Dennisfest, Stony Brook, May 27, 2011

Quantum Riemann surfaces related to Shrődinger equation solutions

Leonid Chekhov (based on papers with B. Eynard and O. Marchal)

- Methods of constructing the $1/N$-expansion in matrix models; "topological recursion"

- General topological recursion; symplectic invariants

- The β-ensemble and Riccati equation: "Quantum" algebraic geometry: holomorphic differentials, A- and B-cycles, symmetric forms

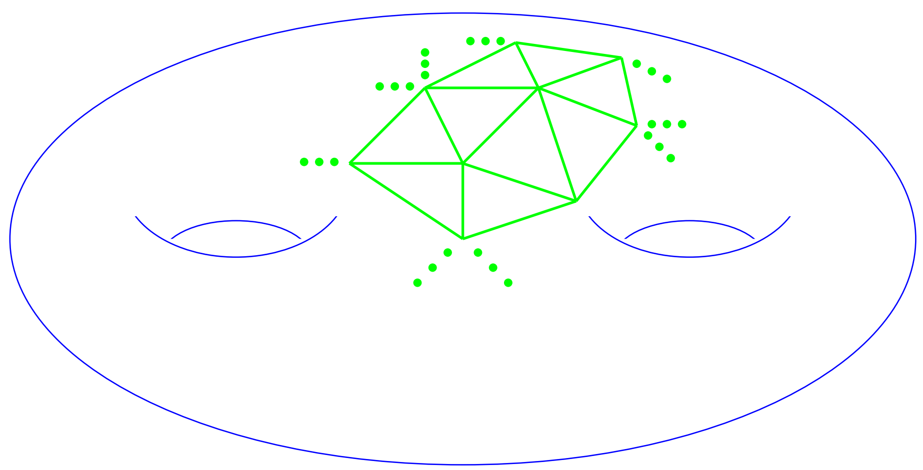
- Fuchsian systems, AGT, and all

Matrix models is a technique for computing "action functionals" and correlation functions appearing in physics and applications. Loosely speaking, main idea is to replace functionals of 2 variables with matrices with two indices.

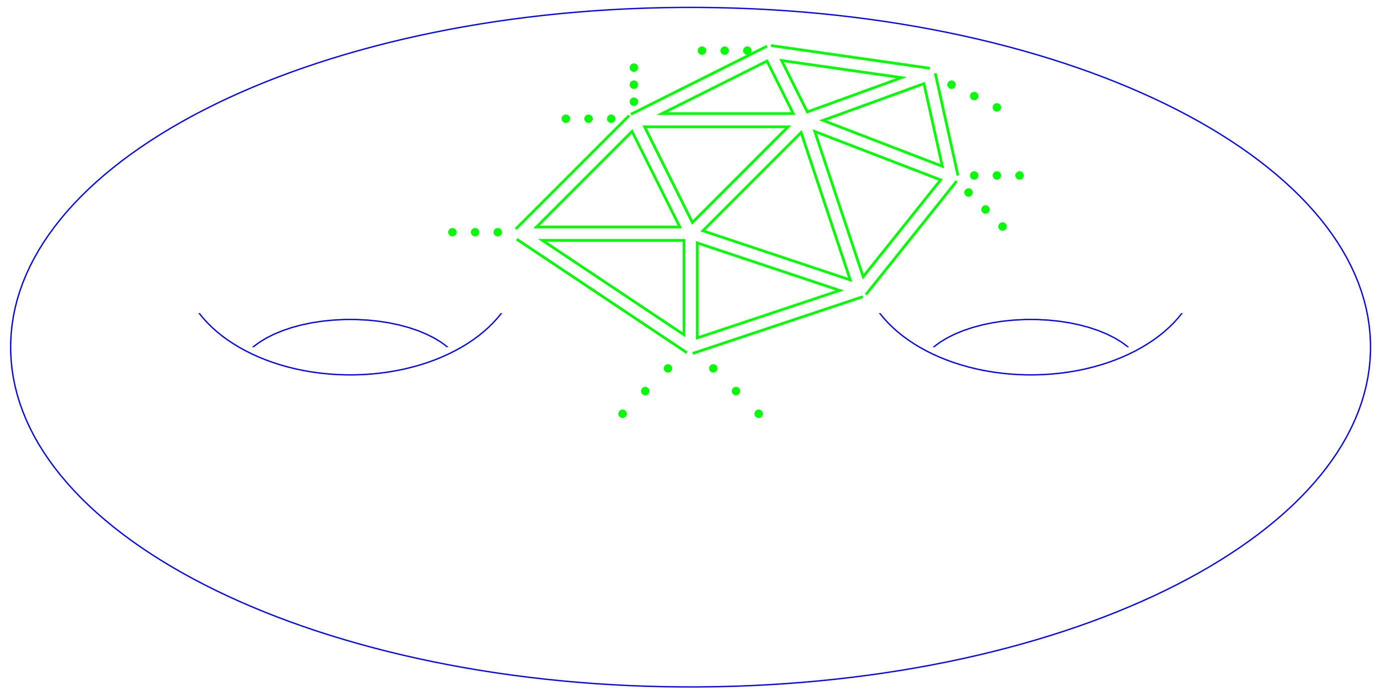
The Einstein action in 2D gravity (over all possible metrics and topologies)

$$\int dg e^{-\kappa \int \sqrt{-g} d^2 x} = e^{\mathcal{F}}$$

is approximated by the sum over **triangulations** of surfaces of all genera,



We represent triangulation by **fat graphs...**



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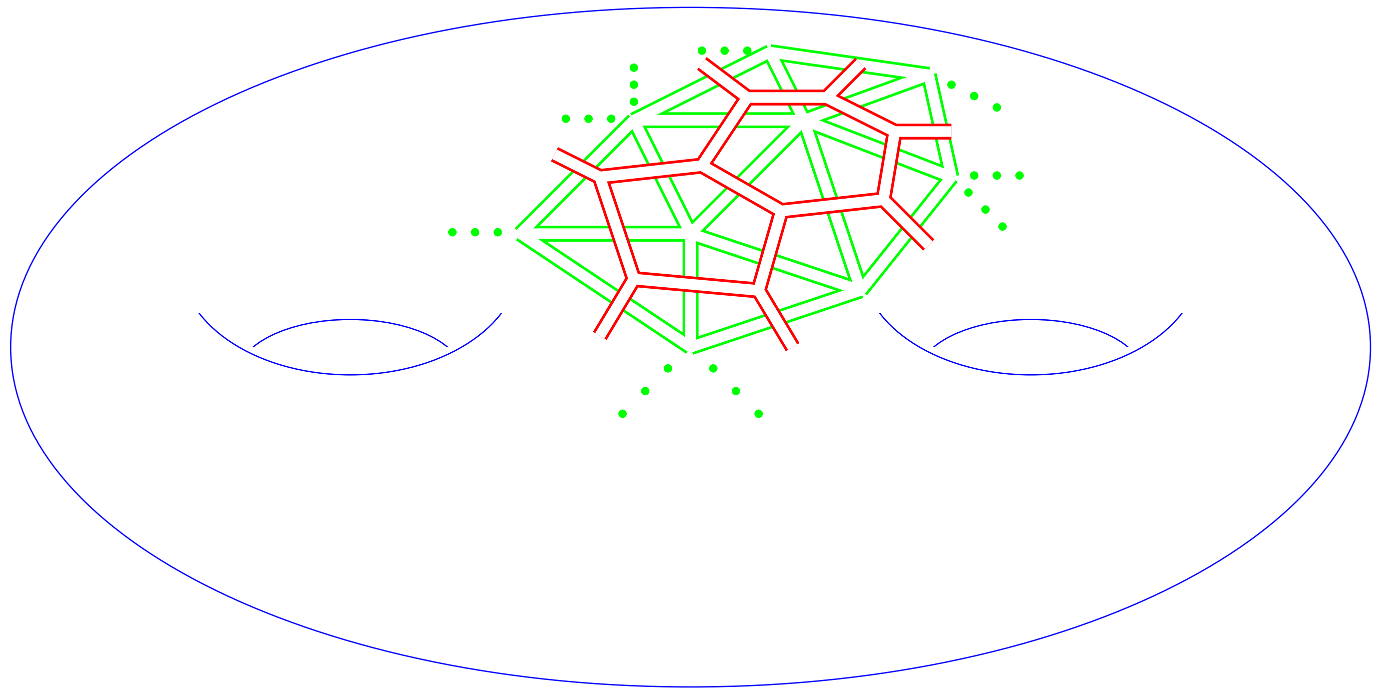
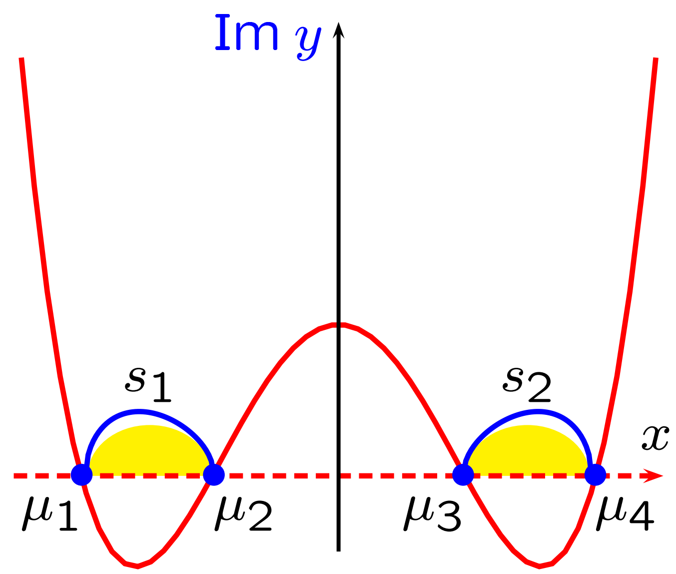


