Kleinian groups and the Sullivan dictionary II

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In this talk we will focus on the history and the applications of

Marden's Tameness Conjecture: *If N is a hyperbolic 3-manifold with finitely generated fundamental group, then N is topologically tame (i.e. homeomorphic to the interior of a compact 3-manifold).*

This conjecture was established in 2004 by Agol and Calegari-Gabai.

In the next talk, Dave Gabai will discuss the proof of this conjecture.

The compact core

(Scott) If $\pi_1(N)$ is finitely generated, there exists a compact submanifold R of N, called the **compact core** such that the inclusion of R into N is a homotopy equivalence.

One may think of the components of $N - R$ as the **ends** of the hyperbolic 3-manifold N. (One may define ends more abstractly, in which case the components of $N - R$ correspond to neighborhoods of the ends of N.)

The Tameness Conjecture predicts that one may choose R so that

$$N - R \cong \partial R \times (0, \infty).$$

Simplifying assumptions

We will assume throughout this talk that Γ is finitely generated and **contains no parabolic elements**. This is equivalent to assuming that $N = \mathbb{H}^3/\Gamma$ **has no cusps**, i.e. every homotopically non-trivial curve is homotopic to a closed geodesic.

In this situation, Ahlfors' Finiteness Theorem assures that $\partial_c N$ is a finite collection of closed hyperbolic surfaces.

The convex core

If $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold, then its convex core $C(N)$ is its smallest convex submanifold such that the inclusion map is a homotopy equivalence.

More explicitly, if we let $CH(\Lambda(\Gamma))$ denote the convex hull of the limit set in $\mathbb{H}^3$, then

$$C(N) \cong CH(\Lambda(\Gamma))/\Gamma$$

.

(Thurston) The intrinsic metric on $\partial C(N)$ is hyperbolic. Moreover, $\partial C(N)$ is a pleated surface.

The complement of the convex core

One may show that

$$N - C(N) \cong \partial C(N) \times (0, \infty).$$

Moreover, in these coordinates the metric on $N - C(N)$ is bilipschitz to the metric

$$\cosh^2 ds^2 + dt^2$$

where t is the real coordinate and ds^2 is the hyperbolic metric on $\partial C(N)$.

Moreover, the convex core $C(N)$ is homeomorphic to the conformal bordification $\hat{N} = N \cup \partial_c N$ (unless $\Lambda(\Gamma)$ lies in a round circle).

In particular, if N is convex cocompact, then $C(N)$ is a compact core for N.

Theorem:(Sullivan) *There exists $K_\infty > 0$ such that if each component of $\Omega(\Gamma)$ is simply connected, e.g. if Γ is quasifuchsian, then $\partial C(N)$ is K_∞-quasiconformal to $\partial_c N$.*

(Epstein-Marden-Markovic) $2.1 < K_\infty \leq 13.88$

(Marden-Markovic) Given $\delta > 0$, there exists $K_\delta > 0$ such that if every geodesic in $\Omega(\Gamma)$ has length at least δ, then $\partial_c N$ is K_δ-quasiconformal to $\partial C(N)$.

(Bridgeman-Canary) Obtain explicit estimates on K_δ.

It follows from these results that $\partial_c N$ and $\partial C(N)$ are uniformly bilipschitz.

Geometrically finite ends

An end of N is **geometrically finite** if it is a component of $N - C(N)$ (or more abstractly, if it has a neighborhood which is disjoint from $C(N)$.)

Since $N - C(N) \cong \partial C(N) \times (0, \infty)$, every geometrically finite end is topologically tame.

If N is convex cocompact, every end is geometrically finite.

A Geometrically infinite hyperbolic 3-manifold

- Let M be a closed hyperbolic 3-manifold which fibers over the circle. (Thurston showed that any mapping torus of a pseudo-Anosov homeomorphism of a closed surface F of genus at least 2 is hyperbolic.)
- Let N be the cover of M associated to the fiber subgroup. Then N is homeomorphic to $F \times \mathbb{R}$. If α is a closed geodesic in M which is homotopic to the fiber, then the pre-images of α are closed geodesics exiting each end.
- Since $C(N)$ contains all the closed geodesics in N, this implies that $C(N) = N$. In particular, $C(N) \cong \hat{N}$ is not compact.
- Greenberg was the first to show that there exist geometrically infinite, finitely generated Kleinian groups. Jorgensen was the first to exhibit finite volume hyperbolic 3-manifolds which fiber over the circle.

