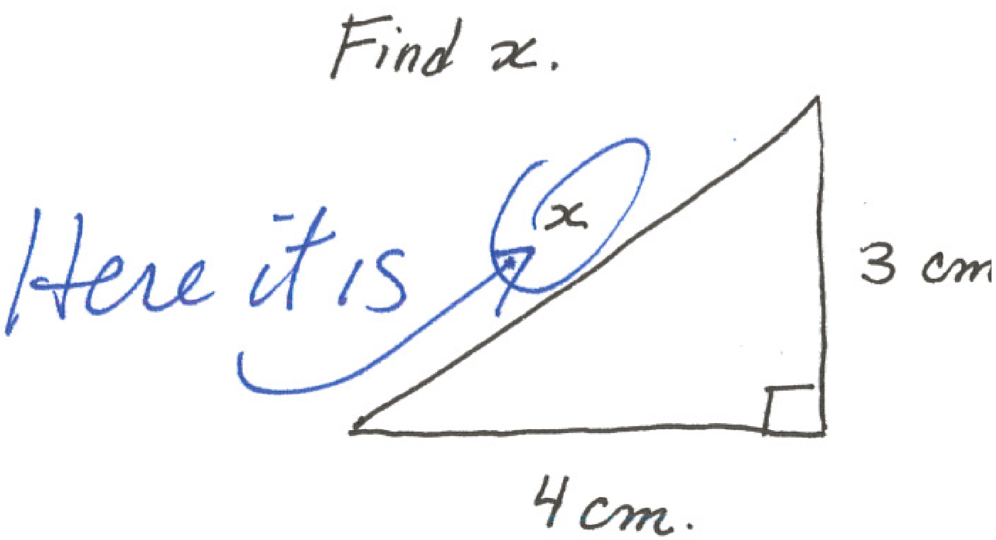
Math Blunders



$$\frac{1}{n}\sin x = ?$$

$$\frac{1}{n}\sin x = six = 6$$

Blunders

Expand $(x+y)^n$

$$= (x + y)^n$$

$$= (x + y)^n$$

$$= (x + y)^n$$

$$\frac{-1}{1} = \frac{1}{-1}$$

$$\sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}}$$

$$\frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}}$$

$$\frac{i}{1} = \frac{1}{i}$$

$$i^2 = 1$$

$$-1 = 1$$

Why 13 x 7 = 28

Math Blunders

13 x 7 = 28

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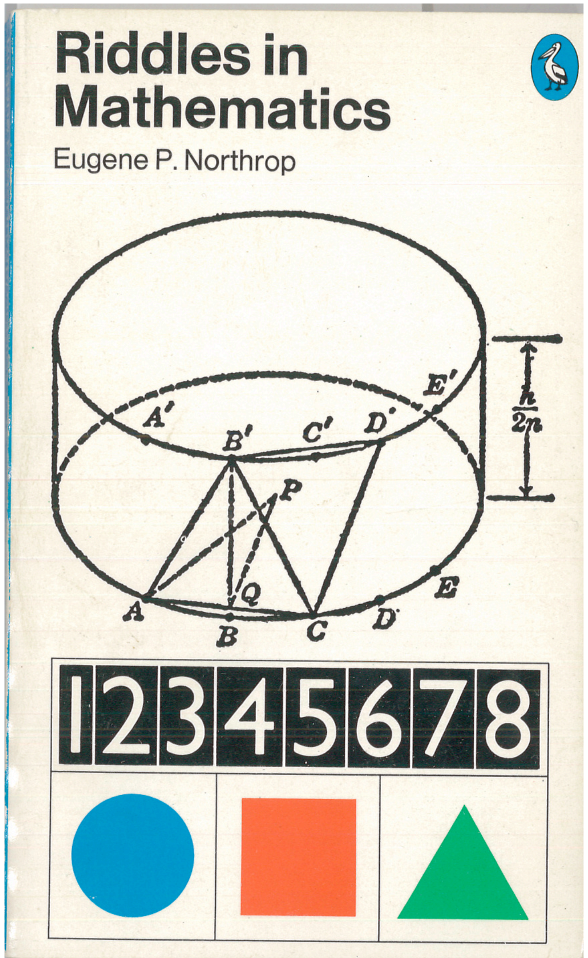
A Book of Fallacies



