SHRINKWRAPPING and the TAMING of HYPERBOLIC 3-MANIFOLDS

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J.H.C.
Whitehead (1935)

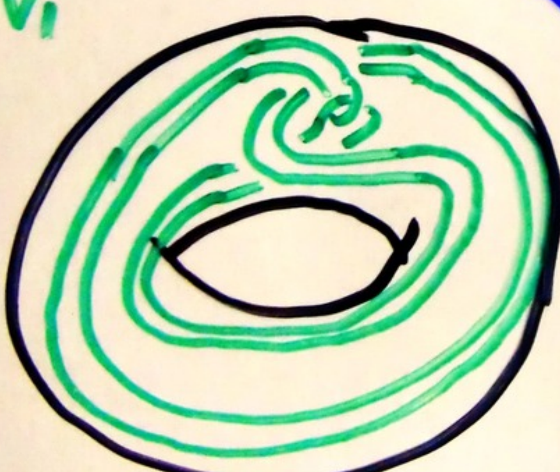
An open contractible 3-manifold $\not\approx \mathbb{R}^3$

$W = \bigcup V_i \qquad V_1 \subset V_2 \subset V_3 \subset \cdots$

$V_i \approx D^2 \times S^1$

V_1

$V_1 \subset V_2$



$V_1 \subset V_3$

$\cdots$

Fact $\pi_1(W - V_1)$ is ∞'ly generated

Proposition If C smooth compact cod-0 submanifold of $\mathbb{R}^3$, then $\pi_1(\mathbb{R}^3 - \mathring{C})$ is finitely generated.

Proof Let $B \subset \mathbb{R}^3$ large 3-ball with $C \subset \mathring{B}$. Then $B - \mathring{C}$ is a compact manifold.

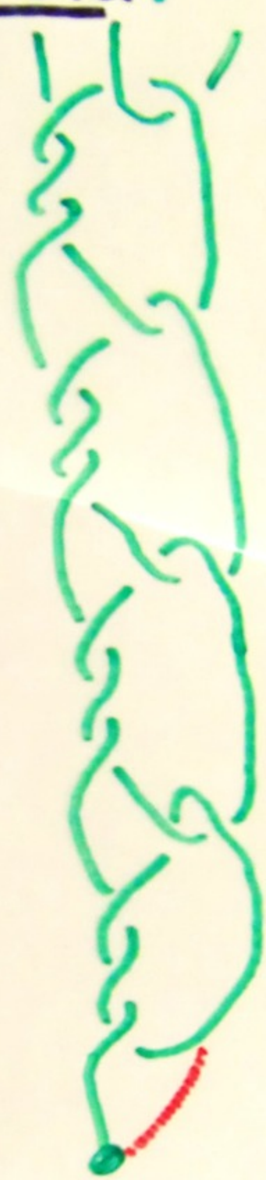
$$\therefore \pi_1(\mathbb{R}^3 - \mathring{C}) = \pi_1(B - \mathring{C})$$

is finitely generated.

Fox Artin Manifold:

$X = \mathbb{R}^3 - \mathring{N}(K)$

K is a properly embedded ray in $\mathbb{R}^3$



K

<u>Fact</u> $\mathring{X} \approx \mathbb{R}^3$

$\partial X \approx \mathbb{R}^2$

$(X, \partial X) \not\approx (\mathbb{R}^3, \mathbb{R}^2)_{Std}$

$= \{(x,y,z) \mid z \geqslant 0\}$

<u>Def</u> Let M be a ∂-manifold, possibly $\partial M \neq \phi$.

M is <u>tame</u> if $\exists$ compact manifold X and a proper embedding

$$i : M \hookrightarrow X$$

with $X = \overline{i(M)}$

So $M \approx X - Y$ where $Y \subset \partial X$ and Y compact

If $\partial M = \phi$ then $M \approx M_1 \cup S \times [0,\infty)$ $\quad$ M_1 compact

$\partial M_1 \sim S \times 0$ $\quad$ S compact surface

Theorem (Agol, Calegari-G 2004)

If N is a complete hyperbolic 3-manifold and $\pi_1(N)$ is finitely generated then N is geometrically and topologically tame.

This gives a positive proof of Marden's tameness conjecture.

Theorem (Canary 1990)

Topologically TAME $\Rightarrow$ Geom. Tame

Partial Results

Marden - N Geom. finite

Thurston - Alg limits of some q. Fuchsian groups

Bonahon - $\pi_1(N)$ Freely indec.

Souto - "Nice" exhaustion

Canary	Bromberg
Minsky	Long
Kleineidam	Reid
Evans	Souto
Ohshika	Myers
Brock	Brin
	Thickston

References

Agol, "Tameness of hyperbolic 3-manifolds"

Calegari–G, "Shrinkwrapping and the taming of hyperbolic 3-manifolds"

T. Soma, "Existence of polygonal wrappings in hyperbolic 3-manifolds"

S. Choi, "The PL methods for hyperbolic 3-manifolds to prove tameness"

B. Bowditch, "Notes on Tameness"

G, "Hyperbolic Geom. & 3-Manifold Top"

If N Complete, hyperbolic then

1) $\pi_1(N) = 1 \Rightarrow N \cong \mathbb{H}^3$

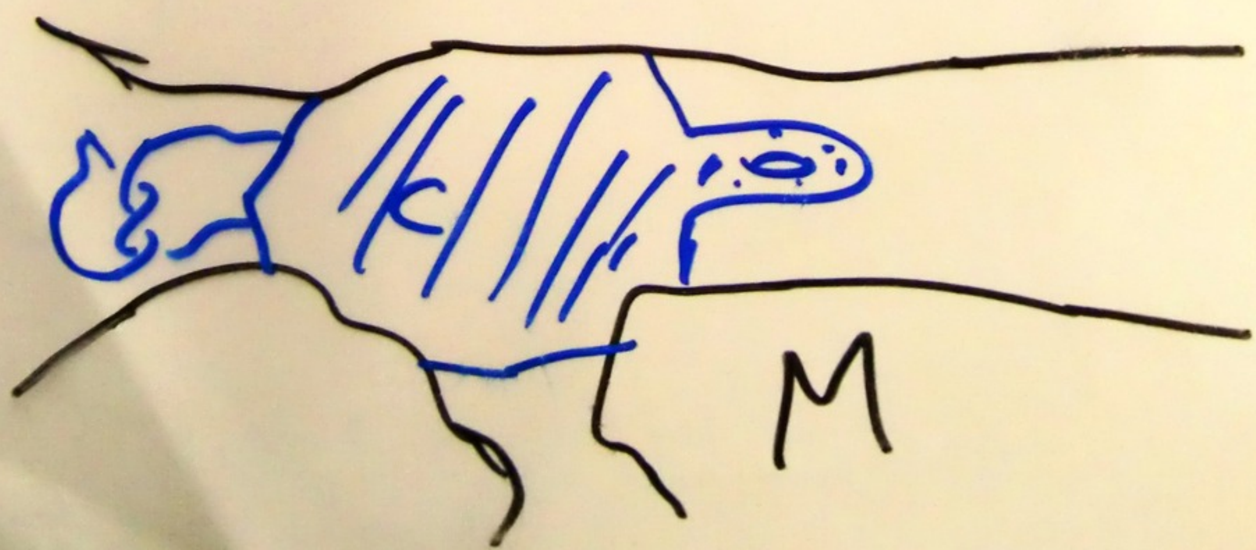
2) $\pi_1(N) = \mathbb{Z} \Rightarrow N \cong \mathbb{H}^3/\langle g \rangle$

