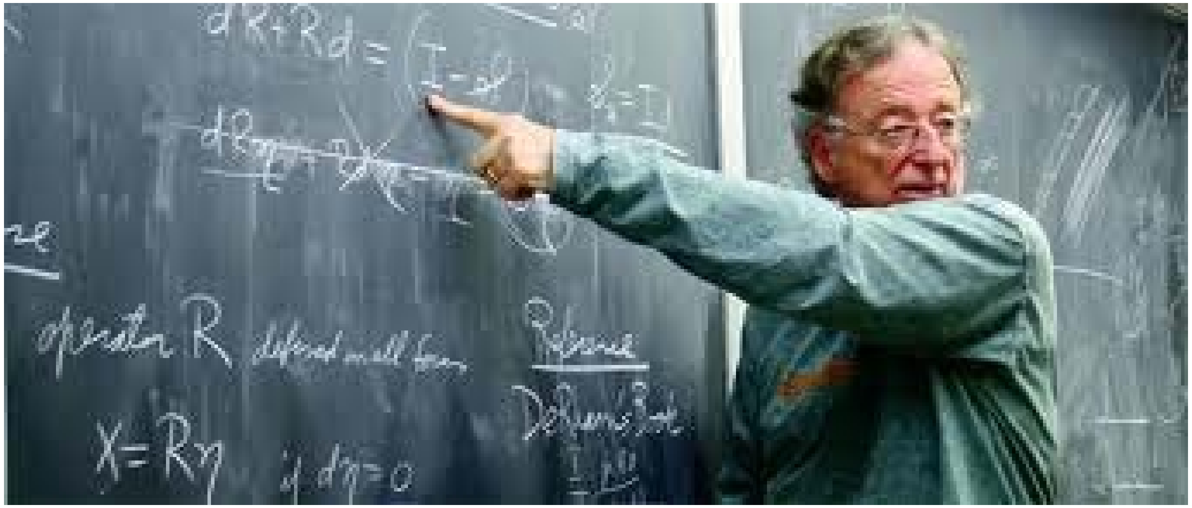

Product Formulas for Measures and Applications to Analysis and Geometry

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“SLE(4)”

The Volume as Multifractal Measure (“Burstiness”): How to Best Represent This Measure?

The Product Formula

- **Theorem (F,K,P)**: A Borel probability measure μ on [0,1] has a unique representation as

$$\prod(1 + a_I h_I) ,$$

where the coefficients a_I are in [-1,+1]. Conversely, if we choose any sequence of coefficients a_I in [-1,+1], the resulting product is a Borel probability measure on [0,1].

Note: For general positive measures, just multiply by a constant. Similar result on $[0,1]^d$. (For $d > 1$ there are choices for the representations.)

Note: See "The Theory of Weights and the Dirichlet Problem for Elliptic Equations" by R. Fefferman, C. Kenig, and J.Pipher (Annals of Math., 1991)

Some Haar-like functions on [0,1]

"The Theory of Weights and the Dirichlet Problem for Elliptic Equations" by R. Fefferman, C. Kenig, and J.Pipher (Annals of Math., 1991). We first define the "L^∞ normalized Haar function" h_I for an interval I of form $[j2^{-n},(j+1)2^{-n}]$ to be of form

$$h_I = -1 \text{ on } [j2^{-n},(j+1/2)2^{-n})$$

and

$$h_I = +1 \text{ on } [(j+1/2)2^{-n},(j+1)2^{-n}).$$

The only exception to this rule is if the right hand endpoint of I is 1. Then we define

$$h_I(1) = +1.$$

Relative Measure

The coefficients a_I are computed simply by computing relative measure ("volume") on the two halves of each interval I. Let L and R = left (resp. right) halves of I. Solve:

$$\mu(R) = \tfrac{1}{2}(1 + a_I)\,\mu(I)$$

$$\mu(L) = \tfrac{1}{2}(1 - a_I)\,\mu(I)$$

Then $-1 \le a_I \le +1$ because μ is nonnegative. $a_I > 0$ $\Rightarrow$ volume increasing, $a_I < 0 \Rightarrow$ volume decreasing.

Topic 1: Telecommunications:

Joint Work With D. Bassu, L. Ness, V. Rokhlin

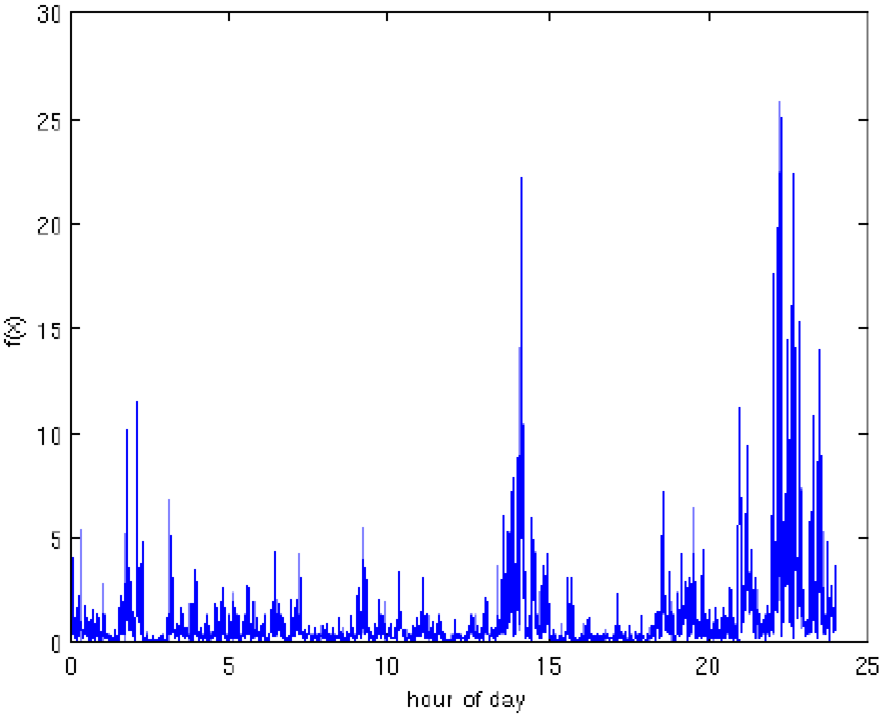
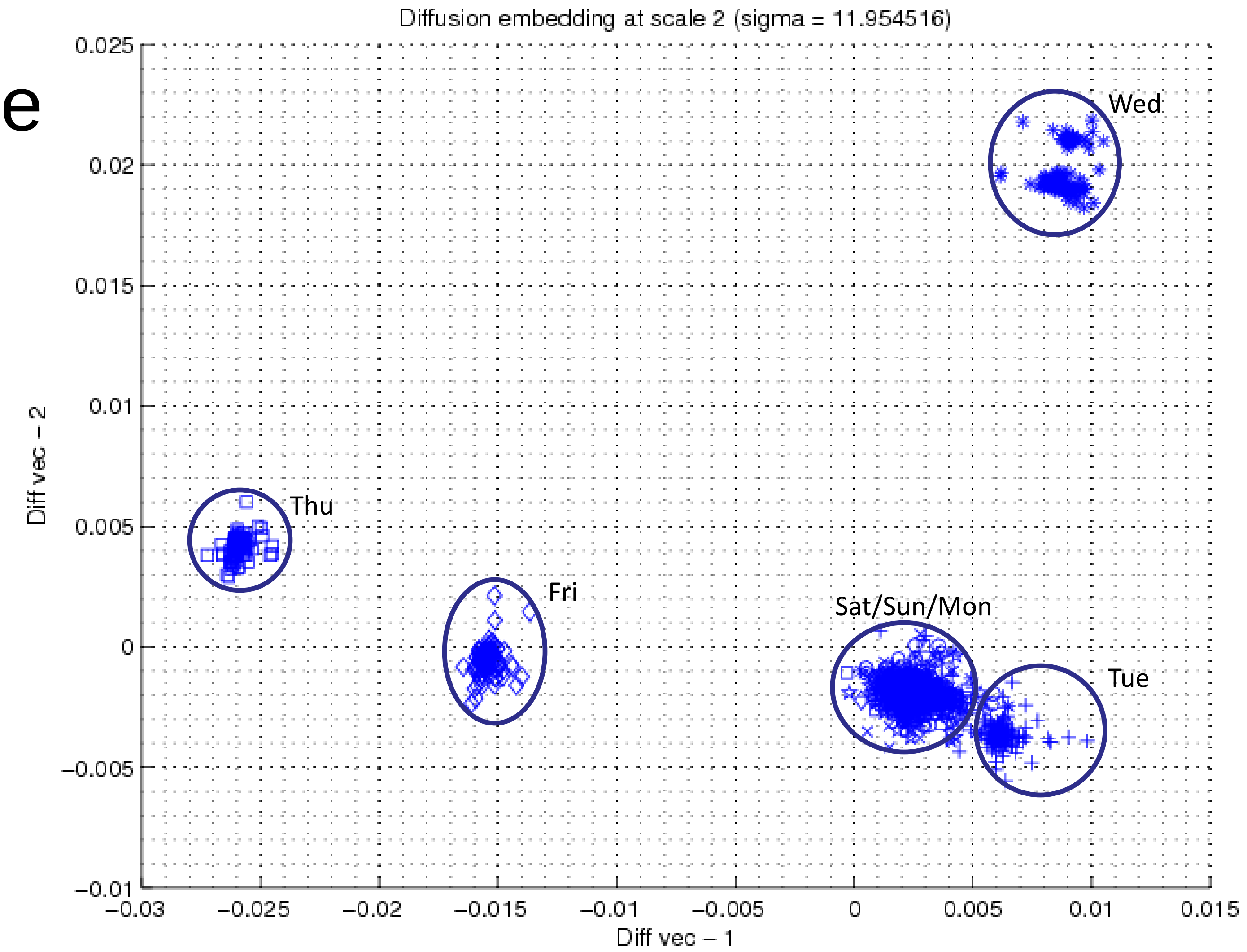


Figure 1 Volume for PDF 1/2(1 + cos(πx))

This is a simulated measure with coefficients chosen randomly from a particular PDF.

Coefficients give Information for Classification

- Daily profile
  - 182 antennas
  - 14 days



Volume from various antennae have had coefficients extracted, embedded by DG.

SLE

"Stochastic Loewner Evolution" or

"Schramm – Loewner Evolution"

