Bridging scales

from microscopic dynamics to macroscopic laws

M. Hairer

Imperial College London

Simons Lectures, Stony Brook, 27/03/2018

Guiding principles in probability

Symmetry

If different outcomes are equivalent (from the perspective of the mechanism causing them), they should have the same probability.

Universality

In many instances, if a random outcome is a consequence of many different sources of randomness, the details of its description should not matter much. (Outcomes of successive coin tosses: de Moivre 1733, Laplace 1812, ...)

Guiding principles in probability

Symmetry

If different outcom … erspective of the mechanism causing … same probability.

Universalit

In many instances, … nsequence of many different sources of … s description should not matter … ve coin tosses: de Moivre 1733, Lapla

Guiding principles in probability

Symmetry

If different outcomes are equivalent (from the perspective of the mechanism causing them), they should have the same probability.

Universality

In many instances, if a random outcome is a consequence of many different sources of randomness, the details of its description should not matter much. (Outcomes of successive coin tosses: de Moivre 1733, Laplace 1812, ...)

Guiding principles in probability

Symmetry

If different outcomes are equivalent (from the perspective of the mechanism causing them), they should have the same probability.

Universality

In many instances, if a random outcome is a consequence of many different sources of randomness, the details of its description should not matter much. (Outcomes of successive coin tosses: de Moivre 1733, Laplace 1812, ...)

Einstein-Smoluchowski-Bachelier

Independently in early 1900s give a semi-heuristic description of Brownian motion.


1. Physics: Effect of water molecules $\Rightarrow$ Brownian motion.
2. Finance: Effect of agents $\Rightarrow$ evolution of stock prices.
3. Mathematics: Heat equation.

Comes with quantitative predictions, verified experimentally by Perrin in 1908 (Nobel prize 1926). Lays foundations for the works of Black & Scholes, 1973 (1997 "Nobel prize" in Economics).

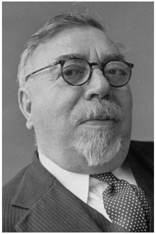
Einstein-Smoluchowski-Bachelier

Independently in early 1900s give a semi-heuristic description of Brownian motion.

1. Physics: Effect of water molecules ⇒ Brownian motion.
2. Finance: Effect of agents ⇒ evolution of stock prices.
3. Mathematics: Heat equation.

Comes with quantitative predictions, verified experimentally by Perrin in 1908 (Nobel prize 1926). Lays foundations for the works of Black & Scholes, 1973 (1997 "Nobel prize" in Economics).

Mathematical description / universality

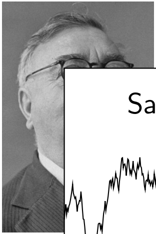


Wiener (late 1920's) provides full mathematical description of Brownian motion.

Went on to become an early researcher in robotics and cybernetics.

Donsker (1951) shows that Brownian motion is "universal" and describes the large-scale behaviour of a multitude of processes with different microscopic descriptions.

Mathematical description / universality

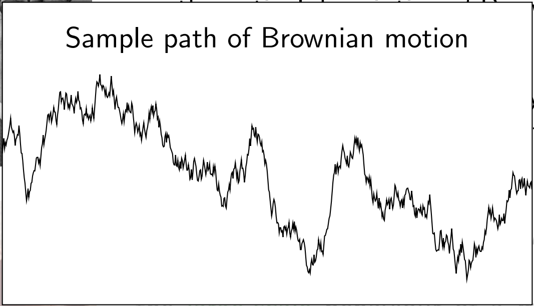


Wiener (late 1920's) provides full

vnian

re-

tics.

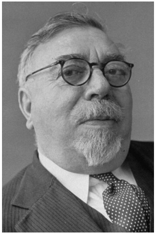


vnian

ribes

the large-scale behaviour of a multi-
tude of processes with different mi-
croscopic descriptions.