the **dual** to which are described by the Hermitian matrix model integrals 't Hooft idea of $1/N$ expansion. We reduce the matrix integral

$$\int_{N\times N} DH\, e^{-N\, tr\, V(H)} \simeq \int_N D\lambda_i\, \Delta(\lambda)^2 e^{-N\sum_{i=1}^N V(\lambda_i)} = e^{\sum_{h=0}^\infty N^{2-2h}\mathcal{F}_h}, \qquad V(x) = \sum_{k=1}^{d+1}\frac{1}{k}t_k x^k$$

to the N-fold integration over the eigenvalues λ_i of H, $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ The genus-h contributions $\mathcal{F}_h$ to the free energy $\mathcal{F}$ come with the factors N^{2-2h}. We assume that as $N \to \infty$, the asymptotic distribution of eigenvalues $\rho(x) = \mathrm{Im}\, y(x)$ spans $n = g+1$ intervals providing the spectral curve—a hyperelliptic Riemann surface possibly with double points.



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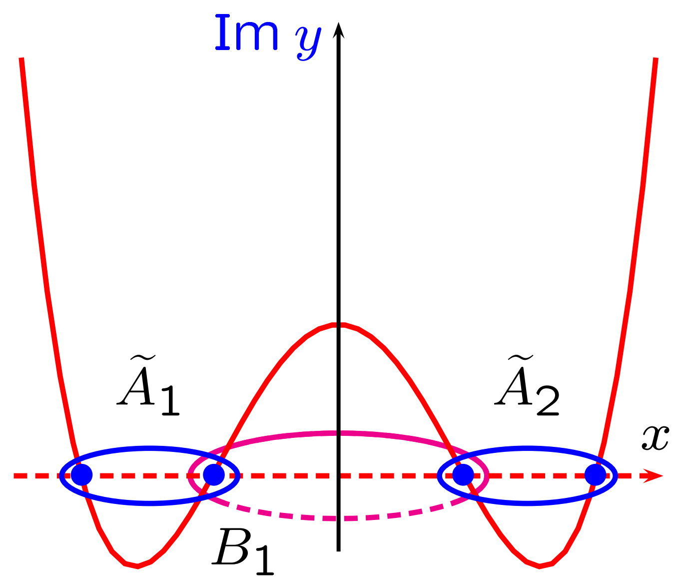


$\mathcal{C}_D = \widetilde{A}_1 \cup \widetilde{A}_2$
contour encircling
only cuts, no other
singular points (∞, etc.)

$\mathcal{F}_0$ – satisfies equations of the Whitham–Krichever hierarchy and WDVV w.r.t. s_i and t_k [I.Krichever'93] [L.Ch., A.Marshakov, A.Mironov, D.Vassiliev]'03

Besides the free energy, objects of main interest are the **correlation functions**, or **resolvents**. **one-point resolvent** is a 1-*differential* $W_1 \in \Omega_1$

$$W_1(x) = \frac{t_0}{N}\left\langle \sum_{i=1}^{N} \frac{1}{x-\lambda_i} \right\rangle dx, \quad W_1(x) = \frac{t_0}{x} + \frac{\partial}{\partial V(x)}\mathcal{F} = \frac{t_0}{x} + \sum_{k=1}^{\infty} \frac{k}{x^{k+1}}\frac{\partial \mathcal{F}}{\partial t_k} dx$$

obtained from $\mathcal{F}$ by the action of the **loop insertion operator**. Correspondingly the t-point resolvents ($t \geq 2$) are symmetric t-differentials $W_t \in \Omega_t$

$$W_t(x_1, \ldots, x_t) = N^{t-2} \left\langle tr\,\frac{1}{x_1 - H} \cdots tr\,\frac{1}{x_t - H} \right\rangle_{\text{conn}} dx_1 \cdots dx_t$$

$$W_t = \frac{\partial}{\partial V(x_1)} \cdots \frac{\partial}{\partial V(x_{t-1})} W_1(x_t)$$

(“conn"means the connected part of a correlation function). All the W’s admit the genus expansions $W_t(x_1, \ldots, x_t) = \sum_{h=0}^{\infty} N^{-2h} W_t^{(h)}(x_1, \ldots, x_t)$.

$$\frac{\partial}{\partial V(\lambda)} : \Omega_t \mapsto \Omega_{t+1}$$

• **Loop equation** expresses invariance of the integral under the change of integration variables $\delta\lambda_i = \epsilon\frac{1}{\lambda_i - x}$ and is exact (generating function for **Virasoro** conditions):

$$W_1^2(x) - V'(x)W_1(x) + \left\langle tr\,\frac{V'(x) - V'(H)}{x - H}\right\rangle + \frac{1}{N^2}W_2(x,x) = 0, \quad W_1(x)\Big|_{x\to\infty} = \frac{t_0}{x} + O(x^{-2}).$$

Disregarding the correction term, for $W_1^{(0)}(x) = y(x) + V'(x)/2$ we obtain **algebraic equation** determining the hyperelliptic **spectral curve**:

$$y^2(x) = \frac{1}{4}V'(x)^2 + P_{d-1}^{(0)}(x) \equiv U_{2d}(x), \qquad y(x) = \sqrt{U_{2d}(x)}.$$

We look for W_t that solve the corresponding loop equations

The **"flat" variables** t_k, s_i are: $t_k = \mathrm{res}_\infty x^{-k} y(x)$, $k \geq 0$; $s_i = \oint_{A_i} y(x)dx$.

Variations w.r.t. the "flat" variables are algebro-geometric objects:

$$\frac{\partial y(x)dx}{\partial s_i} = \omega_i \text{ —canonical holomorphic differential}, \quad \oint_{A_i} \omega_j = \delta_{i,j}.$$

$$\frac{\partial y(x)dx}{\partial t_k} = v_k \text{ —Whitham–Krichever meromorphic diff's}, \ v_k|_{x\to\frac{\infty}{\infty}} = \pm x^{k-1} + O(x^{-2}); \ \oint_{A_i} v_k \equiv 0.$$

We also have **exact** Seiberg–Witten equations— $\dfrac{\partial \mathcal{F}}{\partial s_i} = \oint_{B_i} \dfrac{\partial \mathcal{F}}{\partial V(x)} dx$

2-point correlation function. Universality property.

For P and Q point on the spectral curve, $B(P,Q)$ is the Bergmann bi-differential symmetric in $P \leftrightarrow Q$, canonically normalized, $\oint_{A_i} B(\cdot, Q) = 0$, and such that

$$B(P,Q)\Big|_{P\to Q} = \left(\frac{1}{(\xi(P) - \xi(Q))^2} + O(1) \right) d\xi(P) d\xi(Q),$$

with no other singularities.

Riemann bilinear identities: $\oint_{B_i} B(z,x) = w_i(x)$; $\frac{1}{k} res_{z=\infty} z^{-k} B(z,x) = v_k(x) = \frac{\partial y(x) dx}{\partial t_k}$;
then

$$\frac{\partial}{\partial V(x)} W_1^{(0)}(y) = W_2^{(0)}(x,y) = B(x,\bar{y})$$

is the two-point correlation function. **It depends ONLY on the Riemann surface.**
($\bar{y}$ is the point on the **second** sheet of the hyperelliptic Riemann surface).

3-point correlation functions are originated from Rauch variational formulas: $\frac{\partial}{\partial \mu_\alpha} B(x,z) = \frac{1}{2} B(x,[\mu_\alpha]) B(z,[\mu_\alpha])$,

$$\frac{\partial}{\partial V(x)} W_2^{(0)}(y,z) = W_3^{(0)}(x,y,z) = \oint_{\mathcal{C}_\mathcal{D}} \frac{B(x,\xi)B(y,\xi)B(\bar{z},\xi)}{dy(\xi)d\xi}$$

$$= \sum_{i=1}^{2g+2} \operatorname{res}_{\mu_i} \frac{dE_{\xi,\bar{\xi}}(x)B(y,\xi)B(\bar{z},\xi)}{(y(\xi)-y(\bar{\xi}))d\xi} = \oint_{\mathcal{C}_\mathcal{D}} \frac{dE_{\xi,\bar{\xi}}(x)B(y,\xi)B(\bar{z},\xi)}{(y(\xi)-y(\bar{\xi}))d\xi}, \quad dE_{\xi,\bar{\xi}}(x) = \int_{\bar{\xi}}^{\xi} B(\bullet,x)$$

where $\frac{dE_{\xi,\bar{\xi}}(x)}{(y(\xi)-y(\bar{\xi}))d\xi}$ is the **recursion kernel**: If

