Boundary effect and Turbulence

Claude Bardos-retired

For Dennis Fest.

3 Equations:

Navier-Stokes, Euler and Boltzmann

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0 \quad \nabla \cdot u = 0\,, \quad \nu = \mathcal{R}^{-1} \tag{1}$$

$$\partial_t u + u \cdot \nabla u + \nabla p = 0 \quad \nabla \cdot u = 0\,, \tag{2}$$

$$\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\mathrm{Knudsen}(\epsilon)} \mathcal{B}(F_\epsilon, F_\epsilon) \tag{3}$$

4 Numbers

The Mach number, the Reynolds number, the Knudsen number and the Strouhal number.

The Mach number is the ratio of the characteristic velocity of the Fluid with respect to the sound speed, the Strouhal number gives the time scale.

The above Navier-Stokes is often called incompressible because of the relation

$$\nabla \cdot u = 0 \, .$$

However it is mostly used to describe fluctuations of velocity, density and temperature near equilibrium at low Mach number.

The Reynolds number which appears in Navier-Stokes in general is not the real viscosity of the fluid but a rescaled viscosity adapted to the size of the fluctuations of the velocity therefore it is given by the formula:

$$\mathcal{R} = \frac{UL}{\nu_{\text{physical}}}$$

In all practical applications $\mathcal{R}$ is very large therefore ν is very small.

Bicycle 10^2 Industrial fluids (pipes ships...) 10^4, Wings of airplanes 10^6, Space Shuttle10^8, Weather Forcast, Oceanography 10^{10}, Astrophysic 10^{12}.

It would be natural to study the limit $\nu \to 0$ in equation (1) or even to put $\nu = 0$ and then consider the Euler equation...

Things are not so simple.

The Euler equation derived by Euler (1755) is a fantastic object for mathematics.

- It may in the absence of boundary contribute to the understanding of the Navier-Stokes equation (Clay prize!).

- It is with the natural boundary condition $u \cdot \vec{n} = 0$ locally in time a well posed problem for smooth initial data: $u_0 \in C^{1,\alpha}(\Omega)$ or $u \in H^{\frac{d}{2}+1}(\Omega)$ in any space dimension.

- Not well posed in $C^{0,\alpha}$ on can construct simple example on solutions which at time $t = 0$ belong to $C^{0,\alpha}$ and which do not belong to this space for any other time.

- One has a general construction of “wild solutions " Scheffer, Shnirelman, De Lellis and L. Szkelyhidi. Moreover this construction share much in common with the problem of the isometric imbedding Cohn-Vossen versus Nash-Kuiper.

- There is a notion of critical index

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |u(x,t)|^2 dx + (\nabla : (u\otimes u), u) = 0$$

$$|(\nabla : (u\otimes u), u)| \simeq \int_\Omega |\nabla^{\frac{1}{3}} u|^3 dx \Rightarrow \nabla^{\frac{1}{3}} u \in L^3(\Omega) \Rightarrow \text{ energy conservation}$$

Rigorous proof (with Besov spaces) that solutions more regular than $\frac{1}{3}$ conserve energy. Constantin E Titi.

- It is definitely not a physical object.

If the air around a wing is described as a solenoidal, with no vorticity , solution of the $3d$ Euler equation there is no force on the wing and plane or birds cannot fly.

This is the d'Alembert paradox and one of the main reason for the introduction of the Navier-Stokes equations.

With viscosity they take in account the boundary effect or say the production of vorticity at the boundary.

Euler, D’Alembert, Navier and Stokes

Statistical theory of turbulence $\simeq L^2$ weak convergence.

$$\langle u_\nu \otimes u_\nu \rangle \neq \langle u_\nu \rangle \langle u_\nu \rangle \simeq \overline{u_\nu \otimes u_\nu} \neq \overline{u_\nu} \otimes \overline{u_\nu}$$

$0 < \lim_{\nu \to 0} (u_\nu - \overline{u_\nu}) \otimes (u_\nu - \overline{u_\nu}) = \lim_{\nu \to 0} u_\nu \otimes u_\nu - \overline{u_\nu} \otimes \overline{u_\nu}$ Reynolds S.T.

$0 < \epsilon = \nu \langle |\nabla u_\nu|^2 \rangle \simeq \nu \int_\Omega |\nabla u_\nu|^2 dx$ Kolmogorov hypothesis

$\langle |u(x+r) - u(x)|^2 \rangle^{\frac{1}{2}} \simeq (\nu \langle |\nabla u|^2 \rangle)^{\frac{2}{3}} |r|^{\frac{1}{3}}$ Kolmogorov law

Claim it is in the presence of boundary that the relation between turbulence en energy dissipation is the most evident

Smooth solutions of Navier-Stokes equation with boundary condition

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0 \text{ in } \Omega$$
$$u_\nu \cdot \vec{n} = 0\,, \nu(\partial_{\vec{n}} u_\nu + C(x) u_\nu)_\tau + \lambda u_\nu = 0 \text{ on } \partial\Omega$$
$$\lambda(\nu, x) \geq 0! \;\; C(x) \in C(\mathbb{R}^n \mapsto \mathbb{R}^n)$$