GEOMETRICAL FALLACIES

Paradox 2. To prove that from a point outside a plane an infinite number of perpendiculars can be drawn to the plane.[13]

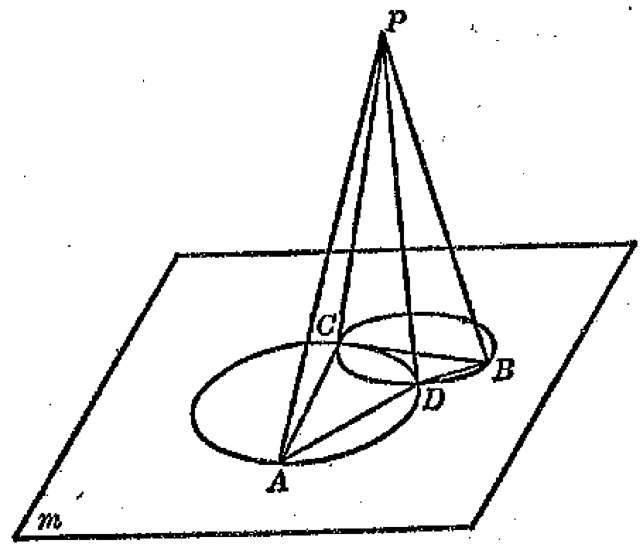
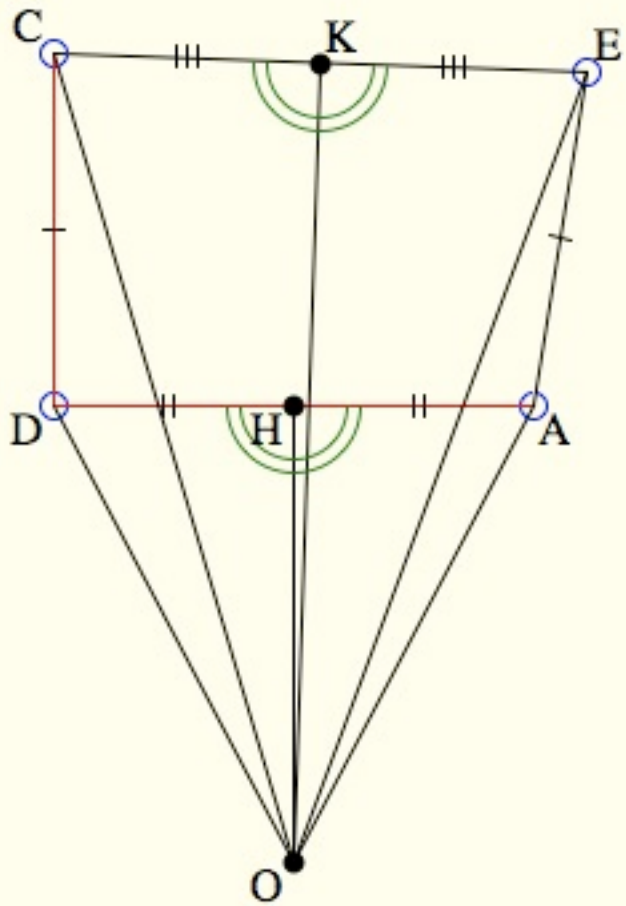


FIG. 73

In Figure 73 let P be any point outside of plane m. Choose any two points A and B in the plane, and on PA and PB as diameters construct two spheres. These spheres will intersect the plane m in two circles. (The intersection of a plane and a sphere is a circle.) And these two circles will intersect at two points, say C and D. Draw PC, PD, AC, AD, BC, and BD.

