Simons Lecture 2: Sup norms, quantum ergodicity, and counting nodal domains of eigenfunctions

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joint work in part with Junehyuk Jung and Chris Sogge
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Nodal set and nodal domains

Let (M, g) be a compact negatively curved surface without boundary, and consider an eigenfunction of the Laplacian,

$$(\Delta + \lambda^2)\varphi_\lambda = 0.$$

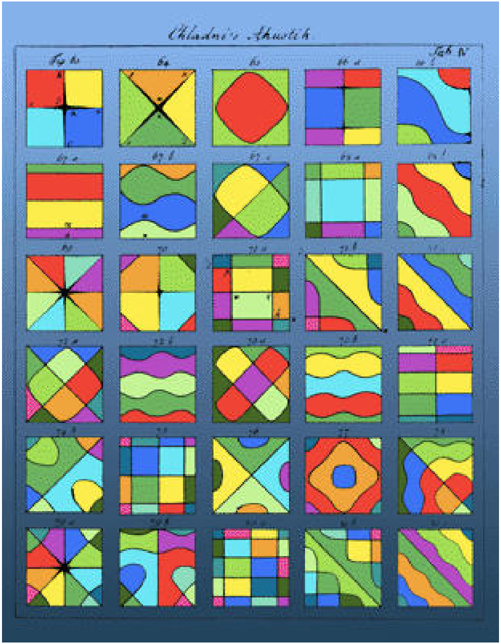
The nodal set of φ_λ is

$$\mathcal{N}_{\varphi_\lambda} = \{x \in M : \varphi_\lambda(x) = 0\}.$$

The nodal domains partition M into disjoint open sets:

$$M \backslash \mathcal{N}_{\varphi_\lambda} = \bigcup_{j=1}^{N(\varphi)} \Omega_j.$$

Chladni images of nodal sets ca. 1800

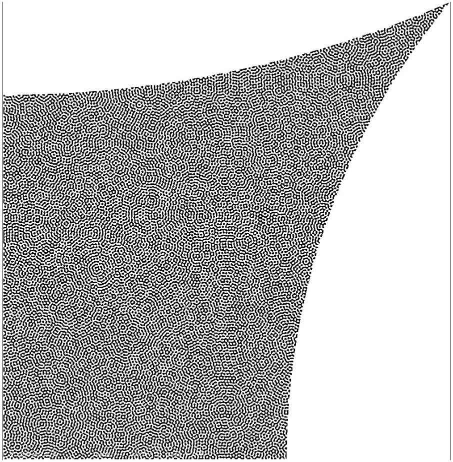


Overview of Lecture 2

- Sketch the proof (with Junehyuk Jung) that the number of nodal domains of an orthonormal basis φ_j of eigenfunctions $\to \infty$ as $j \to \infty$ on a surface of non-positive curvature and concave boundary, or on a surface of negative curvature and an isometric involution σ;
- It is based in part on showing that the eigenfunctions do not extremize sup norms. This is part of a new result with Chris Sogge characterizing surfaces with maximal eigenfunction growth, which will also be sketched.
- It is also based on quantum ergodic restriction theorems (joint with J. Toth).

Ergodic billiards

We expect the dynamics of the geodesic flow to have an important impact on the number of nodal domains. In the case of chaotic geodesic flow, we expect nodal domains to be random.



New results (joint with Junehyuk Jung)

- Let (M, J, σ) be a *real* Riemann surface surface, i.e. a Riemann surface with anti-holomorphic involution σ and with $\text{Fix}(\sigma) \neq \emptyset$. We further assume $M - \text{Fix}(\sigma)$ has two connected components (the "dividing" case). Let g be any negatively curved σ-invariant metric on M. Then, for almost the entire sequence of even or odd eigenfunctions, the number of nodal domains tends to infinity.

- The same statement is true for Dirichlet/Neumann eigenfunctions of any non-positively curved surface with concave boundary.

Real Riemann surface $g = 2$ with dividing real locus: Involution: top-bottom

As this picture indicates, the surfaces in question are complexifications of real algebraic curves. $Fix(\sigma)$ is the underlying real curve.