$\approx \mathring{D}^2 \times S^1$

Scott Core theorem (1974)

If M irreducible 3-manifold
(every S^2 bounds a B^3)
connected, $\pi_1(M)$ finitely gen.
then $\exists$ Compact submanifold
C s.t. $C \hookrightarrow M$
homotopy equivalence

Fact Ends $(M) \xleftrightarrow{1-1}$ Comps. of ∂C



Thick Thin Decomposition

(parabolic free version)

If N complete, parabolic free hyperbolic 3-manifold then $\exists \epsilon > 0$ s.t.

$$N = N_{(0,\epsilon]} \cup N_{[\epsilon,\infty)}$$

where $N_{[\epsilon,\infty)} = \{x \in N \mid \text{inj rad}(x) \geq \epsilon\}$

$$N_{(0,\epsilon]} = \overline{\{x \in N \mid \text{inj rad}(x) < \epsilon\}}$$

and $N_{(0,\epsilon]}$ = Solid geodesic tubes about short geods.

(Margulis tubes)

Geometrically finite end

$S \times [0, \infty)$

Cross sectional area increases exponentially in $t \in [0, \infty)$

Example $T =$  with hyp metric

$T = \mathbb{H}^2/\Gamma \qquad \Gamma \subset \mathrm{Isom}(\mathbb{H}^2)$

$\subset \mathrm{Isom}(\mathbb{H}^3)$

$N = \mathbb{H}^3/\Gamma \approx$  $\times \mathbb{R}$

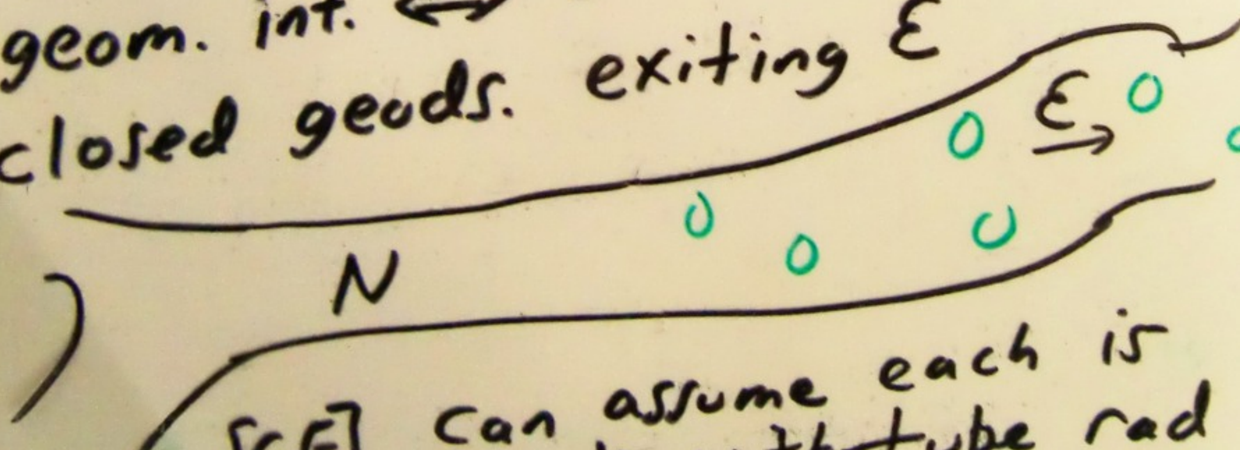
Geom. Finite ends do not satisfy tameness criterian!

Geometrically infinite

– everything else –

Example N fibers over S^1 with fiber S, $\hat{N} = \infty$-cyclic cover to S ends $\hat{N}$ geom. infinite.

Fact (Bonahon) End $\mathcal{E}$ is geom. inf. $\Leftrightarrow$ $\exists$ sequence of closed geods. exiting $\mathcal{E}$



[CG] can assume each is simple with tube rad $\geq h \geq 0.025$

[GMT] $h \geq \log(3)/2$

8.12

Bounded Diameter Lemma

Let S be a cat(-1) surface in N s.t. essential curves of $\text{length}_S \leq \delta$ are essential in N. Then $\exists C$ (depending on $\chi(S), \delta$) s.t. $\text{diam}_N(S) \leq C$ modulo margulis tubes

i.e. if $x, y \in S - N_{(0,\epsilon]}$ then $\exists$ path α from x to y s.t. $\text{length}_N(\alpha - N_{(0,\epsilon]}) \leq C$.

Idea of Proof

Assume S closed simplicial hyp.

$\delta \leq \epsilon$

If $\operatorname{injrad}_S(x) \leq \delta/2 \Rightarrow \operatorname{injrad}_N(x) \leq \frac{\delta}{2} \leq \frac{\epsilon}{2}$

$$\Longrightarrow x \in N_{(0,\epsilon]} \qquad \text{i)}$$

Since S intrinsically ≤ -1 curved, if D an embedded disc in S of radius $\delta/2$ then

$$\operatorname{Area}_S(D) > \pi\left(\frac{\delta}{2}\right)^2 \qquad \text{ii)}$$

Gauss Bonnet $\Rightarrow \operatorname{Area}(S) \leq 2\pi|\chi(S)|$ iii)

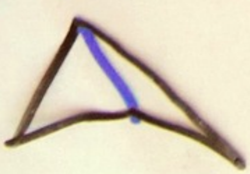
i), ii), iii) $\Longrightarrow S - N_{(0,\epsilon]}$ can be covered by $\frac{16|\chi(S)|}{\delta^2}$ $\frac{\delta}{2}$ balls. $\square$

Examples of Cat(-1) surfaces in a hyperbolic 3-manifold N

① Minimal surface
(i.e. mean curv. = 0)

② Pleated surface – geodesically embedded, except bent along geodesics

③ Simplicial hyperbolic surface
Triangulated surface S
mapping $f : S \longrightarrow N$ s.t.
a) $f|$ 2-simplex totally geod.
b) f has cone angle $\geq 2\pi$ at each vertex.



Theorem (Thurston's Tameness Theorem)

Let W a compact, $\chi(W)<0$, irreducible, atoroidal 3-manifold, $\partial W \neq \phi$. If $\hat{W} \longrightarrow W$ a cover s.t. $\pi_1(\hat{W})$ finitely generated then $\text{int}(\hat{W})$ is tame.