- $\lambda = \infty \Leftrightarrow$ Dirichlet, $(C = 0, \lambda = 0) \Rightarrow u_\nu \cdot \vec{n} = 0$ and $(\partial_{\vec{n}} u_\nu)_\tau = 0$.
- With $S(u_\nu) = \frac{1}{2}(\nabla u_\nu + \nabla^t u_\nu)$ and $u_\nu \cdot \vec{n} = 0$ other similar conditions:

$$(S(u_\nu) \cdot \vec{n})_\tau = (\partial_{\vec{n}} u_\nu)_\tau - (\nabla^t \vec{n} \cdot u_\nu)_\tau \Rightarrow \nu (S(u_\nu) \cdot \vec{n})_\tau + \lambda u_\nu = 0\,,$$
$$(\nabla \wedge u_\nu) \wedge \vec{n} = (\partial_{\vec{n}} u_\nu)_\tau + (\nabla^t \vec{n} \cdot u_\nu)_\tau \Rightarrow \nu (\nabla \wedge u_\nu) \wedge \vec{n} + \lambda u_\nu = 0\,.$$

Trace theorem and energy estimate

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,t)|^2 dx + \nu \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(x)|u_\nu(x,t)|^2 d\sigma$$
$$= \nu \int_{\partial\Omega} C(u_\nu)_\tau u_\nu d\sigma$$
$$\int_\Omega |u_\nu(x,t)|^2 dx \le e^{C\nu t} \int_\Omega |u_0(x)|^2 dx$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,t)|^2 dx + \nu \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(x)|u_\nu(x,t)|^2 d\sigma$$
$$= \nu \int_{\partial\Omega} C(u_\nu)_\tau u_\nu d\sigma \to 0\,.$$

Emphasis on smooth solution of Euler equation and fixed initial data.

- Energy estimate $\Rightarrow$ weak convergence $u_\nu \to \overline{u}$

$$\partial_t \overline{u} + \nabla \overline{u} \otimes \overline{u} + \nabla(\overline{u \otimes u}) - \nabla(\overline{u} \otimes \overline{u}) + \nabla \overline{p} = 0$$
$$0 \leq \lim_{\nu \to 0} (u_\nu - u) \otimes (u_\nu - u) = (\overline{u \otimes u}) - (\overline{u} \otimes \overline{u}) = \mathcal{RT}$$

- $\mathcal{RT}$ Reynold stress tensor of Kolmogorov STT $\langle u \otimes u \rangle - \langle u \rangle \otimes \langle u \rangle$

- $0 = \mathcal{RT} \Leftrightarrow \overline{u}$ weak solution of the Euler equation

- Assume that $u_\nu \to (weak)\overline{u}$ smooth solution of the Euler equation then:

$$\int_\Omega |u(x,0))|^2 dx \geq \lim_{\nu \to 0} \int_\Omega |u_\nu(x,t)|^2 dx \geq \int_\Omega |\overline{u}(x,t)|^2 dx = \int_\Omega |u(x,0))|^2 d$$
$$\Rightarrow \int_0^T (\nu \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(x) |u_\nu(x,t)|^2 d\sigma) dt = 0$$

- *There is a common belief that turbulence would be characterized more by decay of energy than loss of smoothness for the Euler equation. Therefore the issue is the converse of the above statement:*

The notion of dissipative solution for the Euler Equation

$$S(w) = \frac{1}{2}(\nabla w + (\nabla w)^t)\,, \quad \partial_t w + P(w\cdot\nabla w) = E(x,t) = E(w)$$

P Leray Projection $P(w\cdot\nabla w) = w\cdot\nabla w + \nabla q$

in Ω $-\Delta q = \sum_{ij} \partial_{x_i} w \partial_{x_j} w$; On $\partial\Omega$ $\frac{\partial q}{\partial \vec{n}} = w\cdot\nabla w\cdot\vec{n} = -\sum_{i,j} w_i w_j \partial_{x_j} \vec{n}_i\,.$

u a smooth solution:

$$\partial_t u + \nabla\cdot(u\otimes u) + \nabla p = 0\,, \nabla u = 0\,, u\cdot\vec{n} = 0\,,$$
$$\partial_t w + w\cdot\nabla w + \nabla q = E$$
$$\frac{1}{2}\int_\Omega |u(x,t) - w(x,t)|^2 \le \int_0^t \int |(E(x,s), u(x,s) - w(x,s))| dxds$$
$$+ \int_0^t \int_\Omega |(u(x,s) - w(x,s)S(w)u(x,s) - w(x,s))| dxds$$
$$+ \frac{1}{2}\int_\Omega |u(x,0) - w(x,0)|^2 dx\,.$$

Hence the definition of a dissipative solution as a divergence free tangent to the boundary vector field which for any test function w as introduced above satisfies the relation: Hence the stability of dissipative solutions with respect to smooth solutions and in particular the fact that whenever exists a smooth solution $u(x,t)$ any dissipative solution which satisfies $w(.,0) = u(.,0)$ coincides with u for all time.

