Kleinian groups and the Sullivan dictionary I

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Overview

The Sullivan dictionary provides a conceptual framework for understanding the connections between dynamics of rational maps and Kleinian groups. In some cases, very similar proofs can be given of related results in the two field. More commonly, the dictionary suggests loose analogies which motivate research in each area.

In this mini-course, we will begin with a discussion of the Sullivan Dictionary and the quasiconformal deformation theory of Kleinian gropus. We will then focus on three big conjectures, Marden's Tameness Conjecture, Thurston's Density Conjecture and the Bers-Sullivan-Thurston Density conjecture which have been resolved in the last decade. Ken Bromberg, Dave Gabai and Yair Minsky will give talks focussed on these conjectures.

Finally, we will discuss applications of these conjectures and recent developments in the deformation theory of Kleinian groups

Rational maps and Kleinian groups

- A **rational map** is a map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the form $R(z) = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomials.
- A **Kleinian group** is a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{C})$. We identify $\mathrm{PSL}_2(\mathbb{C})$ with both the group of conformal automorphisms of $\widehat{\mathbb{C}}$ and the group of orientation-preserving isometries of $\mathbb{H}^3$. To tie these two pictures together, one may view $\widehat{\mathbb{C}}$ as the boundary at infinity of $\mathbb{H}^3$.
- We will assume throughout the talks that Γ is finitely generated, torsion-free and non-abelian.
- $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold, i.e. a complete Riemannian manifold locally isometric to $\mathbb{H}^3$.

Fatou sets and domains of discontinuity

- If R is a rational map, $z \in F(R)$ if there exists a neighborhood U of z so that $\{f^n|_U\}$ is a normal family. $F(R)$ is called the **Fatou set**.
- The **Julia set** $J(R)$ of a rational map R is the complement of its Fatou set, i.e. $J(R) = \widehat{\mathbb{C}} - F(R)$.
- If Γ is a Kleinian group, $z \in \Omega(\Gamma)$ if there exists a neighborhood U of z so that

$$\{\gamma \in \Gamma \mid \gamma(U) \cap U \text{ is non} - \text{empty}\}$$

  is finite. $\Omega(\Gamma)$ is called the *domain of discontinuity.*
- The **limit set** $\Lambda(\Gamma)$ of a Kleinian group Γ is the complement of its limit set, i.e. $\Lambda(\Gamma) = \widehat{\mathbb{C}} - \Omega(\Gamma)$.

Quasiconformal maps

The dictionary is most precise when the major tool is quasiconformal maps.

- If an orientation-preserving homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is **ACL**(absolutely continuous on lines), then one can define a **Beltrami differential**
$$\mu_\phi = \frac{\phi_z}{\phi_{\bar{z}}} = \frac{\phi_x + i\phi_y}{\phi_x - i\phi_y}$$
which we think of as an element of $L^\infty(\widehat{\mathbb{C}})$. The maps ϕ is **quasiconformal** if
$$||\mu_\phi|| < 1.$$
- If ϕ is differentiable at z, then $d\phi_z$ takes the unit circle to an ellipse with eccentricity
$$\frac{1 + |\mu_\phi(z)|}{1 - |\mu_\phi(z)|}$$
and the pre-image of the major axis make an angle $arg(\mu_\phi(z))$ with the x-axis.

The Measurable Riemann Mapping Theorem

Notice that an equivariant Beltrami differential gives rise to an invariant line field.

A major tool in both fields is:

The Measurable Riemann Mapping Theorem: *If* $\mu \in L^\infty(\widehat{\mathbb{C}})$ *and* $||\mu||_\infty < 1$, *then there exists a quasiconformal map* $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *such that* $\mu_\phi = \mu$. *The quasiconformal map* ϕ *is unique if we require that* $\phi(0) = \phi(0)$, $\phi(1) = 1$ *and* $\phi(\infty) = \infty$, *i.e.* ϕ *is unique up to post-composition with a conformal automorphism. Moreover,* ϕ *depends analytically on* μ.

