Fredholmness of Riemannian exponential maps on diffeomorphism groups

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The Euler equation of ideal hydrodynamics on a Riemannian manifold is

$$u_t + \nabla_u u = -\nabla p, \qquad \operatorname{div} u = 0$$

where u is the velocity field and p is the pressure. The pressure is determined implicitly by $\Delta p = -\operatorname{div}(\nabla_u u)$.

Fluid particles move according to the flow equation

$$\eta_t = u \circ \eta.$$

Hence the Euler equation for particles is

$$\eta_{tt} = -\nabla p \circ \eta.$$

We view this as an *ordinary* differential equation on an infinite-dimensional space. The right side is as smooth as η and u.

The goal is to use finite-dimensional techniques for ODEs to gain a deeper understanding of the equation (well-posedness, stability, etc.)

Arnold discovered that the Euler equation for particles is the geodesic equation on the group $\mathcal{D}_\mu(M)$ of volume-preserving diffeomorphisms, where the Riemannian metric is

$$\langle\langle u \circ \eta, v \circ \eta \rangle\rangle = \int_M \langle u, v \rangle \, d\mu,$$

for divergence-free vector fields u and v. He computed curvature and drew conclusions about Lagrangian stability.

This metric is right-invariant under the group operation of composition. This is why the geodesic equation splits into two first-order equations.

Many other first-order partial differential equations can be expressed as geodesic equations on groups with right-invariant metrics: Camassa-Holm, Korteweg-de Vries, etc. Thus we can try to understand local existence in terms of a Riemannian exponential map, global existence in terms of the Hopf-Rinow theorem, stability in terms of Riemannian curvature, etc.

General geodesic equation on a Lie group with right-invariant metric:

$$\frac{du}{dt} + \mathrm{ad}_u^\star\, u = 0$$
$$\frac{d\eta}{dt} = DR_\eta(u)$$

where $\langle\langle \mathrm{ad}_u^\star\, u, v\rangle\rangle = \langle\langle u, \mathrm{ad}_u\, v\rangle\rangle$ for all $v \in \mathfrak{g}$.

The first equation implies conservation of vorticity:

$$\frac{d}{dt}(\mathrm{Ad}_\eta^\star\, u) = 0$$

which reduces to

- $\mathrm{curl}\, u \circ \eta = \mathrm{curl}\, u_0$ for 2D fluids,
- $\mathrm{curl}\, u \circ \eta = D\eta(\mathrm{curl}\, u_0)$ for 3D fluids.

The geodesic equation can thus be written as

$$\frac{d\eta}{dt} = DL_{\eta^{-1}}^\star\, u_0.$$

Caution!

Although many PDEs can be *formally* derived as geodesic equations, this is only really useful if the equation actually becomes an ODE.

For example, although the Euler equation is rigorously an ODE in a Hilbert space, the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

is not. Thus solutions do not depend smoothly on initial conditions, the sectional curvature is unbounded, etc.

The difficulties arise since the group of volume-preserving maps is only a smooth manifold if we consider diffeomorphisms that are at least C^1 (e.g., of Sobolev class H^s for $s > \frac{1}{2}\dim(M) + 1$). But the Riemannian metric is only L^2, and the volume-preserving maps in the L^2 metric do not form a manifold. The same thing happens with other physically relevant PDEs.

So you get nothing for free.

Infinite dimensions complicate things even in the simplest cases (where the Riemannian distance generates the topology).

Example:

Ellipsoid in Hilbert space:

$$\ell^2 = \{(x_1, x_2, \ldots) \,\Big|\, \sum_k x_k^2 < \infty\}$$

$$M = \{(x_1, x_2, \ldots) \in \ell^2 \,\Big|\, \sum_k a_k x_k^2 = 1\}.$$

Depending on how we choose the positive constants a_k, there are examples where [Grossman]:

1. there is no minimizing geodesic joining the poles
2. there is an convergent sequence of conjugate point locations (i.e., the derivative of the exponential map is not injective—monoconjugate)
3. there is a limiting point for which the exponential map derivative is injective but not surjective—epiconjugate

The Hopf-Rinow theorem generally fails in infinite dimensions. Even if the manifold is metrically complete, there may not be a geodesic joining two points [Atkin], and geodesics may not extend for all time.

The Morse index theorem also generally fails in infinite dimensions.

Exception:

The free loop space ΩM with metric

$$\langle\langle u, v \rangle\rangle = \int_{S^1} \langle u(s), v(s) \rangle + \left\langle \frac{Du}{ds}, \frac{Dv}{ds} \right\rangle ds$$

satisfies the Hopf-Rinow theorem, i.e., it is metrically complete, and there are minimizing geodesics between any two loops.

In addition, the Riemannian exponential map is Fredholm [Misiołek]. That is, its differential is of the form "invertible plus compact" which implies that its kernel and cokernel are both finite-dimensional. Hence conjugate points cannot cluster, and "monoconjugate" is the same thing as "epiconjugate." This works because the curvature operator is compact.