An end $\mathcal{E}$ is **simply degenerate** if it is homeomorphic to $F \times (0, \infty)$ (for a closed surface F) and there exists a sequence of maps $\{h_n : F \to \mathcal{E}\}$ such that

(1) each h_n is homotopic, within $\mathcal{E}$, to a level surface,

(2) the intrinsic metric on $h_n(F)$ has curvature ≤ -1 for all n, and

(3) $\{h_n(F)\}$ leaves every compact subset of N.

Geometrically Tame hyperbolic 3-manifolds

A hyperbolic 3-manifold N with finitely generated fundamental group is **geometrically tame** if every end is either geometrically finite or simply degenerate.

With these definitions, it is clear that a geometrically tame hyperbolic 3-manifold is topologically tame. Thurston originally gave a weaker definition of geometric tameness, which only suffices in the setting where Γ is freely indecomposable, and showed that it implied topological tameness.

It turns out that topological tameness is equivalent to geometric tameness for hyperbolic 3-manifolds. It is geometric tameness which gives rise to the analytical and dynamical applications of the Tameness Conjecture.

History of Marden's Tameness conjecture

- (Thurston) Many limits of freely indecomposable convex cocompact Kleinian groups are geometrically tame.
- (Bonahon) Every hyperbolic 3-manifold with finitely generated, freely indecomposable fundamental group is geometrically tame.
- (Canary) A hyperbolic 3-manifold is geometrically tame if and only if it is topologically tame.
- (Canary-Minsky, Ohshika) Strong limits of topologically tame hyperbolic 3-manifolds are topologicallly tame.
- (Anderson-Canary, Evans, Brock-Bromberg-Evans-Souto, Brock-Souto) All limits of convex cocompact hyperbolic 3-manifolds are topologically tame.

More History

- (Souto) If N can be exhausted by compact cores, then N is topologically tame.
- (Agol, Calegari-Gabai) All hyperbolic 3-manifolds with finitely generated fundamental group are topologically tame.
- (Soma) Gives a nice proof combining ideas from both Agol and Calegari-Gabai.

We now discuss application of the resolution of Marden's Tameness Conjecture.

Ahlfors' Measure Conjecture

Ahlfors' Measure Conjecture: *If Γ is a finitely generated Kleinian group, then either $\Lambda(\Gamma) = \widehat{\mathbb{C}}$ or $\Lambda(\Gamma)$ has measure zero. Moreover, if $\Lambda(\Gamma) = \widehat{\mathbb{C}}$, then Γ acts ergodically on $\widehat{\mathbb{C}}$.*

Ahlfors proved his conjecture for convex cocompact Kleinian groups.

(Sullivan) The area of the Julia set of a hyperbolic rational map is zero.

(Buff,Cheritat) There exist quadratic polynomials with positive area Julia sets, which are not all of $\widehat{\mathbb{C}}$.

Ahlfors' proof in the convex cocompact case

Suppose that $\Lambda(\Gamma)$ has positive measure and is not all of $\widehat{\mathbb{C}}$. Let $\tilde{h} : \mathbb{H}^3 \to [0,1]$ be the harmonic function defined by letting $\tilde{h}(x)$ be the proportion of geodesic rays emanating from x which end in $\Lambda(\Gamma)$. Explicitly,

$$\tilde{h}(x) = \int_{\Lambda(\Gamma)} \left(\frac{1-|x|^2}{|x-\xi|^2} \right)^2 dm$$

where m is Lebesgue measure.

If $x \in \mathbb{H}^3 - CH(\Lambda(\Gamma))$, then there is a plane P_x through x which bounds a half-space which is disjoint from $CH(\Lambda(\Gamma)$. Therefore, at least half the rays beginning at x end in $\Omega(\Gamma)$, so

$$\tilde{h}(x) \leq \frac{1}{2} \text{ on } \mathbb{H}^3 - CH(\Lambda(\Gamma)$$

Since $\Lambda(\Gamma)$ is Γ-equivariant, $\tilde{h}$ descends to a harmonic function $h : N \to [0,1]$ such that $h(x) \leq \frac{1}{2}$ on $N - C(N)$. Since $C(N)$ is compact and $h(x) \leq \frac{1}{2}$ on $\partial C(N)$, the maximum principle for harmonic functions, implies that $h(x) \leq \frac{1}{2}$ on $C(N)$. Therefore,

$$\tilde{h}(x) \leq \frac{1}{2} \text{ on } \mathbb{H}^3.$$

If $\Lambda(\Gamma)$ has positive measure, then it contains a point of density ξ.