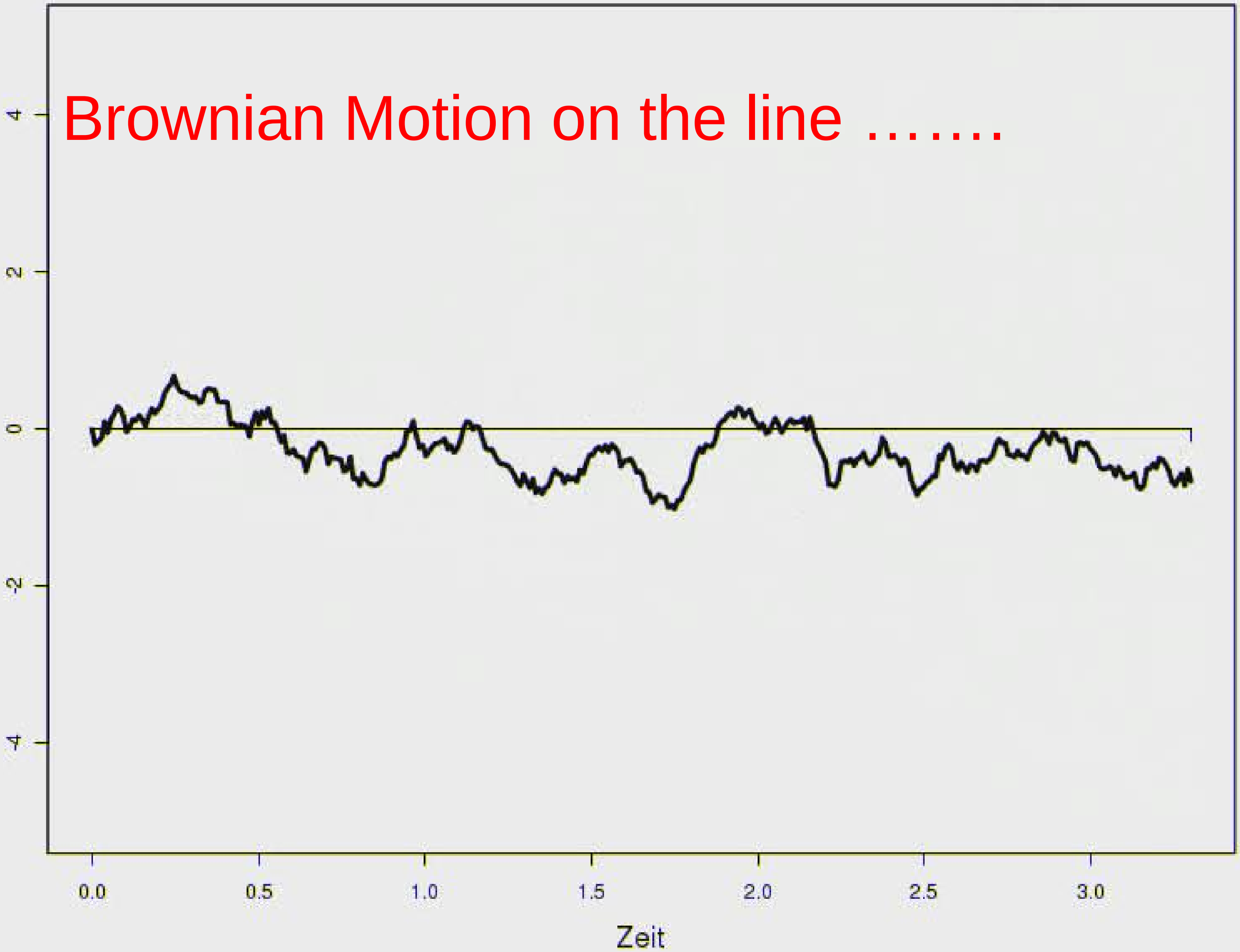
Oded Schramm (1961 – 2008)

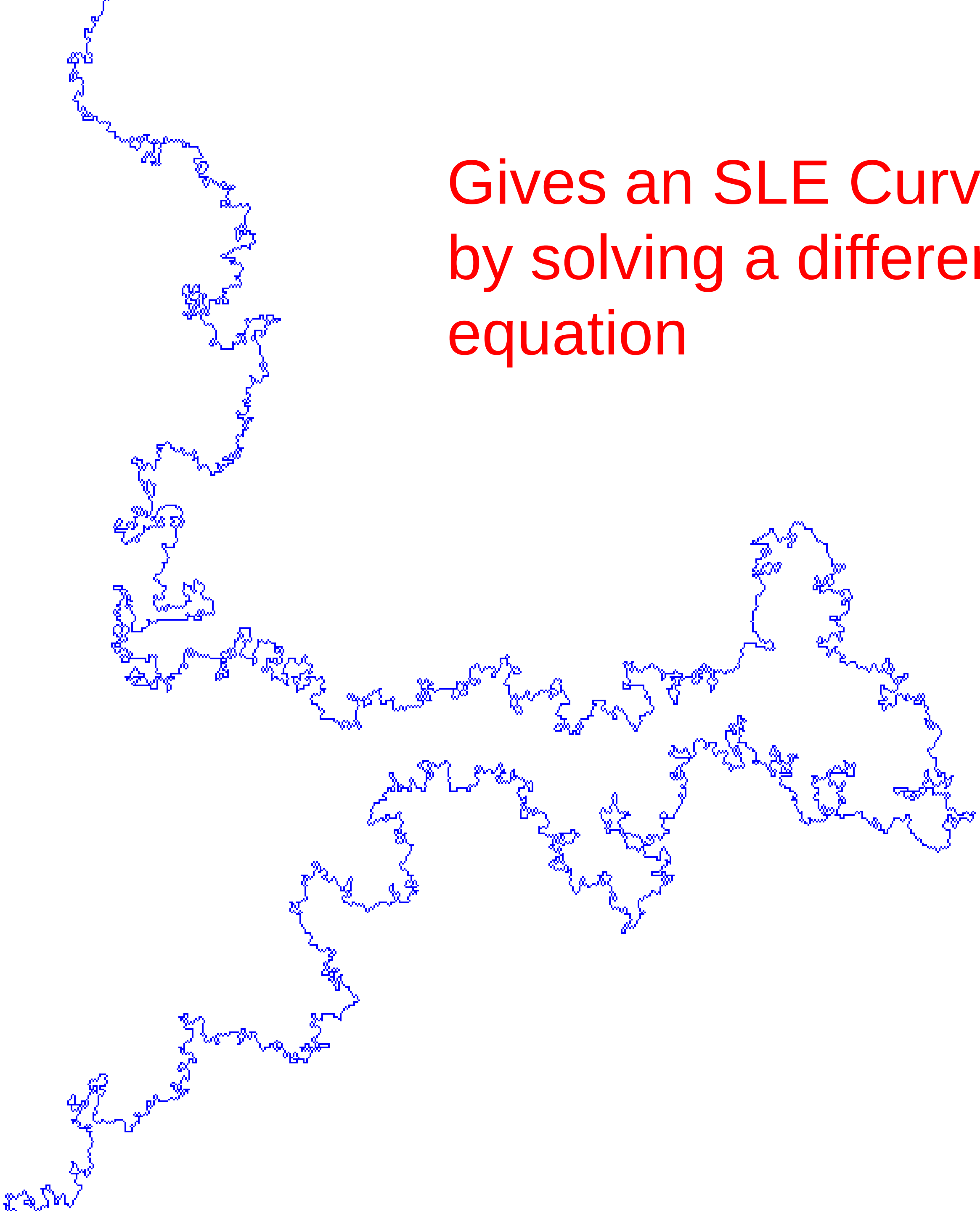
Conformal Mappings from $\mathbb{H}_+$ to $\mathbb{H}_+$:

$$\partial_t F(t,z) = -2/(F(t,z) - B(\kappa t))$$

$$F(0,z) = z$$

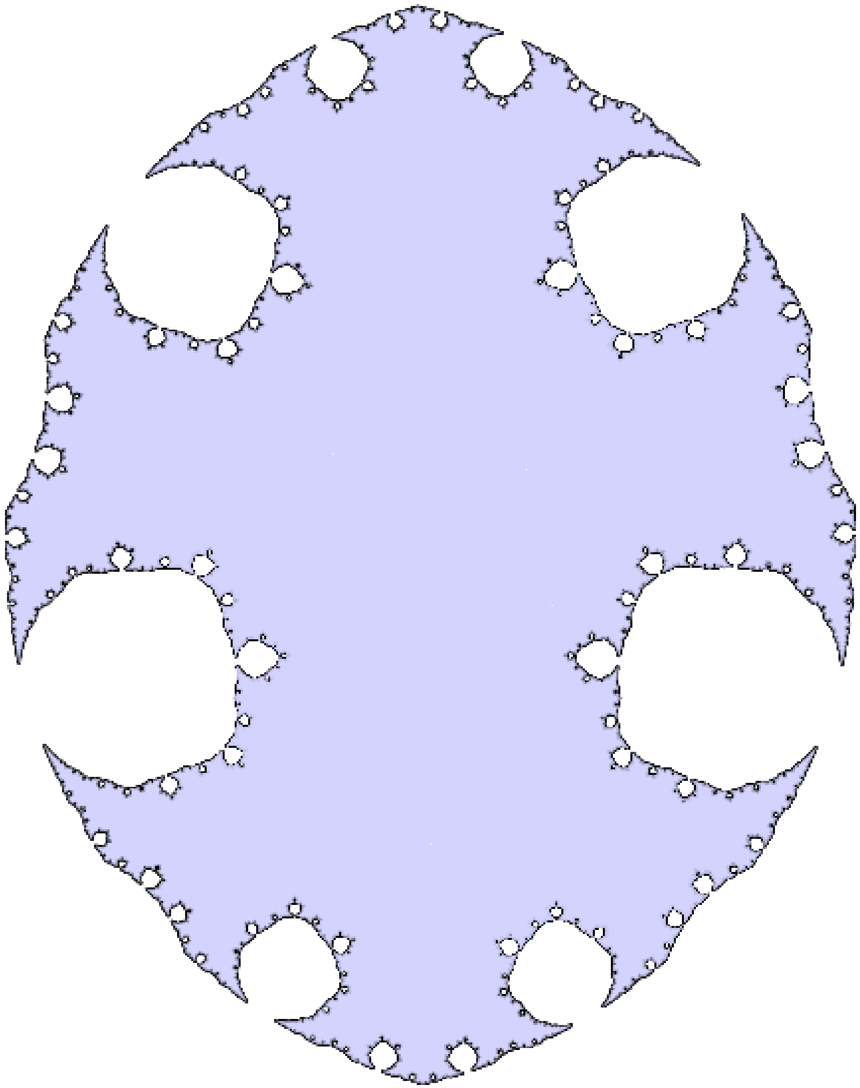
Brownian Motion on the line …….





Gives an SLE Curve
by solving a differential
equation

A Limit Set Produced From A Special "Welding Map".
This method produces "random curves with a group law" (Bers' Theorem for Kleinian groups).



Picture by J. Brock

A Theorem

(Work of K. Astala, PWJ, A. Kupiainen, E. Saksman to appear in Acta Mathematica)

Exponentiate a multiple of the Gaussian Free Field

(= Massless Free Field) after subtracting

infinity (Kahane's Theorem). A.S. can multiply by constant to get derivative

of homeo(S^1). (Here $0<t<2$.)

Theorem (A,J,K,S): The homeo is (a.s.) a welding curve and associated to a curve, which is "rigid" (unique).

One now understands how to get another method of producing SLE from this. (See work of S. Sheffield. The above procedure "must be done twice".)

The Gaussian Free Field

Σ = sum over all $j > 0$ of

$$X_j(\omega)\, j^{-1/2}\cos(j\theta) + X'_j(\omega)\, j^{-1/2}\sin(j\theta) - 1/2j$$

Here X_j , X'_j = Const. times i.i.d. Brownian motion at time t corresponding to kappa)

$$\Phi'(\theta) = \mathrm{const}(\omega)\, e^{\Sigma}$$

Then a.s. Φ is welding with uniqueness for the welding curve (rigid).

J.P. Kahane's Theorem

Inspired by Mandelbrot's Work

Exponentiating the Gaussian Free Field

leads (a.s.) to a Borel measure μ on the circle,

with non zero and finite mass. $(0 < t < 2)$

Multiplying by a constant (normalize) gives

us a derivative of a homeomorphism of the

Circle.

A, J, K, S: "Enemy" for building a curve: We can't let $| a_I | > 1- \varepsilon$ very often. Estimates!

Work with M. Csörnyei: Tangent Fields and Sharpness of Rademacher's Theorem

- **R**: A Lipschitz function F on $\mathbb{R}^d$ is differentiable almost everywhere. Say F maps $\mathbb{R}^d$ to $\mathbb{R}^m$. The most interesting case today is when m = d.
- Is there a converse? Given E a Lebesgue null set, is there F: $\mathbb{R}^d$ to $\mathbb{R}^d$ such that F is not differentiable at all points of E? (Is Rademacher "sharp"?)
- The statement is classical for d = 1.
- For all d > 1, examples show this statement is false if m < d.

A Theorem in Dimension 2

Theorem (G. Alberti , M. Csörnyei, D. Preiss): In dimension 2, for any set E of Lebesgue measure zero, there is a Lipshitz function F from $\mathbb{R}^2$ to $\mathbb{R}^2$ to such that F is nowhere differentiable on E.

The proof uses two ingredients. One is dependent on dimension 2 (combinatorics), the other is not (real variables).

Today: Marianna Csörnyei, PWJ: d Dimensional Version.

This requires a notion from geometric measure theory: Tangent Fields and Cones for Sets.