Number of domains tends to infinity for almost all even/odd eigenfunctions

Theorem

Let (M, J, σ) be a compact real Riemann surface with $Fix(\sigma)$ dividing. Let g be any σ-invariant Riemannian metric. Then for any orthonormal eigenbasis $\{\varphi_j\}$ of $L^2_{even}(Y)$, resp. $\{\psi_j\}$ of $L^2_{odd}(M)$, one can find a density 1 subset A of $\mathbb{N}$ such that

$$\lim_{\substack{j \to \infty \\ j \in A}} N(\varphi_j) = \infty,$$

resp.

$$\lim_{\substack{j \to \infty \\ j \in A}} N(\psi_j) = \infty,$$

For odd eigenfunctions, the conclusion holds as long as $Fix(\sigma) \neq \emptyset$.

Remark on even and odd eigenfunctions

φ_λ is even (resp odd) if $\sigma^* \varphi_\lambda = \pm \varphi_\lambda$.

The even (resp. odd) part of φ_λ is $\frac{1}{2}(\varphi_\lambda \pm \sigma^* \varphi_\lambda)$.

- If the multiplicity of an eigenvalue λ_j^2 of Δ_g is $= 1$, then automatically the eigenfunction is either even or odd.
- For generic σ-invariant negatively curved metrics, all eigenvalues have multiplicity one.
- In the non-generic cases of g for which some eigenvalue has multiplicity > 1, the result only holds for the even or odd eigenfunctions and not necessarily for all eigenfunctions.

Same result Sinai billiards

Theorem

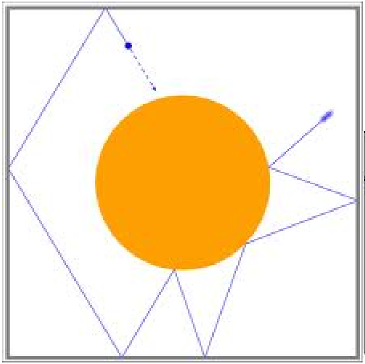
Let (X, g) be a surface with curvature $k \leq 0$ and let D be a small disc in X. Remove the disc to obtain a Sinai-Lorentz billiard $M = X \backslash D$. Then for any orthonormal eigenbasis $\{\varphi_j\}$ of eigenfunctions, one can find a density 1 subset A of $\mathbb{N}$ such that

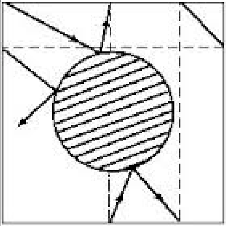
$$\lim_{\substack{j \to \infty \\ j \in A}} N(\varphi_j) = \infty,$$

A density one subset $A \subset \mathbf{N}$ is one for which

$$\frac{1}{N} \#\{j \in A, j \leq N\} \to 1, \quad N \to \infty.$$

Stadium: Ergodic billiards





Analogy

- Fix(σ) of (M, J, σ) is analogous to ∂M for a non-positively curved surface with concave boundary.
- $M \backslash$Fix(σ) is a (pair of) non-positively curved surfaces with boundary;
- $\partial_\nu =$ unit normal to ∂M. Neumann: $\partial_\nu \varphi_j = 0$ on ∂M. Dirichlet $\varphi_j = 0$ on ∂M.
- Even eigenfunctions $\sigma^* \varphi_j = \varphi_j$ are analogous to Neumann eigenfunctions– and are such on $M \backslash$Fix(σ). Odd eigenfunctions are Dirichlet.

Main ideas of the proof

1. Relate number of nodal domains to number of intersection points of nodal set with (i) the boundary ∂M, resp. (ii) $\mathrm{Fix}(\sigma)$. This is purely topological.

2. **Prove that Neumann eigenfunctions have a lot of zeros on ∂M, resp. Dirichlet eigenfunctions have many zeros of $\partial_\nu \varphi_j = 0$ on ∂M. In the (M, σ) setting, even eigenfunctions have many zeros on $\mathrm{Fix}(\sigma)$, odd have any zeros of the normal derivative on $\mathrm{Fix}(\sigma)$. (True for almost any curve if the geodesic flow is ergodic.**

3. To prove (2), we need to show that $||\varphi_j||_{L^\infty} = o(\lambda^{\frac{1}{2}})$. Also that $\int_{\partial M} f \varphi j_k ds = O(\lambda_{j_k}^{-\frac{1}{2}})$ for almost all k.