If $\{x_n\}$ approaches ξ along a geodesic, then $\tilde{h}(x_n) \to 1$.

This contradiction completes the proof.

Observation: If one can show that any harmonic function on a topologically tame hyperbolic 3-manifold achieves its maximum over $C(N)$ on the boundary $\partial C(N)$, this same proof would establish Ahlfors' Measure Conjecture for topologically tame Kleinian groups.

A Minimum principle

Theorem: (Thurston, Canary) If N is a topologically tame hyperbolic 3-manifold and $h : N \to (0, \infty)$ is a positive superharmonic function (i.e. $div(grad\ h) \leq 0$), then

$$\inf_{C(N)} h = \inf_{\partial C(N)} h.$$

If $N = C(N)$, then h must be constant.

Corollary: Ahlfors' Measure Conjecture holds for topologically tame hyperbolic 3-manifolds, hence for all finitely generated Kleinian groups.

Corollary: A topologically tame hyperbolic 3-manifold is strongly parabolic (i.e. admits no non-constant positive superharmonic functions) if and only if $\Lambda(\Gamma) = \widehat{\mathbb{C}}$.

Sketch of proof of minimum principle

- Consider the flow generated by $-grad\ h$. Since $div(-grad\ h) \geq 0$, this flow is volume non-decreasing.
- Since h is positive, the flow moves more and more slowly.
- Neighborhoods of radius one of the negatively curved surfaces in a simply degenerate ends have bounded volume, so act as narrows for the flow.
- Eventually, since the flow is moving increasingly slowly, a flow line takes a longer and longer time to pass through each narrows. Therefore, almost every flow line must eventually turn around and exit the convex core through the boundary of the convex core. (It heads to wide-open spaces where volume grows exponentially.)
- Therefore, the function achieves its infimum over $C(N)$ on the boundary of $C(N)$.

Ergodicity of geodesic flows

We say that the geodesic flow of a manifold N is **ergodic** if every flow-invariant subset of $T^1(N)$ has either zero measure or full measure.

(Sullivan) The geodesic flow of a hyperbolic 3-manifold is ergodic if and only if it does not admit a positive Green's function.

A positive Green's function on N has a pole, but one may use it to obtain a non-constant positive superharmonic function on N. (Alternatively, a Green's function is a function with a single pole which is harmonic elsewhere and the previous argument can be used directly to rule out the possibility of such a function when $\Lambda(\Gamma) = \widehat{\mathbb{C}}$.)

Corollary: The geodesic flow of a topologically tame hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$ is ergodic if and only if $\Lambda(\Gamma) = \widehat{\mathbb{C}}$.

Hausdorff dimension and spectral geometry

- Let $D(\Gamma)$ denote the Hausdorff dimension of $\Lambda(\Gamma)$.
- Let $\lambda_0(N)$ denote the bottom of the spectrum of the Laplacian on N, i.e.

$$\lambda_0(N) = \inf \mathrm{spec}(-div(grad)).$$

- **Theorem:** (Patterson, Sullivan) If $N = \mathbb{H}^3/\Gamma$ is convex cocompact, then

$$\lambda_0(N) = D(\Gamma)(2 - D(\Gamma))$$

unless $D(\Gamma) < 1$, in which case $\lambda_0(N) = 1$. Moreover, if $D(\Gamma) < 1$, then Γ is a Schottky group.

Some ideas involved in Patterson-Sullivan theory

Let $\delta(\Gamma)$ be the critical exponent of the series

$$\sum_{\gamma \in \Gamma} e^{-s\ d(0,\gamma(0))}$$

where 0 is a choice of origin for $\mathbb{H}^3$.

One may use this series to construct a measure ν on $\Lambda(\Gamma)$, called the **Patterson-Sullivan measure**, which transforms , under Γ, like $\delta(\Gamma)$-dimensional Hausdorff measure and then verify that $D(\Gamma) = \delta(\Gamma)$. (In fact, Sullivan proved that this measure is a multiple of δ-dimensional Hausdorff measure.)

The function

$$\tilde{\phi}(x) = \int_{\widehat{\mathbb{C}}} \left(\frac{1 - |x|^2}{|x - \xi|^2} \right)^{\delta} \ d\nu$$

descends to a positive eigenfunction for the Laplacian on N with eigenvalue $\delta(2 - \delta)$.

