An Evening with Euler

William Dunham
Research Associate in Mathematics
Bryn Mawr College

Leonhard Euler
1707 - 1783

1707 – born in Basel, Switzerland

1720 – studied with Johann Bernoulli

1722 – graduated from U. Basel

1727 – to St. Petersburg Academy

1741 – to Berlin Academy

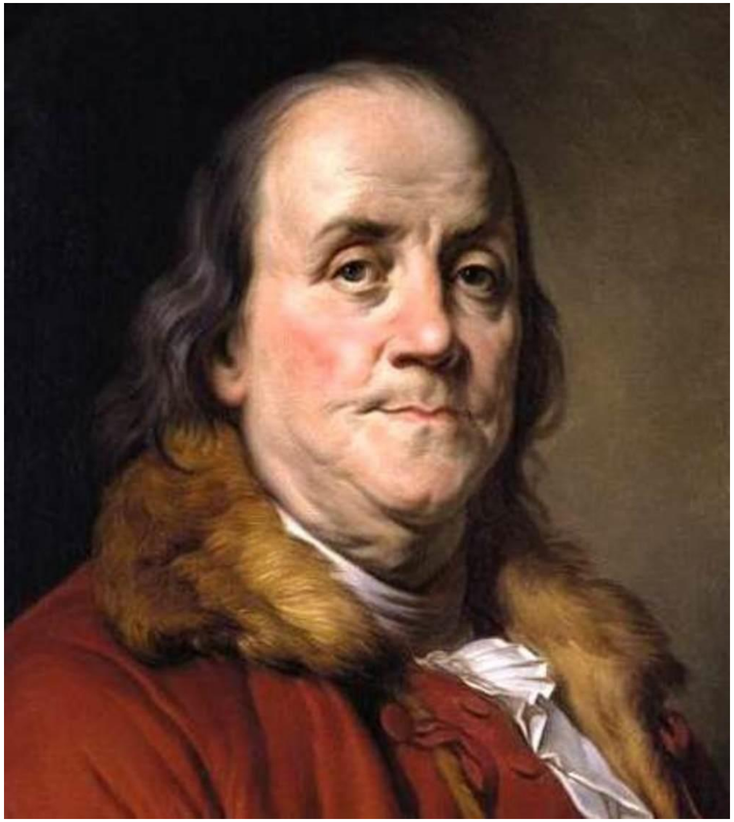
1766 – back to St. Petersburg

1782 – foreign member, AAAS

1783 – died

Euler’s Tomb , St. Petersburg

Leonhard Euler
1707 - 1783



Benjamin Franklin
1706 - 1790

Married to Katharina Gsell; 13 children

Phenomenal memory

In the 1730s, he lost vision in one eye

By 1771, he was essentially blind

In 1775, he produced a paper a *week*!

Euler's mathematics is distinguished by its quantity and quality.

Quantity

1783: Euler died

1783 – 1830: published 159 more papers

1844: 61 new papers

1849: 8 more

1910: Gustav Eneström catalogued Euler's works at 866 books and papers.

The catalogue itself ran to 388 pages!

1911: Swiss Academy began publication of Euler's *Opera Omnia*.

Presently 75 volumes in four series and more than 25,000 pages.

Quality …

Functions (1748)

In the *Introductio in analysin infinitorum*, Euler established that the proper focus of analytic geometry, trigonometry, and calculus was not the curve but …

…the function

and he introduced polynomial functions, logarithmic functions, exponential functions, trigonometric functions, and inverse trig functions

The number e (1748)

Quodsi iam ex hac basi logarithmi construantur, ii vocari solent logarithmi *naturales* seu *hyperbolici*, quoniam quadratura hyperbolae per istiusmodi logarithmos exprimi potest. Ponamus autem brevitatis gratia pro numero hoc 2,71828 18284 59 etc. constanter litteram