Quasiconformal conjugation

Key observation: If Γ is a Kleinian group, $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal map and μ_ϕ is Γ-equivariant, then

$$\phi \Gamma \phi^{-1} = \{\phi\gamma\phi^{-1} \mid \gamma \in \Gamma\}$$

is a Kleinian group. We may define a representation

$$\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$$

by setting $\rho(\gamma) = \phi\gamma\phi^{-1}$ for all $\gamma \in \Gamma$. ρ is said to be **quasiconformally conjugate** to the inclusion map of Γ into $\mathrm{PSL}_2(\mathbb{C})$.

Similarly, if R is a rational map and μ_ϕ is R-equivariant, then $\phi R \phi^{-1}$ is a rational map.

Parallel Theorems

Theorem: (Ahlfors) *If* Γ *is a (finitely generated) Kleinian group, then* $\Omega(\Gamma)$ *has no wandering domains, i.e. if* D *is a component of* $\Omega(\Gamma)$*, then there exists a non-trivial element* $\gamma \in \Gamma$ *such that* $\gamma(D) = D$*.*

Theorem:(Sullivan) *If* R *is a rational map, then* $F(R)$ *has no wandering domains, i.e. if* D *is a component of* $F(R)$*, then the sequence*

$$D, R(D), R^2(D), \ldots, R^n(D), \ldots$$

is eventually periodic.

One may summarize both results by saying that there are **no wandering domains**

Sketch of proofs

- Suppose there exists a wandering domain D in $\Omega(\Gamma)$ (or in $F(R)$).
- Then there exists an infinite-dimensional space of "inequivalent" Beltrami differentials supported on D.
- One may use the group Γ (or the rational map R) to transport the differentials around and obtain an infinite-dimensional space of inequivalent equivariant Beltrami differentials supported on $\Omega(\Gamma)$ (or on $F(R)$).
- The Measurable Riemann Mapping Theorem is then used to obtain an infinite-dimensional space of inequivalent quasiconformal deformations of Γ and hence an infinite dimensional subspace of $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ which is impossible since Γ is finitely generated.

The conformal boundary

- If Γ is a Kleinian group, then one may form the **conformal boundary**

$$\partial_c N = \Omega(\Gamma)/\Gamma$$

  which is naturally a Riemann surface, as it is a quotient of an open subset of $\widehat{\mathbb{C}}$ by a group of conformal automorphisms.
- $\Omega(\Gamma)$ admits a unique hyperbolic metric, called the *Poincaré metric*, which is Γ-invariant, so $\partial_c N$ is also a hyperbolic surface.
- **Ahlfors' Finiteness Theorem:** *If Γ is finitely generated, $\partial_c N$ is a finite type Riemann surface, i.e. it has finite area in its Poincaré metric.*

Convex cocompact hyperbolic 3-manifolds

- $$\hat{N} = N \cup \partial_c N = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$$

  is a bordification of N.
- Γ (or N) is said to be **convex cocompact** if $\hat{N}$ is compact.
- (Sullivan) Γ is convex cocompact if and only if it is hyperbolic on its limit set, i.e. for every $x \in \Lambda(\Gamma)$, there exists $\gamma \in \Gamma$ such that $|\gamma'(x)| > 1$.
- In the setting of complex dynamics, the analogue is the class of hyperbolic rational maps, i.e. maps which are expanding on their Julia set.

- A manifold is *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold.
- Notice that if N is convex cocompact, then it is topologically tame, since it is homeomorphic to the interior of the compact 3-manifold $\hat{N}$.
- **Marden's Tameness Conjecture:** Every hyperbolic 3-manifold with finitely generated fundamental group is topologically tame.
- Marden's Tameness Conjecture was established in 2004 by Agol and Calegari-Gabai.
- It turns out to have important consequences for the geometry, dynamics and algebra of hyperbolic 3-manifolds, e.g. it implies Ahlfors' Measure Conjecture. More on this in lecture 2.

Quasiconformal deformation spaces

- Let $QC(\Gamma)$ denote the set of (conjugacy classes) of representations $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ which are quasiconformally conjugate to the inclusion map, i.e. there exists a quasiconformal map $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\rho(\gamma) = \phi\gamma\phi^{-1}$ for all $\gamma \in \Gamma$.
- $$QC(\Gamma) \subset X(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))//\mathrm{PSL}_2(\mathbb{C})$$
- $X(\Gamma)$ is called the **character variety**.