Proof Thurston's hyperbolization Thm $\Rightarrow$ $W = \mathbb{H}^3/\Gamma$ where Γ is geom. Finite. $\Rightarrow \pi_1(\hat{W}) \subset \Gamma$ is geom. Finite $\square$

see Morgan's paper in Smith Conj Volume

OR Canary "Ends of hyp M^3's" 5.4

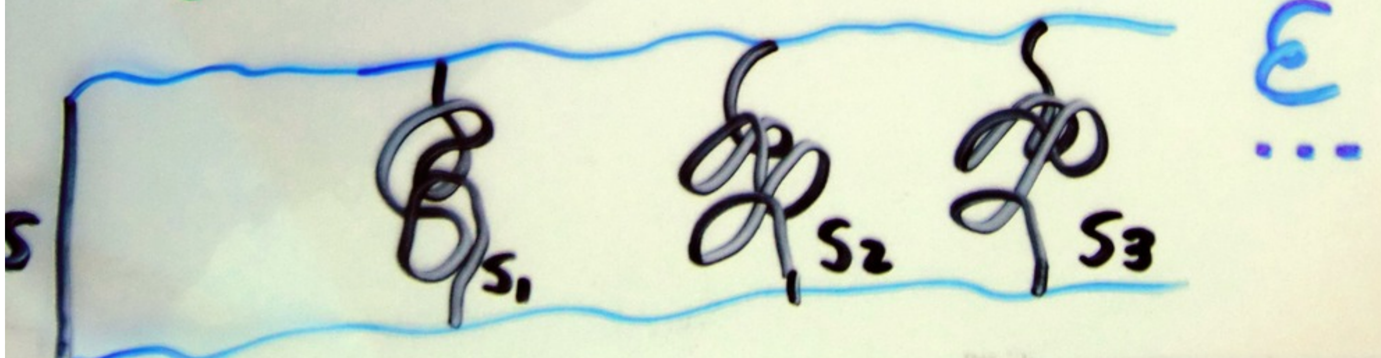
Thurston's Tameness Criterion

$\mathcal{E} \approx S \times [0,\infty)$ if

1) $\mathcal{E}$ h.e to $S \subset \partial\mathcal{E}$

2) $\exists\ S_1, S_2, \dots$ pleated surfaces s.t. S_i's exit $\mathcal{E}$ and $S_i \simeq S$.

Such an end is called geometrically tame as is a geometrically finite end



Idea of proof

Step 1 USE surface interpolation to find a proper map

$$f : S \times [0, \infty) \longrightarrow \mathcal{E}$$

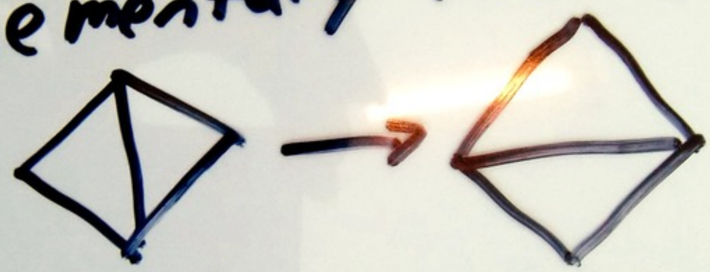
with $f(S \times 0) = S$

Step 2 Show that $f \simeq$ homeo.

Fact (Thurston) If $\exists$ simple closed curves $\gamma_1, \gamma_2, \cdots \subset S$ s.t. $\gamma_i \simeq \gamma_i^*$ geods in $\mathcal{E}$, via homotopy in $\mathcal{E}$, and γ_i^* exit, then $\mathcal{E}$ is geom. tame.

1978 Notes - Chapter 9

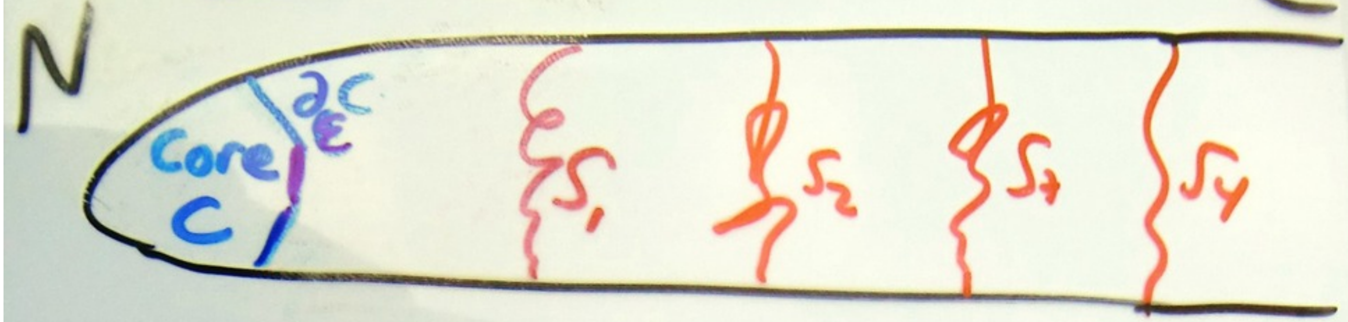
"2) An alternative approach to the construction of F is to make use of a seq. of triangulations of S. Any two ... joined by seq of elementary moves as shown



Although such an approach involves more familiar methods, the author brutally chose to develop extra structure"

W. Thurston

Taming Criterion



① $S_1, S_2, \dots \longrightarrow \mathcal{E}$ $\quad \text{genus}(S_i) \le \text{genus}(\partial_{\mathcal{E}} C)$

② S_i $\quad$ Cat(-1)

③ $[S_i] = [\partial_{\mathcal{E}} C] \in H_2(N - \mathring{C})$

Alg top Exercises (consequences of ① and ③)

① $S_i \hookrightarrow N - \mathring{C}$, $\partial_{\mathcal{E}} C \longrightarrow N - \mathring{C}$

are H_1-surjective

② $\text{genus}(S_i) = \text{genus}(\partial_{\mathcal{E}} C)$

③ S_i is π_1-injective (into $N - \mathring{C}$) on simple closed curves

④ $S_i \hookrightarrow N - \mathring{C}$ is H_1-injective.

2.1

Theorem (Dick Canary)

$\mathcal{E}$ end of N

$\mathcal{E}$ (top) tame, then

$\mathcal{E}$ geom finite or

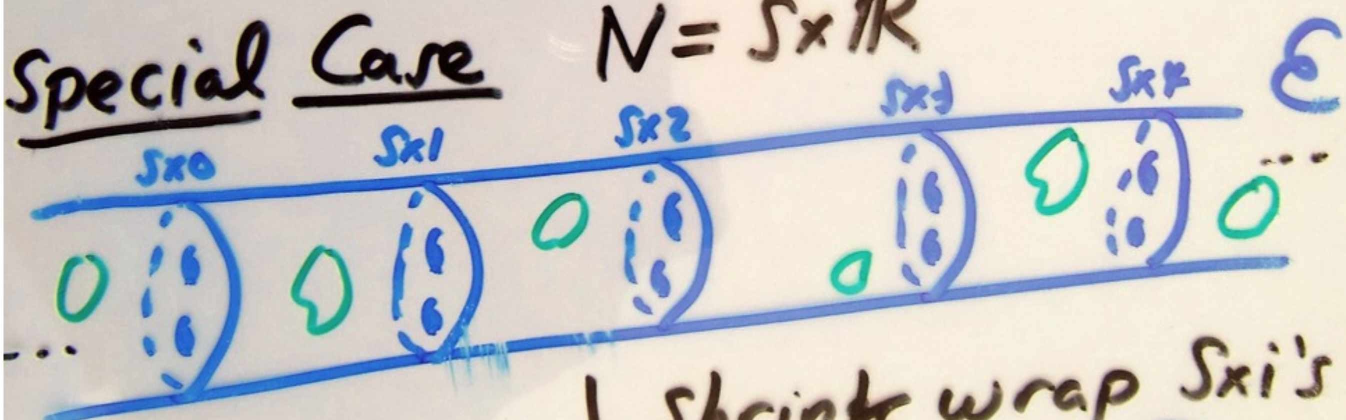
$\mathcal{E}$ satisfies taming criterian.