However it is important to notice that to obtain this property one needs to include in the class of test functions w vector field that may have non 0 tangent to the boundary component.

Convergence of the solution of Navier Stokes to a Dissipative solution.

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p = 0$$

$$\partial_t w - \nu \Delta w + w \cdot \nabla w + \nabla q = E(w)$$

$$\frac{1}{2}\frac{d}{dt}|u_\nu(x,t) - w(x,t)|^2_{L^2(\Omega)} + \nu|\nabla u_\nu(t)|^2_{L^2(\Omega)}$$

$$\leq |(S(w) : (u_\nu - w) \otimes (u_\nu - w))| + |(E(w), u_\nu - w)|$$

$$+\nu(\nabla u, \nabla w)_{L^2(\Omega)} + \nu(\partial_{\vec{n}} u_\nu, u_\nu) + \nu(\partial_{\vec{n}} u_\nu, w)_{L^2(\partial\Omega))}$$

with all types of BC $\leq$

$$|(S(w):(u_\nu - w) \otimes (u_\nu - w))| + |(E(w), u_\nu - w)| + \nu(\partial_{\vec{n}} u_\nu, w)_{L^2(\partial\Omega)} + o(\nu$$

With no boundary convergence (modulo subsequence is always true). No hypothesis on the existence of a solution of Euler...Even with very bad (De Lellis-Szekelyhidi) initial data .

With BOUNDARY

In comparison with:

$$u_\nu \to \text{smooth sol.} \Leftrightarrow \lim_{\nu\to 0}\int_0^T(\nu\int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega}\lambda(\nu)|u_\nu(x,t)|^2 d\sigma)dt = 0$$

$$\int_0^T\int_{\partial\Omega}\lambda(\nu)|u_\nu(x,t)|d\sigma)dt \le \sqrt{\lambda(\nu)T}(\int_0^T\int_{\partial\Omega}\lambda(\nu)|u_\nu(x,t)|^2 d\sigma dt)^{\frac{1}{2}}$$

Theorem Convergence to a dissipative solution:

1 In any case, in particular Dirichlet $(\frac{\partial u_\nu}{\partial \vec{n}})_\tau \to 0$ in $\mathcal{D}'(\partial\Omega\times]0,T[)$

2 For Fourier-Navier $\lambda(\nu)u_\nu \to 0$: in $\mathcal{D}'(\partial\Omega\times]0,T[) \to 0$

3 $\lambda(\nu) \to 0$ or $\lambda(\nu)$ bounded and $\int_{\partial\Omega\times]0,T[}\lambda(\nu)|u_\nu(x,t)|^2 d\sigma dt \to 0$

4 In any case Kato $\lim_{\nu\to 0}\nu\int_0^T\int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}}|\nabla u_\nu(x,t)|^2 dxdt \to 0$

Proof of Kato argument For any $w \in T(\partial\Omega\times]0,T[)$ introduce a sequence $w_\nu(s,\tau,t)$ (in geodesic coordinates near $\partial\Omega$) with

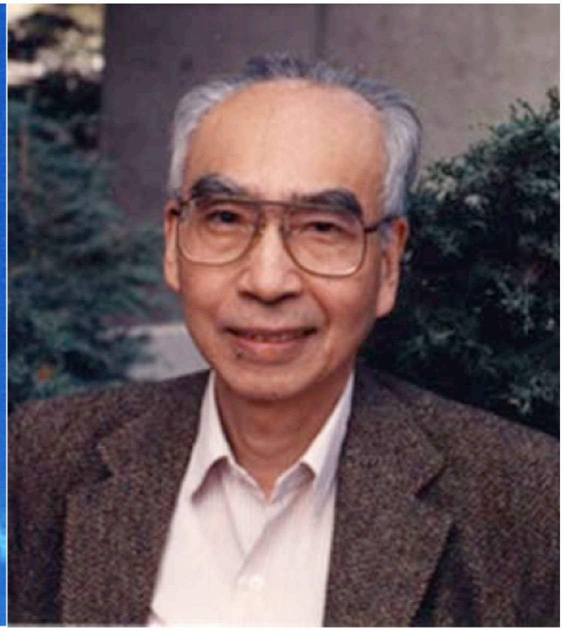
$$\text{support} w_\nu \subset \Omega_\nu\times]0,T[\,, \nabla\cdot w_\nu = 0, \text{ on } \partial\Omega\times]0,T[w_\nu = w$$
$$|\nabla_{\tau,t} w_\nu|_{L\infty} \le C\,, \quad |\partial_s w_\nu|_{L^\infty} \le \frac{C}{\nu}$$

From

$$(0,w_\nu) = ((\partial_t u_\nu + \nabla(u_\nu\otimes u_\nu) - \Delta u_\nu + \nabla p_\nu)w_\nu) =$$
$$-(u_\nu,\partial_t w_\nu) + ((u_\nu\otimes u_\nu) : \nabla w_\nu) + \nu(\nabla u_\nu, \nabla w_\nu) - (\nu\partial_{\vec{n}} u_\nu w)_{L^2(\partial\Omega\times]0,T[}$$
$$\Rightarrow |(\nu\partial_{\vec{n}} u_\nu w)_{L^2(\partial\Omega\times]0,T[)}| = |((u_\nu\otimes u_\nu) : \nabla w_\nu)| + o(\nu)$$

Poincaré estimate and a priori estimate

$$\Rightarrow |((u_\nu\otimes u_\nu) : \nabla w_\nu)| \le C\int_0^T\int_{\Omega_\nu} \nu|\nabla u_\nu|^2 dxdt \to 0$$

Kato: Prandlt..Boundary layer, Kelvin Helmholtz, Von Karman vortex street.