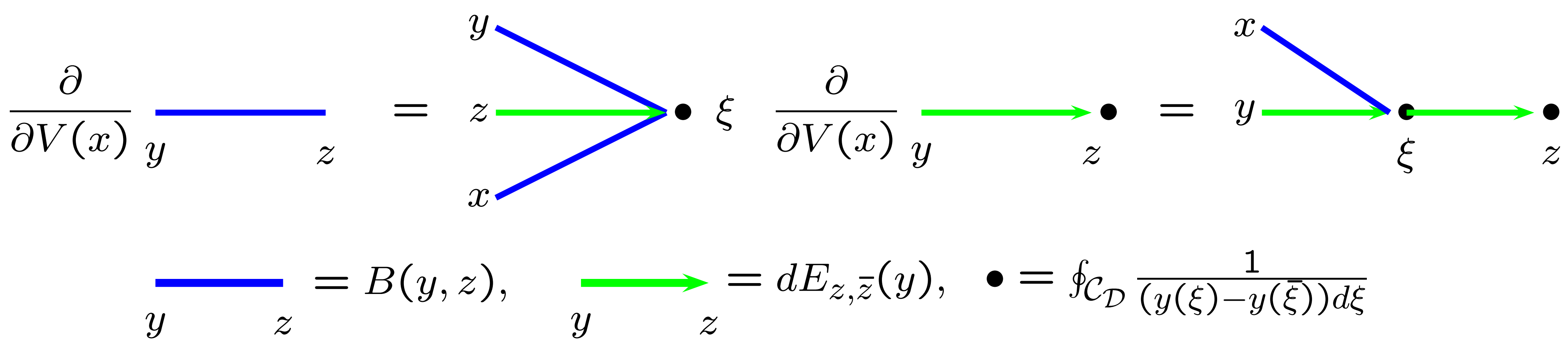
$$(y(\xi)-y(\bar{\xi}))W_{n+1}^{(h)}(\xi,J) + \mathrm{Pol}^{(h)}(\xi;J) = W_{n+2}^{(h-1)}(\xi,\bar{\xi},J) + {\sum_{r,I\subseteq J}}' W_{|I|+1}^{(r)}(\xi,I)W_{n-|I|+1}^{(h-r)}(\bar{\xi},J/I),$$

then

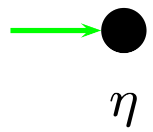
$$W_{n+1}^{(h)}(\xi,J) = \oint_{\mathcal{C}_\mathcal{D}} \frac{dE_{\xi,\bar{\xi}}(x)}{(y(\xi)-y(\bar{\xi}))d\xi}\Big[W_{n+2}^{(h-1)}(\xi,\bar{\xi},J) + {\sum_{r,I\subseteq J}}' W_{|I|+1}^{(r)}(\xi,I)W_{n-|I|+1}^{(h-r)}(\bar{\xi},J/I)\Big].$$

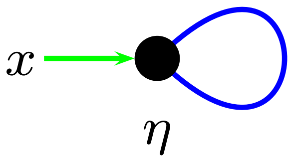
We therefore express 3- and higher-point correlation functions and all its corrections in terms of one- and two-correlation functions in the leading order: $y(x)$ and B(x,z).

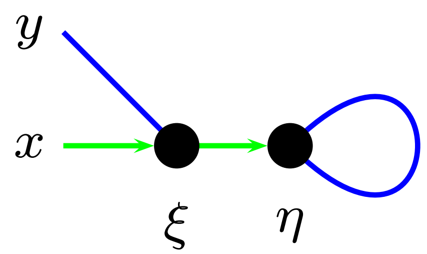
Graphical representation for the action of $\partial/\partial V(x)$:

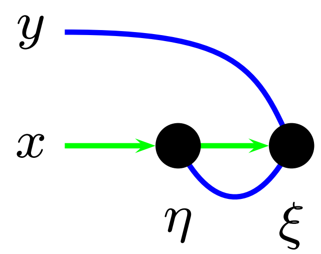


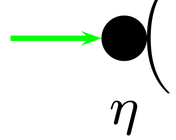
Feynman-like diagrammatic technique **(in terms of standard graphs!)**

$W_1^{(1)}(x) = x$ 

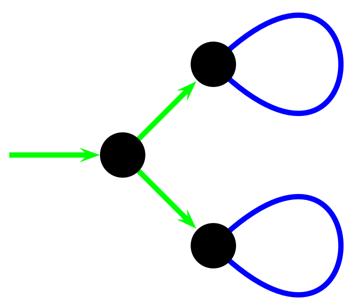
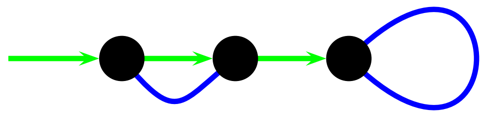
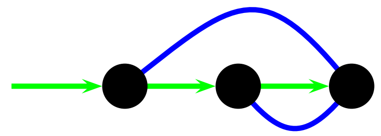
$\Big(W_2^{(0)}(\eta,\eta)\Big) = x$ 

$W_2^{(1)}(x,y) = \frac{\partial}{\partial V(y)} W_1^{(1)}(x) =$ 

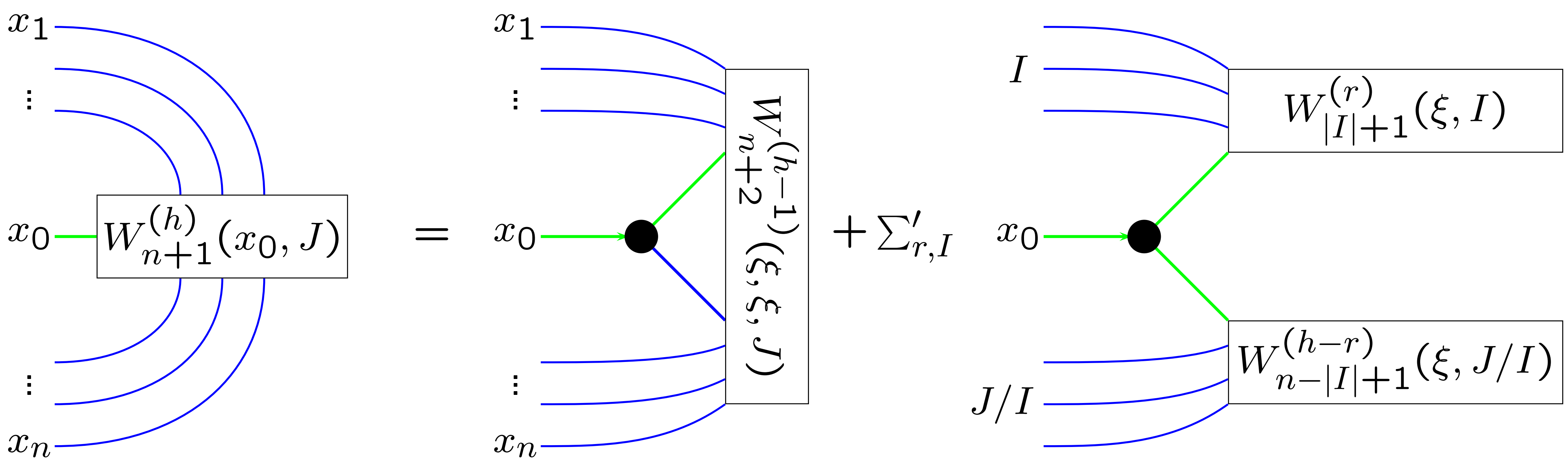
$+$ 

$W_1^{(2)}(x) = x$ 

$\left(\left[W_1^{(1)}(\eta)\right]^2 + W_2^{(1)}(\eta,\eta)\right)$

$= x$  $+x$  $+x$ 

Iterative solution of the loop equation (in the graphic form):



$W_n^{(h)}(J)$ comprises all the diagrams n **external legs and** h **loops** such that

- segregating one variable x_0 we take all the **maximum connected rooted subtrees** composed from **recursion kernels** (arrowed propagators) that starts at the vertex x_0 and does not go to any other external leg; this subtree establishes a **partial ordering** on the set of vertices that determines the order of doing integrals over $\mathcal{C}_D$ at these vertices;

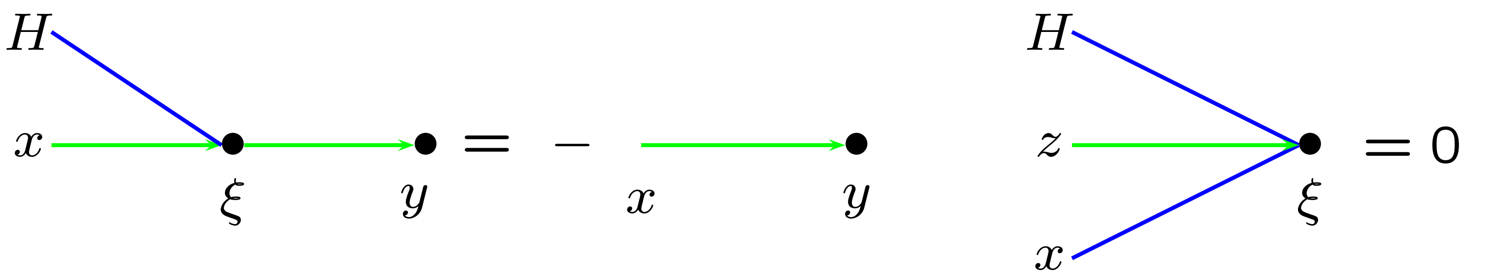
- all other propagators: h inner propagators and $n-1$ remaining external legs are $B(x, y)$ (blue lines); *only vertices comparable in the partial ordering sense can be joint by B(x,y)*.

