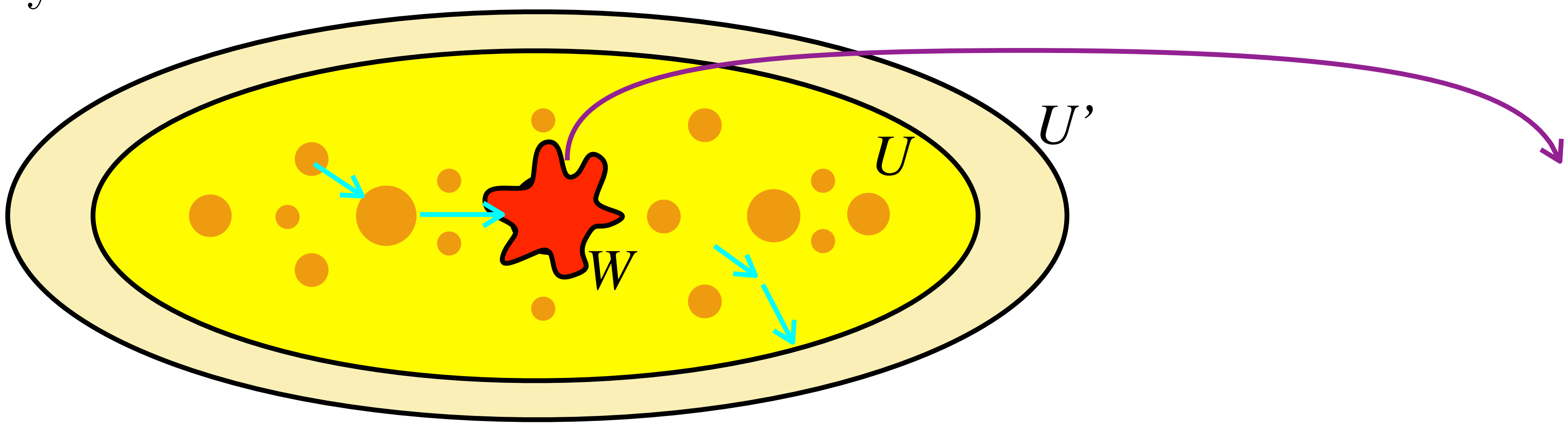
partial conjugacy: $\varphi : U' \to \hat{U}'$, such that $\varphi \circ f = \hat{f} \circ \varphi$ on $U \setminus W$, where W is a puzzle piece containg 0 such that $f^p : W \to f^p(W)$ is a renormalization.





2nd reduction to a critical piece

Theorem: $\forall m$, $\exists K$ for f and $\hat{f}$ as before, $\exists W, \hat{W}$ critical puzzle pieces for f and $\hat{f}$ such that
(a) $f^p : W \longrightarrow f^p(W)$ is a renormalization;
(b) $\exists \varphi : W \longrightarrow \hat{W}$ K-qc preserving the canonical marking on the boundary.



Given φ on W, first construct a map on each piece of a fixed level to its counter part preserving the marking. Then refine these maps by pulling-back these maps by the dynamics. There are three kind of points: eventually land on W, eventually land on $U' \setminus U$ and the rest. The 1st kind will have the same bound K as the above theorem; the 2nd kind also have a bound by $K_0(m)$; the 3rd kind has measure 0.

In general, the boundary of W can be very complicated!

Interpretation in the universal Teichmüller space

Take any qc-extension $\varphi_0 : W \to \hat{W}$ of the canonical marking. $(K(\varphi_0)$ large!)

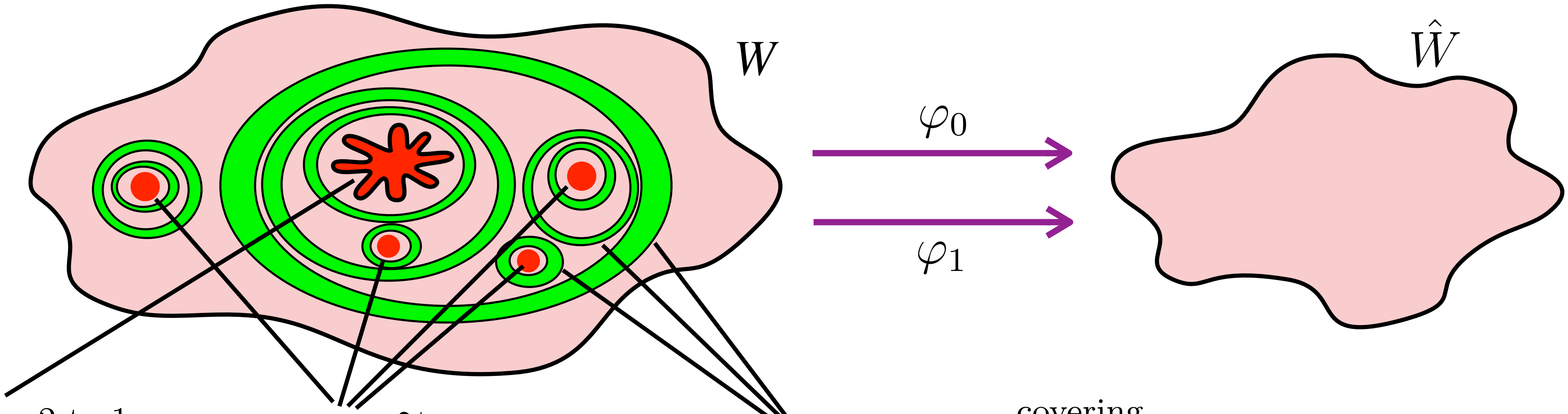
$$Teich(W) = \{\varphi : W \to \varphi(W) \text{ quasiconformal}\}/\sim$$