Now think of a plane passed through P, A, and C. (Three points determine a plane.) This plane will intersect the sphere about PA in a circle, so that $\angle PCA$ will be inscribed in a semicircle. Hence $\angle PCA$ is a right angle. (An angle inscribed in a semicircle is a right angle.) $\angle PCB$ is a right angle for the same reason. Therefore

Rouse Ball's fallacy



Source: Cut the knot, Bolgomolny

A ladder that's faster than light!

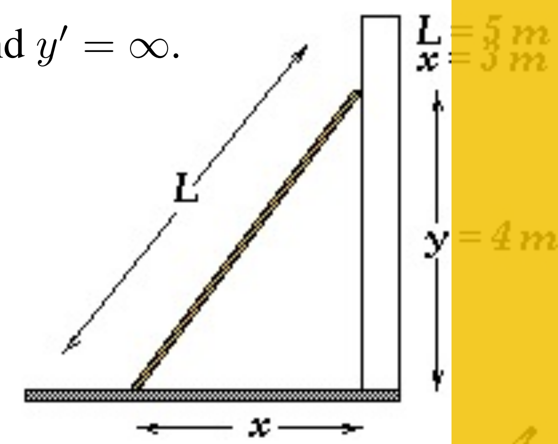
A ladder of length L leans against a wall. The bottom has distance x from the wall and the latter rests against the wall at height y, so that

$$x^2 + y^2 = L^2$$

The bottom of the ladder is pulled away from the wall at constant velocity x' The downward velocity of the ladder is by calculus:

$$2xx' + 2yy' = 0, \text{ or } y' = -xx'/y.$$

If x' is constant. At $x = L$, $y = 0$, and $y' = \infty$.



Berkeley gave a famous criticism of Newton's calculus:

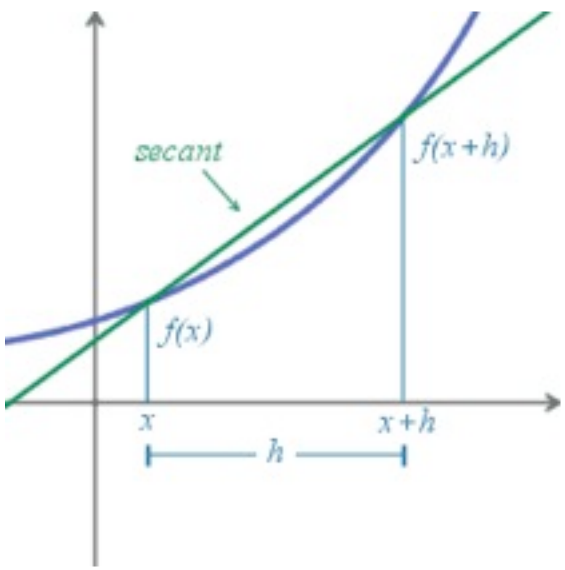
If an increment is zero, then you cannot divide by it; and if it is nonzero, then it cannot give the exact answer.

"However useful it may have been in practice, the concept of infinitesimal could scarcely withstand logical scrutiny. Derided by Berkeley in the 18th century as 'ghosts of departed quantities', in the 19th century execrated by Cantor as 'cholera-bacilli' infecting mathematics, and in the 20th roundly condemned by Bertrand Russell as 'unnecessary, erroneous, and self-contradictory', these useful, but logically dubious entities were believed to have been finally supplanted in the foundations of analysis by the limit concept which took rigorous and final form in the latter half of the 19th century. By the beginning of the 20th century, the concept of infinitesimal had become, in analysis at least, a virtual 'unconcept'.