For $\lambda(\nu) \to \infty (= \infty)$ Kato theorem is the only existing result, leaving open the final issue.

To consolidate the fact that this is the correct point of view and that the boundary condition is the good one one can argue that the introduction of a microscopic derivation based on the Boltzmann equation leads to the same results.

The Boltzmann Equation with accomodation

$F_\epsilon(x,v,t)$ solution in $\Omega \times \mathbb{R}^n_v$ of the (rescaled in time) Boltzmann equation:

$$\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\text{Knudsen}(\epsilon)} \mathcal{B}(F_\epsilon, F_\epsilon)$$

with Maxwell Boundary Condition for $v \cdot \vec{n} < 0$ in term of $v \cdot \vec{n} > 0$

$$F_\epsilon^-(x,v) = (1-\alpha(\epsilon))F_\epsilon^+(x,v^*) + \alpha(\epsilon) M(v) \sqrt{2\pi} \int_{v\cdot\vec{n}<0} |v\cdot\vec{n}| F_\epsilon^+(x,v)dv$$

$$0 \le \alpha(\epsilon) \le 1\,, v^* = v - 2(v\cdot\vec{n})\vec{n} = \mathcal{R}(v)\,,$$

$$M(v) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|v|^2}{2}} \quad \Lambda(\phi) = \sqrt{2\pi}\int_{\mathbb{R}^n_v} (v\cdot\vec{n})_+ \phi(v) M(v) dv.$$

$$\Lambda(1) = 1(\text{proba!}) \quad F_\epsilon^-(x,v) = (1-\alpha(\epsilon))F_\epsilon^+(x,\mathcal{R}(v)) + \alpha(\epsilon)\Lambda(\frac{F_\epsilon}{M})\,.$$

Analysis Theorem

- Scaling and convergence low Mach number:

$$\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon^{1+q}} \mathcal{B}(F_\epsilon, F_\epsilon)$$

$$F_\epsilon^-(x,v) = (1-\alpha(\epsilon)) F_\epsilon^+(x, \mathcal{R}(v)) + \alpha(\epsilon) \Lambda(\frac{F_\epsilon}{M})$$

$$F_\epsilon = G_\epsilon M(v) = (1 + \epsilon g_\epsilon) M(v)$$

$$M_{1,\epsilon u^{in},1} = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v-\epsilon u^{in}|^2}{2}} \qquad M = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v|^2}{2}}$$

• **Theorem** In a periodic box $\mathbf{T}^3$ (no boundary and $q > 0$ Saint-Raymond (2003) Let F_ϵ be a family of renormalized solutions in $\Omega \times \mathbb{R}^n_v$ of the Boltzmann equation:

$$\epsilon \partial_t F_\epsilon + v \nabla_x F_\epsilon = \frac{1}{\epsilon^{1+q}} \mathcal{B}(F_\epsilon, F_\epsilon)$$

with initial data

$$F_\epsilon(x, v, 0) = M_{1,\epsilon u^{in},1} = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v-\epsilon u^{in}|^2}{2}}, \ \ \nabla \cdot u^{in}(x) = 0.$$

Then the family $(\frac{1}{\epsilon} \int v F_\epsilon dv)$ is relatively compact in $w\!-\!L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^3))$ and each of its limit points is a dissipative solution of the $3d$ Euler equation.

• For $q = 0$, $u_\epsilon = \frac{1}{\epsilon}\int_{\mathbb{R}^n_v} vF_\epsilon dv$ converges to a Leray solution of Navier-Stokes with the boundary condition:

$$u \cdot \vec{n} = 0 \quad \text{and} \quad \nu((\nabla u + \nabla^t u) \cdot n)_\tau + \lambda u = 0$$
$$\lambda = \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \frac{\alpha(\epsilon)}{\epsilon} \quad \text{Dirichlet} \Leftrightarrow \lim_{\epsilon \to 0} \frac{\alpha(\epsilon)}{\epsilon} = \infty\,.$$

With no boundary: Formal proof B. Golse Levermore (1991), Complete results with Di Perna Lions solution Golse Saint Raymond (2009).

With boundary effect:

Aoki, Inamuro, Onishi (1979) Stationary solution linearized regime and Hilbert expansion; Masmoudi Saint Raymond for Mischler solutions towards Leray solutions. General formal proof C.B., Golse, Paillard.