Quasiconformal stability

Theorem:(Marden) *If* Γ *is convex cocompact, then* Γ *is quasiconformally stable, i.e.* $QC(\Gamma)$ *contains an open neighborhood in* $X(\Gamma)$ *of the inclusion map. Moreover,* $QC(\Gamma)$ *is an open subset of* $X(\Gamma)$.

Sketch of proof: If Γ is convex cocompact, it has a finite-sided fundamental polyhedron P. If ρ is near the inclusion map, one may "wiggle" P to obtain a fundamental polyhedron P_ρ for $\rho(\Gamma)$. Therefore, $\rho(\Gamma)$ is also convex cocompact and there exists a biLipschitz homeomorphism

$$h : N \to N_\rho = \mathbb{H}^3/\rho(\Gamma)$$

in the homotopy class determined by ρ. h lifts to a bilipschitz map

$$\tilde{h} : \mathbb{H}^3 \to \mathbb{H}^3$$

which conjugates Γ to $\rho(\Gamma)$.

Structural Stability

The bilipschitz map $\tilde{h}$ extends to a quasiconformal map

$$\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

which conjugates Γ to $\rho(\Gamma)$ as desired.

Definition: A Kleinian group is **structurally stable** if there exists an open neighborhood in $X(\Gamma)$ of the inclusion map consisting of faithful representations, i.e. if every representation which is close enough to the inclusion is injective.

Corollary: Convex cocompact Kleinian groups are structurally stable.

Characterizing structurally stable Kleinian groups

Theorem: (Sullivan) If a finitely generated Kleinian group Γ is structurally stable, it is convex cocompact.

Sketch of Proof: (for experts) The λ-Lemma may be used to show that any holomorphically varying 1-parameter family of faithful representations consists of quasiconformally conjugate representations. Since $X(\Gamma)$ is an algebraic variety, all representations near the inclusion map lie in a holomorphic disk including the inclusion map. Therefore, $QC(\Gamma)$ has the same dimension as $X(\Gamma)$. We will soon see that $QC(\Gamma)$ is a quotient of $\mathcal{T}(\partial_c N)$. (This proof uses Sullivan's Rigidity Theorem which asserts that there are no invariant line fields supported on the limit set of a finitely generated Kleinian group.) Therefore, using a calculation of the dimension of $X(\Gamma)$ due to Thurston, we conclude that $\chi(\partial_c N) = 2\chi(\Gamma)$ and N has no cusps. Therefore, all the ends of N are convex cocompact, so N itself is convex cocompact.

Structurally stable rational maps

A rational map R is *structurally stable* if R is topologically conjugate to all rational maps in some neighborhood of R (in the space of rational maps of the same degree).

Theorem: (Mañé-Sad-Sullivan) *The set of structurally stable rational maps is open and dense.*

Conjecture: Structurally stable rational maps are hyperbolic.

(Mañé-Sad-Sullivan) It suffices to prove that no invariant line fields are supported on the Julia set of a rational map.

Bers-Sullivan-Thurston Density Conjecture

Conjecture:(Bers-Sullivan-Thurston) Every finitely generated Kleinian group (which contains no rank two free abelian subgroups) is a limit of convex cocompact Kleinian groups. More formally, if Γ is a finitely generated Kleinian group (which contains no rank two free abelian subgroups), then there exists a sequence $\{\rho_n : \Gamma \to \mathrm{PSL}_2(\mathbb{C})\}$ of discrete, faithful representations with convex cocompact image converging to the inclusion map.

If Γ contains a rank two free abelian subgroup, then it is a limit of geometrically finite Kleinian groups (all of whose parabolic elements lie in a rank two free abelian subgroup).

Ken Bromberg will discuss the proof(s) of this conjecture.

Parameterizing quasiconformal deformation spaces

Quasiconformal Parameterization Theorem: (Bers, Kra, Maskit, Sullivan) *If* Γ *is a freely indecomposable Kleinian group, or, more generally, if every component of* $\Omega(\Gamma)$ *is simply connected, then*

$$QC(\Gamma) \cong \mathcal{T}(\partial_c N).$$

We recall that the Teichmüller space $\mathcal{T}(\partial_c N)$ is the collection of pairs (X, ψ) where X is a Riemann surface and $\psi : \partial_c N \to X$ is a quasiconformal map, where $(X, \psi) \sim (X', \psi')$ if and only if there exists a conformal map $j : X \to X'$ which is homotopic to $\psi' \circ \psi^{-1}$.