Symplectic invariants (free-energy terms) $\mathcal{F}_h$ [L.Ch., B.Eynard]'05

The new operator $H_\bullet$ is **inverse** to the loop insertion operator, $H_\bullet : \Omega_{t+1} \mapsto \Omega_t$,

$$H \cdot \varphi := \mathop{res}_{\infty_x} V(x)\varphi(x) - \mathop{res}_{\infty_{\bar{x}}} V(x)\varphi(x) + t_0 \int_{\infty_x}^{\infty_{\bar{x}}} \varphi(x)dx + \sum_{i=1}^{g} s_i \oint_{B_i} \varphi(x)dx,$$

Action of $H_\bullet$:



$$H_x B(x,z) = -y(z)dz, \qquad H_x dE_{\xi,\bar{\xi}}(x)dx = \int_{\xi}^{\bar{\xi}} y(\rho)d\rho$$

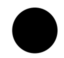
$$\frac{\partial}{\partial V} \circ H_\bullet - H_\bullet \circ \frac{\partial}{\partial V} = \mathrm{Id}, \qquad H_z W_2^{(h)}(x,z) = -(2h-1)W_1^{(h)}(x),$$

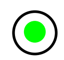
$$\mathcal{F}_h = \frac{1}{2-2h} H_x \cdot W_1^{(h)}(x).$$

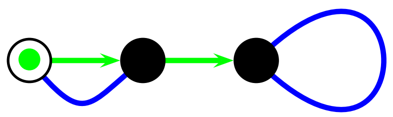
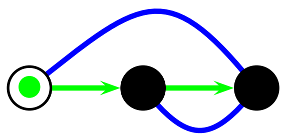
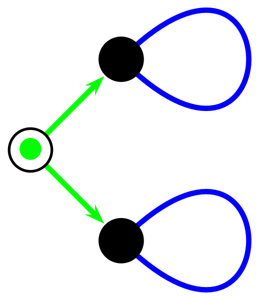
We then have the diagrammatic expression for $\mathcal{F}_h$ with $h \geq 2$; for example

$= B(x,z)dxdz$

$= \int^z d\xi B(x,\xi)dx$


$= \sum_j \mathrm{res}\Big|_{\mu_j} \frac{1}{y(x)dx}$


$= \sum_j \mathrm{res}\Big|_{\mu_j} \frac{\int^x y(\xi)d\xi}{y(x)dx}$

$2 \cdot \mathcal{F}_2 = 2$  $+2$  $+1$ 

[green arrows indicate the order of taking residues]

Same technique works for

—finding $\mathcal{F}_h$ in the two-matrix model (here Σ is an arbitrary algebraic curve) [B.Eynard, L.Ch., N.Orantin]'06

$$\int DH_1\, DH_2\, e^{-N\, tr(V_1(H_1)+V_2(H_2)+H_1H_2)}$$

—topological recursion for holomorphic anomalies; KP hierarchies (generalized Kontsevich model); plane partitions; generating functions for Hurwitz numbers and more [B. Eynard, N. Orantin, M. Mariño et al]

Generating function for simple Hurwitz numbers [Bouchard, M. Mariño]: Lambert function $x = ye^y$ (has just one branching point)

The (Wigner) β-ensemble model

$$\int_N dx_i \, |\Delta(x)|^{2\beta} \, e^{-\frac{N\sqrt{\beta}}{t_0}\sum_{i=1}^N V(x_i)} \qquad \beta = \begin{cases} 1/2 - \text{orthogonal matrices} \\ 1 - \text{Hermitian matrices} \\ 2 - \text{symplectic matrices} \end{cases},$$

For arbitrary β **and any potential for which V' is a rational function** [this includes the AGT-conjecture case], we know the answer for $\mathcal{F}_{g,k}$, where

$$\mathcal{F} = \sum_{g,k=0}^{\infty} N^{2-2g-k}(\sqrt{\beta} - \sqrt{\beta}^{-1})^k \mathcal{F}_{g,k}.$$

A general procedure of finding $\mathcal{F}_{g,k}$ in the β eigenvalue model using Feynman-like diagrams was developed in [B.Eynard, L.Ch.]'06

There is no obvious interpretation in terms of "triangulations" of Riemann surfaces, which exists for $\beta = 1$ (the Hermitian matrix model).

Quantum surfaces = nonperturbative solutions of the β-eigenvalue model[L.Ch.,B.Eynard,O.Marchal]'09-10 TMΦ February 2011.

- The **loop equation** again expresses invariance under the change of integration variables $\delta x_i = \epsilon \frac{1}{x_i - x}$ and is **exact**

$$W_1^2(x) - V'(x)W_1(x) + \left\langle tr\, \frac{V'(x) - V'(H)}{x - H} \right\rangle + \frac{1}{N}(\sqrt{\beta} - \sqrt{\beta}^{-1})W_1'(x) + \frac{1}{N^2}W_2(x,x) = 0.$$

We incorporate the term with $W_1'(x)$ into the leading order. This results in **resummation** of the asymptotic series for $\mathcal{F}_{g,k}$ in k.

For $W_1^{(0)}(x) = y(x) + V'(x)/2$ we obtain **Riccati equation** determining the **spectral curve**:

$y^2(x) + \hbar y'(x) = \frac{1}{4}V'(x)^2 + P_{n-1}(x) \equiv U(x)$, where we identify $\hbar = (\sqrt{\beta} - \sqrt{\beta}^{-1})/N$.

Solution is $y(x) = \hbar\psi'(x)/\psi(x)$, where $\psi(x)$ solves the **Schrödinger equation**

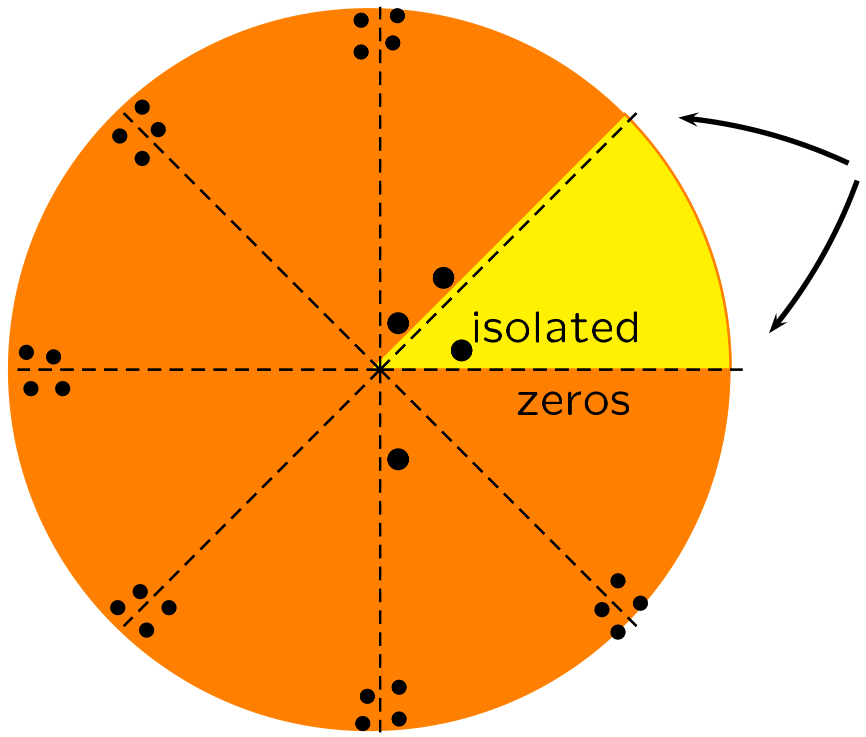
$$\hbar^2\psi''(x) = U(x)\psi(x).$$

with $V'(x) = 2\sqrt{U(x)}_+$.