Tangent Fields in Dimension d

Now let E be a set of Lebesgue measure $\varepsilon > 0$, and let x be a point in E. Then x has a "good" tangent cone for E of angle $\pi - \delta$ if any Lipschitz curve Γ (with respect to the axis of the cone) has

$$\mathrm{Length}(E \cap \Gamma) \leq C(\delta)\, \varepsilon^{-\delta + 1/d}.$$

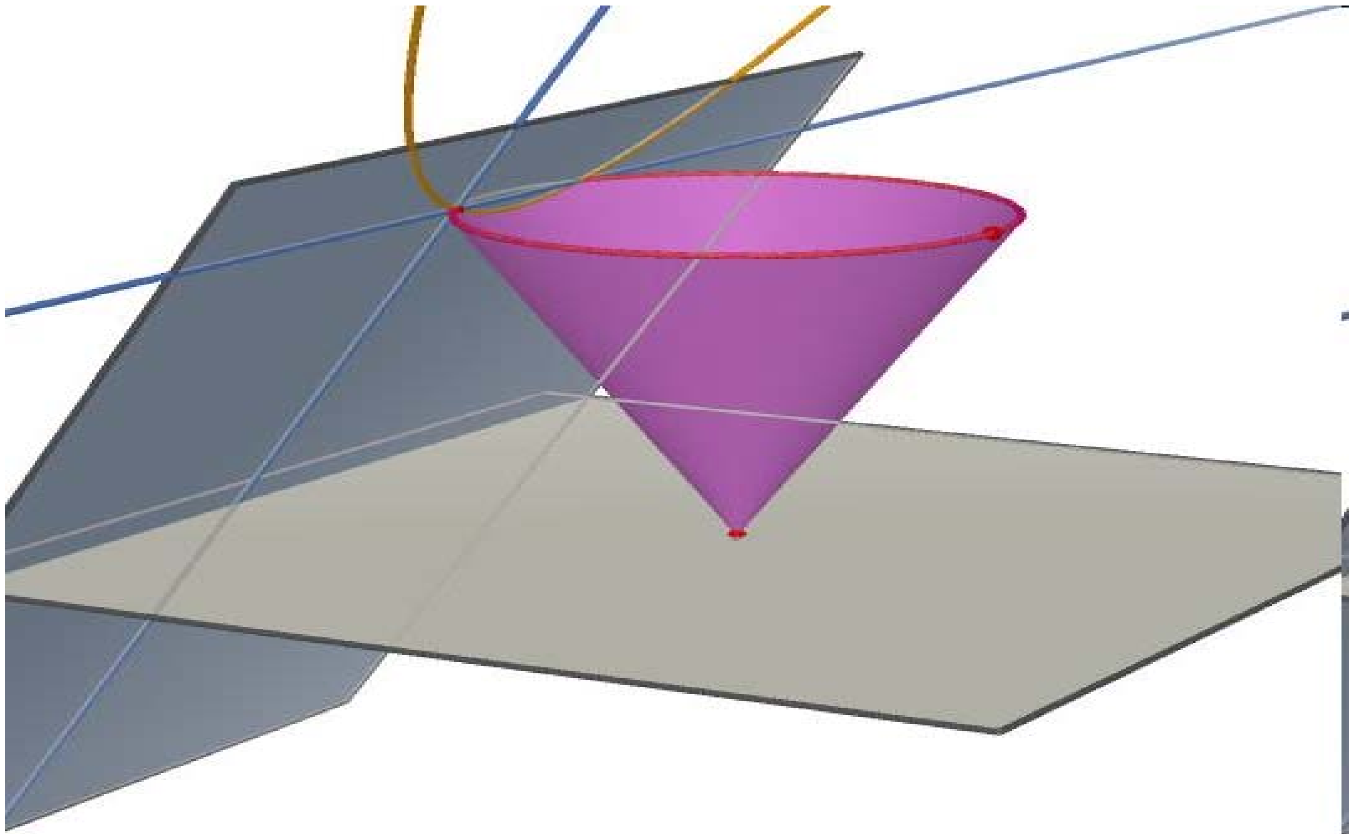
This means E hits Γ in a "small set" with a certain control. (This estimate is supposed to be sharp: use a **BOX** as E. There $\delta = 0$.)

For null sets, demand $\mathrm{Length}(E \cap \Gamma) = 0$

What is a tangent field here?