Mathematical description / universality



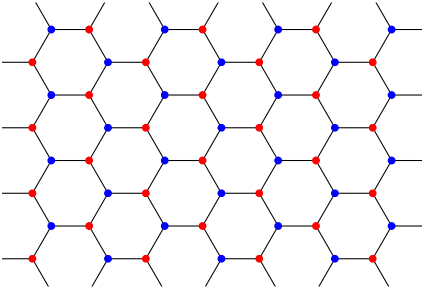
Wiener (late 1920's) provides full mathematical description of Brownian motion.

Went on to become an early researcher in robotics and cybernetics.

Donsker (1951) shows that Brownian motion is "universal" and describes the large-scale behaviour of a multitude of processes with different microscopic descriptions.

What about two dimensions?

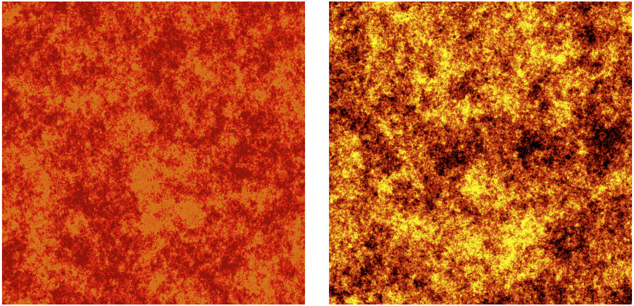
Two dimensional analogue of random walk:



Random function $h\colon Grid \to \mathbf{Z}$ such that $|h(x) - h(y)| = 1$ for $x \sim y$. What does h look like at very large scales?

Free field

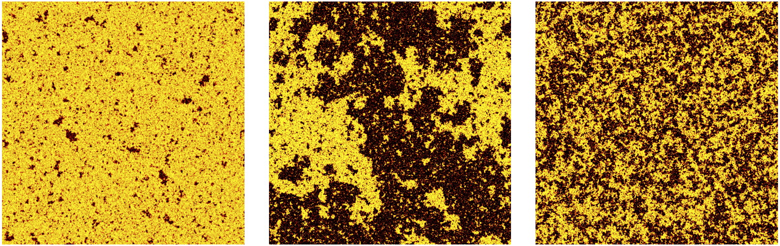
Large scale behaviour should be described by "free field", Gaussian generalised function with $\mathbf{E}h(x)h(y) = -\log|x-y|$. No proof yet! (But for similar models, see Borodin, Johansson, Kenyon, Okounkov, Peled, Toninelli, etc.)



Formally, $\mathbf{P}(dh) \propto \exp(-\int |\nabla h|^2 \, dx)$ "dh".

Beyond "free" systems

Ising model: state space $\sigma\colon \Lambda \to \{\pm 1\}$. Probability to see σ proportional to $\exp(\beta \sum_{x\sim y} \sigma_x \sigma_y)$.



At critical temperature, one has a non-Gaussian scaling limit (rigorous proofs only over last few years), conjectured to be universal for many phase transition models.

Some properties of these objects

In general: Gaussianity not expected when interactions are present.

Scale invariance holds for such scaling limits essentially by definition. Markov property in space(-time) natural for systems with local interactions. Translation invariance and Rotation invariance holds as soon as limit is canonical in some sense. Leap of faith: conformal invariance.

Two dimensions: conformal invariance gives infinite-dimensional symmetry group. Consequence: a lot is known explicitly for a one-parameter family of conformally invariant / covariant objects called conformal field theories. (From probability perspective, see SLE, QLE, CLE, ...)

Some properties of these objects

In general: Gaussianity not expected when interactions are present.

Scale invariance holds for such scaling limits essentially by definition. Markov property in space(-time) natural for systems with local interactions. Translation invariance and Rotation invariance holds as soon as limit is canonical in some sense. Leap of faith: conformal invariance.

Two dimensions: conformal invariance gives infinite-dimensional symmetry group. Consequence: a lot is known explicitly for a one-parameter family of conformally invariant / covariant objects called conformal field theories. (From probability perspective, see SLE, QLE, CLE, ...)

Some properties of these objects

In general: Gaussianity not expected when interactions are present.