• We **cannot** satisfy asymptotic conditions $W_1^{(0)}(x) \sim t_0/x + O(x^{-2})$ in all directions if we take just one solution $\psi(x)$, so define $W_1^{(0)}(x)$ **sectorwise**:

$$y(\overset{\alpha}{x}) := W_1^{(0)}(\overset{\alpha}{x}) = \hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \frac{V'(x)}{2}, \text{ for } x \in S_\alpha.$$

• **Stokes Sectors** We choose the function $\psi_\alpha(x)$ to be a unique solution of the Schrödinger equation that decreases at the αth sector bounded by the lines $\mathrm{Re}\, V(x) = 0$,



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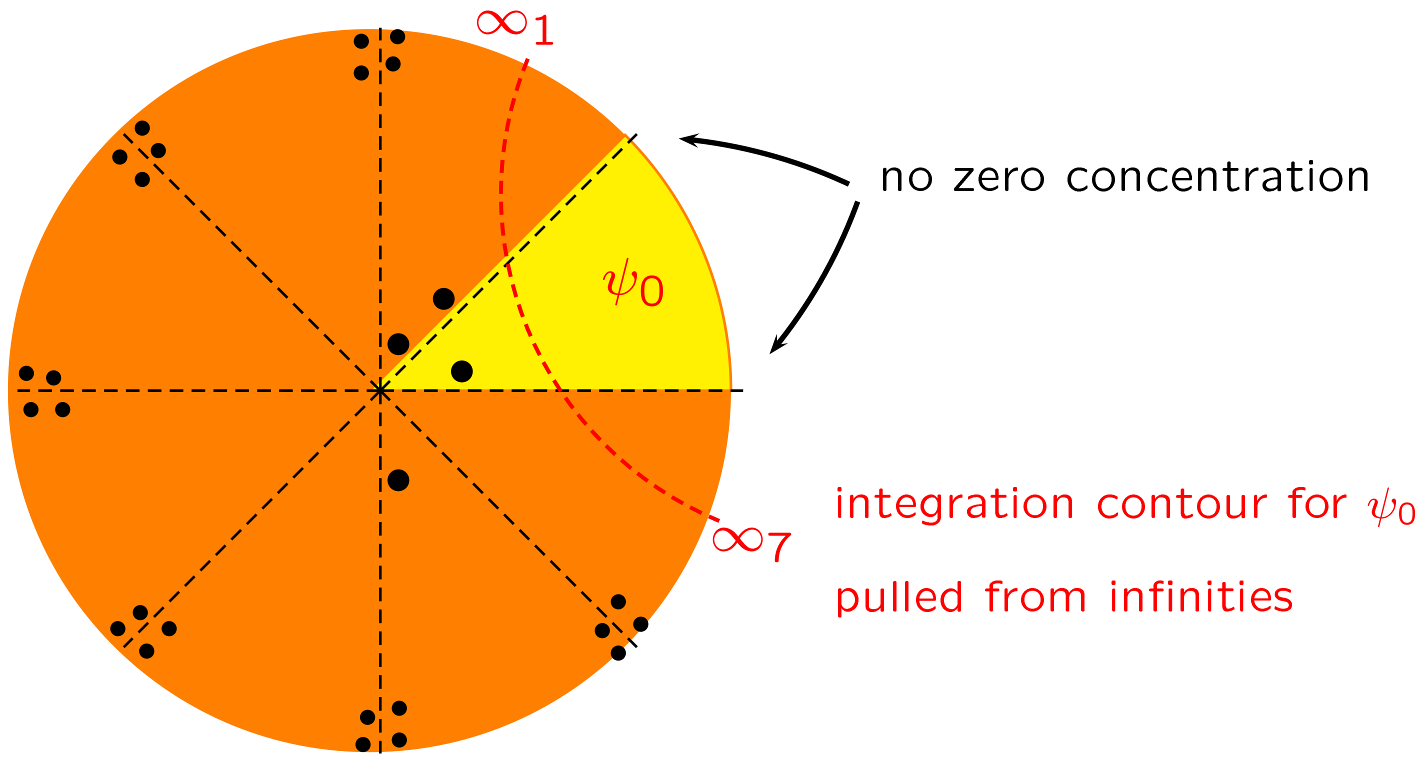
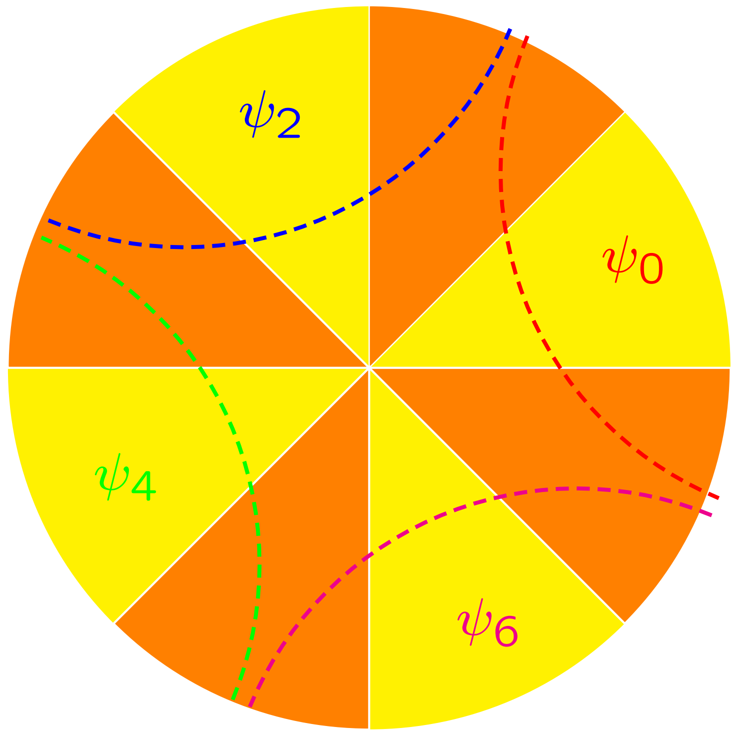


The contour $\mathcal{C}_D$ and the set of A- and B-cycles

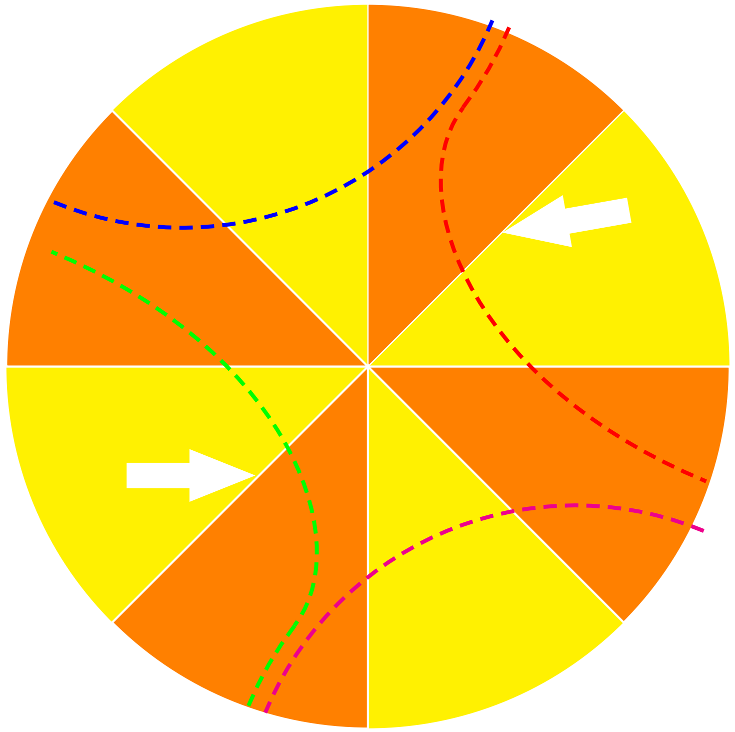
- The contour $\mathcal{C}_D$

The integration contour $\mathcal{C}_D$: $\oint_{\mathcal{C}_D}$ — the analogue of $\mathrm{res}|_\infty$



$$\oint_{\mathcal{C}_D} f(x)dx \equiv \sum_\alpha \int_{\infty_{\alpha-1}}^{\infty_{\alpha+1}} f(\overset{\alpha}{x})dx$$

To obtain A-**cycles**, we "protrude" integration contours to make them running between infinities "in pairs":



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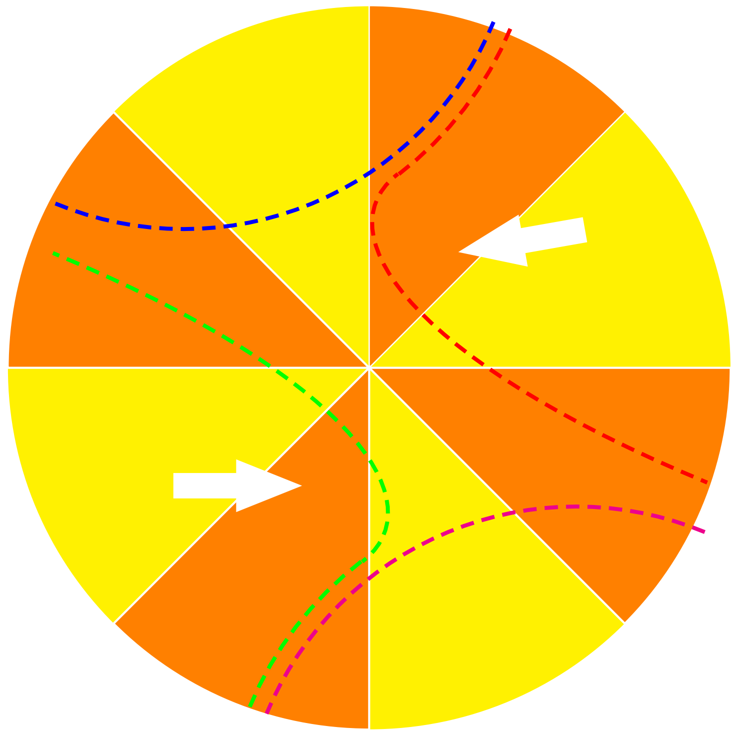

