Lattès Maps and Combinatorial Expansion

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where $\bar{A}$ is a map of a torus $\mathcal{T}$ that is a quotient of an affine map of the complex plane, and Θ is a finite-to-one holomorphic map.

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Some Properties of Lattès Maps

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- measure of maximal entropy is absolutely continuous w.r.t. Lebesgue measure (Zdunik '90)
- only rational maps that admit an "invariant line field" on their Julia set (Conjecture)

Review: Expanding Thurston maps

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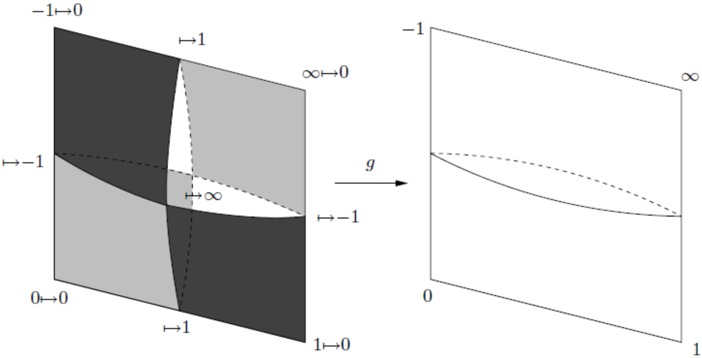
Example: Lattès maps are expanding Thurston maps.

Review: Expanding Thurston maps



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0-level cell decomposition: $\mathrm{post}(f) =$ 0-vertices,
closure of connected components of $\mathcal{C} \setminus \mathrm{post}(f) =$ 0-edges
closure of connected components of $\mathbb{S}^2 \setminus \mathcal{C} =$ 0-tiles

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f^{-1}(0-edges) $=$ 1-edges, f^{-1}(0-tiles) $=$ 1-tiles

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1-level cell decomposition: $f^{-1}(\operatorname{post}(f)) =$ 1-vertices,
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$\vdots$

n-level cell decomposition: $f^{-n}(\operatorname{post}(f)) = n$-vertices,
f^{-n}(0-edges) $= n$-edges, f^{-n}(0-tiles) $= n$-tiles

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Let $D_n = D_n(f, \mathcal{C})$ be the minimum number of n-tiles needed to join two non-adjacent 0-edges.

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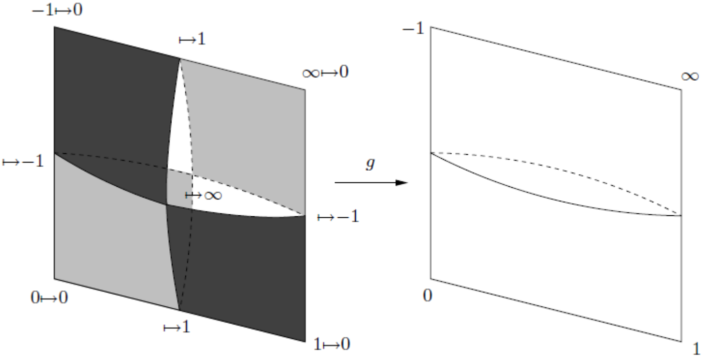
Proposition (Y '11)

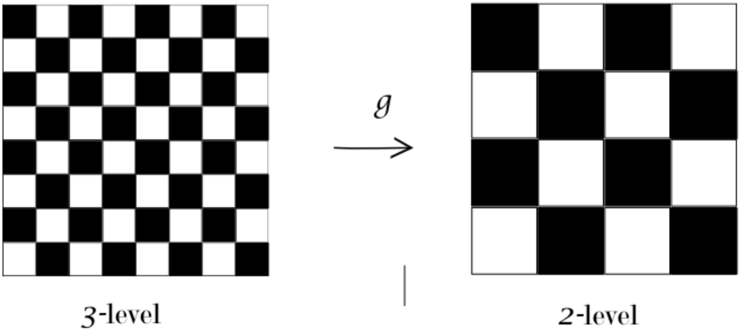
Let f be a Thurston map without periodic critical points and let $\mathcal{C} \supseteq \mathrm{post}(f)$ be a Jordan curve. Then there exists a constant $C > 0$ such that