4. We also need to show that $\varphi_j^2 ds \to 1$ in the weak sense of measures on ∂M (Neumann case) or $\lambda_j^{-2} |\partial_\nu \varphi_j|^2 ds \to 1$ (Dirichlet case). "Quantum ergodic restriction theorems"

From zeros on $\gamma = \partial M, \text{Fix}(\sigma)$ to nodal domains

If

1. $\int_\gamma f \varphi_{\lambda_j} ds = O(\lambda_j^{-\frac{1}{2}} (\log \lambda_j)^{1/4})$,
2. $\int_\gamma f \varphi_{\lambda_j}^2 ds \geq 1$,
3. $||\varphi_j||_{L^\infty} \leq C \frac{\lambda_j^{\frac{1}{2}}}{\sqrt{\log \lambda_j}}$,

there must exist an unbounded number of sign changes of $\varphi_{\lambda_j}|_\gamma$ as $j \to \infty$. Indeed, for any arc $\beta \subset \gamma$,

$$| \int_\beta \varphi_{\lambda_j} ds| \leq C \lambda_j^{-1/2} (\log \lambda)^{1/4}$$

and

$$\int_\beta |\varphi_{\lambda_j}| ds \geq ||\varphi_{\lambda_j}||_\infty^{-1} ||\varphi_{\lambda_j}||^2_{L^2(\beta)} \geq C \lambda_j^{-1/2} (\log \lambda_j)^{\frac{1}{2}},$$

and this is a contradiction if $\varphi_{\lambda_j} \geq 0$ on β.

Remainder of talk

In the remainder, I emphasize the two main analytical ingredients of independent interest:

1. Geometry of sup norms of eigenfunctions: why eigenfunctions are never of "maximal size" on non-positively curved surfaces with concave boundary, or on negatively curved surfaces;

2. Quantum ergodicity and quantum ergodic restriction theorems: why restrictions of eigenfunctions to ∂M or to $\mathrm{Fix}(\sigma)$ oscillate very rapidly and weakly smear out to their average values of 1.

A word on the topological argument

For the sake of completeness, let us first sketch the topological argument. Let $\gamma = \partial M$, resp. $\mathrm{Fix}(\sigma)$.
One can modify the nodal set $\mathcal{N}_{\varphi_j}$ ($\mathcal{N}_{\varphi_j} \cup \gamma$, when φ_j is even) to give it the structure of a one-dimensional CW complex):

1. For each embeded circle which does not intersect γ, we add a vertex.
2. Each singular point $\varphi_j(p) = d\varphi_j(p) = 0$ is a vertex.
3. If $\gamma \not\subset \mathcal{N}_{\varphi_\lambda}$, then each intersection point in $\gamma \cap \mathcal{N}_{\varphi_j}$ is a vertex.
4. Edges are the arcs of $\mathcal{N}_{\varphi_j}$ ($\mathcal{N}_{\varphi_j} \cup \gamma$, when φ_j is even) which join the vertices listed above.

This way, we obtain a graph embeded into the surface M.

Graph embedded in a surface

An embedded graph G in a surface M is a finite set $V(G)$ of vertices and a finite set $E(G)$ of edges which are simple (non-self-intersecting) curves in M such that any two distinct edges have at most one endpoint and no interior points in common.

The *faces* f of G are the connected components of $M \backslash V(G) \cup \bigcup_{e \in E(G)} e$. The set of faces is denoted $F(G)$.

An edge $e \in E(G)$ is *incident* to f if the boundary of f contains an interior point of e. Every edge is incident to at least one and to at most two faces; if e is incident to f then $e \subset \partial f$.

The faces are not assumed to be cells and the sets $V(G), E(G), F(G)$ are not assumed to form a CW complex.

Euler inequality

Now let $v(\varphi_\lambda)$ be the number of vertices, $e(\varphi_\lambda)$ be the number of edges, $f(\varphi_\lambda)$ be the number of faces, and $m(\varphi_\lambda)$ be the number of connected components of the graph. Then by Euler's formula

$$v(\varphi_\lambda) - e(\varphi_\lambda) + f(\varphi_\lambda) - m(\varphi_\lambda) \geq 1 - 2g_M \qquad (1)$$

where g_M is the genus of the surface.

