Kleinian groups and the Sullivan dictionary III

Dick Canary

University of Michigan

June 3, 2011

Deformation Spaces of Hyperbolic 3-manifolds

- Let M be a compact, orientable, atoroidal 3-manifold with boundary.
- **Simplifying assumption:** The boundary ∂M of M contains no tori.
- Let $AH(M)$ denote the space of (conjugacy classes of) discrete faithful representations $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$.
- If $\rho \in AH(M)$, then

$$N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$$

is a hyperbolic 3-manifold and there exists a homotopy equivalence

$$h_\rho : M \to N_\rho$$

such that $(h_\rho)_* = \rho$.

- $$AH(M) \subset X(M) = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbb{C}))//\mathrm{PSL}_2(\mathbb{C})$$

The interior of $AH(M)$

- Conversely, given a homotopy equivalence $h : M \to N = \mathbb{H}^3/\Gamma$ from M to a hyperbolic 3-manifold, one obtains a discrete, faithful representation

$$\rho = h_* : \pi_1(M) \to \pi_1(N) = \Gamma \subset \mathrm{PSL}_2(\mathbb{C}).$$

- So, $AH(M)$ is the space of marked hyperbolic 3-manifolds homotopy equivalent to M.
- (Marden, Sullivan) The interior $\mathrm{int}(AH(M))$ of $AH(M)$ consists exactly of the convex cocompact representations, i.e. representations such that N_ρ (or $\rho(\pi_1(M))$) is convex cocompact, since $\rho \in AH(M)$ is structurally stable if and only if $\rho(\pi_1(M))$ is convex cocompact, and convex cocompact representations are quasiconformally stable.

Marked homeomorphism type

- Associated to a convex cocompact representation, there is a well-defined marked compact 3-manifold $(\hat{N}_\rho, h_\rho)$.
- Let $\mathcal{A}(M)$ denote the space of marked compact 3-manifolds homotopy equivalent to M, i.e. pairs (M', h') where M' is a compact 3-manifold and $h : M \to M'$ is a homotopy equivalence. We say two pairs (M_1, h_1) and (M_2, h_2) are equivalent if there exists an orientation-preserving homeomorphism $j : M_1 \to M_2$ such that j is homotopic to $h_2 \circ h_1^{-1}$.
- We define

$$\Theta : \mathrm{int}(AH(M)) \to \mathcal{A}(M)$$

by letting $\Theta(\rho) = (\hat{N}_\rho, h_\rho)$.

Components of $\mathrm{int}(AH(M))$

- (Thurston) Θ is surjective.
- **Marden's Isomorphism Theorem:** *If $\rho_1, \rho_2 \in \mathrm{int}(AH(M))$, then ρ_1 is quasiconformally conjugate to ρ_2 if and only if $\Theta(\rho_1) = \Theta(\rho_2)$.*
- **Idea of Proof:** If ρ_1 and ρ_2 are quasiconformally conjugate, then the quasiconformal conjugacy on $\widehat{\mathbb{C}}$ can be extended to a bilipschitz conjugacy on $\mathbb{H}^3$, so $\hat{N}_{\rho_1}$ and $\hat{N}_{\rho_2}$ are homeomorphic.

  On the other hand, if $\hat{N}_{\rho_1}$ and $\hat{N}_{\rho_2}$ are homeomorphic, one may upgrade the homeomorphism to a bilipschitz homeomorphism from N_{ρ_1} to N_{ρ_2}. This homeomorphism lifts to $\mathbb{H}^3$ and extends to a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ conjugating ρ_1 to ρ_2.

Parameterization of $\mathrm{int}(AH(M))$

So, components of $\mathrm{int}(AH(M))$ are in one-to-one correspondence with marked homeomorphism types in $\mathcal{A}(M)$ and each component is a quasiconformal deformation space. We parameterized quasiconformal deformation spaces in our first talk, so we obtain:

Parameterization Theorem: If $\pi_1(M)$ is freely indecomposable, then

$$\mathrm{int}(AH(M)) \cong \coprod_{(M',h')\in\mathcal{A}(M)} \mathcal{T}(\partial M').$$

(Canary-McCullough) In this setting, $\mathcal{A}(M)$ is finite, so $\mathrm{int}(AH(M))$ is homeomorphic to a finite union of open balls.

In general,

$$\mathrm{int}(AH(M)) \cong \coprod_{(M',h')\in\mathcal{A}(M)} \mathcal{T}(\partial M')/Mod_0(M').$$

(McCullough,Canary-McCullough) Typically, if $\pi_1(M)$ is freely decomposable, $\mathcal{A}(M)$ is infinite and $\mathrm{Mod}_0(M')$ is infinitely generated.

Recall that $\mathrm{Mod}_0(M')$ is the group of isotopy classes of orientation-preserving homeomorphisms of $\partial M'$ which extend to homemorphisms of M' which are homotopic to the identity.