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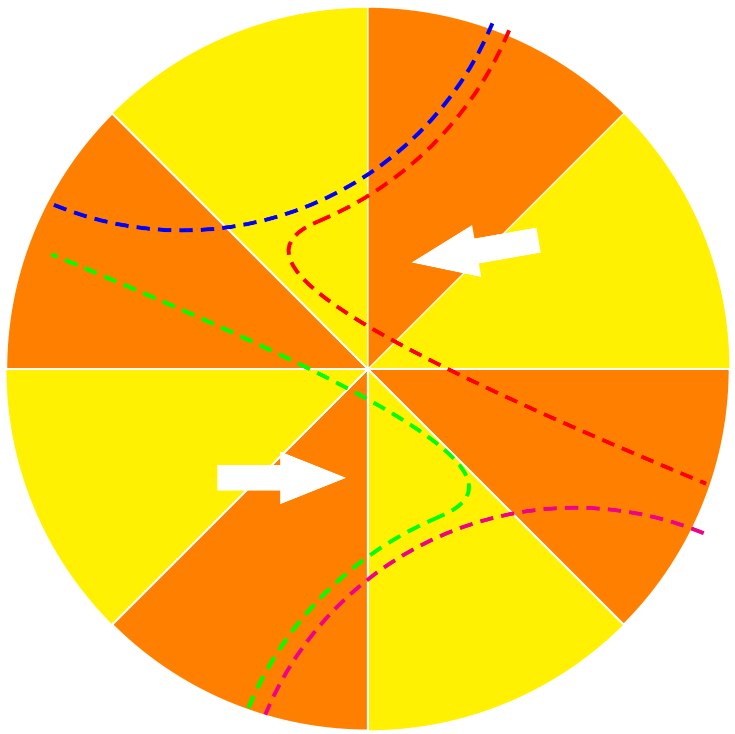

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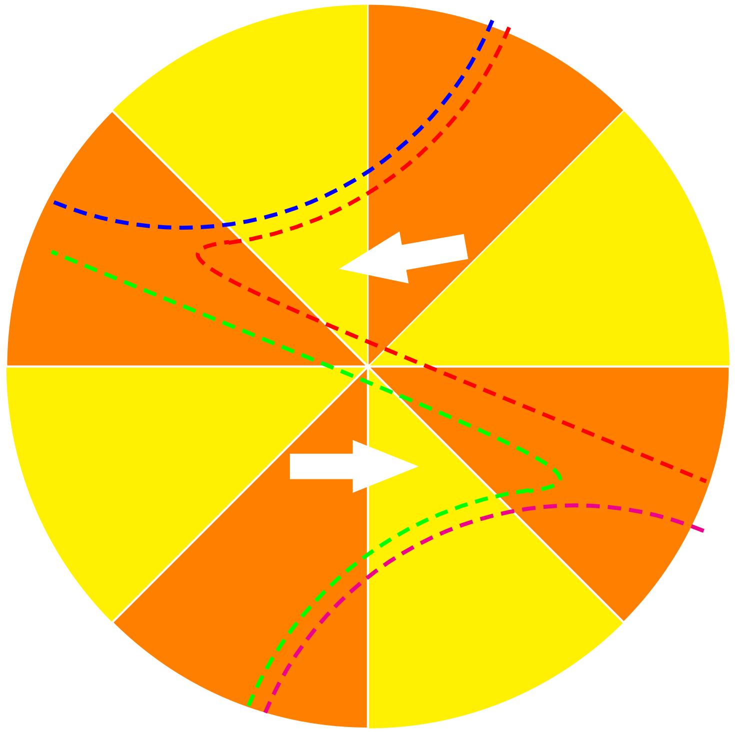

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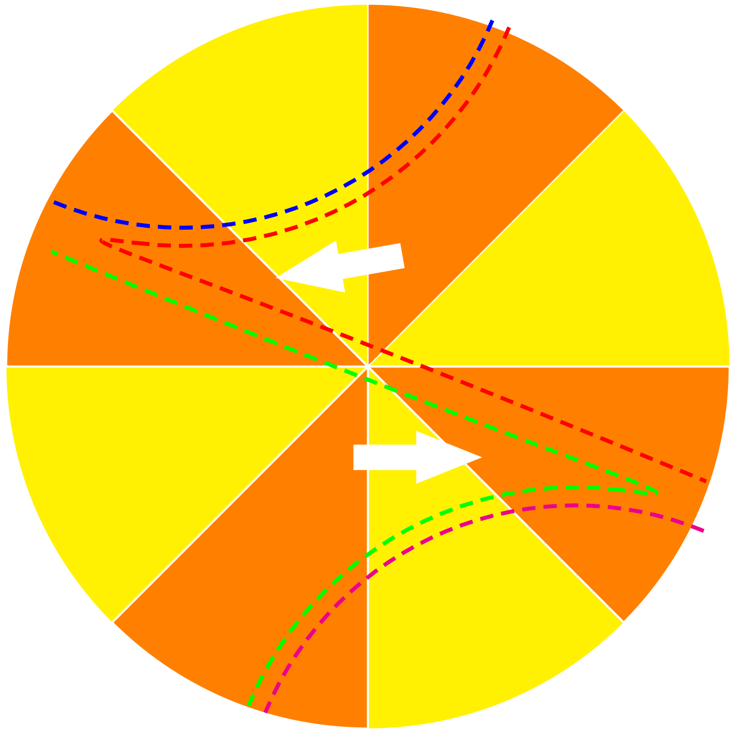
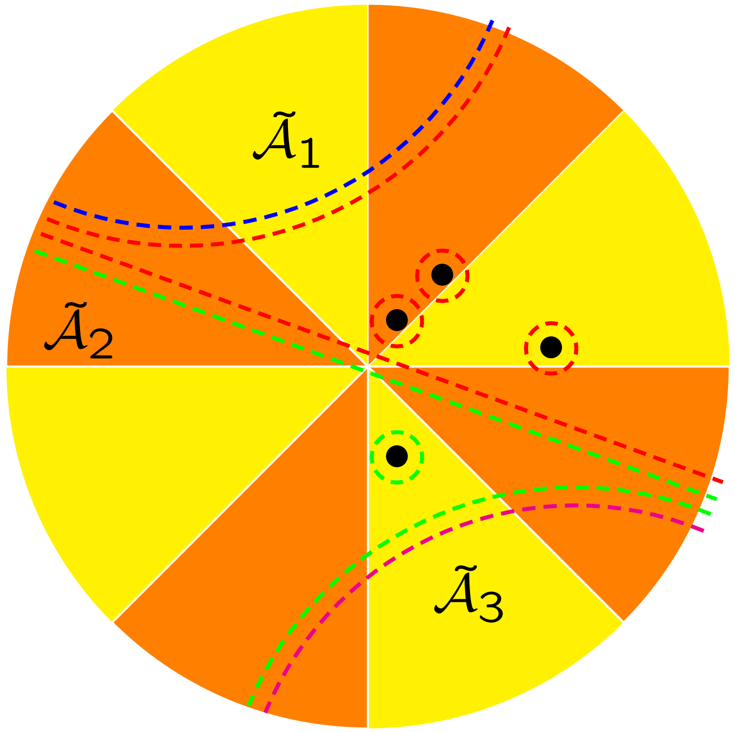
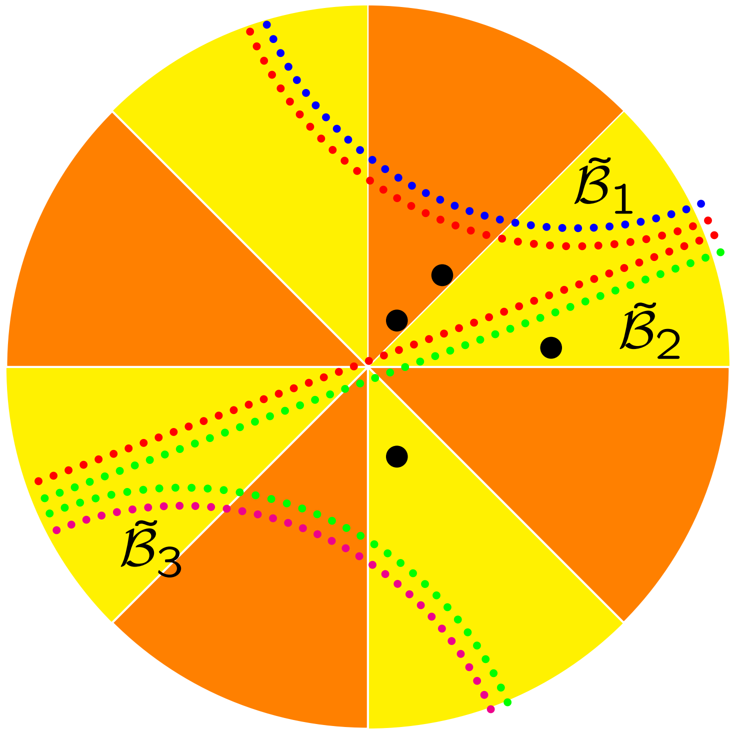


To obtain A-**cycles**, we "protrude" integration contours to make them running between infinities "in pairs":



$$\oint_{\tilde{\mathcal{A}}_\alpha} f(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\tilde{\alpha}_-}}^{\infty_{\tilde{\alpha}_+}} \left(f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x}) \right) dx + \sum_{s_i^{(\alpha_\pm)}(\alpha)} \underset{}{res}\, f(\overset{\alpha_\pm}{x})$$

B**-cycles** are, as usual, "dual" to A-cycles:



$$\oint_{\tilde{\mathcal{B}}_\alpha} f(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\alpha_-}}^{\infty_{\alpha_+}} (f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x}))\, dx,$$

Variations of the resolvent w.r.t. "flat" coordinates, which comprise

- **filling fractions**

$$\epsilon_i = \frac{1}{2i\pi}\oint_{\tilde{\mathcal{A}}_i} y(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\tilde{i}_-}}^{\infty_{\tilde{i}_+}} \Big(y(\overset{i_+}{x}) - y(\overset{i_-}{x})\Big)dx,$$

The difference $y(\overset{i_+}{x}) - y(\overset{i_-}{x}) = \frac{\text{Wron}_{i_+,i_-}}{\psi_{i_+}(x)\psi_{i_-}(x)}$ decreases exponentially in sectors where the both solutions ψ_{i_+} and ψ_{i_-} increase **so we can integrate it with any polynomial function**.