(originally stated differently)

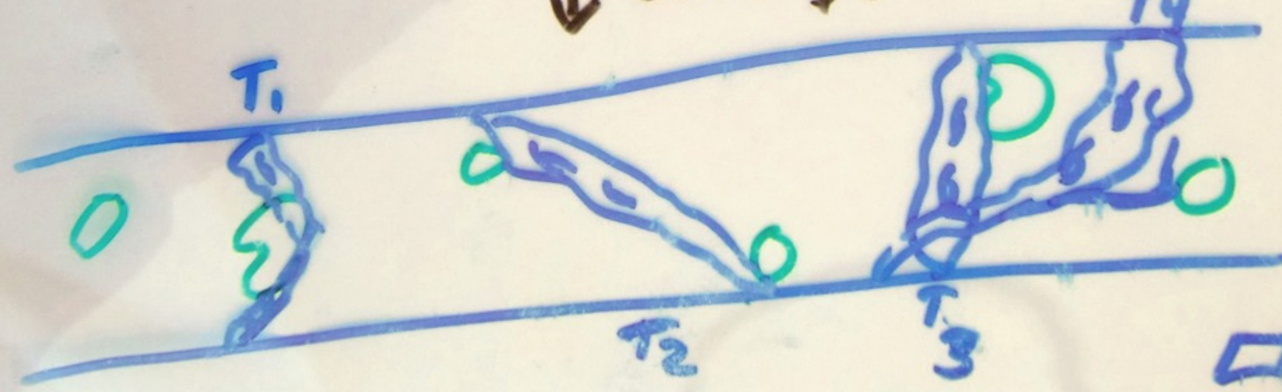
Proof If $\mathcal{E}$ geom. infinite

let $\delta_1, \delta_2, \ldots$ simple geods exiting $\mathcal{E}$.

Special Case $N = S \times \mathbb{R}$



$\downarrow$ shrink wrap $S\times i$'s



$\square$ Y.M.

Shrinkwrapping (Calegari_G)

A method for finding Cat(-1) surfaces in hyperbolic M^3's.

"Smooth" version [CG]

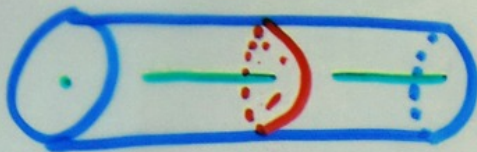
"wrapped" version - Soma

soma + ϵ = PL version

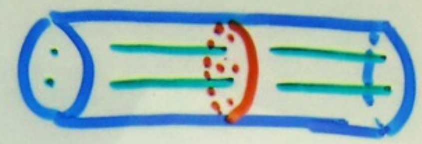
DEF Let Δ collection of geodesics in N.

The surface T is 2-inc. rel Δ if each essential compressing disc D has

$$|D \cap \Delta| \geq 2$$



1-Compression



2-Compression

Compression

Theorem (shrinkwrapping)

Given embedded $S \subset N^3_{hyp}$

S closed, Δ locally finite

geods in N s.t.

i) S separates Δ, $S \cap \Delta = \phi$

ii) S is 2-inc. rel Δ

then $\exists\, F : S \times [0,1] \longrightarrow N$

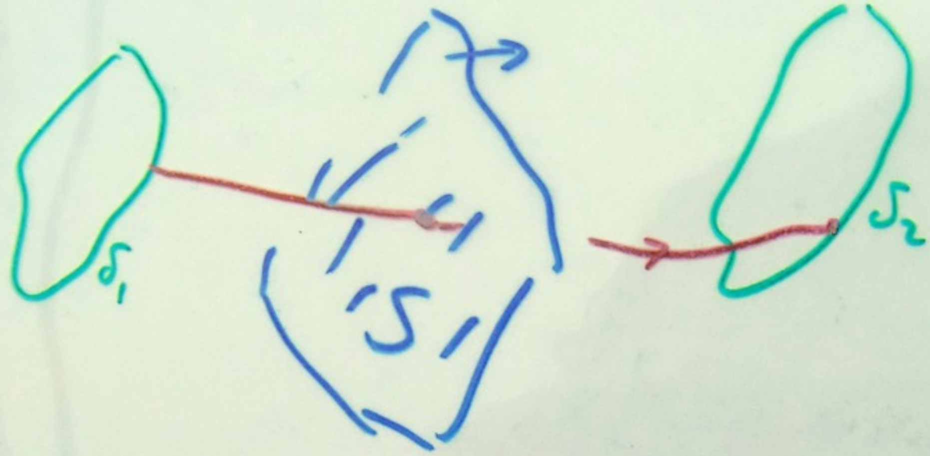
$F|S \times 0 = S$

$F|S \times 1$ cat(-1)

$F|S \times [0,1) \cap \Delta = \phi$

3.175

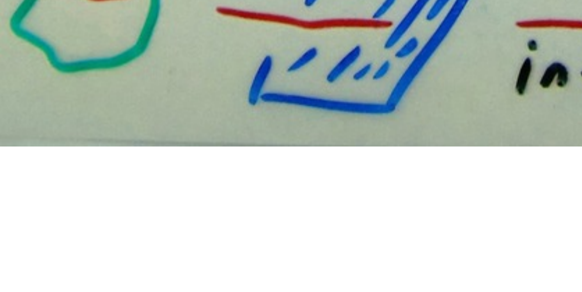
Geodesics Trap Surfaces



If S separates δ_1 from δ_2 $\Rightarrow$
$\langle S, \alpha \rangle = 1$ where α path from δ_1 to δ_2