Application to nodal sets: odd case

The Euler inequality gives a lower bound for the number of nodal domains from a lower bound on the number of points where $\partial_\nu \varphi_j = 0$ and changes sign on γ (odd case, Dirichlet), resp. numbers of sign-change zeros on γ (even, Neumann case).

Lemma

For an odd eigenfunction ψ_j, let $\Sigma_{\psi_j} = \{x : \psi_j(x) = d\psi_j(x) = 0\}$. Then

$$N(\psi_j) \geq \#\left(\Sigma_{\psi_j} \cap \gamma\right) + 2 - 2g_M,$$

and for an even eigenfunction φ_j,

$$N(\varphi_j) \geq \frac{1}{2}\#\left(\mathcal{N}_{\varphi_j} \cap \gamma\right) + 1 - g_M.$$

Recap on main ideas of the proof

1. Relate number of nodal domains to number of intersection points of nodal set with (i) the boundary ∂M, resp. (ii) Fix(σ), the fixed point set of the involution σ. This is purely topological.

2. Prove that eigenfunctions have a lot of zeros on ∂M resp. Fix(σ) – this is true for almost any curve on M when the geodesic flow is ergodic.

3. To prove (2), we need to show that $||\varphi_j||_{L^\infty} = o(\lambda^{\frac{1}{2}})$. Also that $\int_{\partial M} f \varphi_j ds = O(\lambda_j^{-\frac{1}{2}})$.

4. We also need to show that $\varphi_j^2 ds \to 1$ in the weak sense of measures on ∂M (Neumann case) or $\lambda_j^{-2} |\partial_\nu \varphi_j|^2 ds \to 1$ (Dirichlet case). "Quantum ergodic restriction theorems"

Sizes of eigenfunctions

We consider the sizes of L^2 normalized eigenfunctions

$$\Delta\varphi_\lambda = -\lambda^2\varphi_\lambda, \quad \int_M |\varphi_\lambda|^2 dV = 1$$

of compact Riemannian manifolds (M, g).

How should we measure the size? The simplest way is by L^p norms

$$||\varphi_\lambda||_{L^p}^p = \int_M |\varphi_\lambda|^p dV.$$

Measuring eigenfunction growth

The eigenvalues $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \uparrow \infty$ form a discrete increasing sequence, and we denote an orthonormal basis of eigenfunctions with increasing eigenvalue by $\{\varphi_{\lambda_j}(x)\}$, or by φ_j for short.

The eigenvalues may have multiplicity > 1 and L^p norms depend very much on which basis we pick. So we consider the maximal value the L^p norm in the eigenspace (minimal is also of much interest).

Denote the eigenspaces by

$$V_\lambda = \{\varphi : \Delta\varphi = -\lambda^2\varphi\}.$$

We measure the growth rate of L^p norms by

$$L^p(\lambda, g) = \sup_{\varphi \in V_\lambda : ||\varphi||_{L^2} = 1} ||\varphi||_{L^p}. \tag{2}$$

Universal upper bounds on L^∞ norms of eigenfunctions

One of the simplest L^p norms to consider is the L^∞ norm,

$$||\varphi_\lambda||_{L^\infty} = \sup_M |\varphi_\lambda(x)|.$$

There exist a bound which holds for all (M, g): A classical result of Avakumovic, Levitan, H"ormander states:

$$||\varphi_\lambda||_{L^\infty} \leq C_g \lambda^{m-1}, \;\; (m = \dim M).$$

It is sharp in the sense that there exist (M, g) and sequences of eigenfunctions saturating (achieving) the bound. But such (M, g) are very rare.

Sogge universal upper bounds on L^p norms of eigenfunctions

Theorem

(Sogge, 1985)

$$\sup_{\varphi \in V_\lambda} \frac{\|\varphi\|_p}{\|\varphi\|_2} = O(\lambda^{\delta(p)}), \quad 2 \leq p \leq \infty \tag{3}$$

where

$$\delta(p) = \begin{cases} n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases} \tag{4}$$

Note that there is a 'phase transition' at $p = \frac{2(n+2)}{n-1}$. Again, there exist (M, g) and sequences of eigenfunctions for which they are sharp, but again such (M, g) are very rare.