- **times of the potential**

$$t_k = \oint_{\mathcal{C}_D} y(x)x^{-k}dx, \quad k = 0, 1, \ldots.$$

For **any** infinitesimal polynomial variation $U(x) \to U(x) + \delta U(x)$ we have

$$\hbar\delta y(\overset{\alpha}{x})' + 2y(\overset{\alpha}{x})\delta y(\overset{\alpha}{x}) = \delta U(x); \qquad \delta y(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)}\int_{\infty_\alpha}^{x} \psi_\alpha^2(x')\delta U(x')dx'.$$

- First kind functions $w_k(\overset{\alpha}{x})$. Let h_k, $k = 1, \ldots, d-1$, be a basis of polynomials of degree $\leq d-2$. Then

$$\delta_{\epsilon_k} y(\overset{\alpha}{x}) = w_k(\overset{\alpha}{x}) = \frac{1}{\hbar\,\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} h_k(x')\,\psi_\alpha^2(x')\,dx'$$

with **the same** polynomial $h_k(x')$ for all the sheets and with the canonical normalization

$$\oint_{\mathcal{A}_\alpha} w_k(x)\,dx = \delta_{k,\alpha} \quad k, \alpha = 1, \ldots, d-1;$$

$w_k(\overset{\alpha}{x})$ has double poles with **no residues** at the zeroes of ψ_α and behaves like $O(1/x^2)$ inside all the sectors including the sector S_α (**so it can be integrated over any cycle!**)

- The Riemann matrix of periods

$$\tau_{j,i} \overset{\text{def}}{=} \oint_{\mathcal{B}_j} w_i(x)dx$$

is symmetric [proof is not direct, however...]

- Meromorphic differentials $\delta_{t_k} y(\overset{\alpha}{x}) = v_k(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x') h_{d-1+k}(x')dx'$ such that

$$\frac{\partial t_p}{\partial t_k} = \delta_{k,p} = \oint_{\mathcal{C}_\mathcal{D}} v_k \cdot x^{-p} dx; \qquad 0 = \frac{\partial \epsilon_i}{\partial t_k} = \oint_{\mathcal{A}_i} v_k(x)dx.$$

- The recursion kernel $K(\overset{\alpha}{x}, z)$. Introduce

$$\hat{K}(\overset{\alpha}{x}, z) = \frac{1}{\hbar}\frac{1}{\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x')\, \frac{dx'}{x'-z}$$

The recursion kernel $K(\overset{\alpha}{x}, z)$ reads

$$K(\overset{\alpha}{x}, z) = \hat{K}(\overset{\alpha}{x}, z) - \sum_{j=1}^{d-1} v_j(\overset{\alpha}{x}) C_j(z), \quad \hbar C_\alpha(z) = \oint_{\mathcal{A}_\alpha} \hat{K}(x, z), \;\; \alpha = 1, \dots, g$$

- The "quantum" Bergman kernel $B(\overset{\alpha}{x}, \overset{\beta}{z})$

$$B(\overset{\alpha}{x}, \overset{\beta}{z}) = \frac{\hbar}{2}\partial_z \left(\psi_\beta^2(z)\, \partial_z \frac{K(\overset{\alpha}{x}, z)}{\psi_\beta^2(z)} \right).$$

$B(\overset{\alpha}{x}, \overset{\beta}{z})$ *is an analytical function of* x *and* z *in the whole complex plane (no cuts) with the double pole with zero residue at* $x = z$ *for* $\alpha = \beta$.

The kernel B satisfies the loop equations *in the both variables*.

The properties of $B(\overset{\alpha}{x},\overset{\beta}{z})$ are analogous to those of the Bergmann kernel:

- For every $\alpha = 1,\dots,g$: $\oint_{\mathcal{A}_i} B(x,\overset{\beta}{z})\,dx = 0, \quad \oint_{\mathcal{A}_j} B(\overset{\alpha}{x},z)\,dz = 0$;
- $\oint_{\mathcal{B}_j} B(\overset{\alpha}{x},z)\,dz = 2i\pi w_j(\overset{\alpha}{x})$;
- $B(\overset{\alpha}{x},\overset{\beta}{z})$ is symmetric, $B(\overset{\alpha}{x},\overset{\beta}{z}) = B(\overset{\beta}{z},\overset{\alpha}{x})$.

Corollary The period matrix $\tau_{k,\alpha}$ is symmetric: $\tau_{k,\alpha} = \oint_{\mathcal{B}_k}\oint_{\mathcal{B}_\alpha} B(z,x)dz\,dx$.

Since $\frac{\partial}{\partial t_r} y(\overset{\alpha}{x}) = v_r(\overset{\alpha}{x})$, we can define the loop insertion operator $\frac{\partial}{\partial V(\xi)} := \sum_{r=1}^{\infty} r\xi^{-r-1}\frac{\partial}{\partial t_r}$

$$\frac{\partial}{\partial V(\xi)} y(\overset{\alpha}{x}) = \sum_{r=1}^{\infty} \xi^{-r-1}\oint_{\xi > \mathcal{C}_D} B(\overset{\alpha}{x},z)z^r dz = B(\overset{\alpha}{x},\overset{\beta}{\xi})$$

We identify $B(\overset{\alpha}{x},\overset{\beta}{z})$ with the **two-point correlation function**.

3-point correlation function

$$\frac{\partial^3}{\partial V(x)\partial V(y)\partial V(z)}\mathcal{F}_0 = \oint_{\mathcal{C}_D} K(x,\xi)B(y,\xi)B(z,\xi)(d\xi)^{-1}$$

and it admits **the same diagrammatic representation** as in the one-matrix model. But we have no apparent residue formula, so **what about WDVV?**

Seiberg–Witten relations are **exact**:

$$\frac{\partial \mathcal{F}_h}{\partial \epsilon_i} = \oint_{B_i} \frac{\partial \mathcal{F}_h}{\partial V(\xi)} d\xi.$$

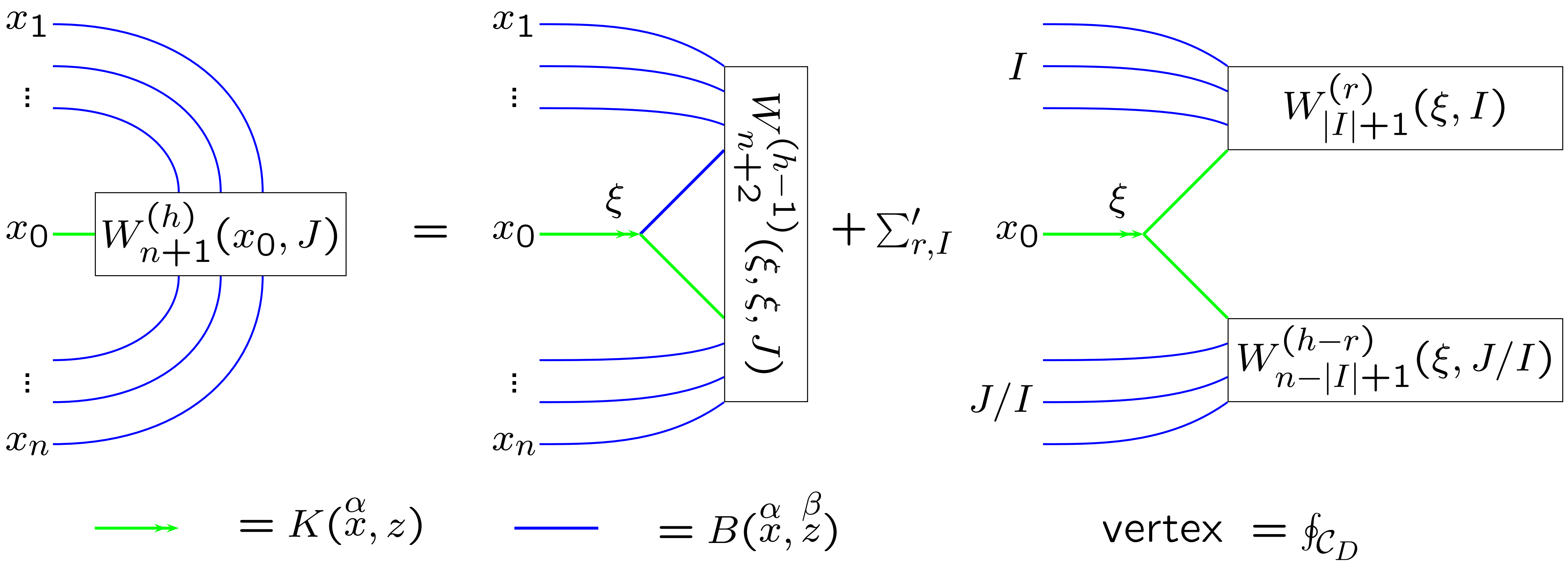
or

$$\frac{\partial \mathcal{F}_h}{\partial \epsilon_i} = \oint_{B_i} W_1^{(h)}(\xi) d\xi,$$

where $W_1^{(h)}(\xi)$ is to be defined by the recursion procedure, as in the standard matrix model.