(McCullough) $\mathrm{Mod}_0(M')$ is called the **twist group** since it it is generated by Dehn twists about compressible curves

Examples

- If $M = F \times I$, then $|\mathcal{A}(M)| = 1$, since every homotopy equivalence between $F \times I$ and a compact 3-manifold is homotopic to an orientation-preserving homeomorphism. Moreover,

$$\mathrm{int}(AH(F \times I)) = QF(F) \cong \mathcal{T}(F) \times \mathcal{T}(\bar{F})$$

- If M is acylindrical, then $|\mathcal{A}(M)| = 2$, since every homotopy equivalence from M to a compact 3-manifold is homotopic to a homeomorphism, and

$$\mathrm{int}(AH(M)) = \mathcal{T}(\partial M) \coprod \mathcal{T}(\partial \bar{M}).$$

- $|\mathcal{A}(M_n)| = (n-1)!$ (see blackboard.)

Introductory Bumponomics

- (Anderson-Canary) Any two components of $\mathrm{int}(AH(M_n))$ bump, i.e have intersecting closures.
- This implies that homeomorphism type is not locally constant on $AH(M)$.
- (Anderson-Canary-McCullough) If M has freely indecomposable fundamental group, one can characterize exactly which components of $\mathrm{int}(AH(M))$ bump. Roughly, two components bump if their marked homeo types differ by removing primitive solid torus components of the characteristic submanifold and reattaching the complementary pieces in a different order. (This operation is called a primitive shuffle.)
- Example: There exist manifolds M_n' such that $\mathrm{int}(AH(M_n'))$ has $(n-1)!$ components and no two components bump.

The bumping construction for M_n

- Form $\hat{M}_n$ by removing the core curve of the solid torus and let $\hat{N}$ be a hyperbolic 3-manifold homeomorphic to $\mathrm{int}(\hat{M}_n)$.
- If M_n' is homotopy equivalent to M_n, there exists an immersion $h : M_n' \to \hat{N}$ such that the cover of $\hat{N}$ associated to $h_*(\pi_1(M_n'))$ is homeomorphic to $\mathrm{int}(M_n')$.
- Let N_i be the result of hyperbolic $(1,i)$-Dehn surgery on $\hat{N}$ (topologically N_i is obtained by attaching a solid torus to $\hat{M}_n$ such that the meridian is glued to a $(1,i)$-curve.) Each $\hat{N}_i$ is homeomorphic to $\mathrm{int}(M_n)$. There is a natural associated map $r_i : \hat{N} \to N_i$.
- If $N_i = \mathbb{H}^3/\Gamma_i$ and $\hat{N} = \mathbb{H}^3/\Gamma_\infty$, then Γ_i converges to Γ_∞. Moreover, $(r_i)_* : \Gamma \to \Gamma_i$ converges to the identity map.
- If $\rho_i = (h \circ r_i)_*$ and $\rho = h_*$, then $\rho_i \to \rho$, but $N_{\rho_i} \cong \mathrm{int}(M_n)$ for all i and $N_\rho \cong \mathrm{int}(M_n')$.

Intermediate bumponomics

- (McMullen) $QF(F) = \mathrm{int}(AH(F))$ self-bumps, i.e. there exists a point $\rho \in \partial QF(F)$, such that any sufficiently small neighborhood of ρ in $AH(F \times I)$ has disconnected intersection with $QF(F)$.
- An embedded annulus A in M is a **primitive essential annulus** if $\pi_1(A)$ is a maximal abelian subgroup of $\pi_1(M)$ and A is not homotopic (rel boundary) into ∂M.
- (Bromberg-Holt) If M contains a primitive essential annulus, then every component of $\mathrm{int}(AH(M))$ self-bumps.
- (Bromberg, Magid) $AH(F \times I)$ is not locally connected.

Conjectural bumponomics

- **Conjecture:** (Bromberg) If M has non-empty boundary, then $AH(M)$ is not locally connected.
- In pictures of one-dimensional deformation spaces, e.g. Bers slices of punctured torus groups, the boundary appears to be quite fractal.
- **Conjecture:** (McMullen) The Hausdorff dimension of the boundary of a Bers slice of punctured torus groups lies strictly between 1 and 2.
- **Theorem:** (Shishikura) The boundary of the Mandelbrot set has Hausdorff dimension 2.
- **Problem:** Explore the fractal nature of $\partial AH(M)$ in general.
- **Problem:** Understand how the components of $\mathrm{int}(AH(M))$ bump when $\pi_1(M)$ is freely decomposable.

A cool argument

Theorem: (Thurston) If N is convex cocompact and $\partial_c N$ is non-empty, and N_0 is a cover of N with finitely generated fundamental group, then N_0 is convex cocompact.

Proof: If $N = \mathbb{H}^3/\Gamma$, then $N_0 = \mathbb{H}^3/\Gamma_0$ and $\Gamma_0 \subset \Gamma$.