Scale invariance holds for such scaling limits essentially by definition. Markov property in space(-time) natural for systems with local interactions. Translation invariance and Rotation invariance holds as soon as limit is canonical in some sense. Leap of faith: conformal invariance.

Two dimensions: conformal invariance gives infinite-dimensional symmetry group. Consequence: a lot is known explicitly for a one-parameter family of conformally invariant / covariant objects called conformal field theories. (From probability perspective, see SLE, QLE, CLE, ...)

Crossover regimes

Consider models that converge to a Gaussian fixed point when "zooming in" and a non-Gaussian FP when "zooming out".
Described by simple "normal form" equations:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi - C\;, \qquad (\text{KPZ}; d = 1)$$
$$\partial_t \Phi = -\Delta\big(\Delta\Phi + C\Phi - \Phi^3\big) + \nabla\xi\;. \quad (\Phi^4; d = 2,3)$$

Here ξ is space-time white noise (think of independent random variables at every space-time point).

KPZ: universal model for weakly asymmetric interface growth.
Φ^4: universal model for phase coexistence near mean-field.

Problem: red terms ill-posed, requires $C = \infty$!!

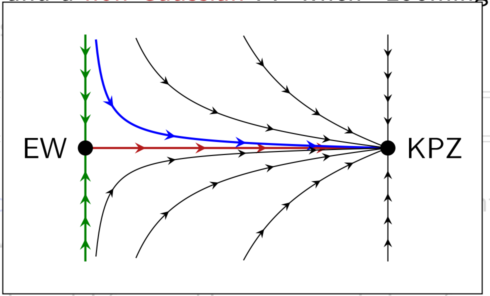
Crossover regimes

Consider models that converge to a Gaussian fixed point when "zooming in" and a non-Gaussian FP when "zooming out".

Described by s

$\partial_t h =$ … $= 1)$

$\partial_t \Phi =$ … $= 2, 3)$



Here ξ is spac … nt random variables at ev

KPZ: universal model for weakly asymmetric interface growth.

Φ^4: universal model for phase coexistence near mean-field.

Problem: red terms ill-posed, requires $C = \infty$!!

Crossover regimes

Consider models that converge to a Gaussian fixed point when "zooming in" and a non-Gaussian FP when "zooming out". Described by simple "normal form" equations:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi - C \;, \qquad (\text{KPZ};\, d = 1)$$

$$\partial_t \Phi = -\Delta\big(\Delta\Phi + C\Phi - \Phi^3\big) + \nabla\xi \;. \quad (\Phi^4;\, d = 2,3)$$

Here ξ is space-time white noise (think of independent random variables at every space-time point).

KPZ: universal model for weakly asymmetric interface growth.

Φ^4: universal model for phase coexistence near mean-field.

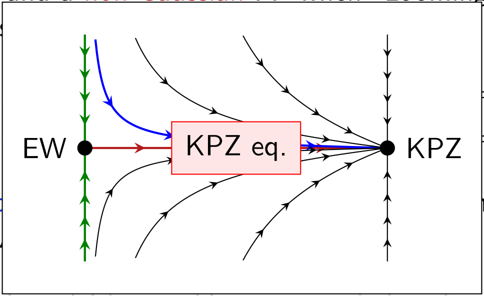
Problem: red terms ill-posed, requires $C = \infty$!!

Crossover regimes

Consider models that converge to a Gaussian fixed point when “zooming in” and a non-Gaussian FP when “zooming out”. Described by s

$\partial_t h =$ … $= 1)$

$\partial_t \Phi =$ … $= 2, 3)$



Here ξ is spac … nt random variables at ev

KPZ: universal model for weakly asymmetric interface growth.

Φ^4: universal model for phase coexistence near mean-field.

Problem: red terms ill-posed, requires $C = \infty$!!

Crossover regimes

Consider models that converge to a Gaussian fixed point when "zooming in" and a non-Gaussian FP when "zooming out". Described by simple "normal form" equations:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi - C \;, \qquad (\text{KPZ}; d = 1)$$
$$\partial_t \Phi = -\Delta\big(\Delta\Phi + C\Phi - \Phi^3\big) + \nabla\xi \;. \quad (\Phi^4; d = 2, 3)$$

Here ξ is space-time white noise (think of independent random variables at every space-time point).