- Diagrammatic representation for correlation functions (solutions of the loop equation) is formally the same as the one in the original matrix model

Recurrent relation:



Perturbative part of $\mathcal{F}_0$

For the one- and two-matrix models:

$$\text{Pert } \mathcal{F}_0 = \sum_{i=1}^{n} \frac{1}{2} s_i^2 \log s_i; \qquad \frac{\partial^3 \mathcal{F}_0}{\partial s_i \partial s_j \partial s_k} = \frac{\delta_{i,j,k}}{s_i} + \text{reg.}$$

Perturbative part of $\mathcal{F}_0$ in the quantum case follows from the Seiberg–Witten relations:

$$\frac{\partial^3}{\partial \epsilon_i \partial \epsilon_j \partial \epsilon_k} \mathcal{F}_0 = \oint_{B_i} dx \oint_{B_j} dy \oint_{B_k} dz \frac{\partial^3}{\partial V(x) \partial V(y) \partial V(z)} \mathcal{F}_0$$

In the quantum surface case (for $\hbar = 1$),

$$\frac{\partial^3 \mathcal{F}_0}{\partial \epsilon_i \partial \epsilon_j \partial \epsilon_k} = \hbar^{-1} \delta_{i,j,k} \Big(\log \Gamma(\epsilon_i/\hbar)\Big)'' + \text{reg.} \qquad \text{Pert } \mathcal{F}_0 = \hbar^2 \sum_{i=1}^{d} [\int \log \Gamma](\epsilon_i/\hbar);$$

So we have poles of the **second** order at $\epsilon_i = 0, -\hbar, -2\hbar, \ldots$ (not the **first**-order pole $1/s_i$ as in the matrix-model case), but the **same** singular behavior $\simeq \frac{1}{2}\epsilon_i^2 \log \epsilon_i$ at large positive ϵ_i. This expression coincides with Superpotential of Nekrasov and Shatashvili for (quantum) TBA.

Rational potentials (related to the AGT conjecture)

$$V' = \sum_{\alpha=1}^{n} \frac{b_\alpha}{x-\mu_\alpha}, \qquad U(x) = \sum \frac{b_\alpha^2}{(x-\mu_\alpha)^2} + \sum \frac{c_\alpha}{x-\mu_\alpha}, \quad \sum c_\alpha = 0$$

$\psi_\alpha \sim (x-\mu_\alpha)^{-|\ell_\alpha|} \to 0$ as $x \to \mu_\alpha,$ everything else remains the same...

$$w_j(\overset{\alpha}{x}) = \frac{1}{\hbar \psi_\alpha^2(x)} \int_{\mu_\alpha}^{x} \psi_\alpha^2(x') \frac{h_j(x')}{\prod_{\gamma=1}^{n}(x'-\mu_\gamma)} dx', \quad \deg h_j \le n-3.$$

AGT: Liouville $\to$ CFT $\to$ β-ensembles

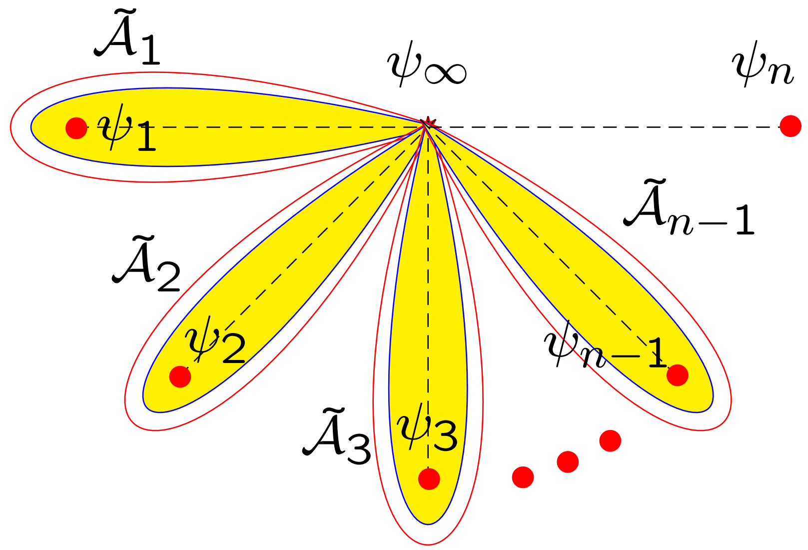
CFT central charge: $c = 1 + 6\Big(\sqrt{\beta} - \sqrt{\beta^{-1}}\Big)^2.$

Nekrasov–Shatashvili super YM instanton counting $\beta = \varepsilon_1/\varepsilon_2$, $N = (\varepsilon_1\varepsilon_2)^{-1/2}$, i.e.,

$$(\sqrt{\beta} - \sqrt{\beta}^{-1})/N = \varepsilon_1 - \varepsilon_2, \quad N^{-2} = \varepsilon_1\varepsilon_2,$$

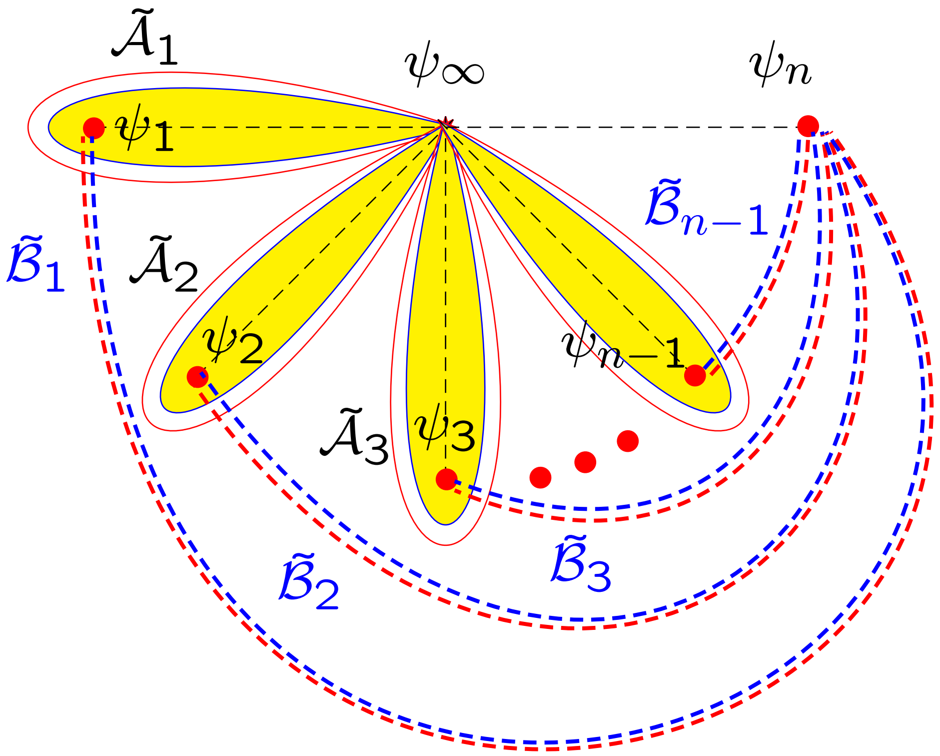
so we calculate these functions in the limit of $\varepsilon_2 \to 0$ keeping ε_1 arbitrary.

System of A- and B-cycles



$$\epsilon_\alpha = \int_{\mathcal{A}_\alpha} \left(\frac{\psi_n'}{\psi_n} - \frac{\psi_\alpha'}{\psi_\alpha} \right) = \int_{\mathcal{A}_\alpha} \frac{\psi_n'}{\psi_n}$$ as ψ_α'/ψ_α has trivial monodromy around μ_α

System of A- and B-cycles



$$\epsilon_\alpha = \int_{\mathcal{A}_\alpha} \left(\frac{\psi_n'}{\psi_n} - \frac{\psi_\alpha'}{\psi_\alpha} \right) = \int_{\mathcal{A}_\alpha} \frac{\psi_n'}{\psi_n} \quad \text{as } \psi_\alpha'/\psi_\alpha \text{ has trivial monodromy around } \mu_\alpha$$

$$\frac{\partial \mathcal{F}_0}{\partial \epsilon_\alpha} = \oint_{B_\alpha} y(\xi) d\xi = \int_{\mu_\alpha}^{\mu_n} \left(\frac{\psi_\alpha'(\xi)}{\psi_\alpha(\xi)} - \frac{\psi_n'(\xi)}{\psi_n(\xi)} \right) d\xi \quad \text{— SW relations}$$

Problems, perspectives...

- constructing symplectic invariants: no clear analogue of H-operator exists at the moment (only some guesses);

- generalization to higher-order ODEs for the function ψ (general “quantum” algebraic surfaces);

- isomonodromic (quantum) τ-functions: under construction

- SLE ?? (talk by S.Smirnov on May 26th)