The Formal Proof

Start from the moment equation in g_ϵ multiply by $w\,, \nabla\cdot w = 0\,, w\cdot\vec{n} = 0\,.$

$$\partial_t\langle v g_\epsilon\rangle + \nabla_x \frac{1}{\epsilon}\langle A(v) g_\epsilon\rangle = 0\,, A(v) := v^{\otimes^2} - \frac{1}{3}|v|^2$$

$$\partial_t \int_\Omega w\cdot\langle v g_\epsilon\rangle + \int_{\partial\Omega} w\otimes\vec{n} : \frac{1}{\epsilon}\langle A(v) g_\epsilon\rangle d\sigma - \int_\Omega \nabla w : \frac{1}{\epsilon}\langle A g_\epsilon\rangle dx = 0$$

$$\int_{\partial\Omega} w\otimes\vec{n} : \frac{1}{\epsilon}\langle A(v) g_\epsilon\rangle d\sigma = \frac{\alpha(\epsilon)}{\epsilon}\int_{\partial\Omega}\langle (v\cdot w)(v\cdot\vec{n}) g_\epsilon\rangle \simeq \frac{\alpha}{\epsilon\sqrt{2\pi}}\int_{\partial\Omega} u\cdot w d\sigma$$

$$-\int_\Omega \nabla w : \frac{1}{\epsilon}\langle A g_\epsilon\rangle dx \simeq -\int_\Omega \nabla w : (u^{\otimes^2} - \nu\epsilon^q\Sigma(u))dx$$

$$-\int_\Omega \nabla w : (u^{\otimes^2} - \nu\epsilon^q\Sigma(u))dx = \int_\Omega w(-\nu\epsilon^q\nu\Delta u + \nabla(u\otimes u))dx$$

$$-\int_{\partial\Omega} \epsilon^q\nu\Sigma(u)\vec{n}\cdot w d\sigma$$

Energy Balance Versus Entropy Dissipation:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,t)|^2 dx + \nu\int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(\nu)|u_\nu(x,t)|^2 d\sigma = o(\nu)$$

$$\epsilon\partial_t \int_{\mathbb{R}^n_v} (F_\epsilon \log(\frac{F_\epsilon}{M}) - F_\epsilon + M)dv + \nabla_x \int_{\mathbb{R}^n_v} v((F_\epsilon \log(\frac{F_\epsilon}{M}) - F_\epsilon + M)dv$$

$$-\frac{1}{\epsilon^{q+1}}\int_v \mathcal{B}(F_\epsilon, F_\epsilon)\log(F_\epsilon)dv = 0\,.$$

For renormalized solutions:

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) + \frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dv dv_1 d\sigma + \frac{1}{\epsilon^3}\int_{\partial\Omega} \mathrm{DG} \leq 0$$

$$\mathrm{DE}(F)(v,v_1,\sigma) = \frac{1}{4}(F'F'_1 - FF_1)\log(F'F'_1 - FF_1)b(|v-v_1|,\sigma)$$

$$\mathrm{DG}(F) = \int_{\mathbb{R}^3_v} v\cdot\vec{n}H(F_\epsilon|M)d\sigma dv$$

The Darrozes-Guiraud local entropy

$$h(z) = (1+z)\log(1+z) - z)$$

$$\sqrt{2\pi}\mathrm{DG} = \int_{\mathbb{R}^3_v} v \cdot \vec{n} H(F_\epsilon | M) d\sigma dv =$$

$$\sqrt{2\pi}\int_{\mathbb{R}^3_v} v \cdot \vec{n} H(M(1+\epsilon g_\epsilon)|M) dv = \sqrt{2\pi}\int_{\mathbb{R}^3_v} v \cdot \vec{n} M(v) h(1+\epsilon g_\epsilon) dv$$

$$= \sqrt{2\pi}\int_{\mathbb{R}^3_v} (v \cdot \vec{n})_+ M(v) h(\epsilon g_\epsilon(v)) dv - \sqrt{2\pi}\int_{\mathbb{R}^3_v} (v \cdot \vec{n})_+ M(v) h(\epsilon g_\epsilon(\mathcal{R}v)) d$$

$$= \Lambda(h(\epsilon g_\epsilon)) - \Lambda(h[(1-\alpha(\epsilon))\epsilon g_\epsilon + \alpha(\epsilon)\Lambda(\epsilon g_\epsilon)])$$

$$\geq \alpha(\epsilon)\left[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))\right] \geq 0$$

Hence the final entropy estimate:

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) + \frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma$$
$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))]d\sigma \leq 0\,.$$

Compare formally to energy with $g_\epsilon = (F_\epsilon - M)/M \to u\cdot v$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,t)|^2dx + \nu\int_\Omega |\nabla u_\nu|^2dx + \int_{\partial\Omega}\lambda(\nu)|u_\nu(x,t)|^2d\sigma \to 0$$
$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) \to \frac{1}{2}\frac{d}{dt}\int_\Omega |u(x,t)|^2dx$$
$$\frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma \simeq \epsilon^q\nu\int_\Omega |\nabla u + \nabla^\perp u|^2dx$$
$$\frac{1}{\epsilon^2}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))]d\sigma \simeq \int_{\partial\Omega}|u_\epsilon(x,t)|^2d\sigma$$
$$\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}} \simeq \lambda(\epsilon^q\nu)$$