–Stanford Encyclopedia of Philosophy

Continuity and Infinitesimals

Bell, John L.,



$$\frac{\Delta y}{\Delta x}$$

A Berry strange conclusion

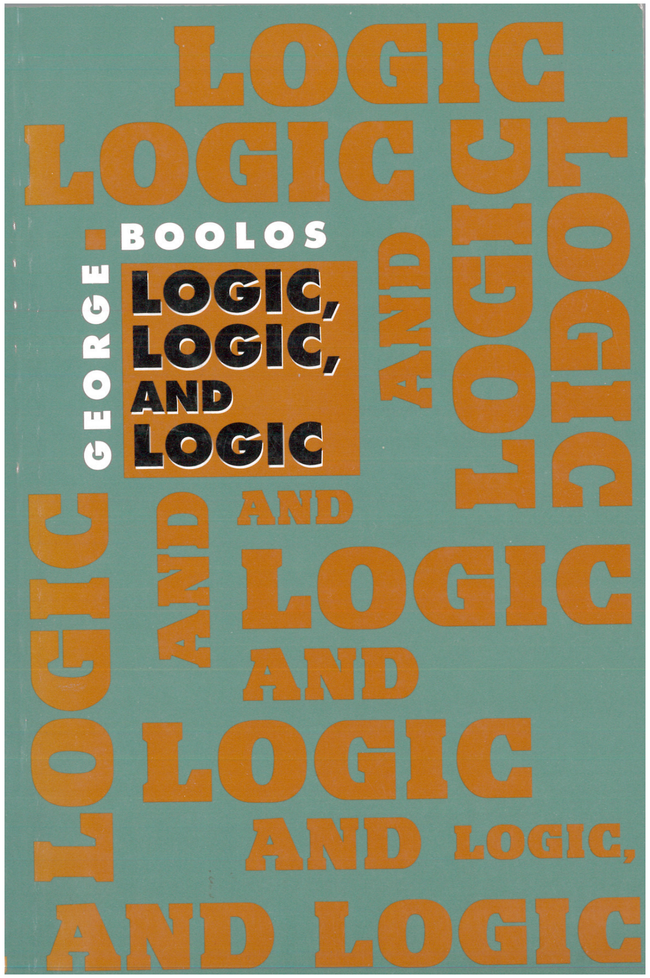
Every natural number can be unambiguously specified in fourteen words or less.

Proof by contradiction. Assume for a contradiction that there is a natural number that cannot be unambiguously specified in fourteen words or less.

Then there must be a smallest such number.

That number is "the smallest natural number that cannot be unambiguously specified in fourteen words or less."

This is an unambigous specification in fourteen words, contrary to its assumed property. Therefore no such number exists.



Boolos gave a proof of Gödel's incompleteness theorem based on the Berry paradox.

A mathematician faced with error

A programmer faced with error

2011, Henrik Kniberg Lean from the Trenches.

Report on software development for the Swedish national police authority. (When a motorist gets pulled over, it goes directly into the computer system this group designed.)

“If a bug is found, . . . , we have a decision to make ‘Is this bug more important than any of the other top thirty bugs in the bug tracker?’ . . . If not, then we ignore the new bug.” (p. 47) “If a bug is unlikely to be fixed (because it didn’t make top thirty), we are honest about that from start, instead of building up false expectations.” (page 49).

The book that describes itself as the "bestselling software testing book of all time" states that "testers shouldn't want to to verify that a program runs correctly."

Another book on software testing states "Don't insist that every bug be fixed . . . When the programmer fixes a minor bug, he might create a more serious one."

Former Intel President Andy Grove "I have come to the conclusion that no microprocessor is ever perfect; they just come closer to perfection."

About one bug per hundred lines of computer code makes it to market without detection.

Consequences of Computer Bugs

Math Blunders

- A library patron is fined $40 trillion for an overdue book.
- A dentist in San Diego is delivered 16,000 tax forms.
- A textbook on the "Making of the fly" sells for $23 million on Amazon.com. (The price dropped back down to $79.99.)

- The Intel Pentium division bug eventually cost Intel $500 million.
- The bug causing the explosion of the Ariane 5 rocket cost hundreds of millions of dollars.
- The front page of the NYT reported on March 24, that BATS, a major new electronic stock exchange, just opened. However, “software bug in one of its computer systems” caused havoc and eventually all of the trades executed by the had to be canceled.

Unjustified trust in computers?