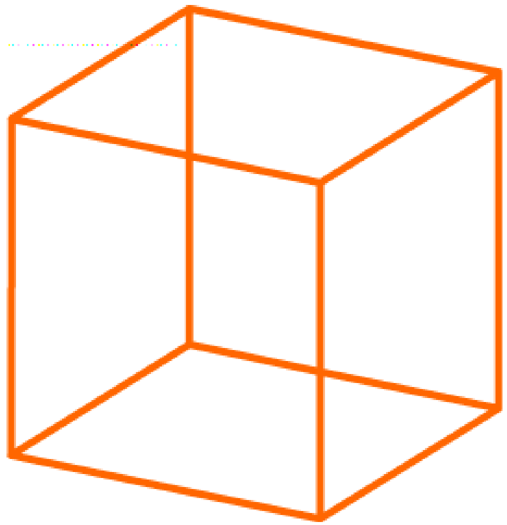
$$e,$$

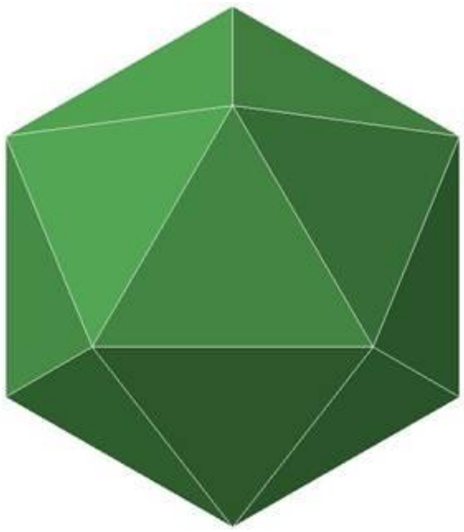
quae ergo denotabit basin logarithmorum naturalium seu hyperbolicorum [1]), cui respondet valor litterae $k = 1$; sive haec littera e quoque exprimet summam huius seriei

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc. in infinitum.}$$

Euler's Polyhedral Formula (1752)

$$V + F = E + 2$$





V = # vertices

F = # faces

E = # edges

$V = 8$

$F = 6$

$E = 12$

"I find it surprising that these general results in solid geometry have not previously been noticed by anyone, so far as I am aware."

The Basel Problem (1734)

In 1689, Jakob Bernoulli challenged the mathematical community to find the *exact* sum of the infinite series

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{k^2}+\ldots=\frac{\pi^2}{6}$$

Geometry

For any triangle, consider:

the intersection of the altitudes (*orthocenter*)

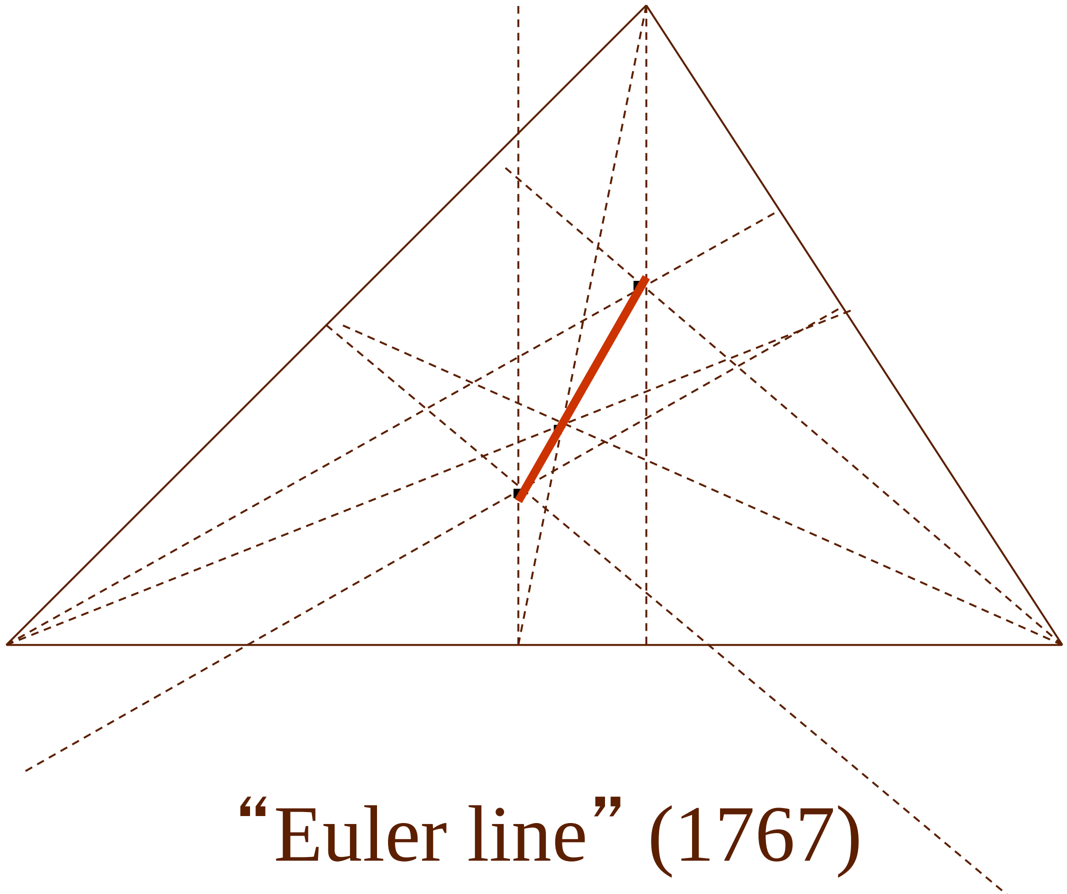
the intersection of the medians (*centroid*)