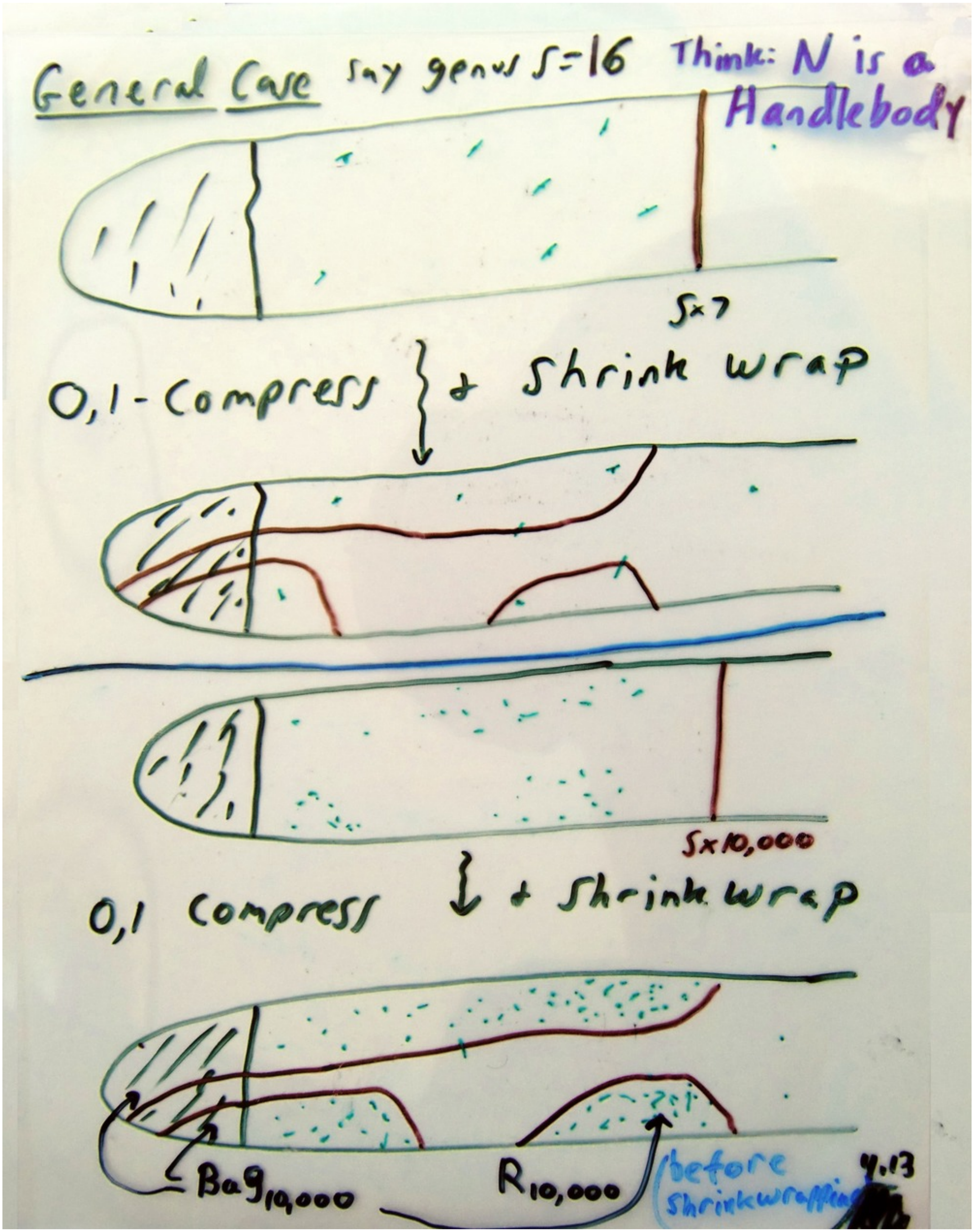
$\Rightarrow S' \cap \alpha \neq \emptyset$, where $S' =$ shrinkwrapped S

$\Rightarrow \max\{ d(x,\alpha) \mid x \in S' \}$ unit bounded modulo $N_{\leq \epsilon}$



Similarly for infinite Rays

4.11


General Case say genus S=16
Think: N is a Handlebody
S×7
0,1-Compress
+ Shrink wrap
S×10,000
0,1 Compress
+ Shrink wrap
Bag$_{10,000}$
R$_{10,000}$
(before shrinkwrapping

Special Case R_i connected all i.

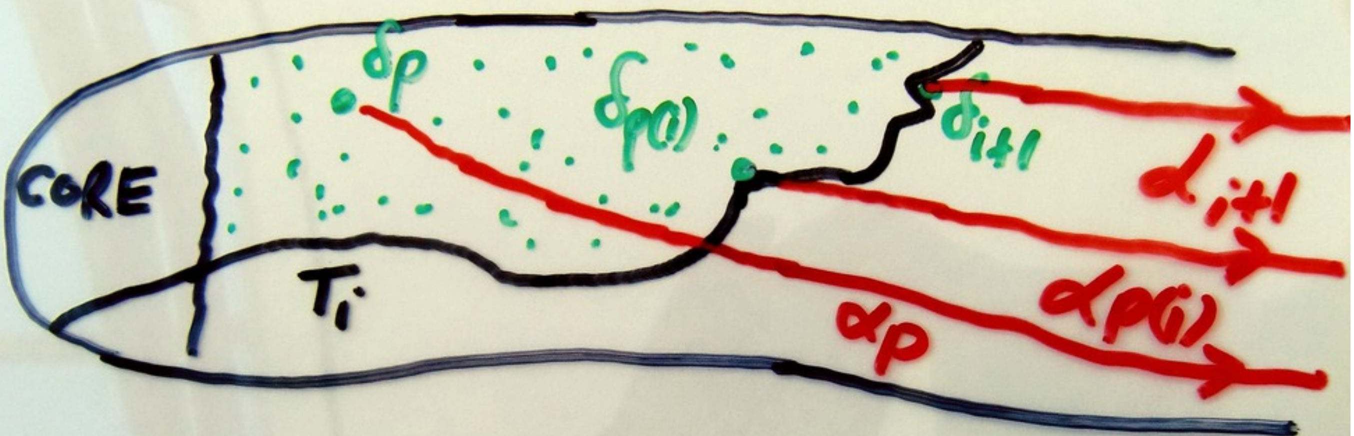
Proof After passing to subseq. suppose $\delta_p \in Bag_i$ $i \gg 0$

Let $\delta_{p(i)}$ "furthest out" $\delta_j \in Bag_i$

Let α_i $i=1,2,\cdots$ locally finite rays from δ_i's to ∞.

$R_i \xrightarrow{\text{Shrinkwrap}} T_i$

(Shrinkwrap R_i w.r.t. δ_j's in Bag_i <u>and</u> δ_{i+1}).



geodesics trap surfaces

$\Rightarrow \alpha_{p(i)} \cap T_i \neq \phi$

Bounded diameter Lemma

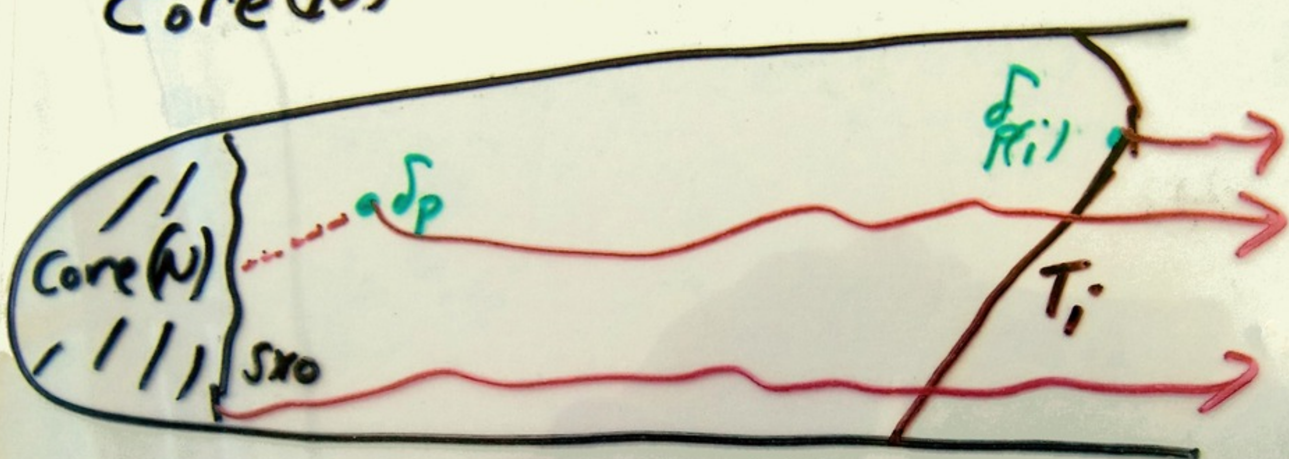
$\Rightarrow$ dist $(\alpha_{p(i)}, T_i)$ unif bounded modulo margulis tubes

$\Rightarrow$ T_i's exit $\mathcal{E}$ as $i \to \infty$

$\langle \alpha_p, R_i \rangle = 1 \implies$

$\langle \alpha_p, T_i \rangle = 1 \implies$

T_i homologically separates core(N) from ∞



$\implies p : T_i \longrightarrow S \times 0$ (Projection from product structure) is deg ≥ 1

$\implies$ genus$(T_i) \geq$ genus$(S \times 0)$

$\implies$ genus$(T_i) =$ genus$(S \times 0)$

$\implies S_i$ did not compress

$p \simeq$ homeo T_i exiting cat(-1) seq.