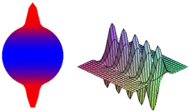
Extremals

The upper bounds are sharp in the class of all (M, g) and are saturated on the round sphere:

- For $p > \frac{2(n+1)}{n-1}$, zonal (rotationally invariant) spherical harmonics saturate the L^p bounds. Such eigenfunctions also occur on surfaces of revolution.
- For L^p for $2 \leq p \leq \frac{2(n+1)}{n-1}$ the bounds are saturated by highest weight spherical harmonics, i.e. Gaussian beam along a stable elliptic geodesic. Such eigenfunctions also occur on surfaces of revolution.

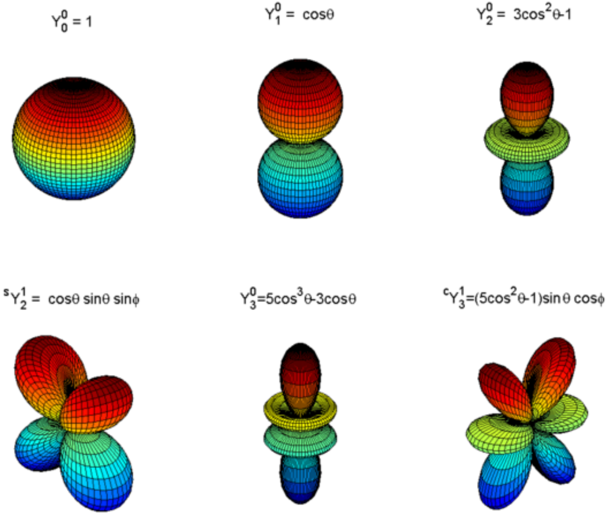
(i)Zonal eigenfunction (ii) Gaussian beam

The left image is a zonal spherical harmonic of degree N on S^2: it has high peaksof height $\sqrt{N}$ at the north and south poles. The right image is a Gaussian beam: its height along the equator is $N^{1/4}$ and then it has Gaussian decay transverse to the equator.



The zonal has high L^p norm due to its high peaks on balls of radius $\frac{1}{N}$. The balls are so small that they do not have high L^p norms for small p. The Gaussian beams are not as high but they are relatively high over an entire geodesic.

Graphics of spherical haromonics



(M, g) with maximal eigenfunction growth

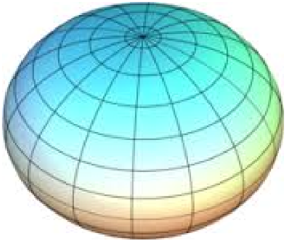
Definition: Say that (M, g) has maximal L^p eigenfunction growth if it possesses a sequence of eigenfunctions $\varphi_{\lambda_{j_k}}$ which saturates the L^p bounds. When $p = \infty$ we say that it has maximal sup norm growth.

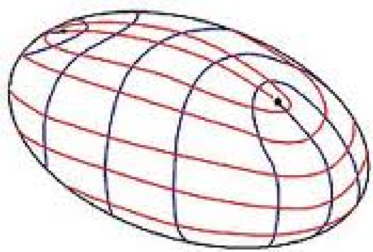
Poles and self-focal points

- The zonal spherical harmonics on S^2 or a surface of revolution peak at the "poles". We call a point $p \in (M, g)$ a pole if every geodesics leaving p is a closed geodesic: the geodesic $\gamma_{p,\xi}$ at p in direction ξ returns to p at a fixed time T and $\gamma'_{p,\xi}(0) = \gamma'_{p,\xi}(T)$. The fixed points of a surface of revolution are poles; all points of S^2 are poles.

- We call a point p a *self-focal point* or *blow-down point* if all geodesics leaving p loop back to p at a common time T. They do not have to be closed geodesics. Examples: umbilic points of ellipsoids; foci of ellipses;

Pole of a surface of revolution and umbilic points on an ellipsoid





Known examples of (M, g) with maximal eigenfunction growth

- The only known examples of (M, g) with maximal eigenfunction growth for high L^p norms have poles. Sogge/Z prove this for real analytic surfaces. In fact, the only known examples are zonal eigenfunctions on surfaces of revolution (and their higher dimensional analogues).

- The only known examples of eigenfunctions saturating low L^p bounds are Gaussian beams. It is a very interesting problem to prove that any (M, g) with maximal L^p growth of eigenfunctions for low p must have a stable elliptic geodesic. We will not discuss them further in this talk.