the intersection of the ^ bisectors (*circumcenter*)

Intersection of altitudes

Intersection of medians

Intersection of ^ bisectors

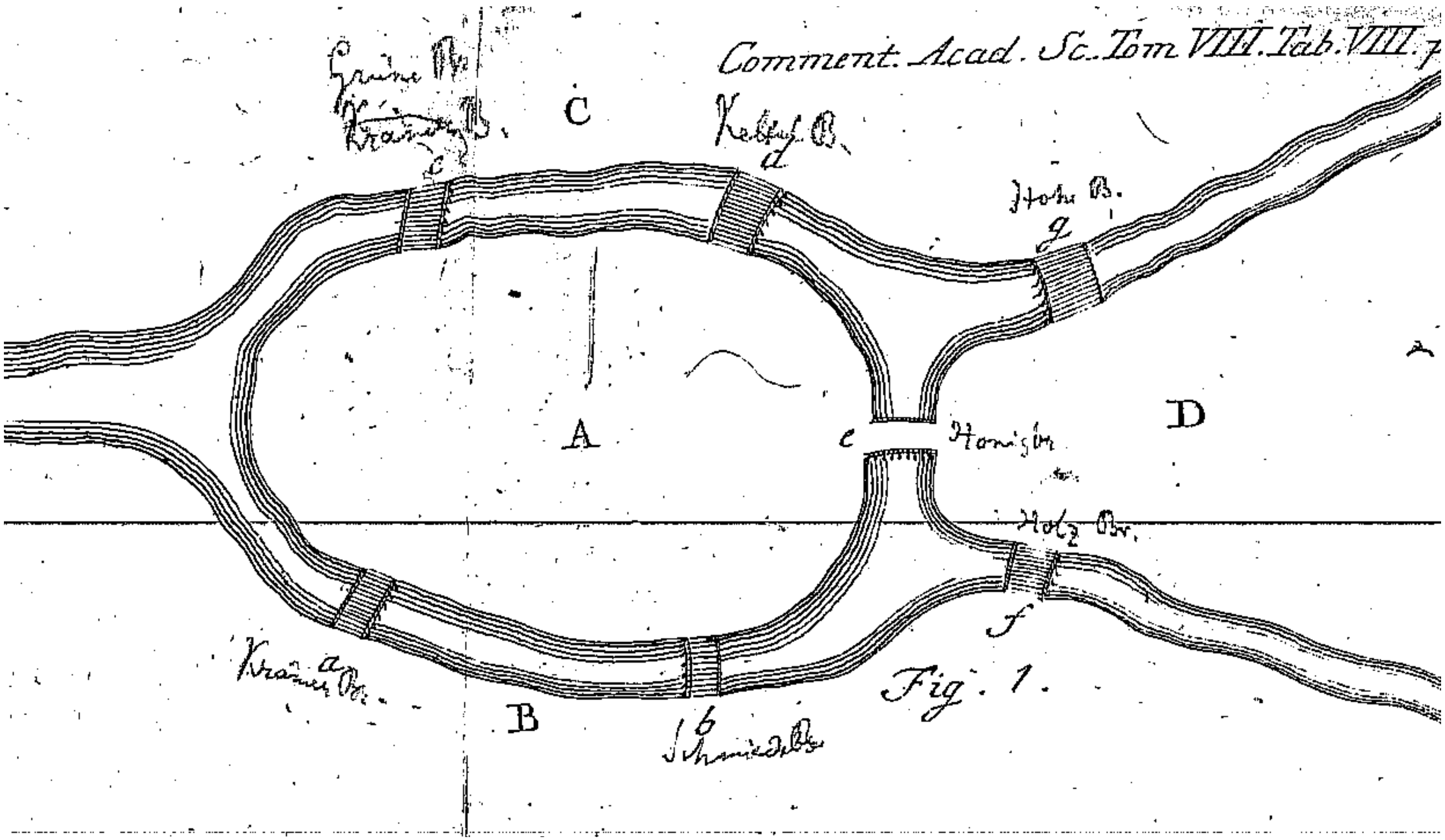


“Euler line” (1767)

SOLVTIO PROBLEMATIS AD GEOMETRIAM SITVS PERTINENTIS.

AVCTORE
Leonh. Eulero.

Bridges of Königsberg (1736)



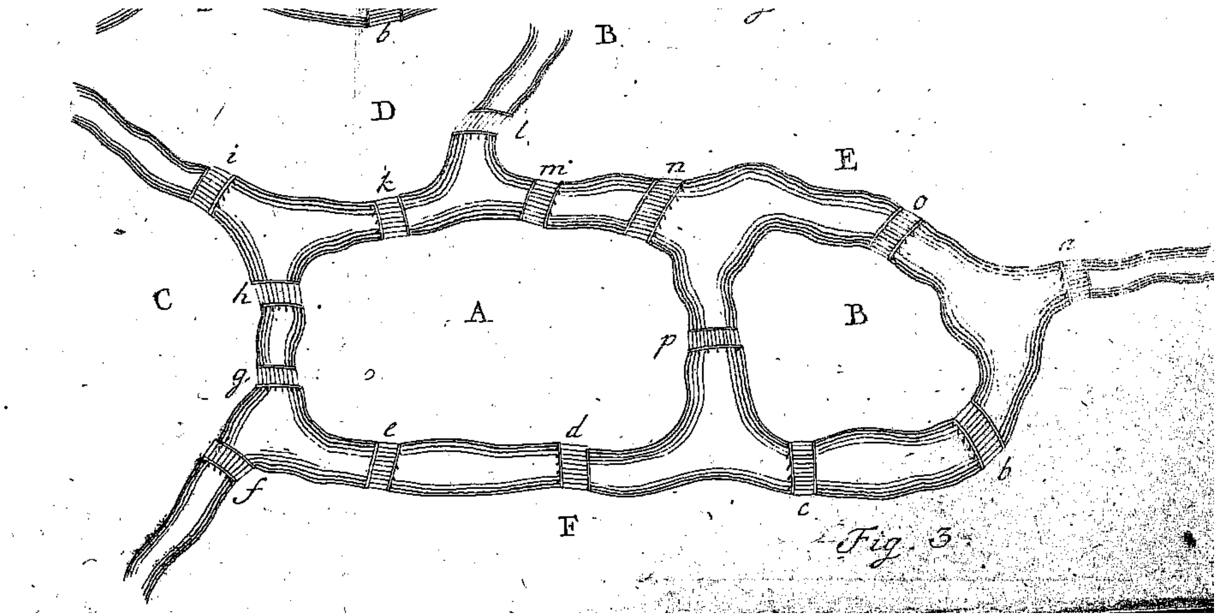


Fig. 3

“ … this solution bears little relationship to mathematics, and I do not understand why to expect a mathematician to produce it, rather than anyone else, for the solution is based on logic alone.”

Number Theory

Def: Whole numbers M and N are **amicable** if each is the sum of the proper divisors of the other.