But what about the Flash Crash on Wall Street that brought a 600 point plunge in the Dow Jones in just 5 minutes at 2:41 pm on May 6, 2010? According to the New York Times [NYT10], the flash crash started when a mutual fund used a computer algorithm "to sell $4.1 billion in futures contracts." The algorithm was designed to sell "without regard to price or time.…[A]s the computers of the high-frequency traders traded [futures] contracts back and forth, a 'hot potato' effect was created." When computerized traders backed away from the unstable markets, share prices of major companies fluctuated even more wildly. "Over 20,000 trades across more than 300 securities were executed at prices more than 60% away from their values just moments before" [SEC10] Throughout the crash, computers followed algorithms to a T, to the havoc of the global economy.

The near-implosion of Knight Capital Group Inc. in early August sent shock waves through rival firms. . . . Knight, one of the nation's largest handlers of share orders for retail and institutional investors, lost $440 million from a 40-minute burst of trading because of faulty software.

"It's terrifying," said Mark Gorton, chief executive of Tower Research LLC, which is among the biggest high-frequency trading businesses in the U.S. . . . "You almost can't know there's no bug, anywhere in your system, ever."

"It's pretty clear to us that the Knight Capital episode really instilled some fear among financial-service firms," [SEC Chairman] Ms. Shapiro said.

quoted from *Rapid-Fire Traders' Big Fear: Themselves'* Wall Street Journal, Sept 2, 2012

Bugs as mathematical blunders

Our experience with computers is that once given a consistent set of instructions, they compute consistently. It's just hard to give them a consistent set. – Georges Gonthier

```
minimum of three variables A, B, C:

 if A < B and A < C then
   Min := A;
 elsif B < A and B < C then
   Min := B;
 else
   Min := C;
 end if;
```

Source: Mark Adams, aircraft guidance software

Mathematical Certainty
Myth and Reality

Proof that $1 + 1 = 2$

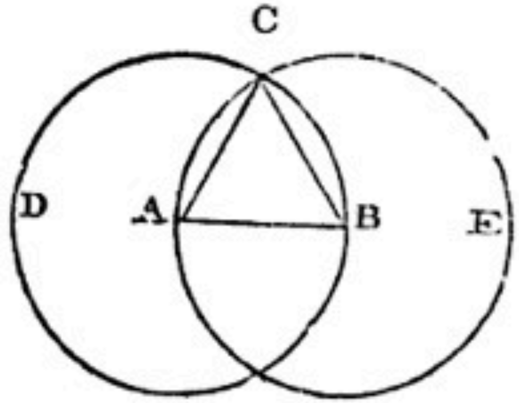
$$
\begin{aligned}
1 + 1 &= 1 + \mathrm{SUC}\ 0 \\
&= \mathrm{SUC}\ (1 + 0) \\
&= \mathrm{SUC}\ 1 \\
&= 2
\end{aligned}
$$

Math Blunders

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line; it is required to describe an equilateral triangle upon it.



From the centre A, at the distance AB, describe (3. Postulate.) the circle BCD, and from the centre B, at the distance BA, describe the circle ACE; and from the point C, in which the circles cut one another, draw the straight lines (2. Post.) CA, CB to the points A, B; ABC shall be an equilateral triangle.

Because the point A is the centre of the circle BCD, AC is equal (15. Definition.) to AB; and because the point B is the centre of the circle ACE, BC is equal to BA: but it has been proved that CA is equal to AB; therefore CA, CB are each of them equal to AB; but things which are equal to the same are equal to one another; (1st. Axiom.) therefore CA is equal to CB; wherefore CA, AB, BC are equal to one another; and the triangle ABC is therefore equilateral, and it is described upon the given straight line AB. Which was required to be done.

PROP. II. PROB.