Geometrically infinite hyperbolic 3-manifolds

- (Sullivan,Tukia) If Γ is convex cocompact, then $D(\Gamma) < 2$. (The Julia set of a hyperbolic rational map has dimension less than 2.)
- (Bishop-Jones) If Γ is finitely generated and geometrically infinite, then $\delta(\Gamma) = 2$. (First examples by Sullivan.)
- (Canary) If N is topologically tame, but not geometrically finite, then $\lambda_0(N) = 0$.
- **Idea:** The surfaces $h_n(F)$ exiting an end may be used to show that the Cheeger constant of N is 0 and hence, by a result of Buser, that $\lambda_0(N) = 0$.
- Therefore, if $N = \mathbb{H}^3/\Gamma$ is topologically tame, then

$$\lambda_0(N) = D(\Gamma)(2 - D(\Gamma))$$

unless $D(\Gamma) < 1$, in which case Γ is a Schottky group.

Hausdorff dimension and the convex core

(Burger-Canary,Canary) If Γ is a convex cocompact hyperbolic 3-manifold and $D(\Gamma) > 1$, then

$$\frac{K_1}{Vol(C_1(N))^2} \leq 2 - D(\Gamma) \leq \frac{K_2|\chi(\partial C(N))|}{Vol(C(N))}$$

where K_1 and K_2 are (computable) constants and $Vol(C(N))$ denotes the volume of the convex core and $Vol(C_1(N))$ denotes the volume of the neighborhood of radius one of the convex core.

Moral: Limit set fuzzy $\iff$ convex core thick

One may apply the results of the previous slide to check that this remains true for topologically tame, geometrically infinite hyperbolic 3-manifolds.

A purely topological application

Conjecture: (Simon) *Let M be a compact, irreducible 3-manifold and let N be a cover of N with finitely generated fundamental group, then the interior of N is topologically tame.*

The Tameness Theorem, combined with Perelman's resolution of Thurston's Geometrization Conjecture, together with work of Simon and an argument of Long and Reid, give a full proof of Simon's Conjecture.

Group-theoretic applications

One may combine work of Canary, Susskind and Thurston with the Tameness Theorem to show:

Theorem: *If M is a closed hyperbolic 3-manifold, then $\pi_1(M)$ has the finitely generated intersection property (i.e. the intersection of any two finitely generated subgroups is finitely generated) if and only if M does not have a finite cover which fibers over the circle.*

Other group theoretic applications are to separability properties of fundamental groups of hyperbolic 3-manifolds (e.g. Agol-Long-Reid) and to the pro-normal topology on $\pi_1(M)$, e.g.

Theorem: (Glasner-Souto-Storm) If M is a closed hyperbolic 3-manifold and H is a maximal subgroup of $\pi_1(M)$ (i.e. is contained in no proper subgroup of $\pi_1(M)$), then H is either finite index or infinitely generated.

The curve complex

- If F is a closed surface, then the vertices of $\mathcal{C}(F)$ are (isotopy classes of) essential simple closed curves on F.
- A collection $\{[\alpha_0],[\alpha_1],\ldots,[\alpha_n]\}$ span a n-simplex if the curves have disjoint representatives.
- The curve complex was introduced by Harvey and studied extensively by Harer, Ivanov and others.
- (Masur-Minsky) $\mathcal{C}(F)$ is a Gromov hyperbolic metric space (although notice that it isn't locally compact.)
- (Klarreich) $\partial_\infty\mathcal{C}(F)$ may be identified with the space of filling geodesic laminations on F

The ending lamination of a simply degenerate end

- Let $h_n(F)$ be a sequence of "hyperbolic" surfaces exiting a simply degenerate end $\mathcal{E}$ with a neighborhood homeomorphic to $F \times (0, \infty)$.
- (Bers) There exists K such that every $h_n(F)$ has an essential simple closed curve β_n of length at most K.
- We may think of $\{\beta_n\}$ as a sequence of vertices in the curve complex $\mathcal{C}(F)$.
- (Thurston, Bonahon, Canary, Klarreich) $\{\beta_n\}$ converges to a well-defined point $\lambda \in \partial_\infty \mathcal{C}(F)$, which we call the **ending lamination** of the simply degenerate end $\mathcal{E}$.

Thurston's Ending Lamination Conjecture: *A hyperbolic 3-manifold N with finitely generated fundamental group is determined (up to isometry) by its homeomorphism type and its ending invariants, which consist of the conformal boundary* $\partial_c N$ *and the ending laminations of the simply degenerate ends.*

(Minsky, Brock-Canary-Minsky) Thurston's Ending Lamination Conjecture holds for topologicallly tame hyperbolic 3-manifolds.

Combining this result with the Tameness Theorem one obtains a complete proof of Thurston's Ending Lamination Conjecture.