In general,

$$QC(\Gamma) \cong \mathcal{T}(\partial_c N)/\mathrm{Mod}_0(\partial_c N).$$

where $\mathrm{Mod}_0(\partial_c N)$ is the group of isotopy classes of homeomorphisms of $\partial_c N$ which extend to homeomorphisms of $\hat{N}$ which are homotopic to the identity.

Sketch of proof

- If $\rho \in QC(\Gamma)$, let

$$N_\rho = \mathbb{H}^3/\rho(\Gamma) \text{ and } \partial_c N_\rho = \Omega(\rho(\Gamma))/\rho(\Gamma).$$

- If $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal and $\rho(\gamma) = \phi\gamma\phi^{-1}$ for all $\gamma \in \Gamma$, then $\phi(\Omega(\Gamma)) = \Omega(\rho(\pi_1(M)))$ and ϕ descends to a quasiconformal homeomorphism

$$\bar{\phi} : \partial_c N \to \partial_c N_\rho.$$

- We define the Ahlfors-Bers map

$$AB : QC(\Gamma) \to \mathcal{T}(\partial_c N_\rho)$$

by letting

$$AB(\rho) = (\partial_c N_\rho, \bar{\phi}).$$

AB is surjective

- If $(X,\psi) \in \mathcal{T}(\partial_c N_\rho)$, then $\psi : \partial_c N_\rho \to X$ lifts to a quasiconformal homeomorphism
$$\tilde{\psi} : \Omega(\rho) \to \tilde{X}$$
where $\tilde{X}$ is the universal cover of X.
- Let μ denote the Beltrami differential of $\tilde{\psi}$. Set $\mu = 0$ on $\Lambda(\Gamma)$, to obtain $\mu \in L^\infty(\widehat{\mathbb{C}})$. The Measurable Riemann Mapping Theorem produces a quasiconformal map $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\mu_\phi = \mu$.
- Since μ was Γ-equivariant, $\phi\Gamma\phi^{-1}$ is a Kleinian group and we define $\rho \in QC(\Gamma)$ by setting $\rho(\gamma) = \phi\gamma\phi^{-1}$. Then
$$AB(\rho) = (X,\psi)$$
so AB is surjective.

AB is injective

- Suppose that ϕ_1 and ϕ_2 are quasiconformal maps so that
$$\rho_i(\gamma) = \phi_i \gamma \phi_i^{-1}.$$
- If $AB(\rho_1) = AB(\rho_2)$, then $\bar{\phi}_2 \circ \bar{\phi}_1^{-1}$ is homotopic to a conformal map, so we may assume that $\phi_2 \circ \phi_1^{-1}$ is conformal on $\Omega(\rho_1(\Gamma))$.
- If Γ is convex cocompact, then Ahlfors proved that $\Lambda(\rho_1(\Gamma))$ has measure zero, so we may conclude that $\phi_2 \circ \phi_1^{-1}$ is conformal on all of $\widehat{\mathbb{C}}$, which implies that ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}_2(\mathbb{C})$.
- In general, we consider the Beltrami differential of $\phi_2 \circ \phi_1^{-1}$, which is trivial on $\Omega(\rho_1(\Gamma))$. Sullivan's Rigidity Theorem implies that no Beltrami differential is supported on $\Lambda(\rho_1(\Gamma))$, so the Beltrami differential is trivial on $\widehat{\mathbb{C}}$, so again $\phi_2 \circ \phi_1^{-1}$ is conformal and ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}_2(\mathbb{C})$.

Quasifuchsian Kleinian groups

- Suppose that Γ is a **Fuchsian group**, i.e. a subgroup of $\mathrm{PSL}_2(\mathbb{R})$, and $X = H/\Gamma = S$ is a closed Riemann surface, where H is the upper half-plane.
- Then Γ also acts on $\widehat{\mathbb{C}}$, $\Omega(\Gamma) = H \cup \bar{H}$,

$$N = \mathbb{H}^3/\Gamma \cong X \times \mathbb{R} \text{ and } \partial_c N = (H \cup \bar{H})/\Gamma = X \cup \bar{X}.$$

- The quasiconformal parameterization asserts, in this case, that

$$QC(\Gamma) \cong \mathcal{T}(X) \times \mathcal{T}(\bar{X}).$$

- A Kleinian group is called **quasifuchsian** if it is quasiconformally conjugate to a Fuchsian group.
- A quasifuchsian Kleinian group is determined by its conformal boundary and any pair of conformal structures on a surface and its complex conjugate arise as the conformal boundary of a quasifuchsian Kleinian group.