Since N is convex cocompact, its convex core $C(N)$ is compact. Therefore, the diameter D of $C(N)$ is finite. It follows that if $z \in CH(\Lambda(\Gamma))$, then

$$d(z, \partial CH(\Lambda(\Gamma)) \leq D.$$

Since $\Gamma_0 \subset \Gamma$, $\Lambda(\Gamma_0) \subset \Lambda(\Gamma)$, so

$$CH(\Lambda(\Gamma_0)) \subset CH(\Lambda(\Gamma))$$

Therefore, if $z \in CH(\Lambda(\Gamma_0))$, then

$$d(z, \partial CH(\Lambda(\Gamma_0)) \leq D.$$

It follows that if $x \in C(N_0)$, then

$$d(x, \partial C(N_0)) \leq D.$$

Since Γ_0 contains no parabolics, Ahlfors' Finiteness Theorem implies that $\partial C(N_0)$ is compact.

It follows that $C(N_0)$ has bounded diameter, and since $C(N_0)$ is closed in N_0, that $C(N_0)$ is compact. Therefore, N_0 is convex cocompact as claimed.

If we combine this with Thurston's Geometrization Theorem, one obtains.

Corollary: (Thurston) If M is compact, irreducible and atoroidal and ∂M is empty, then every cover of M with finitely generated fundamental group is topologically tame.

The outer automorphism group

$$\mathrm{Out}(\pi_1(M)) = \mathrm{Aut}(\pi_1(M))/\mathrm{Inn}(\pi_1(M))$$

acts naturally on the character variety

$$X(M) = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbb{C}))//\mathrm{PSL}_2(\mathbb{C})$$

via

$$\alpha(\rho) = \rho \circ \alpha^{-1}$$

where $\rho \in X(M)$ and $\alpha \in \mathrm{Out}(\pi_1(M))$.

Proper discontinuity on $\mathrm{int}(AH(M))$

- $AH(M)$ is preserved by the action.
- If C is a component of $\mathrm{int}(AH(M))$ and $\Theta(C) = (M', h')$, then the stabilizer of C is identified with a subgroup of $\mathrm{Mod}(\partial M')/\mathrm{Mod}_0(M')$.
- Since $C \cong \mathcal{T}(\partial M')/\mathrm{Mod}_0(M')$ and $\mathrm{Mod}(\partial M')$ acts properly discontinuously on $\mathcal{T}(\partial M')$, $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $\mathrm{int}(AH(M))$.
- **Question:** Is $\mathrm{int}(AH(M))$ a maximal domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ on $X(M)$?

Two extreme cases

- (Johannson) $\mathrm{Out}(\pi_1(M))$ is finite if and only if M is acylindrical.
- So, $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on all of $X(M)$ if and only if M is acylindrical.
- **Conjecture:**(Goldman) If $M = F \times I$, then $QF(F) = \mathrm{int}(AH(F \times I))$ is a maximal domain of discontinuity for the action of $\mathrm{Out}(\pi_1(F))$. Moreover, $\mathrm{Out}(\pi_1(F))$ acts ergodically on $X(F \times I) - QF(F)$.
- (Minsky,Lee) Every point in $\partial QF(F)$ is a limit of fixed points of infinite order elements in $\mathrm{Out}(\pi_1(F))$, so cannot lie in any domain of discontinuity for $\mathrm{Out}(\pi_1(F))$. The proof relies on
- (McMullen, Canary-Culler-Hersonsky-Shalen) There is a dense set of representation in $\partial QF(F)$ whose images contain parabolic elements, i.e. cusps are dense in the boundary of quasifuchsian space. This holds for any M.

Typically one can find a bigger domain of discontinuity

- (Minsky) If H_g is the handlebody of genus g, then there is a domain of discontinuity for $\mathrm{Out}(\pi_1(H_g)) = \mathrm{Out}(\pi_1(F_g))$ which contains both $\mathrm{int}(AH(H_g))$ and points in $\partial AH(H_g)$.
- (Canary-Storm) If $\pi_1(M)$ is freely indecomposable, but is not an interval bundle, then there is a domain of discontinuity for $\mathrm{Out}(\pi_1(M))$ which contains both $\mathrm{int}(AH(M))$ and points in $\partial AH(M)$.
- (Lee) If M is a twisted interval bundle, then there is a domain of discontinuity for $\mathrm{Out}(\pi_1(M))$ which contains both $\mathrm{int}(AH(M))$ and points in $\partial AH(M)$.

Is there a domain of discontinuity which contains $AH(M)$?

- (Canary-Storm) $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on some open neighborhood of $AH(M)$ in $X(M)$ if and only if M contains no primitive essential annuli.
- There exist compact, atoroidal, irreducible 3-manifolds such that M contains no primitive essential annuli, yet M is not acylindrical. If time permits, an example will appear on the blackboard.

Happy Birthday Dennis!