From a given point to draw a straight line equal to a given

Almgren's text

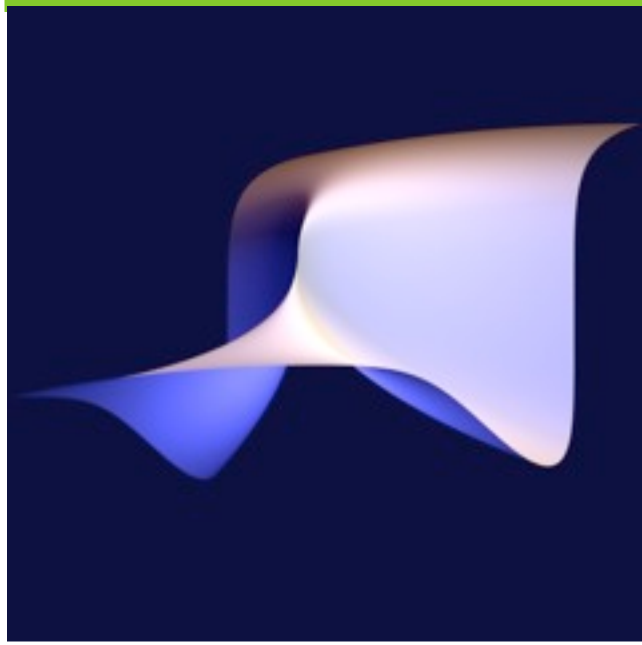
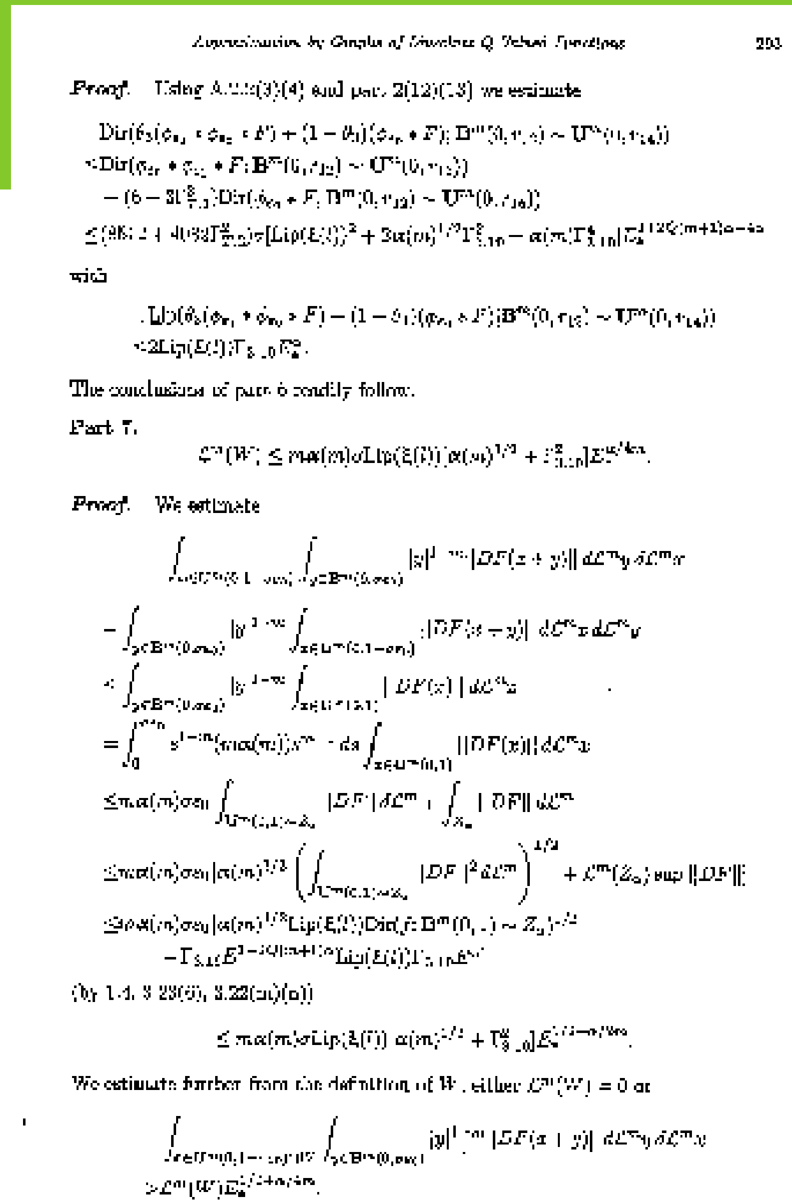


Image: http://www.freigeist.cc/gallery/sofa.jpg



Proof. Using ... and part 2(12)(13) we estimate

with

The conclusions of part 6 readily follow.