Ex: $M = 220$ and $N = 284$

Proper divisors of 220:

$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$

Proper divisors of 284:

$1 + 2 + 4 + 71 + 142 = 220$

Brief History of Amicable Numbers:

ca. 300 BCE – Greeks knew 220 and 284

9th C – Thabit ibn Qurra's rule

1636 – Fermat found 17,296 and 18,416

1638 – Descartes found
9,363,584 and 9,437,056

Euler, in a 1750 paper, …

De Numeris Amicabilibus.

Definitio.

§. I.

Bini Numeri vocantur amicabiles, si ita sint comparati, ut summa partium aliquotarum unius aequalis sit alteri numero, & vicissim summa partium aliquotarum alterius priori numero aequetur.

Sic isti numeri 220 & 284 sunt amicabiles; prioris enim 220 partes aliquotae junctim sumtae: 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 faciunt 284: & hujus numeri 284 partes aliquotae: 1 + 2 + 4 + 71 + 142 producunt priorem numerum 220.

Scholion.

§. II. *Stifelius*, qui primus hujusmodi numerorum mentionem fecit, casu hos duos numeros 220 & 284 contemplatus ad hanc speculationem deductus videtur; analysin enim ineptam existimat, cujus ope plura istiusmodi numerorum paria inveniantur. Cartesius vero analysin ad hoc negotium accommodare est conatus, regulamque tradidit, qua tria talium numerorum paria efficuit, neque praeter ea Schotenius, qui multum in hac investigatione desudasse videtur, plura eruere valuit. Post haec tempora nemo fere Geometrarum ad hanc quaestionem magis evolvendam operam impendisse reperitur. Cum autem nullum sit dubium quin analysis quoque ex hac parte incrementa non contemnenda sit consecutura, si methodus aperiatur, qua multo plura hujusmodi numerorum paria investigare liceat, haud abs re fore arbitror, si methodos quasdam huc spectantes, in quas forte incidi, communicavero. In hunc finem autem sequentia praemittere necesse est.