First characterization of (M, g) of maximal eigenfunction growth

THEOREM

(Sogge-Z, 2002) Suppose (M, g) is a C^∞ Riemannian manifold with maximal eigenfunction growth, i.e. having a sequence $\{\varphi_{\lambda_{j_k}}\}$ of eigenfunctions which achieves (saturates) the bound $||\varphi_\lambda||_{L^\infty} \leq \lambda^{(n-1)/2}$.
Then there must exist a point $x \in M$ for which the set

$$\mathcal{L}_x = \{\xi \in S^*_x M : \exists T : \exp_x T\xi = x\} \tag{5}$$

of directions of geodesic loops at x has positive measure in $S^*_x M$. Here, exp is the exponential map, and the measure $|\Omega|$ of a set Ω is the one induced by the metric g_x on $T^*_x M$. For instance, the poles x_N, x_S of a surface of revolution (S^2, g) satisfy $|\mathcal{L}_x| = 2\pi$.

Real analytic (M, g) of maximal eigenfunction growth

THEOREM

(Sogge-Z, 2014) Suppose (M, g) is a real analytic Riemannian surface without boundary and with maximal eigenfunction growth, i.e. having a sequence $\{\varphi_{\lambda_{j_k}}\}$ of eigenfunctions which achieves (saturates) the bound $||\varphi_\lambda||_{L^\infty} \leq \lambda^{1/2}$.

Then there must exist a pole i.e. a point $x \in M$ such that every geodesic through x is closed. In particular, M must be a topological S^2.

Moreover: If (M, g) is C^∞, non-positively curved with concave boundary, and then the sup norms of Cauchy data $(\lambda^{-1}\partial_\nu\varphi, \varphi)|_{\partial M}$ of Dirichlet or Neumann eigenfunctions is always $o(\lambda^{1/2})$.

Higher dimensions (2013-2014)

THEOREM

(Z, Sogge 2013) If (M, g) *is real analytic and has maximal eigenfunction growth, then it possesses a self-focal point whose first return map* Φ_x *has an invariant function* $f \in L^2(S^*_x M, \mu_x)$*, where* μ_x *is the Euclidean surface measure. Equivalently, it has an invariant measure* $|f|^2 d\mu_x$ *which is absolutely continuous with respect to* $d\mu_x$

Invariant means that $f \circ \Phi_x = f$.

Geometry problems

- Do there exist real analytic (M, g) with $\dim \geq 3$ which have twisted self-focal points (all geodesics loop back but almost none are closed)?

- In 2D if $f : S^1 \to S^1$ is a reversible diffeo (conjuagate to its inverse) and preserves an L^1 measure $dd\theta$ then $f^2 = id$. Is this true in higher dims?

Second ingredient: quantum ergodic restriction theorem

(Hans Christianson, John Toth, Andrew Hassell, and S. Z):

Theorem

Let γ be either ∂M for the surface with boundary or $Fix(\sigma)$ for the surface with involution. Then, for a subsequence of Neumann eigenfunctions of density one,

$$\langle f \varphi_j|_\gamma, \varphi_j|_\gamma \rangle_{L^2(\gamma)}$$

$$\to \frac{4}{2\pi Area(M)} \int_\gamma f(s) ds.$$

Similarly for normal derivatives of Dirichlet eigenfunctins. Cauchy data of eigenfunctions to γ are quantum ergodic along γ. This is part of a much more general result.

Quantum ergodic restriction theorem for (M, J, σ, g)

Theorem

Assume that (M, g) has an orientation reversing isometric involution with separating fixed point set H. Let γ be a component of H. Let φ_j be the sequence of even ergodic eigenfunctions. Then,

$$\langle f \varphi_j|_\gamma, \varphi_j|_\gamma \rangle_{L^2(\gamma)}$$

$$\to_{j \to \infty} \frac{4}{2\pi Area(M)} \int_\gamma f(s) ds.$$

Similarly for normal derivatives of odd eigenfunctions.

General quantum ergodic restriction problem

Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions

$$\Delta \varphi_j = -\lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

of the Laplacian on a (mainly compact) Riemannian manifold (M, g), with our without boundary. If $\partial M \neq \emptyset$ we impose standard BC. Suppose:
Assume that the geodesic flow $G^t : S_g^* M \to S_g^* M$ is ergodic.

Theorem Let $H \subset M$ be a hypersurface. Then the Cauchy data of eigenfunctions is **always** quantum ergodic along H.