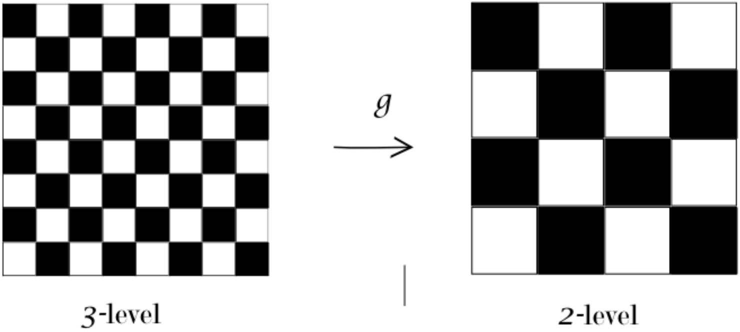
$$D_n = D_n(f, \mathcal{C}) \leq C \deg(f)^{n/2}$$

for all $n \geq 0$.

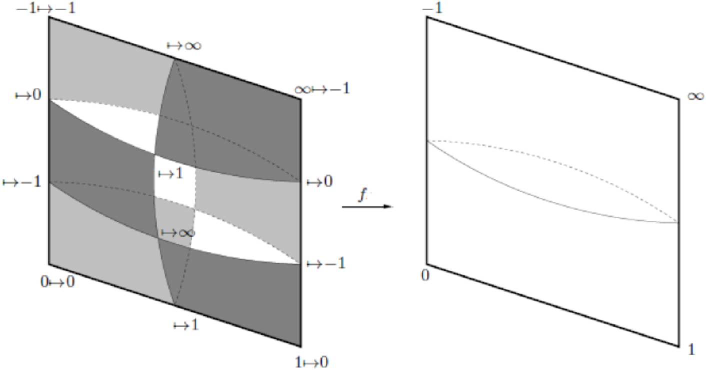


$-1 \mapsto 0$
$\mapsto 1$
$\infty \mapsto 0$
$\mapsto -1$
$\mapsto \infty$
$\mapsto -1$
$0 \mapsto 0$
$\mapsto 1$
$1 \mapsto 0$
g
-1
∞
0
1

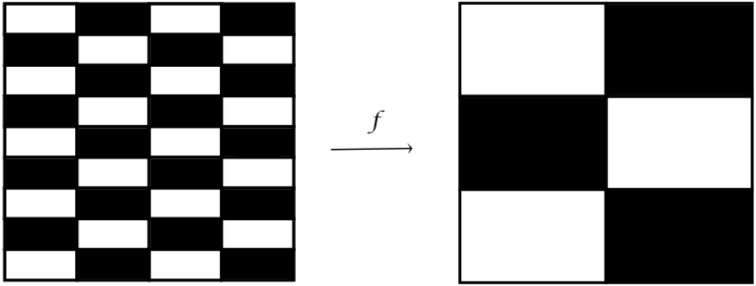

g
3-level
2-level

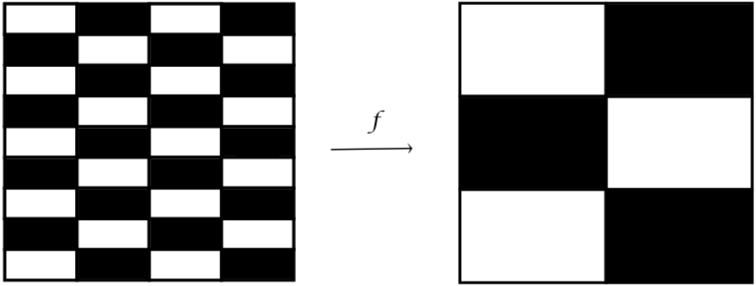


$$D_n = 2^n = (\deg g)^{n/2}$$



−1↦−1
↦∞
∞↦−1
↦0
↦−1
↦1
↦0
↦∞
↦−1
0↦0
↦1
1↦0
f
−1
∞
0
1


f



$$D_n = 2^n < 6^{n/2} = (\deg f)^{n/2}$$

Main Theorem

Theorem (Y '11)

A map $f : \mathbb{S}^2 \to \mathbb{S}^2$ is topologically conjugate to a Lattès map ***iff*** *the following conditions hold:*

- *f is an expanding Thurston map;*
- *f has no periodic critical points;*
- *there exists $c > 0$ such that $D_n \geq c(\deg f)^{n/2}$ for all $n > 0$.*

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- the Hausdorff measure w.r.t. d is absolutely continuous with respect to the Lebesgue measure (using Heinonen-Koskela '98)
- R is a Lattès map (using Zdunik '90, Meyer '09)

Thank you!