Entropic convergence to a regular Euler solution $\Rightarrow$

$$\frac{1}{\epsilon^{q+4}}\int_{\Omega}\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma$$
$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma \to 0$$

Theorem Sufficient condition for the convergence to Euler:

1 $\quad \lim_{\epsilon\to 0}\frac{\alpha(\epsilon)}{\epsilon} = 0$

2 $\frac{\alpha(\epsilon)}{\epsilon} \leq C < \infty$ and $\frac{1}{\epsilon^2}\int_{\partial\Omega\times]0,T[}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma dt \to 0$

Conjecture (Kato!)

$$\frac{1}{\epsilon^{q+4}}\int_0^T\int_{\Omega\cap\{d(x,\partial\Omega)\leq\epsilon^q)\}}\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma dt \to 0$$

Proof uses Laure Saint Raymond argument. Simpler assuming local conservation of moment. Focus on the terms coming from the boundary. Introduce a divergence free tangent to the boundary smooth vector fields $w(x,t)$.

$$\frac{1}{\epsilon^2}H(M_{(1,\epsilon u_0,1)}|M_{(1,\epsilon w,1)}) = \frac{1}{2}\int_\Omega |u_{in} - w(x,0)|^2 dx$$

$$\frac{1}{\epsilon^2}H(F_\epsilon|M_{(1,\epsilon w,1)})(t) = \frac{1}{\epsilon^2}H(F_\epsilon|M)(t) + \int_{\Omega\times\mathbb{R}^3_v}(\frac{w^2}{2} - \frac{v}{\epsilon}w)F_\epsilon(t,x,v)dxdv$$

$$\begin{aligned}&\frac{1}{2\epsilon^2}\frac{d}{dt}\int_\Omega\int F_\epsilon(t,x,v)(\epsilon^2 w^2 - 2\epsilon v\cdot w)dxdv\\ &= \int_\Omega\int \partial_t w\cdot(w - \frac{1}{\epsilon}v)F_\epsilon(t,x,v)dxdv\\ &+ \int_\Omega\left(\frac{w^2}{2}\partial_t\int F_\epsilon(t,x,v)dv - \frac{w}{\epsilon}\cdot\int \partial_t F_\epsilon(t,x,v)vdv\right)dx\,.\end{aligned}$$

For $\partial_t \int F_\epsilon(t,x,v)dv$ and $\partial_t \int F_\epsilon(t,x,v)vdv$ use the local conservation laws :

In the first term appears the conservation of mass:

$$\int_{\partial\Omega}\int_{\mathbb{R}^d_v} v\cdot\vec{n}F_\epsilon(t,x,v)dvd\sigma = 0 :$$

$$\int_\Omega \frac{1}{2}w^2\partial_t \int F_\epsilon(t,x,v)dx = -\frac{1}{\epsilon}\int_\Omega \frac{1}{2}w^2\nabla_x\cdot\int vF_\epsilon(t,x,v)vdvdx$$
$$= \frac{1}{\epsilon}\int_\Omega\int (v\cdot\nabla_x w)\cdot wF_\epsilon(t,x,v)dvdx$$
$$-\frac{1}{\epsilon}\int_{\partial\Omega} d\sigma\frac{1}{2}w^2\int_{\mathbb{R}^d_v} v\cdot\vec{n}F_\epsilon(t,x,v)dv =$$
$$= \int_\Omega\int \frac{1}{\epsilon}(v\cdot\nabla_x w)\cdot wF_\epsilon(t,x,v)dvdx\,.$$

In the second term appear the boundary effects:

$$-\int_\Omega \frac{w}{\epsilon}\cdot\int \partial_t F_\epsilon(t,x,v)v dv = \int_\Omega\int_{\mathbb{R}^3}\frac{w}{\epsilon^2}\cdot\int \nabla_x F_\epsilon(t,x,v)v\otimes v dv =$$
$$-\frac{1}{\epsilon^2}\int_\Omega\int (v\cdot\nabla_x)w\cdot vF_\epsilon(t,x,v)dvdx + \int_{\partial\Omega}\frac{1}{\epsilon^2}\int F_\epsilon(t,x,v)(w\cdot v)(\vec{n}\cdot v)dvd$$

Since w is tangent to the boundary one has for $x\in\partial\Omega$:

$$\frac{1}{\epsilon^2}\int F_\epsilon(t,x,v)(w\cdot v)(\vec{n}\cdot v) =$$
$$\frac{\alpha(\epsilon)}{\epsilon^2}\int F_\epsilon(t,x,v)(w\cdot v)(\vec{n}\cdot v)_+ dv =$$
$$\frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^2}\Lambda(\epsilon g_\epsilon(x,v,t)(w\cdot v))\,.$$

Therefore one obtains:

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon|M_{(1,\epsilon w,1)})(t) +$$
$$\frac{1}{\epsilon^{4+q}}\mathsf{DE}(F_\epsilon) + \frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^3}\int_{\partial\Omega}\Big[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))\Big]d\sigma$$
$$\leq \int_\Omega\int(\partial_t w + w\cdot\nabla w)(w-\frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv -$$
$$\int_\Omega\int(w-\frac{v}{\epsilon})\nabla_x w(w-\frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv$$
$$+\frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^2}\int_{\partial\Omega}\Lambda(\epsilon g_\epsilon(x,v,t)(w\cdot v))d\sigma\,.$$

The exotic terms coming from the boundary are

$$\text{Good} \quad \frac{1}{\sqrt{2\pi}} \frac{\alpha(\epsilon)}{\epsilon^3} \int_{\partial\Omega} \left[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v))) \right] d\sigma$$

$$\text{Bad} \quad \frac{1}{\sqrt{2\pi}} \frac{\alpha(\epsilon)}{\epsilon^2} \int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(x, v, t)(w \cdot v)) d\sigma \,.$$

The bad has to be balanced by the good.

Proposition

$$\forall \eta > 0$$

$$\int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v))d\sigma \leq (\frac{1}{\eta} + \frac{\eta C(w)}{\epsilon}) \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon \Lambda g_\epsilon) d\sigma$$
$$+ C_2 \eta \int_{\partial\Omega} \int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma$$

With $\eta = 2\epsilon$

$$\frac{\alpha(\epsilon)}{\epsilon^2} \int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v))d\sigma$$
$$\leq (1 + 2\epsilon C(w)) \frac{\alpha(\epsilon)}{2\epsilon^3} \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon \Lambda g_\epsilon) d\sigma$$
$$+ C_2 \frac{\alpha(\epsilon)}{\epsilon} \int_{\partial\Omega} \int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma$$

With $\frac{\alpha(\epsilon)}{\epsilon} \to 0$

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon|M_{(1,\epsilon w,1)})(t) \leq \int_\Omega \int (\partial_t w + w \cdot \nabla w)(w - \frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv$$
$$- \int_\Omega \int (w - \frac{v}{\epsilon})\nabla_x w(w - \frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv + o(\epsilon)$$

Then (cf. Saint Raymond) for

$$u = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\int_{\mathbb{R}^3_v} vF_\epsilon(x,v,t)dv$$

$$\frac{1}{2}\partial_t \int_\Omega |u(x,t) - w(x,t)|^2 \; + \; \int (u(x,t) - w(x,t)S(w)u(x,t) - w(x,t))dx$$
$$\leq \; \int (E(x,t), u(x,t) - w(x,t))dx\,.$$

Proof of the Proposition 2 steps

- Symmetry: $\Lambda(\Lambda(g_\epsilon)(w \cdot v)) = 0$

Legendre duality between

$$l(\epsilon g_\epsilon - \Lambda(\epsilon g_\epsilon)))$$
$$= h((\epsilon g_\epsilon - \Lambda(\epsilon g_\epsilon)) + \Lambda(\epsilon g_\epsilon)) - h(\Lambda(\epsilon g_\epsilon)) - h'(\Lambda(\epsilon g_\epsilon))(g_\epsilon - \Lambda(\epsilon g_\epsilon))$$

and its Legendre transform:

$$l^*(p) = (1 + \Lambda(\epsilon g_\epsilon))(e^p - p - 1)$$

$$(\epsilon g_\epsilon(t,x,v) - \Lambda(\epsilon g_\epsilon))(w\cdot v)) = \frac{1}{\eta}(\epsilon g_\epsilon(t,x,v) - \Lambda(\epsilon g_\epsilon))(\eta w\cdot v))$$

$$\leq \frac{1}{\eta}\Big(h((\epsilon g_\epsilon - \Lambda(\epsilon g_\epsilon)) + \Lambda(\epsilon g_\epsilon)) - h(\Lambda(\epsilon g_\epsilon)) - h'(\Lambda(\epsilon g_\epsilon))(g_\epsilon - \Lambda(\epsilon g_\epsilon))\Big)$$

$$+(1+\Lambda(\epsilon g_\epsilon))\frac{(e^{\eta|w||v|} - \eta|w||v| - 1)}{\eta}$$

$\Lambda(h'(\Lambda(\epsilon g_\epsilon))(g_\epsilon - \Lambda(\epsilon g_\epsilon))) = 0$ Proba!

$$\Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v))) \leq \frac{1}{\eta}(\Lambda(h(\epsilon g_\epsilon)) - h((\Lambda(\epsilon g_\epsilon)) + \eta C(w)(1 + \Lambda(\epsilon g_\epsilon))$$

- Step 2

$$\int_{\partial\Omega}(1+\Lambda(\epsilon g_\epsilon)d\sigma$$
$$\leq C_1\int_{\partial\Omega}\Lambda(h(\epsilon g_\epsilon)-h(\epsilon\Lambda g_\epsilon)d\sigma+C_2\int_{\partial\Omega}\int_{\mathbb{R}^3}F_\epsilon(v\cdot\vec{n}_x)^2dvd\sigma$$