KPZ: universal model for weakly asymmetric interface growth.

Φ^4: universal model for phase coexistence near mean-field.

Problem: red terms ill-posed, requires $C = \infty$!!

Crossover regimes

Consider models that converge to a Gaussian fixed point when "zooming in" and a non-Gaussian FP when "zooming out". Described by simple "normal form" equations:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi - C \;, \qquad (\text{KPZ}; d = 1)$$
$$\partial_t \Phi = -\Delta\big(\Delta\Phi + C\Phi - \Phi^3\big) + \nabla\xi \;. \quad (\Phi^4; d = 2, 3)$$

Here ξ is space-time white noise (think of independent random variables at every space-time point).

KPZ: universal model for weakly asymmetric interface growth.

Φ^4: universal model for phase coexistence near mean-field.

Problem: red terms ill-posed, requires $C = \infty$!!

A general theorem

Joint with Y. Bruned, A. Chandra, I. Chevyrev, L. Zambotti.

Consider a system of semilinear stochastic PDEs of the form

$$\partial_t u_i = \mathcal{L}_i u_i + G_i(u, \nabla u, \ldots) + F_{ij}(u)\xi_j \;, \qquad (\star)$$

with elliptic $\mathcal{L}_i$ and stationary random (generalised) fields ξ_j that are scale invariant with exponents for which $(\star)$ is subcritical.

Then, there exists a canonical family $\Phi_g \colon (u_0, \xi) \mapsto u$ of "solutions" parametrised by $g \in \mathfrak{R}$, a finite-dimensional nilpotent Lie group built from $(\star)$. Furthermore, the maps Φ_g are continuous in both of their arguments.

A general theorem

Joint with Y. Bruned, A. Chandra, I. Chevyrev, L. Zambotti.

Consider a system of semilinear stochastic PDEs of the form

$$\partial_t u_i = \mathcal{L}_i u_i + G_i(u, \nabla u, \ldots) + F_{ij}(u)\xi_j \;, \qquad (\star)$$

with elliptic $\mathcal{L}_i$ and stationary random (generalised) fields ξ_j that are scale invariant with exponents for which $(\star)$ is subcritical.

Then, there exists a canonical family $\Phi_g \colon (u_0, \xi) \mapsto u$ of "solutions" parametrised by $g \in \mathfrak{R}$, a finite-dimensional nilpotent Lie group built from $(\star)$. Furthermore, the maps Φ_g are continuous in both of their arguments.

Some clarifications

Canonicity

Family $\{\Phi_g : g \in \mathfrak{R}\}$ is canonical, but parametrisation only canonical modulo shifts: action of $\mathfrak{R}$ on (F, G) such that

$$\Phi_{g\tilde g}^{(F,G)} = \Phi_g^{\tilde g(F,G)} \;.$$

For smooth ξ, one has a classical solution map $\Phi^{(F,G)}$ and $\Phi_g^{(F,G)} = \Phi^{(g\circ\hat g(\xi))(F,G)}$.

Continuity

Measure $\mathcal{S}$ of "size" of noise. Take ξ_n with $\sup_n \mathcal{S}(\xi_n) < \infty$ and $\xi_n \to \xi$ weakly in probability. Then $\Phi_g(\cdot, \xi_n) \to \Phi_g(\cdot, \xi)$ in some $\mathcal{C}^\alpha$, locally uniformly in time and initial condition, in probability. However, $\xi \mapsto \hat g(\xi)$ not continuous, not even defined!

Some clarifications

Canonicity

Family $\{\Phi_g : g \in \mathfrak{R}\}$ is canonical, but parametrisation only canonical modulo shifts: action of $\mathfrak{R}$ on (F, G) such that

$$\Phi_{g\tilde{g}}^{(F,G)} = \Phi_g^{\tilde{g}(F,G)} \ .$$

For smooth ξ, one has a classical solution map $\Phi^{(F,G)}$ and $\Phi_g^{(F,G)} = \Phi^{(g \circ \hat{g}(\xi))(F,G)}$.