For other Riemannian manifolds (e.g., diffeomorphism groups with right-invariant metrics), proving Hopf-Rinow is a lot harder. For example, on the volumorphism group with weak L^2 metric (giving the equations of fluid mechanics) we know

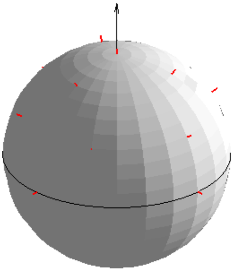
- The manifold is not metrically complete. In three dimensions, the metric completion is the space of all measurable volume-preserving maps [Shnirelman], which is not known to be a manifold. In two dimensions, the metric completion is unknown.
- Existence of minimizing geodesics between is unknown, although work of Brenier and Shnirelman gives answers among generalized flows (where particles may split).
- Extension of geodesics for all time is known to work in two dimensions, but is famously unknown in three dimensions. Constantin has shown that global existence for the 3D Euler equation would imply global existence for 3D Navier-Stokes (for small viscosity), solving a Millennium problem.

So global questions are hard, because we lack compactness.

Local questions are easier. For diffeomorphism groups with weak right-invariant metrics, there is a simple criterion for the exponential map to be Fredholm (which implies conjugate points are discrete along a geodesic and of finite order). This is now well-understood for fluids [Ebin-Misiołek-P] and for other geodesic continuum equations like Camassa-Holm [Misiołek-P].

In particular, Fredholmness is *true* for fluids in two dimensions and *false* for fluids in three dimensions.

Example:
On S^2, rigid rotation is a (steady) solution of the Euler equations. A small (Lagrangian) perturbation is introduced at every point, which grows but then shrinks again; at the end we have a conjugate point.



In general for the unit-speed rotation geodesic on S^2 the conjugate points happen at times

$$\left\{ \frac{kn(n+1)\pi}{m} \,\middle|\, m, n, k \in \mathbb{N}, m \leq n \right\}$$

which is a discrete subset of $[\pi, \infty)$.

For the unit-speed left-invariant field geodesic on S^3, the conjugate points happen at time

$$\left\{ \frac{n\pi}{m} \,\middle|\, m, n \in \mathbb{N}, m \geq n \right\}$$

which is dense in $[\pi, \infty)$.

This is proved by analyzing the Jacobi equation, the linearized geodesic equation. In general the Jacobi equation is

$$\frac{D^2 J}{dt^2} + R(J, \dot{\eta})\dot{\eta} = 0$$

along a geodesic η, and the differential of the exponential map is given by

$$(d \exp_{\mathrm{id}})_{(tv)}(tw) = J(t),$$

where J is the Jacobi field with $J(0) = 0$ and $\dot{J}(0) = w_0$ along the geodesic η with $\eta(0) = \mathrm{id}$ and $\dot{\eta}(0) = v$.

If the curvature operator were compact, we could use this directly. This rarely happens however.

It is much easier to incorporate right-invariance, which splits the Jacobi equation in the same way that it splits the geodesic equation.

Linearize: $\delta\eta = y \circ \eta$ and $\delta u = z$, so that

$$
\begin{aligned}
\frac{d\eta}{dt} &= DR_\eta u & \Longrightarrow \quad & \frac{dy}{dt} - \mathrm{ad}_u\, y = z \\
\frac{du}{dt} + \mathrm{ad}_u^\star\, u &= 0 & \Longrightarrow \quad & \frac{dz}{dt} + \mathrm{ad}_u^\star\, z + \mathrm{ad}_z^\star\, u = 0.
\end{aligned}
$$

Rewrite as

$$
\begin{aligned}
\frac{d}{dt}(\mathrm{Ad}_{\eta^{-1}}\, y) &= \mathrm{Ad}_{\eta^{-1}}\, z \\
\frac{d}{dt}(\mathrm{Ad}_\eta^\star\, z) + \mathrm{Ad}_\eta^\star\, \mathrm{ad}_z^\star\, u &= 0.
\end{aligned}
$$

Then let $y = \mathrm{Ad}_\eta v$ and $z = \mathrm{Ad}_\eta w$, so that

$$\frac{dv}{dt} = w$$

$$\frac{d}{dt}(\mathrm{Ad}_\eta^\star \mathrm{Ad}_\eta w) + \mathrm{Ad}_\eta^\star \mathrm{ad}_{\mathrm{Ad}_\eta w}^\star u = 0.$$

Finally conservation of vorticity is $\mathrm{Ad}_\eta^\star u = u_0$, which implies $\mathrm{Ad}_\eta^\star \mathrm{ad}_{\mathrm{Ad}_\eta w}^\star u = \mathrm{ad}_w^\star u_0$, so the Jacobi equation for the Jacobi field $J = \delta y = DL_\eta(v)$ is

$$\frac{d}{dt}\left(\mathrm{Ad}_\eta^\star \mathrm{Ad}_\eta \frac{dv}{dt}\right) + \frac{d}{dt} \mathrm{ad}_v^\star u_0 = 0.$$

So if for an initial velocity u_0 the operator $v \mapsto \mathrm{ad}_v^\star u_0$ is compact, then we can deduce the solution operator is Fredholm. This comes from expressing it as the integral of the positive-definite operator $\mathrm{Ad}_\eta^\star \mathrm{Ad}_\eta$ plus a compact perturbation.