$\sqrt{4.16}$

<u>Topological Proposition</u> Let M be an irreducible homotopy handlebody

$\gamma_1, \gamma_2, \cdots$ p.d. loc. finite s.c.c.

$\gamma_i \neq *$. After passing to subseq. and allowing γ_1 to have multiple comps.

$\exists W^{irr.}$ open $\begin{matrix}\pi_1 \\ H_1\end{matrix}$ injective in M

exhausted by $W_1 \subset W_2 \subset \cdots$ s.t.

① If $\Gamma_i = \gamma_1 \cup \cdots \cup \gamma_i$, then $\Gamma_i \subset W_i$,

∂W_i connected, 2 inc. rel Γ_i

② $\exists$ core C for W

$C = B^3 \cup$ 1-handles (i.e. thickened graph)

Γ_i can be homotoped into C

via homotopy in W_i $\square$

Schematic Picture
E
W4
γ4
W3
γ3
W2
γ2
W1
γ1
D
M
W = ∪Wi
D = Core W
5.2

Proof Based on theory of <u>end reduction</u> developed by Brin-Thickston (1980's)

Addendum to Prop.

If M is hyperbolic γ_i's simple geodesics then each ω_i is atoroidal (No embedded π_1-inj tori)

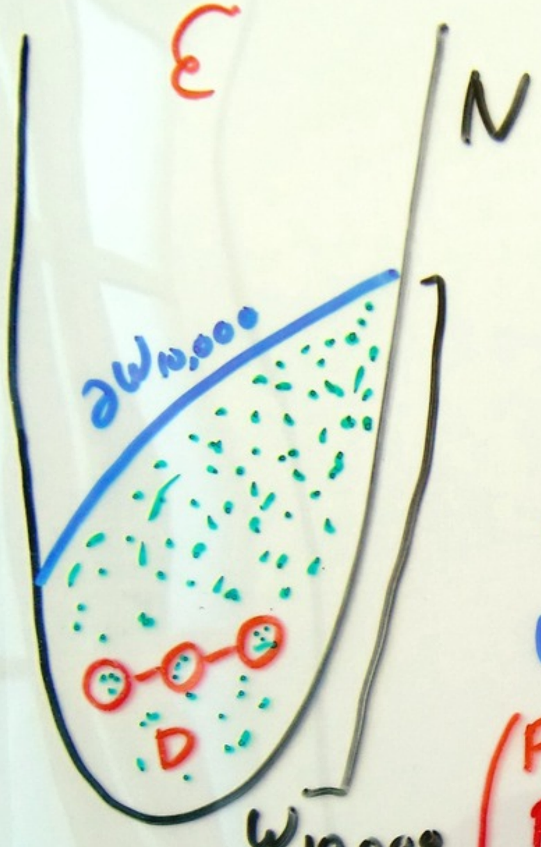
Exercise

Any Torus in M bounds a solid torus or cube with knotted hole

Tube

Convolutube

How to find surfaces $T_1, T_2, \cdots$ in N satisfying Taming Criterion



Given N, exiting geods. $\delta_1, \delta_2, \cdots$ apply topological prop. to produce $W_1 \subset W_2 \subset \cdots$ $\mathcal{W} = \cup W_i \subset N$

(Abuse Notation by passing to subseq. of δ_i's and reindexing and calling $\delta_1, \delta_2, \cdots$ the result.)

⁘ denotes $\Delta_1 = \delta_1 \cup \cdots \cup \delta_{10,000}$

<u>Note</u>: $\operatorname{rank} \pi_1(D) \le \operatorname{rank} \pi_1(N)$ Since D is a core of $\mathcal{W}$ which H_1, π_1 injects in N.

Pass to the $\pi_1(D)$ Covering $\widehat{\omega}_i$ of ω_i

Each δ_j $j \le i$ has a canonical lift plus many other lifts.

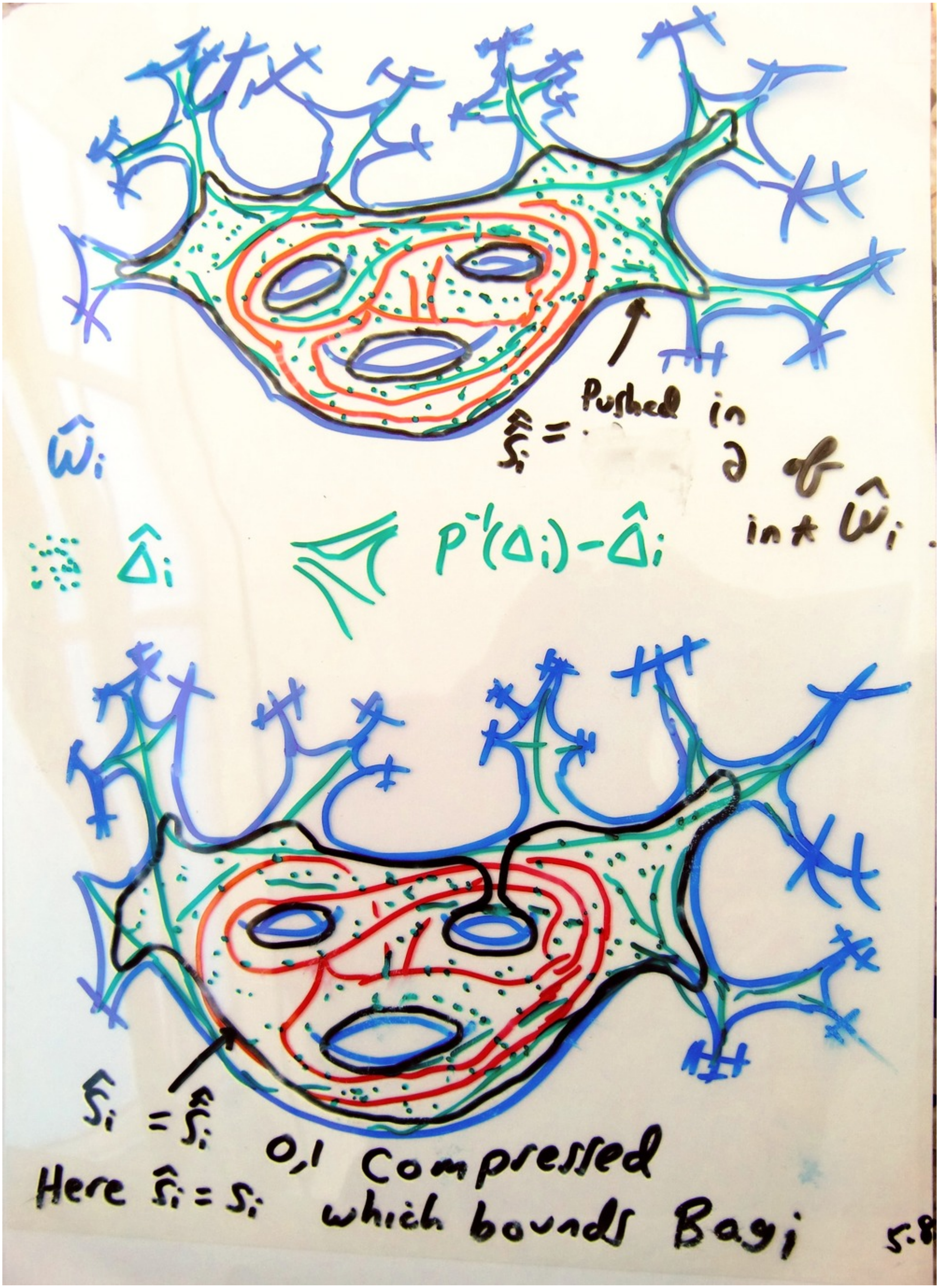
$\hat{\hat{S}}_i$ = manifold boundary of int $\widehat{\omega}_i$ pushed slightly into int $\widehat{\omega}_i$