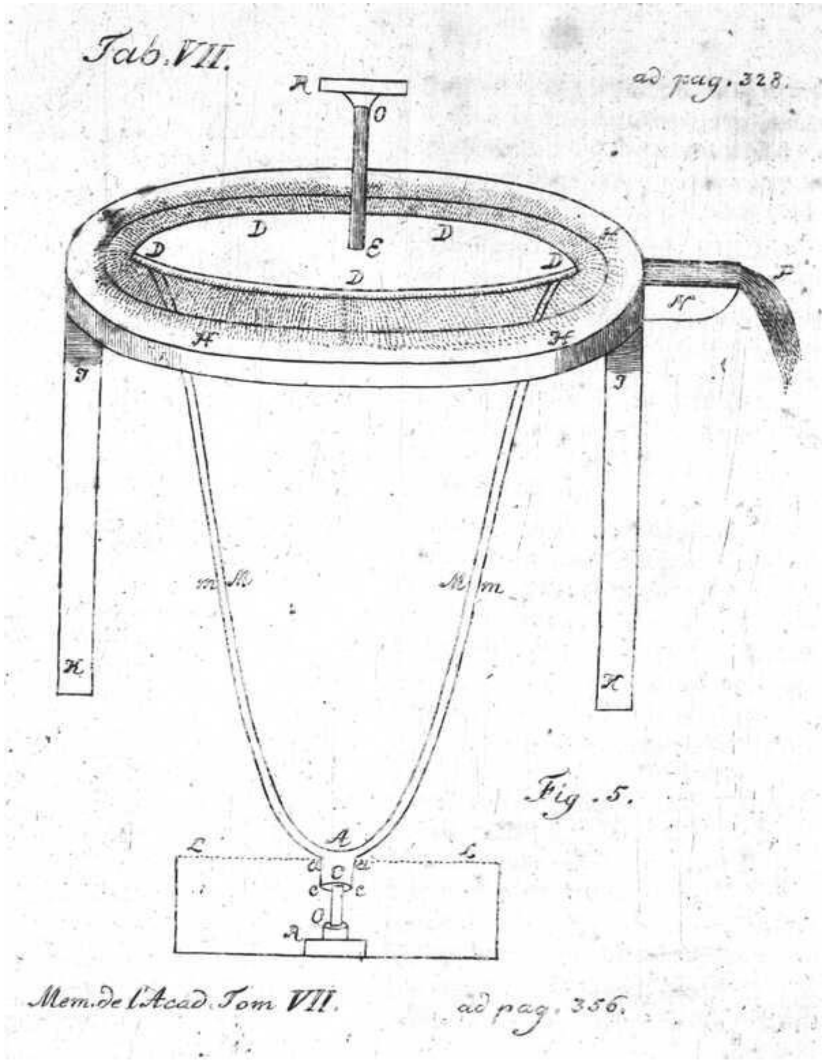
Hy-

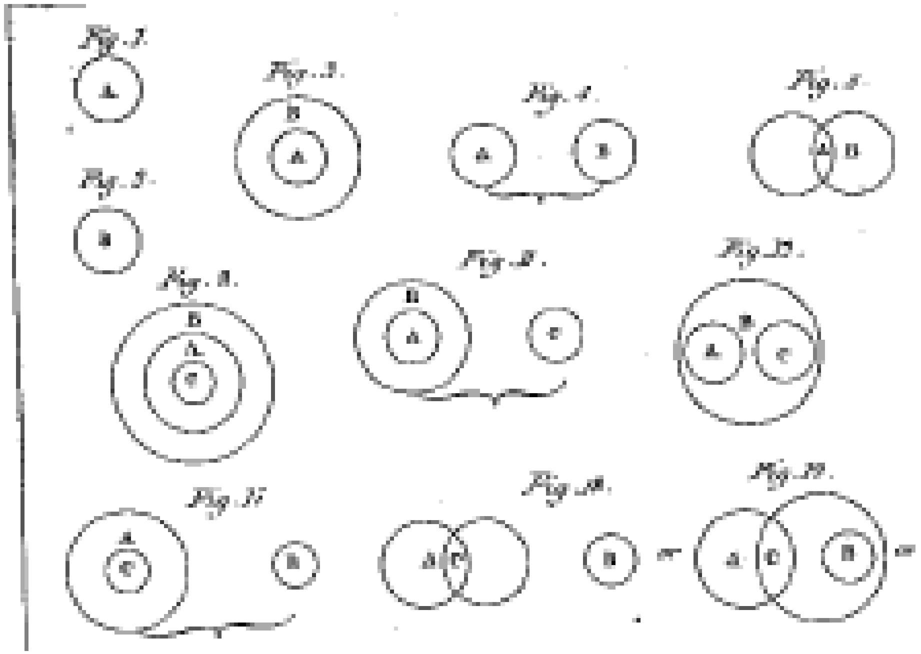
Euler, in a 1750 paper, …

… found 58 more .

AMICABILIBUS !

Theory of Machines





Venn Diagram

Euler Diagram

A Morsel (1781)

Find four different whole numbers, the sum of any two of which is a perfect square.

1 , 3 , 6

Euler gave:

18530 , 38114 , 45986 , 65570

How?

Another Morsel

Factoring polynomials into first and second degree factors:

$$x^2 - 1 = (x+1)(x-1)$$

$$x^4 - 1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)$$

Nicholas Bernoulli asserted that there was no such factorization of the 4th degree polynomial

$$x^4 - 4x^3 + 2x^2 + 4x + 4$$

$$= x^2 - \left(2 + \sqrt{4 + 2\sqrt{7}}\right)x + \left(1 + \sqrt{4 + 2\sqrt{7}} + \sqrt{7}\right)$$

$$\times$$

$$x^2 - \left(2 - \sqrt{4 + 2\sqrt{7}}\right)x + \left(1 - \sqrt{4 + 2\sqrt{7}} + \sqrt{7}\right)$$

The picture can't be displayed.