Continuity

Measure $\mathcal{S}$ of "size" of noise. Take ξ_n with $\sup_n \mathcal{S}(\xi_n) < \infty$ and $\xi_n \to \xi$ weakly in probability. Then $\Phi_g(\cdot, \xi_n) \to \Phi_g(\cdot, \xi)$ in some $\mathcal{C}^\alpha$, locally uniformly in time and initial condition, in probability. However, $\xi \mapsto \hat{g}(\xi)$ not continuous, not even defined!

Construction of Φ_g

Crucial remark: Locally, near any space-time point z, solution looks like a linear combination of functions / distributions $\Pi_z\tau$ such that, for each index τ, $\Pi_z\tau$ is scale-invariant with exponent $\deg\tau$.

Deterministic analogue: solutions to parabolic PDEs are smooth, so are locally a linear combination of $(\Pi_z X^k)(\tilde z) = (\tilde z - z)^k$, scale-invariant with exponent $|k|$.

Methodology: Work in spaces of distributions locally described by $\Pi_z H$ for continuous coefficient-valued functions H and look for a fixed point problem for H.

Construction of Φ_g

Crucial remark: Locally, near any space-time point z, solution looks like a linear combination of functions / distributions $\Pi_z \tau$ such that, for each index τ, $\Pi_z \tau$ is scale-invariant with exponent $\deg \tau$.

Deterministic analogue: solutions to parabolic PDEs are smooth, so are locally a linear combination of $(\Pi_z X^k)(\tilde{z}) = (\tilde{z} - z)^k$, scale-invariant with exponent $|k|$.

Methodology: Work in spaces of distributions locally described by $\Pi_z H$ for continuous coefficient-valued functions H and look for a fixed point problem for H.

Construction of Φ_g

Crucial remark: Locally, near any space-time point z, solution looks like a linear combination of functions / distributions $\Pi_z \tau$ such that, for each index τ, $\Pi_z \tau$ is scale-invariant with exponent $\deg \tau$.

Deterministic analogue: solutions to parabolic PDEs are smooth, so are locally a linear combination of $(\Pi_z X^k)(\tilde{z}) = (\tilde{z} - z)^k$, scale-invariant with exponent $|k|$.

Methodology: Work in spaces of distributions locally described by $\Pi_z H$ for continuous coefficient-valued functions H and look for a fixed point problem for H.

Example / problem

Solution to

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + \sigma(h)\xi \ ,$$

locally given by $h(\tilde z) \approx \big(\Pi_z H(z)\big)(\tilde z)$ with

$$\begin{aligned} H = h\,\mathbf{1} &+ \sigma(h) + (\sigma\sigma')(h) + (f\sigma^2)(h) + h'\,X \\ &+ 2(f\sigma^2\sigma')(h) + 2(f^2\sigma^3)(h) + (f'\sigma^3)(h) \\ &+ \frac{1}{2}(\sigma^2\sigma'')(h) + (\sigma(\sigma')^2)(h) + (f\sigma^2\sigma')(h) \\ &+ (f'\sigma)(h)h' + 2(f\sigma)(h)h' + \dots \end{aligned}$$

Problem: $\Pi_z = G \star (\partial_x G \star \xi)^2$ is divergent!

Example / problem

Solution to

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + \sigma(h)\xi \ ,$$

locally given by $h(\tilde z) \approx \big(\Pi_z H(z)\big)(\tilde z)$ with

$$H = h\,\mathbf{1} + \sigma(h) + (\sigma\sigma')(h) + (f\sigma^2)(h) + h'\,X$$
$$+ 2(f\sigma^2\sigma')(h) + 2(f^2\sigma^3)(h) + (f'\sigma^3)(h)$$
$$+ \frac{1}{2}(\sigma^2\sigma'')(h) + (\sigma(\sigma')^2)(h) + (f\sigma^2\sigma')(h)$$
$$+ (f'\sigma)(h)h' + 2(f\sigma)(h)h' + \ldots$$

Problem: $\Pi_z = G \star (\partial_x G \star \xi)^2$ is divergent!