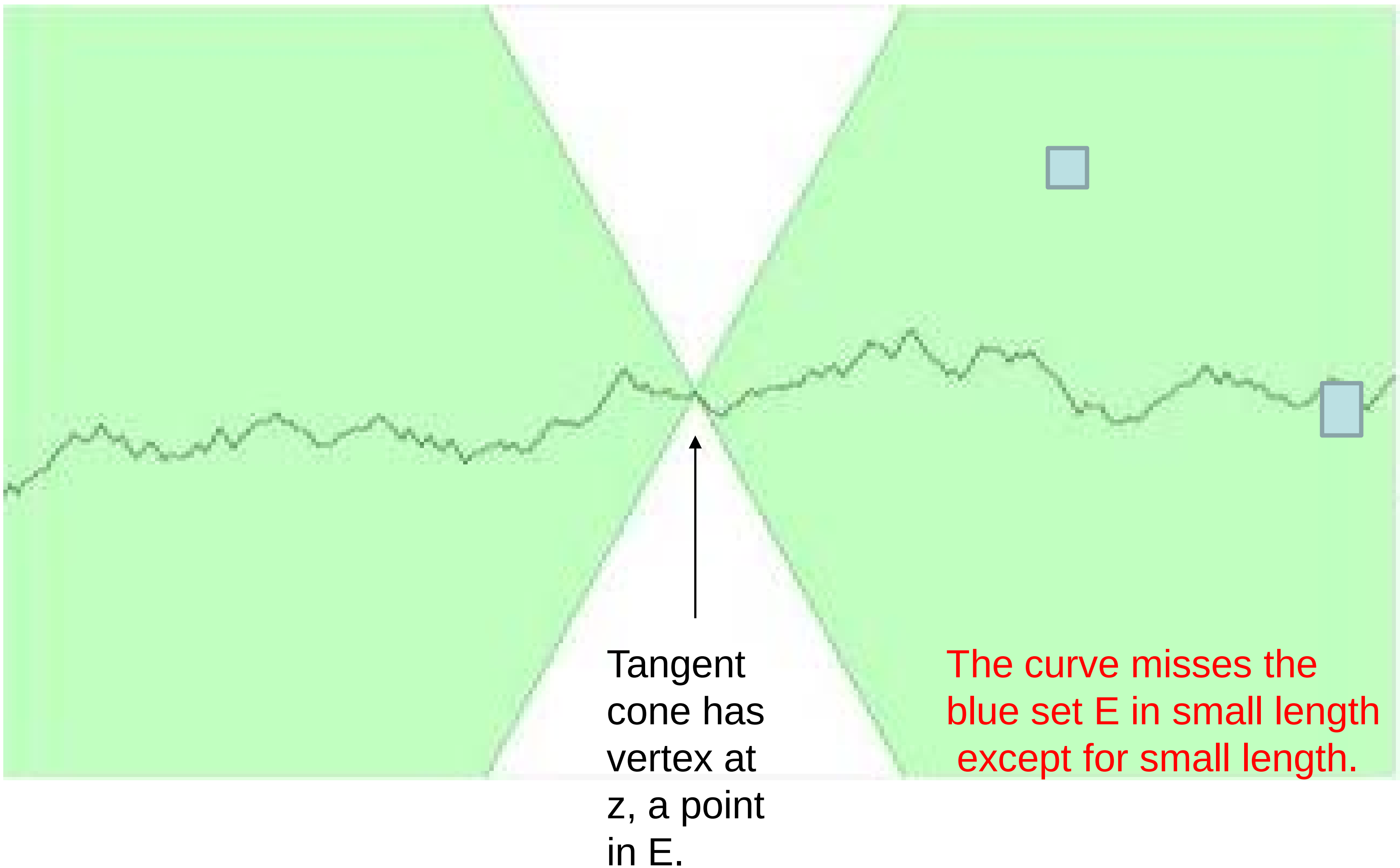
- If G is a differentiable function then at each point x (in the corresponding surface) we have a tangent PLANE to the surface. Another way of saying this is that at x, for a double sided cone oriented in the correct direction, of opening $\pi - \delta$, the cone intersects the surface “in a very small set”, i.e. only close to x.

Tangent Plane and Cone



The usual picture from elementary geometry of surfaces:
The cone misses the surface locally.

Lipschitz Curve Inside “Tangent Cone”: $|F(x) – F(x')| \leq M|x-x'|$



The curve intersects a box of measure ε in length ~ $\varepsilon^{1/d}$ = sidelenght of box.

This should be the "worst case": E = box.

Some Combinatorics

There is a combinatorial problem whose solution would make part of the proof much easier.

Corollary to Erdös-Szekeres Theorem (1930's): In $D = 2$, a set of N points contains at least $N^{1/2}$ points lying on a good Lipschitz curve.

Open Problem: In $D > 2$ can we have

$N^{(D-1)/D}$ points lying on a good Lipschitz surface? (It can't be exactly correct.)

C,J prove a local, measure theoretic version of E-S in higher dimensions.

("Old" + New)Tangent Field Results

- G. Alberti , M. Csörnyei, D. Preiss (ACP, to appear)
- **Thm**: (ACP): Existence of tangent field implies existence of $\mathbb{R}^d$ to $\mathbb{R}^d$ Lipschitz function nowhere diff. on set a Lebesgue null set E.
- **Thm** (A,C,P): Every measurable set of finite measure has a tangent field in dim = 2. (Implies Rademacher is sharp in dimension 2.)
- **Thm 1:** (M. Csörnyei, PWJ): Any measurable set of finite measure has a "narrow" cone tangent field (for any dim d). (Split E into a controlled number of subsets, allow narrow cones.)
- **Thm 2:** (M. Csörnyei, PWJ): Any Lebesgue null set has a tangent field (for any dim d).
- **Cor.** (From {M.Csörnyei, PWJ} + {A,C,P}) In any dimension Rademacher is sharp.

Some Ideas of the Proof

N.B. We prove a "local" theorem, about subsets of the unit cube. This localization is required by the form of our proof, and we don't know if it is necessary. But null sets are just null sets. (Local implies global.)

For simplification, we outline the proof in dim = 2. Let Q be a dyadic square, Q = I x J, where I, J are intervals of the same size. For an interval I let L, R denote the left, right halves of J. Construct the product for $\mu = \varepsilon^{-1} 1_E(x) =$

$$\prod(1 + 1_Q a_I h_I(x)) \prod(1 + 1_Q a_L h_L(y)) (1 + 1_Q a_R h_R(y))$$

$= \Pi_1$ times $\Pi_2 = \varepsilon^{-1}$ on E.

Model Case

E is a subset of $[0,1]x[0,1]$, of measure $= \boldsymbol{\varepsilon}$.

Then

$1_E(x,y) = \boldsymbol{\varepsilon}\, \Pi_1$ times Π_2 .

Therefore for each $z = (x,y)$ in E, either

$\Pi_1(z) \geq \boldsymbol{\varepsilon}^{-1/2}$ or $\Pi_2(z) \geq \boldsymbol{\varepsilon}^{-1/2}$.

This divides E into E_x, E_y according to whether the first case happens or not.

Assume the first case.

Look at E_x

If we integrate on a horizontal line L,

$$\int_L \Pi_1(x,y)\,dx = 1,$$

because on L, Π_1 is just a product corresponding to some probability measure.

Since

$\Pi_1(z) \geq \boldsymbol{\varepsilon}^{-1/2}$ on E_x,

$$\text{length}(E_x \cap L) \leq \boldsymbol{\varepsilon}^{1/2}.$$

Now Perturb

- By perturbing the Haar functions we can control integrals on lines with small slope. Here we must throw away a small piece of E_x.
- Working much harder, perturb and control integrals over Lipschitz perturbations of horizontal curves. (Strangely, the problem, which is deterministic, becomes "random" because the space of Lipschitz curves is infinite dimensional.)
- Now work harder for the full theorem. Lots of estimates. Today we outline how to control cones with very small angle, not $\pi - \delta$.