$\hat{S}_i = \hat{\hat{S}}_i$ maximally 0,1 compressed

S_i = component of $\hat{S}_i$ bounding $B_{a}g_i$. S_i's chosen s.t $\exists p$ with for $i = i_1, i_2, \cdots$ $\hat{\delta}_p \subset B_{a}g_i$ and $\lim_{j \to \infty} P(i) \longrightarrow \infty$ where for $i = i_1, i_2, \cdots$ $\delta_{p(i)} \in B_{a}g_i$

5.7

Compare with Proof of Canary's theorem



Pushed in
$\hat{\hat{S}}_i =$ ∂ of int $\hat{W}_i$.
$\hat{W}_i$
$\hat{\Delta}_i$
$P^{-1}(\Delta_i) - \hat{\Delta}_i$
$\hat{S}_i = \hat{\hat{S}}_i$ 0,1 Compressed
Here $\hat{S}_i = S_i$ which bounds Bag_i
5.8

P_i = shrinkwrapped S_i w.r.t $\hat{\delta}_i \in Bag_i$

T_i = projection of P_i into N

Technical Problem Shrinkwrapping occurs in $\hat{\omega}_i$. Shrinkwrapped surface might want to jump out of $\hat{\omega}_i$.

Original Sol'n (Calegari-G) First shrinkwrap ∂W_i in N. If s'wraped $\partial W_i \cap \Delta_i = \phi$, then $\partial \hat{\omega}_i$ smooth, mean curv. 0, hence acts as barrier, for shrinkwrapping in $\hat{\omega}_i$. Otherwise we do a limit argument.

5.9

We show that $T_1, T_2, \ldots$

(or rather $T_{i_1}, T_{i_2}, \ldots$)

satisfy the taming criterion.

i.e. a) $\text{genus}(T_i) \leq \text{genus}(\partial \text{Core})$

b) each T_i is Cat(−1)

c) T_i's exit

d) T_i's homolg. separate

a), b) hold by construction.

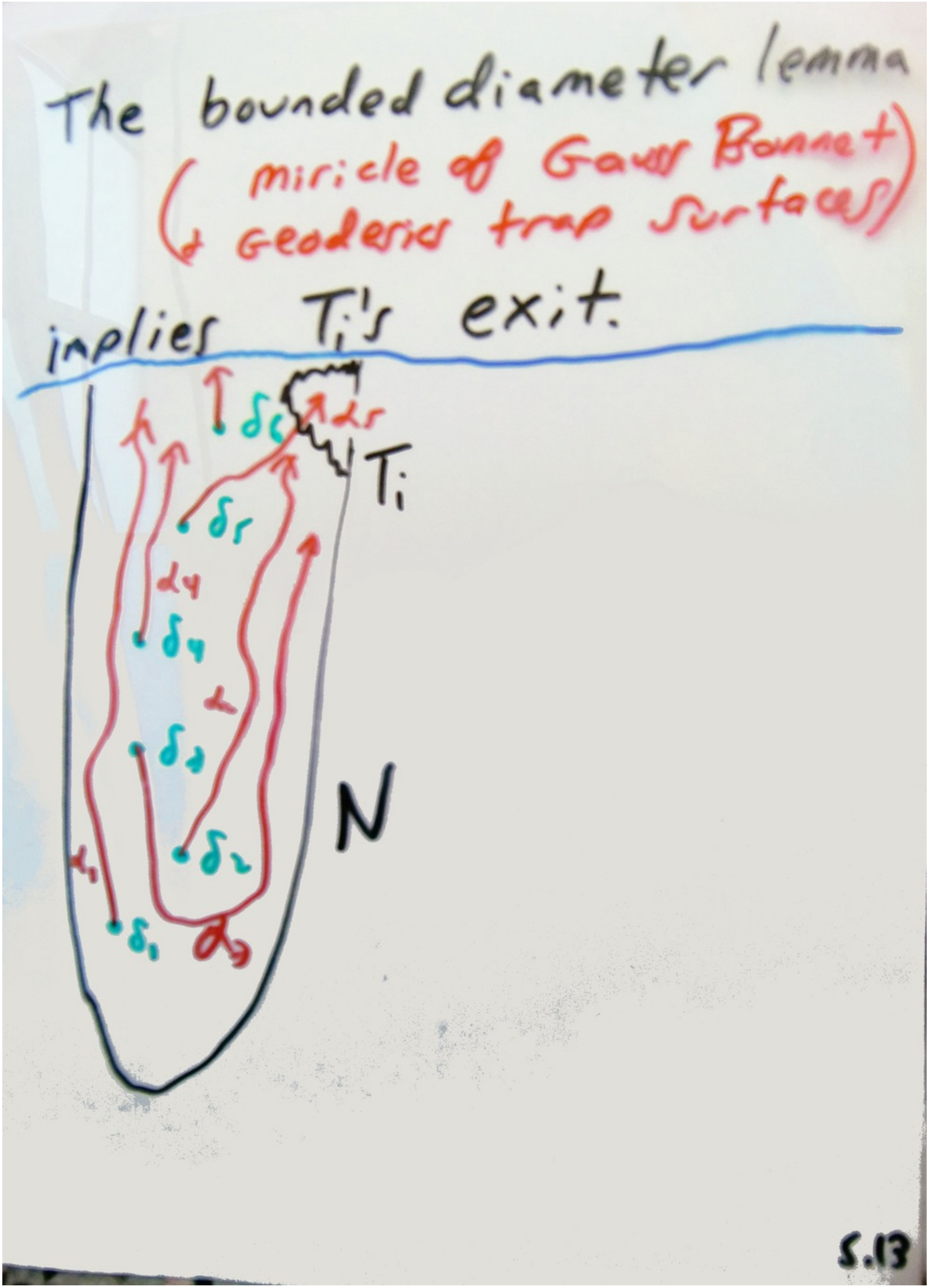
T_i's exit Let $\alpha_1, \alpha_2, \dots$ be a locally finite collection of proper rays in N s.t. $\forall j$ α_j starts at δ_j. If $j \leq i$ Let $\hat{\alpha}_j^i$ denote lift of α_j to $\bar{Y}_i$ starting at $\hat{\delta}_j$.

Since $\hat{\delta}_{p(i)} \in Bag_i$, $\langle S_i, \hat{\alpha}_j^i \rangle = 1$ where $j = p(i)$.