Euler's Identity (1748)

$$e^{ix} = \cos x + i \sin x$$

The picture can't be displayed.

$$e^{+v\sqrt{-1}} = \cos. v + \sqrt{-1}. \sin. v$$

Euler's Identity

Theorem: $e^{ix} = \cos x + i \sin x$

Proof:

Let $i = \sqrt{-1}$ and integrate:

$$\int \frac{dz}{\sqrt{1+z^2}} = \ln\left(z + \sqrt{1+z^2} \right)$$

Let $z = i\,y \quad \Rightarrow \quad dz = i\,dy$

Denying the validity of such procedures…

"… shatters the foundation of all analysis, which consists principally in the generality of the rules and operations which are deemed true, whatever the nature which one supposes for the quantities to which they are applied"

– Euler

Euler's Identity

Let $i = \sqrt{-1}$ and integrate:

$$\int \frac{dz}{\sqrt{1+z^2}} = \ln\left(z + \sqrt{1+z^2} \right)$$

Let $z = i\,y \quad \Rightarrow \quad dz = i\,dy$

Then $$\int \frac{i\,dy}{\sqrt{1+(iy)^2}} = \ln\left(iy + \sqrt{1+(iy)^2} \right)$$

$$\int \frac{i\,dy}{\sqrt{1+(iy)^2}} = \ln\left(iy + \sqrt{1+(iy)^2} \right)$$

$$i\int \frac{dy}{\sqrt{1-y^2}} = \ln\left(\sqrt{1-y^2} + iy \right)$$

Denying the validity of such procedures…

"… shatters the foundation of all analysis, which consists principally in the generality of the rules and operations which are deemed true, whatever the nature which one supposes for the quantities to which they are applied"

– Euler

$$\int \frac{i\,dy}{\sqrt{1+(iy)^2}} = \ln\left(iy + \sqrt{1+(iy)^2} \right)$$

$$i\int \frac{dy}{\sqrt{1-y^2}} = \ln\left(\sqrt{1-y^2} + iy \right)$$

Now let $y = \sin x \quad \Rightarrow \quad dy = \cos x\, dx$

$$i\int \frac{\cos x\, dx}{\sqrt{1-\sin^2 x}} = \ln\left(\sqrt{1-\sin^2 x} + i\sin x \right)$$

$$i\int \frac{\cos x\, dx}{\cos x} = \ln(\cos x + i\sin x)$$

$$i\int \frac{\cos x\, dx}{\cos x} = \ln(\cos x + i\sin x)$$

$$i\int dx = \ln(\cos x + i\sin x)$$

$$e^{ix} = e^{\ln(\cos x + i\sin x)}$$

$$e^{ix} = \cos x + i\sin x$$

Wow !

AMICABILIBUS !

$$e^{ix} = \cos x + i \sin x$$

Let $x = \pi$:

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i \cdot 0 = -1$$

So, $e^{i\pi} + 1 = 0$

"The most beautiful formula in mathematics"

$$e^{i\pi} + 1 = 0$$

Made the mathematician Euler a hero.

From the real to complex,

With our brains in great flex

He led us with zest but no fearo.

– W. C. Willig

Evaluate: $\sqrt{-1}^{\sqrt{-1}} = i^i$

Let $x = \pi/2$ in Euler's identity:

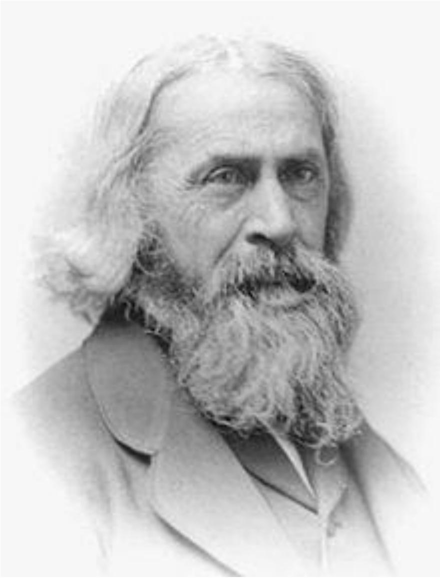
$$e^{i(\pi/2)} = \cos(\pi/2) + i\sin(\pi/2) = 0 + 1 \cdot i = i$$

So $i^i = \left(e^{i\pi/2}\right)^i = e^{i^2\pi/2} = e^{-\pi/2}$

Hence $i^i = 1/\sqrt{e^{\pi}}$

$$i^i = 1/\sqrt{e^\pi}$$

Benjamin Peirce:



“We have no idea what this equation means, but we may be sure that it means something very important.”

Talent is doing easily what others find difficult.

Genius is doing easily what others find *impossible*.

Way to Go, Uncle Leonhard!