Technical Tools

1. A_1 weights
2. BMO
3. Fourier Magic (Special Kernels, “Holomorphy”)
4. “Off Diagonal” estimates from Cauchy Integrals on Lipschitz Curves and the estimates from TSP
5. Hoeffding’s Inequality and Martingale Square Functions

The product Π Redefined

Let be the collection of all possible translations of the standard dyadic grid in $\mathbb{R}^d$. (Only cubes of length $\leq$ 1.) This gives a probability space of grids. For each such grid G, form the product Π_G as before.

Then Π_G (x) ~ $\boldsymbol{\varepsilon}^{-1}$ on E. We now define Π by the geometric mean:

$$\Pi = \exp\{\mathcal{E}_G(\log(\Pi_G (x))\},$$

where $\mathcal{E}_G$ is the expectation over the grids.

Then Π(x) ~ $\boldsymbol{\varepsilon}^{-1}$ on E.

A Technical Adjustment: A_1 weights

Instead of the characteristic function of E, use $w(x) = (M(1_E)(x))^{1-\delta}$ for some small δ. (M = H.L. Maximal function)Then by Coifman-Rochberg, w(x) is an A_1 Weight, with norm bounded by C(δ).

Forming the product as before (but with the new probability measure $(||w||_1)^{-1} w(x)$,We get $\Pi(x) \sim \boldsymbol{\varepsilon}^{-1+\delta}$ on E. Splitting as before we obtain:

$$\Pi_1(z) \geq \boldsymbol{\varepsilon}^{-1/2 + \delta/2} \text{ on } E_x.$$

And we have (due to the A_1 weight's bound)

L^∞ bounds on pieces of the log.

Directionality and Scales

As before we write $\Pi_G = \Pi_{1,G}$ times $\Pi_{2,G}$ (x,y) (Haar functions), so $\Pi_{1,G}$ "sees" the x direction.

Again we divide E into E_x, E_y according to which product is large. Now

$$\log(\Pi_{1,G}(x)) = \Sigma_n \log(\Pi_{1,G,n}(x)),$$

Where the expectation $(= \mathcal{E}_{G,n})$uses only cubes of scale 2^{-n}, i.e.

$$\log(\Pi_{1,G,n}(x)) = \mathcal{E}_{G,n}(\log(\Pi_{1,G,n}(x))$$

Then $E_G(\log(\Pi_{1,G,n}(x))\}$, is a nice Lipschitz function, or so we would like. (A$_1$ used here!)

Philosophy and Obstacles

1. The log of any subproduct is a $BMO(\mathbb{R}^d)$ function of bounded norm. ☺
2. John-Nirenberg thus gives pointwise bounds except on a set of very small measure. ☺
3. But this log(subproduct) is completely out of control on a curve and is not even integrable there. ☹
4. We can perturb our initial product so that it's log is better behaved on curves. ☺
5. But even this will have problems related to #3. ☹ (We need finer estimates.)

Reasons to be Perturbed

1. We form a new product where we replace Haar functions h with perturbations $h^{\sim}$. The functions $h^{\sim}$ have integral = 0 on lines in a small cone and decay rapidly. Only “small changes” from the original product. (J.-N.)
2. A Lipschitz curve is well approximated in an L^2 sense by lines. (“Beta numbers”)
3. While log(subproduct) is still not very good on the curve, it is “mostly under control” on E intersect the curve. (Local Hoeffding replaces J.-N.)
4. So we can still work with subproducts on pieces of the curve.

Products and Ratios

We build three different products:

1. $\prod_1$ is built directly from E using Haar functions "in the x direction". (As before)
2. $\prod_\sim$ is built by replacing Haar functions from step 1 with special perturbed "Haar-like" functions.
3. $\prod_\Gamma$ (ONLY DEFINED ON Γ) replaces Haar functions in $\mathbb{R}^d$ with Haar functions respecting grids of dyadic intervals on a Lipschitz curve Γ.

Philosophy: Integrate after throwing away a very small piece of E on Γ

$$\int_{E_\Gamma \cap \Gamma} \Pi_1(x,y)\, dx$$

$$= \int_{E_\Gamma \cap \Gamma} (\Pi_1 / \Pi_\sim(z))(\Pi_\sim / \Pi_\Gamma)\Pi_\Gamma \, dx$$

$$\sim \int_{E_\Gamma \cap \Gamma} \Pi_\Gamma \, dx \leq \int \Pi_\Gamma \, dx = 1$$

This gives us the correct estimate.

Estimates for $\prod_1 / \prod_\sim$

Log $(\prod_1 / \prod_\sim)$ **= (Sum over scales = 2^{-n})=**

$\Sigma_n \Sigma_Q \, \mathcal{E}_{G,Q}(\log(1 + a_{Q'}(h_{Q'}(x)/(1 + a_{Q'}(h^\sim_{Q'}(x\,))$

Where the expectation $\mathcal{E}_{G,Q}$ is over Q' having center in Q, a STANDARD dyadic cube. Now the log in the sum is $=$

$a_{Q'}(h_{Q'}(x) - (h^\sim_{Q'}(x\,)) +$

$O(|a_{Q'}|^2 |h_{Q'}(x) - h^\sim_{Q'}(x\,)|(|h_{Q'}(x)| + |(h^\sim_{Q'}(x\,)|)$

The first term is a BMO($\mathbb{R}^d$) function, the second a square function.

BMO($\mathbb{R}^d$) Estimates

$\Sigma_n \Sigma_Q \, \mathcal{E}_{G,Q}(a_{Q'}(h_{Q'}(x) - (h^{\sim}_{Q'}(x\,) =$

$\Sigma_n \Sigma_Q \, (\boldsymbol{\Psi}_Q - \boldsymbol{\Psi}^{\sim}_Q) = \Sigma_n(\boldsymbol{\Psi}_n - \boldsymbol{\Psi}^{\sim}_n\,)$.