$\Rightarrow P_i \cap \hat{\alpha}_j^i \neq \emptyset$

$\Rightarrow T_i \cap \alpha_j \neq \emptyset$

i.e. $\forall i$ $T_i \cap \alpha_{p(i)} \neq \emptyset$ and $p(i) \to \infty$ as $i \to \infty$.

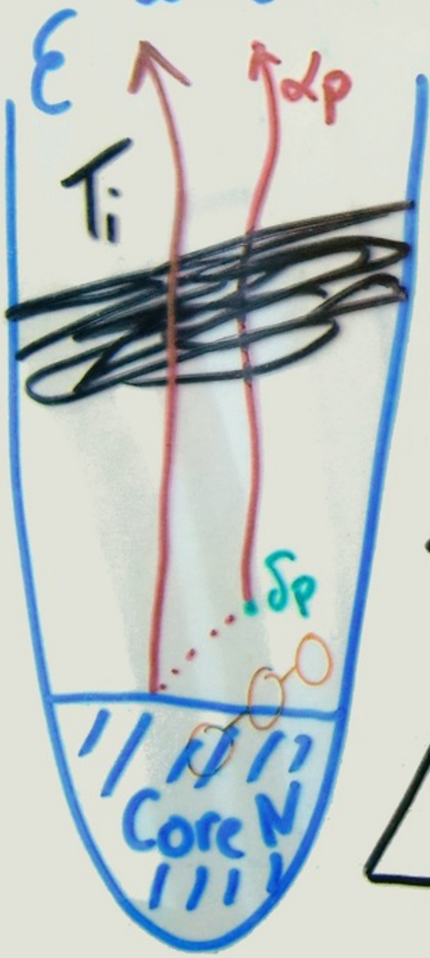

The bounded diameter lemma
(miricle of Gauss Bonnet
+ geodesics trap surfaces)
implies T_i's exit.
δ_1
δ_2
δ_3
δ_4
δ_5
δ_6
α_1
α_3
α_4
α_5
T_i
N
5.13

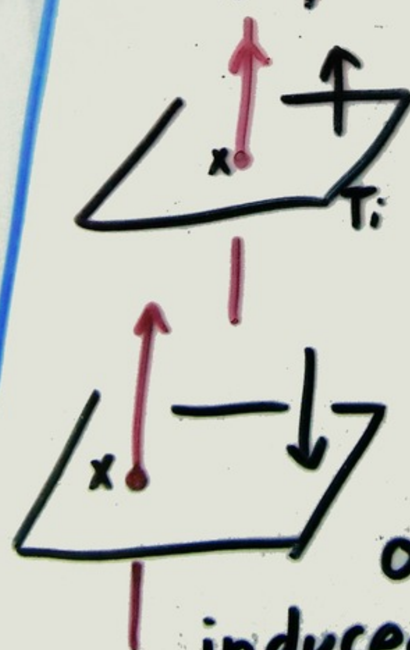
T_i is Homologically Separating
For i large

For i very large, $d(T_i, \delta_p)$ large

hence $[T_i] = n[\partial(\mathrm{Core}\,N)] \in H_2(N - \mathrm{int}(\mathrm{Core}\,N))$

where $n = \langle \alpha_p, T_i \rangle = \sum_{x \in T_i \cap \delta_p} \sigma(x)$ where

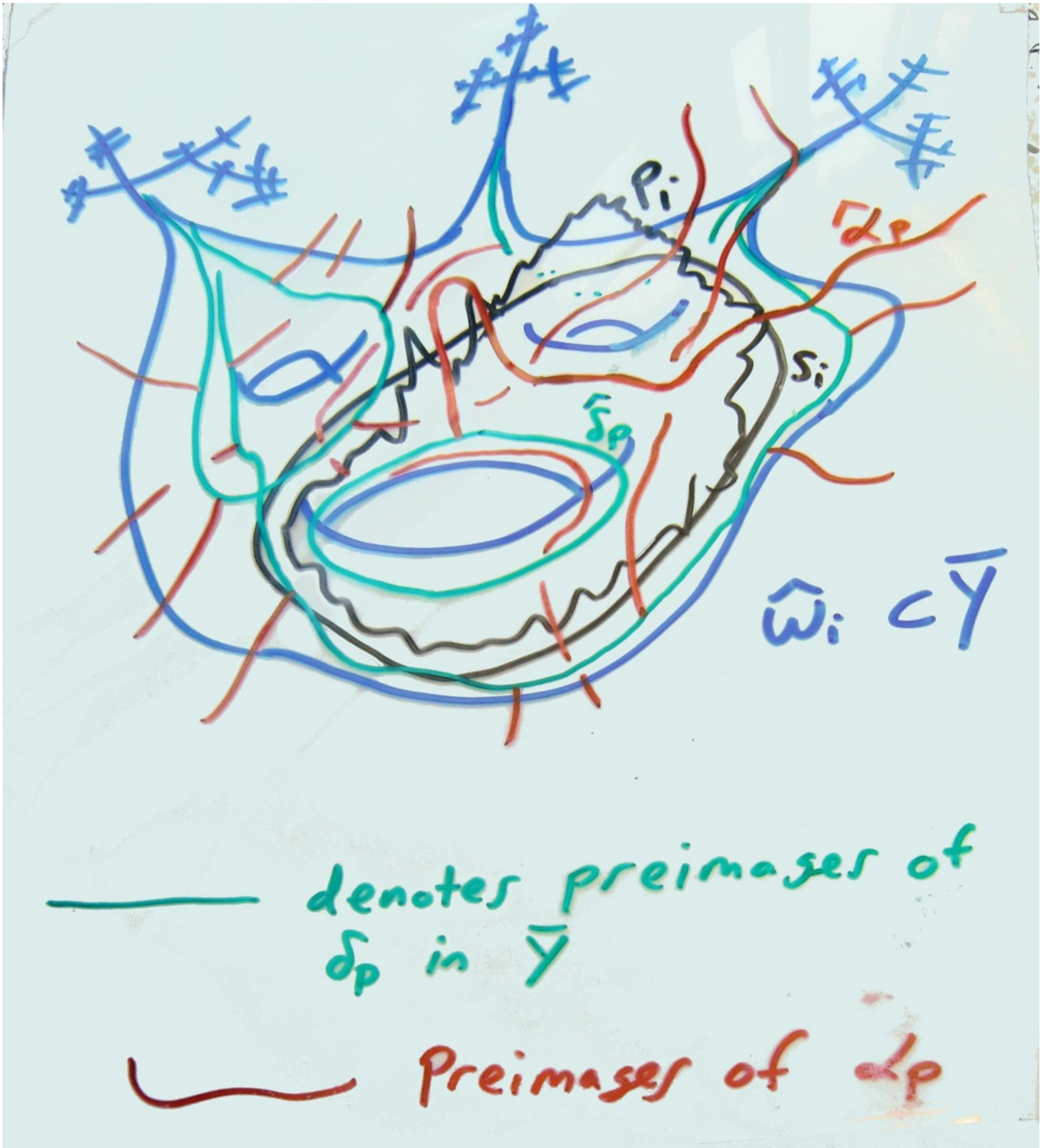




$\sigma(x) = 1$

$\sigma(x) = -1$

orientation on T_i induced from outward orientation on P_i


P_i
$\tilde{\alpha}_p$
S_i
$\tilde{\delta}_p$
$\hat{\omega}_i \subset \bar{Y}$
denotes preimages of δ_p in $\bar{Y}$
Preimages of α_p