- Let M be a compact, orientable, atoroidal 3-manifold, e.g. $F \times [0,1]$.
- Let $AH(M)$ denote the space of (conjugacy classes of) discrete faithful representations $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$.
- If $\rho \in AH(M)$, then $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is a hyperbolic 3-manifold and there exists a homotopy equivalence $h_\rho : M \to N_\rho$ such that $(h_\rho)_* = \rho$.
- $$AH(M) \subset X(M) = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbb{C}))//\mathrm{PSL}_2(\mathbb{C})$$
- So, $AH(M)$ is the space of marked hyperbolic 3-manifolds homotopy equivalent to M.

The interior of $AH(M)$ and the Density Conjecture

(Marden, Sullivan) The interior $\mathrm{int}(AH(M))$ of $AH(M)$ consists exactly of the convex cocompact representations, i.e. representations such that N_ρ (or $\rho(\pi_1(M))$) is convex cocompact.

Bers-Sullivan-Thurston Density Conjecture:

$$AH(M) = \overline{\mathrm{int}(AH(M))}$$

- Associated to a convex cocompact representation, there is a well-defined marked compact 3-manifold $(\hat{N}_\rho, h_\rho)$.
- Let $\mathcal{A}(M)$ denote the space of marked compact 3-manifolds homotopy equivalent to M, i.e. pairs (M', h') where M' is a compact 3-manifold and $h : M \to M'$ is a homotopy equivalence. We say two pairs (M_h, h_1) and (M_2, h_2) are equivalent if there exists an orientation-preserving homeomorphism $j : M_1 \to M_2$ such that j is homotopic to $h_2 \circ h_1^{-1}$.
- We define

$$\Theta : \mathrm{int}(AH(M)) \to \mathcal{A}(M)$$

by letting $\Theta(\rho) = (\hat{N}_\rho, h_\rho)$.

Components of $\mathrm{int}(AH(M))$

- (Thurston) Θ is surjective.
- **Marden's Isomorphism Theorem:** *If $\rho_1, \rho_2 \in \mathrm{int}(AH(M))$, then ρ_1 is quasiconformally conjugate to ρ_2 if and only if $\Theta(\rho_1) = \Theta(\rho_2)$.*
- **Idea of Proof:** If ρ_1 and ρ_2 are quasiconformally conjugate, then the quasiconformal conjugacy on $\widehat{\mathbb{C}}$ can be extended to a bilipschitz conjugacy on $\mathbb{H}^3$, so $\hat{N}_{\rho_1}$ and $\hat{N}_{\rho_2}$ are homeomorphic.

  On the other hand, if $\hat{N}_{\rho_1}$ and $\hat{N}_{\rho_2}$ are homeomorphic, one may upgrade the homeomorphism to a bilipschitz homeomorphism from N_{ρ_1} to N_{ρ_2}. This homeomorphism lifts to $\mathbb{H}^3$ and extends to a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ conjugating ρ_1 to ρ_2.

So, components of $\mathrm{int}(AH(M))$ are in one-to-one correspondence with marked homeomorphism types in $\mathcal{A}(M)$ and each component is a quasiconformal deformation space.

Parameterization Theorem: If $\pi_1(M)$ is freely indecomposable, then

$$\mathrm{int}(AH(M)) \cong \coprod_{(M',h')\in\mathcal{A}(M)} \mathcal{T}(\partial M').$$

(Canary-McCullough) In this setting, $\mathcal{A}(M)$ is finite, so $\mathrm{int}(AH(M))$ is homeomorphic to a finite union of open balls.

In general,

$$\mathrm{int}(AH(M)) \cong \coprod_{(M',h')\in\mathcal{A}(M)} \mathcal{T}(\partial M')/Mod_0(M').$$

(McCullough,Canary-McCullough) Typically, if $\pi_1(M)$ is freely decomposable, $\mathcal{A}(M)$ is infinite and $\mathrm{Mod}_0(M')$ is infinitely generated.