Proof With $G_\epsilon = F_\epsilon/M$ and $c=\int(v\cdot\vec{n})_+^2\wedge 1Mdv$

$$c\int_{\partial\Omega}(1+\Lambda(\epsilon g_\epsilon)d\sigma=\int_{\partial\Omega}\Lambda(G_\epsilon)\int(v\cdot\vec{n})_+^2\wedge 1dvM(v)d\sigma_x$$
$$=I_1+I_2$$
$$\int_{\partial\Omega}\int_{\mathbb{R}^3_v}\Lambda(G_\epsilon)\mathbf{1}_{|G_\epsilon/\Lambda(G_\epsilon)-1|>\beta}(v\cdot\vec{n})_+^2\wedge 1M(v)d\sigma_xdv$$
$$+$$
$$\int_{\partial\Omega}\int_{\mathbb{R}^3_v}\Lambda(G_\epsilon)\mathbf{1}_{|G_\epsilon/\Lambda(G_\epsilon)-1|\leq\beta}(v\cdot\vec{n})^2\wedge 1M(v)d\sigma_xdv$$

$h(z) = (z+1)\log(z+1) - z\,,\, h(z) \geq h(|z|)$ and h is increasing on $\mathbb{R}_+$

$$I_1 \leq \frac{1}{h(\beta)} \int_{\partial\Omega} \int_{\mathbb{R}^3_v} \Lambda(G_\epsilon) h\left(|G_\epsilon / \Lambda(G_\epsilon) - 1| \right) (v \cdot \vec{n})^2_+ \wedge 1 M(v) d\sigma_x dv$$

$$\leq \frac{1}{h(\beta)} \int_{\partial\Omega} \int_{\mathbb{R}^3_v} \Lambda(G_\epsilon) h\left(G_\epsilon / \Lambda(G_\epsilon) - 1 \right) (v \cdot \vec{n})_+ M(v) d\sigma_x dv$$

$$\leq \frac{1}{h(\beta)} \int_{\partial\Omega} \int_{\mathbb{R}^3_v} \left(G_\epsilon \log\left(\frac{G_\epsilon}{\Lambda(G_\epsilon)}\right) - G_\epsilon + \Lambda(G_\epsilon) \right) (v \cdot \vec{n})_+ M(v) d\sigma_x dv$$

$$= \frac{1}{h(\beta)} \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon \Lambda(g_\epsilon)) d\sigma$$

For I_2 with $\beta < 1$

$$|G_\epsilon/\Lambda(G_\epsilon) - 1| \leq \beta \Rightarrow (\Lambda(G_\epsilon)) \leq \frac{1}{1-\beta} G_\epsilon$$

Hence

$$I_2 = \int_{\partial\Omega}\int_{\mathbb{R}^3_v} \Lambda(G_\epsilon) \mathbf{1}_{|G_\epsilon/\Lambda(G_\epsilon)-1|\leq\beta} (v\cdot\vec{n})^2 \wedge 1 M(v) d\sigma_x dv$$
$$\leq \frac{1}{1-\beta}\int_{\partial\Omega}\int_{\mathbb{R}^3_v} G_\epsilon (v\cdot\vec{n})^2_+ \wedge 1 M(v) d\sigma_x dv$$
$$\leq \frac{1}{1-\beta}\int_{\partial\Omega}\int_{\mathbb{R}^3_v} F_\epsilon (v\cdot\vec{n})^2_+ d\sigma_x dv$$

Use trace theorems introduced by Mischler!!!

Proposition With $\frac{\alpha(\epsilon)}{\epsilon} \to \lambda < \infty$ the convergence to zero of the Darrozes Guiraud entropy implies the convergence to a dissipative solution.

Proof Just show that in this case the term

$$\frac{\alpha(\epsilon)}{\epsilon^2} \int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w \cdot v)))d\sigma$$

goes to zero.

Starting from

$$\forall \eta > 0$$

$$\int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v)))d\sigma \leq (\frac{1}{\eta} + \frac{\eta C(w)}{\epsilon}) \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon\Lambda g_\epsilon)d\sigma$$
$$+C_2\eta \int_{\partial\Omega}\int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma$$

With

$$\frac{\alpha(\epsilon)}{\epsilon^3}\int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon\Lambda g_\epsilon)d\sigma = d(\epsilon) \to 0 \quad \eta = \epsilon D(\epsilon)\,, \ \frac{d(\epsilon)}{D(\epsilon)} \to 0$$

$$\frac{\alpha(\epsilon)}{\epsilon^2}\int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v)))d\sigma \leq (\frac{d(\epsilon)}{D(\epsilon)} + C(w)\epsilon d(\epsilon) D(\epsilon))$$
$$+ \quad C_2 \frac{\alpha(\epsilon)}{\epsilon} D(\epsilon) \int_{\partial\Omega}\int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma\,.$$

And the conclusion follows.

Thanks for the invitation,

Thanks for listening

And Happy Birthday Dennis.