Example / problem

Solution to

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + \sigma(h)\xi \ ,$$

locally given by $h(\tilde z) \approx \big(\Pi_z H(z)\big)(\tilde z)$ with

$$H = h\,\mathbf{1} + \sigma(h) + (\sigma\sigma')(h) + (f\sigma^2)(h) + h'\,X$$

$$+\, 2(f\sigma^2\sigma')(h) + 2(f^2\sigma^3)(h) + (f'\sigma^3)(h)$$

$$+\, \frac{1}{2}(\sigma^2\sigma'')(h) + (\sigma(\sigma')^2)(h) + (f\sigma^2\sigma')(h)$$

$$+\, (f'\sigma)(h)h' + 2(f\sigma)(h)h' + \ldots$$

Problem: $\Pi_z = G \star (\partial_x G \star \xi)^2$ is divergent!

Example / problem

Solution to

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + \sigma(h)\xi\ ,$$

locally given by $h(\tilde z) \approx \big(\Pi_z H(z)\big)(\tilde z)$ with

$$\begin{aligned} H = h\,\mathbf{1} &+ \sigma(h) + (\sigma\sigma')(h) + (f\sigma^2)(h) + h'\,X \\ &+ 2(f\sigma^2\sigma')(h) + 2(f^2\sigma^3)(h) + (f'\sigma^3)(h) \\ &+ \frac{1}{2}(\sigma^2\sigma'')(h) + (\sigma(\sigma')^2)(h) + (f\sigma^2\sigma')(h) \\ &+ (f'\sigma)(h)h' + 2(f\sigma)(h)h' + \dots \end{aligned}$$

Problem: $\Pi_z = G \star (\partial_x G \star \xi)^2$ is divergent!

Example / problem

Solution to

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + \sigma(h)\xi \;,$$

locally given by $h(\tilde z) \approx \big(\Pi_z H(z)\big)(\tilde z)$ with

$$\begin{aligned} H = h\,\mathbf{1} &+ \sigma(h) + (\sigma\sigma')(h) + (f\sigma^2)(h) + h'\,X \\ &+ 2(f\sigma^2\sigma')(h) + 2(f^2\sigma^3)(h) + (f'\sigma^3)(h) \\ &+ \frac{1}{2}(\sigma^2\sigma'')(h) + (\sigma(\sigma')^2)(h) + (f\sigma^2\sigma')(h) \\ &+ (f'\sigma)(h)h' + 2(f\sigma)(h)h' + \ldots \end{aligned}$$

Problem: $\Pi_z = G \star (\partial_x G \star \xi)^2$ is divergent!

Example / problem

Solution to

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + \sigma(h)\xi \;,$$

locally given by $h(\tilde z) \approx \big(\Pi_z H(z)\big)(\tilde z)$ with

$H = h\,\mathbf{1} + \sigma(h)$  $+ (\sigma\sigma')(h)$  $+ (f\sigma^2)(h)$  $+ h'\,X$

$+ 2(f\sigma^2\sigma')(h)$  $+ 2(f^2\sigma^3)(h)$  $+ (f'\sigma^3)(h)$ 

$+ \frac{1}{2}(\sigma^2\sigma'')(h)$  $+ (\sigma(\sigma')^2)(h)$  $+ (f\sigma^2\sigma')(h)$ 

$+ (f'\sigma)(h)h'$  $+ 2(f\sigma)(h)h'$  $+ \dots$

Problem: Π_z  $= G \star (\partial_x G \star \xi)^2$ is divergent!