Easy estimate:

$||\Sigma_n \boldsymbol{\Psi}_n\,||_{BMO} \leq C$ (Here BMO($\mathbb{R}^d$))

With more work:

$||\Sigma_n(\boldsymbol{\Psi}_n - \boldsymbol{\Psi}^{\sim}_n\,)||_{BMO} \leq C\,\Upsilon$ (Ditto)

John Nirenberg on the "Bad Set":

$|\{x: |\Sigma_n(\boldsymbol{\Psi}_n - \boldsymbol{\Psi}^{\sim}_n\,)| > C'\,\Upsilon \log(1/\boldsymbol{\varepsilon})\}| << \boldsymbol{\varepsilon}$

Let E' be the subset of E that is not "bad".

Life on $E'_x \cap \Gamma$

Here $|\Pi_1 / \Pi_\sim| \leq \boldsymbol{\varepsilon}^{-\delta}$.

We would Like to have:

$||\log(\Pi_\sim / \Pi_\Gamma)||_{BMO(\Gamma)} \leq C\,\Upsilon.$

Note this is BMO on the curve Υ.

☹ BUT IT IS NOT. ☹

Closer analysis needed.

.

Hoeffding's Inequality on Γ

On Γ, represent the ratio as a Haar series (on Γ)!

$\log(\Pi_{\sim} / \Pi_{\Gamma})(x) = \Sigma_I \, \alpha_I \, h_I(x)$,

Where we sum over all "dyadic intervals"

I in the curve Γ. (Badly divergent!)

Careful calculation: most x in E'_x are ok:

$\Sigma_{I \ni x} \, |\alpha_I|^2 \leq C' \Upsilon \log(1/\epsilon)$.

Hoeffding's inequality:

$|\{x \in E'_x : |\log(\Pi_{\sim}/\Pi_{\Gamma})(x)| > C'' \Upsilon \log(1/\epsilon)\}| << \epsilon$

So we can throw out a "garbage set" on Γ.

Hoeffding's Inequality

Hoeffding's theorem states that on [0,1], a function given by a Haar series,

$$F(x) = \Sigma\ \alpha_I\, h_I(x)\ ,$$

where

$$\Sigma_{I \ni x} |\alpha_I|^2 \leq 1 \text{ for all } x,$$

satisfies

$$|\{x : |F(x)| \geq \lambda\}| \leq 2 \exp\{-\lambda^2/2\}.$$

"Proof":

$$\cosh(x) \leq \exp\{x^2/2\}$$

Integrating over Γ: The idea

$$\epsilon^{-1/2}\ |E'_x \cap \Gamma| \leq \int_{E_\Gamma \cap \Gamma} \Pi_1(x,y)\, dx \qquad (+\ \boldsymbol{\varepsilon}^{+\text{power}})$$

$$= \int_{E_\Gamma \cap \Gamma} (\Pi_1 / \Pi_\sim(z))(\Pi_\sim / \Pi_\Gamma)\Pi_\Gamma\, dx$$

$$\leq \boldsymbol{\varepsilon}^{-2\delta} \int_{E_\Gamma \cap \Gamma} \Pi_\Gamma\, dx \ \leq \boldsymbol{\varepsilon}^{-2\delta} \int_{\Gamma} \Pi_\Gamma\, dx \ = \boldsymbol{\varepsilon}^{-2\delta}$$

So $|E'_x \cap \Gamma| \leq 2\ \boldsymbol{\varepsilon}^{1/2\ -\ 2\delta}$ (If Ratios $\leq \boldsymbol{\varepsilon}^{-\delta}$)

So we must estimate ratios.

(Baron) Charles Jean de la Vallée Poussin

$$V_n = \frac{S_n + S_{n+1} + \ldots + S_{2n-1}}{n}$$

The $\mathbb{R}^d$ de la Vallée Poussin kernel has L^1 norm > 1 and is not Schwarz class. But easy variants are Schwartz class, have L^1 norm < 1 +δ, and have Fourier Transform $\equiv$ 1 near 0, $\equiv$ 0 near infinity.

Kernels, Perturbation, and Hoeffding

Let K be a nice kernel with:

K^ = 0 for |x| < α

K^ = 0 for |x|, |y| > 1/α

Definition of perturbation: h~ = K* h

Then $\int_L$ **h~**(x,y) dx = 0

if we integrate over lines L of slope less than α^2**. (Fourier Transform vanishes in a cone.) This is the crucial requirement.**

Estimates, Also for Kernels K in L^1

Let Ψ_Q, $\Psi_{Q'}$ be a functions of mean value zero, supported on 3Q, 3Q'. Then if $L(Q) = 2^{-n}$, $L(Q') = 2^{-(n+j)}$, standard arguments show

$$|\int \Psi_Q(x)\,\Psi_{Q'}(x)\,dx\,| \leq$$

$$C2^{-j}\,||1_{3Q'}(x)\,2^{-n}\nabla\Psi_Q\,||_2\,||\,\Psi_{Q'}\,||_2$$

Corollary of this is:

$$|<\Psi_n,\Psi_{n+j}>| \leq C'2^{-j}\,||\,2^{-n}\nabla\Psi_n\,||_2\,||\,\Psi_{n+j}\,||_2$$

Here Ψ_m is the sum of terms Ψ_{Q^*} where $L(Q^*) = 2^{-m}$, and we assume all cubes on that scale have finite overlap. Same for $K*\Psi_n$.

“Off Diagonal” Corollaries (by C.S.)

1. $|\langle \Sigma_n \Psi_n, \ \Sigma_{j \geq k} \Psi_{n+j} \rangle| \leq$

   $C2^{-k} \ || \ \Sigma_n \ (2^{-n} \nabla \Psi_n) \ ||_2 \ || \ \Sigma_n \ \Psi_n \ ||_2$

2. We need later: Let $F_n = \Psi_n - \Psi^{\sim}_n$ or good L^1 kernels K_n convolved with that.

   $\Sigma_n |\langle F_n, \Sigma_{j \geq k} F_{n+j} \rangle|$

   $\leq \ C2^{-k} \ (\Sigma_n \ (||2^{-n} \nabla \ F_n||_2)^2 \ )^{1/2}$

   times $(\Sigma_n (|| \ F_n \ ||_2)^2 \ )^{1/2}$

   $\leq C' 2^{-k} \ \Sigma_n \ (||2^{-n} \nabla \ F_n||_2)^2$