$$= \{\mu_\varphi = \frac{\bar\partial\varphi}{\partial\varphi} \text{ Beltrami differential}\}/\sim$$

$$\varphi \sim \psi \iff \exists h : \varphi(W) \to \psi(W) \text{ conformal such that } h = \psi \circ \varphi^{-1} \text{ on } \partial\varphi(W)$$

$$\boxed{\text{Want: } d([\varphi_0], [id]) = d([\mu_{\varphi_0}], [0]) \leq C(\text{depending only on } m).}$$

2nd refinement: pull-back construction within W



$V_0 \xrightarrow{\text{2 to 1}} W$ $\quad$ $V_i \xrightarrow{\simeq} W \ (i > 0)$ $\quad$ good annuli $\xrightarrow{\text{covering}} A_N(0)$

worse than $K(\varphi_0)$ $\quad$ $K(\varphi_0)$ $\quad$ $K_1(m)$ $\quad$ $K_1(m)$

escape to $U' \setminus U$ $\quad$ rest (measure 0)

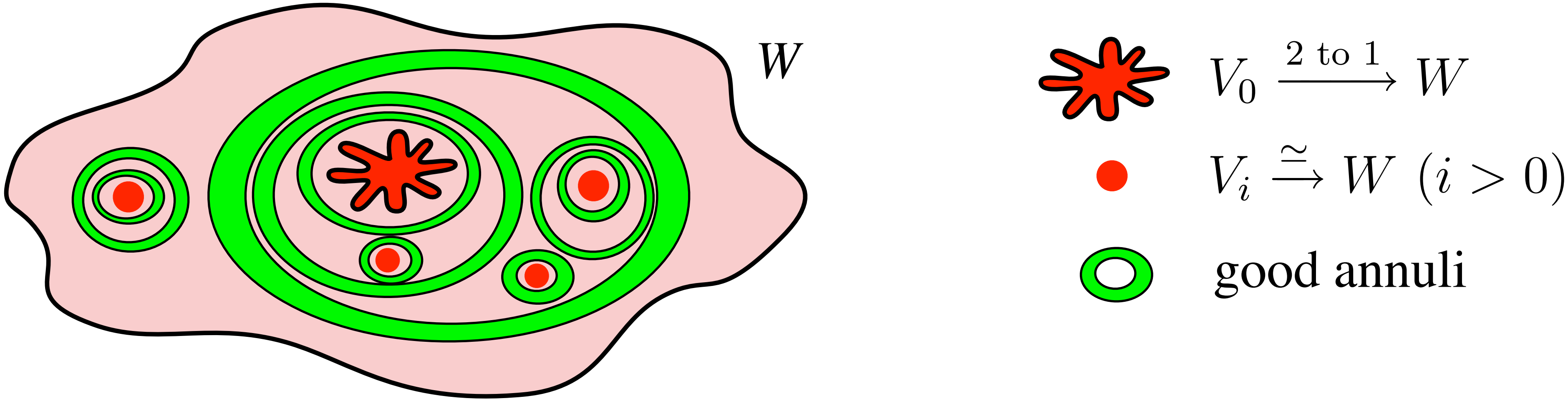
$K_0(m)$

For real maps, $N = 2$ and the base annulus $A^f_N(0)$ has a $K_1(m)$-qc map respecting the boundary marking. This induces $K_1(m)$-qc maps on good annuli.

By combinatorial a priori bound,

$$\sum_{A\,:\,\text{good annulus surrounding } V_i} mod(A) \geq m' = m'(m) \quad (i = 0, 1, \dots).$$

$[\varphi_0] = [\varphi_1]$ in $Teich(W)$



Define $\Theta : Teich(W) \to Teich(W)$ by

$$\Theta([\mu]) = \begin{cases} (f^{n_i})^*(\mu) & \text{on } V_i \quad (i > 0) \quad (f^{n_i} : V_i \xrightarrow{\simeq} W) \\ \mu_{\varphi_1} & \text{on the rest} \quad \text{(including } V_0) \end{cases}$$

Then Θ is well-defined and we have:

(a) $\Theta([\mu_{\varphi_0}]) = [\mu_{\varphi_1}] = [\mu_{\varphi_0}]$;

(b) $d(\Theta([0]), [0]) \leq C$(depending only on m);

(c) $d(\Theta([\varphi]), \Theta([\psi])) \leq \lambda d([\varphi], [\psi])$, where $\lambda < 1$ depends only on m.

$$||\nu||_{Teich} = \sup\left\{Re \iint q\nu : \iint |q| = 1\right\}$$

Modulus-area inequality holds for the area form defined by $|q|$.

Hence $d(\Theta([\varphi_0]), [0]) \leq \frac{C}{1-\lambda}$ (depending only on m).

Happy Birthday, Dennis!