Steps of proof

1. Show that (h, h') depend continuously on the data $\{\Pi_z \tau \,:\, z, \tau\}$ in suitable topology enforcing some natural algebraic relations.
2. Replace $\Pi_z \tau$ by "renormalised version" $\Pi_z^g \tau$ such that algebraic relations between the $\Pi_z^g \tau$'s remain unchanged. (Determines the group $\mathfrak{R}$.)
3. Choose g such that $\mathbf{E}\Pi_z^g \tau = 0$ for $\deg \tau \leq 0$. (Determines the element g from the law of ξ.)
4. Show stability / continuity of $\xi \mapsto \Pi_z^{g(\xi)}$.
5. Show that the substitution $\Pi \mapsto \Pi^g$ is equivalent to replacing (f, σ) by $g(f, \sigma)$ for suitable action of $\mathfrak{R}$.

Steps of proof

1. Show that (h, h') depend continuously on the data $\{\Pi_z \tau \,:\, z, \tau\}$ in suitable topology enforcing some natural algebraic relations.
2. Replace $\Pi_z \tau$ by "renormalised version" $\Pi_z^g \tau$ such that algebraic relations between the $\Pi_z^g \tau$'s remain unchanged. (Determines the group $\mathfrak{R}$.)
3. Choose g such that $\mathbf{E}\Pi_z^g \tau = 0$ for $\deg \tau \leq 0$. (Determines the element g from the law of ξ.)
4. Show stability / continuity of $\xi \mapsto \Pi_z^{g(\xi)}$.
5. Show that the substitution $\Pi \mapsto \Pi^g$ is equivalent to replacing (f, σ) by $g(f, \sigma)$ for suitable action of $\mathfrak{R}$.

Steps of proof

1. Show that (h, h') depend continuously on the data $\{\Pi_z\tau \,:\, z, \tau\}$ in suitable topology enforcing some natural algebraic relations.
2. Replace $\Pi_z\tau$ by "renormalised version" $\Pi_z^g\tau$ such that algebraic relations between the $\Pi_z^g\tau$'s remain unchanged. (Determines the group $\mathfrak{R}$.)
3. Choose g such that $\mathbf{E}\Pi_z^g\tau = 0$ for $\deg\tau \le 0$. (Determines the element g from the law of ξ.)
4. Show stability / continuity of $\xi \mapsto \Pi_z^{g(\xi)}$.
5. Show that the substitution $\Pi \mapsto \Pi^g$ is equivalent to replacing (f, σ) by $g(f, \sigma)$ for suitable action of $\mathfrak{R}$.

Steps of proof

1. Show that (h, h') depend continuously on the data $\{\Pi_z\tau \,:\, z, \tau\}$ in suitable topology enforcing some natural algebraic relations.
2. Replace $\Pi_z\tau$ by "renormalised version" $\Pi_z^g\tau$ such that algebraic relations between the $\Pi_z^g\tau$'s remain unchanged. (Determines the group $\mathfrak{R}$.)
3. Choose g such that $\mathbf{E}\Pi_z^g\tau = 0$ for $\deg\tau \le 0$. (Determines the element g from the law of ξ.)
4. Show stability / continuity of $\xi \mapsto \Pi_z^{g(\xi)}$.
5. Show that the substitution $\Pi \mapsto \Pi^g$ is equivalent to replacing (f, σ) by $g(f, \sigma)$ for suitable action of $\mathfrak{R}$.

Steps of proof

1. Show that (h, h') depend continuously on the data $\{\Pi_z \tau : z, \tau\}$ in suitable topology enforcing some natural algebraic relations.
2. Replace $\Pi_z \tau$ by "renormalised version" $\Pi_z^g \tau$ such that algebraic relations between the $\Pi_z^g \tau$'s remain unchanged. (Determines the group $\mathfrak{R}$.)
3. Choose g such that $\mathbf{E}\Pi_z^g \tau = 0$ for $\deg \tau \leq 0$. (Determines the element g from the law of ξ.)
4. Show stability / continuity of $\xi \mapsto \Pi_z^{g(\xi)}$.
5. Show that the substitution $\Pi \mapsto \Pi^g$ is equivalent to replacing (f, σ) by $g(f, \sigma)$ for suitable action of $\mathfrak{R}$.

Thank you very much for your attention

(See you tomorrow...)

Thanks

Thank you very much

for your attention

(See you tomorrow...)