For volumorphisms in the L^2 metric:

- In two dimensions we compute (assuming $H^1(M) = 0$, but it works in general) that

$$\mathrm{ad}_w^\star u_0 = \operatorname{sgrad} \Delta^{-1} \langle w, \nabla \operatorname{curl} u_0 \rangle,$$

  so the operator is compact.

- In three dimensions we compute that

$$\mathrm{ad}_w^\star u_0 = \operatorname{curl} \Delta^{-1} [w, \operatorname{curl} u_0],$$

  so the operator is not compact.

The only possible drawback is needing to actually establish Fredholmness in the H^s space, which involves checking for compactness in ad-star and then using commutator estimates. This almost always works, except in one case: surfaces with boundary. There, weak Fredholmness (of the differential in L^2) is known but strong Fredholmness (of the differential in H^s) is unknown.

Generalizations:

- Shnirelman has shown that the exponential map on $\mathcal{D}_\mu(\mathbb{T}^2)$ is quasiruled, which means it is close enough to a linear map that a degree can be defined for it. The conjecture is that this leads to surjectivity of the Riemannian exponential map (i.e., that any area-preserving map in two dimensions can be reached from the identity by a geodesic). This is known not to be true in three dimensions [Shnirelman].
- The paper [Misiołek, P] shows that the exponential map of the Camassa-Holm and EPDiff equations, and more generally any Sobolev H^r metric on a diffeomorphism group for r large enough, is Fredholm. Relation to global existence is not clear: the Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

is known to have solutions which blow up, although the Lagrangian version has global solutions. This is related to the fact that composition for diffeomorphisms is not a smooth map in Sobolev spaces: reducing to the Euler equation sometimes results in loss of smoothness.

Example:

The periodic Hunter-Saxton equation

$$u_{txx} + uu_{xxx} + 2u_x u_{xx} = 0$$

corresponds to geodesics on the round sphere.

The coadjoint operator is

$$w \mapsto \operatorname{ad}^\star_w u_0 = (\partial_x^2)^{-1}(u_0''' w + 2u_0'' w' - \mu(u_0'' w'))$$

which is *compact*.

So the exponential map should be Fredholm. Yet the exponential map on the round sphere is not Fredholm: conjugate point of infinite order.

However solutions of the Hunter-Saxton equation always blow up before they reach any conjugate point.

For the three-dimensional Euler equation, the failure of Fredholmness means it is very easy to find conjugate points. We just need to solve a finite-dimensional ODE along a Lagrangian path γ,

$$\frac{D^2 J}{dt^2} + \nabla^2 p(J) + R(J, \dot{\gamma})\dot{\gamma} = 0,$$

where $\nabla^2 p$ is the Hessian of the pressure function evaluated along the path.

Alternatively we can solve the left-translated version

$$\frac{d}{dt}\left(D\eta^{\dagger} D\eta \frac{dj}{dt}\right) + \omega_0 \times \frac{dj}{dt} = 0;$$

Here ω_0 is the initial vorticity.

Solving one of these equations *at any point* will give a conjugate point on the volumorphism group [P, 2006]. (This doesn't work in two dimensions since it would contradict Fredholmness, which makes conjugate points harder to find.)

This can be used to give geometric insight into blowup because of the Beale-Kato-Majda criterion, which says that a solution of the 3D Euler equation blows up at time T if and only if

$$\int_0^T \sup_{x\in M} |D\eta(t,x)(\omega_0(x))|\, dt = \infty.$$

If we assume the order can be interchanged, then this is a condition on blowup of vorticity along a single Lagrangian path, which means it should be expressible in terms of conjugate points.

In fact we can prove [P, 2010] that if there is such a path along which vorticity blows up, then typically we will have a sequence of conjugate point locations approaching the blowup time, i.e., times t_n such that $t_n \nearrow T$ with $\eta(t_n)$ conjugate to $\eta(t_{n+1})$ for every n. The exceptions can be characterized concretely in terms of eigenvalues of the stretching matrix.

The consequence is that we get some real geometric information about blowup just by studying conjugate points. The existence of a conjugate point means that the geodesic fails to locally minimize between the endpoints, which is a geometric condition that makes sense even in the L^2 topology on volume-preserving maps.

To understand this, we need to have a better way of understanding fluids in L^2. Approximations may be helpful: for example, we could consider particles moving with the constraint of preserving volumes of cubes, a finite-dimensional geometric model. Alternatively we can consider flows which preserve volumes of finitely many sets or integrals of finitely many functions. Unfortunately we lose the group structure in such models. There are also group approximations which do not have a Riemannian geometric structure, but we probably need both.