Part 7.

Proof. We estimate

We estimate further from the definition of W, either $\mathcal{L}^m(W) = 0$ or

Math Blunders

The length of unusually long proofs has increased with time. As a rough rule of thumb, 100 pages in 1900, or 200 pages in 1950, or 500 pages in 2000 is unusually lo

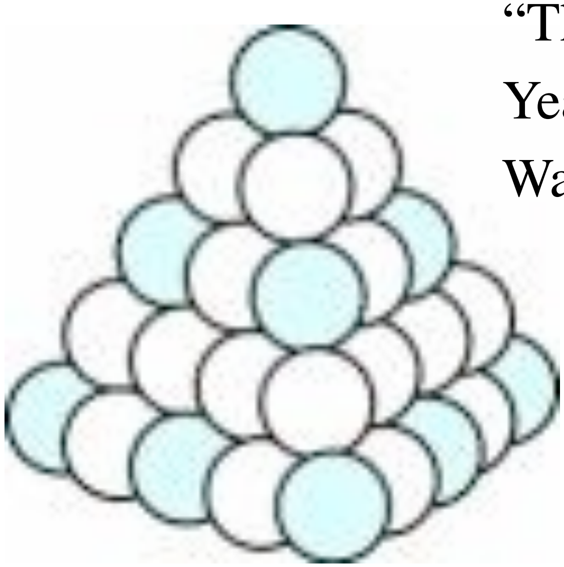
- 1799 The Abel–Ruffini theorem was nearly proved by Paolo Ruffini, but his proof, spanning 500 pages, was mostly ignored and later, in 1824, Niels Henrik Abel p pages
- 1890 Killing's classification of simple complex Lie algebras, including his discovery of the exceptional Lie algebras, took 180 pages in 4 papers.
- 1894 The ruler-and-compass construction of a polygon of 65537 sides by Johann Gustav Hermes took over 200 pages.
- 1905 Lasker–Noether theorem Emmanuel Lasker's original proof took 98 pages, but has since been simplified: modern proofs are less than a page long.
- 1963 Odd order theorem This was 255 pages long, which at the time was over 10 times as long as what had previously been considered a long paper in group theory.
- 1964 Resolution of singularities Hironaka's original proof was 216 pages long; it has since been simplified considerably down to about 10 or 20 pages.
- 1966 Discrete series representations of Lie groups. Harish-Chandra's construction of these involved a long series of papers totaling around 500 pages. His later work on the Plancherel theorem for semisimple groups added another 150 pages to these.
- 1968 the Novikov-Adian proof solving Burnside's problem on finitely generated infinite groups with finite exponents negatively. The three-part original paper is more than 300 pages long. (Britton later published a 282 page paper attempting to solve the problem, but his paper contained a serious gap.)
- 1960–1970 Fondements de la Géometrie Algébrique, Éléments de géométrie algébrique and Séminaire de géométrie algébrique. Grothendieck's work on the foundations of algebraic geometry covers many thousands of pages. Although this is not a proof of a single theorem, there are several theorems in it whose proofs depend on hundreds of earlier pages.
- 1974 N-group theorem Thompson's classification of N-groups used 6 papers totaling about 400 pages, but also used earlier results of his such as the odd order theorem, which bring to total length up to more than 700 pages.
- 1974 Ramanujan conjecture and the Weil conjectures. While Deligne's final paper proving these was "only" about 30 pages long, it depended on background results in algebraic geometry and etale cohomology that Deligne estimated to be about 2000 pages long.
- 1974 4-color theorem. Appel and Haken's proof of this took 741 pages, and also depended on long computer calculations.
- 1974 The Gorenstein–Harada theorem classifying finite groups of sectional 2-rank at most 4 was 464 pages long.
- 1976 Eisenstein series Langlands's proof of the functional equation for Eisenstein series was 337 pages long.
- 1983 Trichotomy theorem Gorenstein and Lyons's proof for the case of rank at least 4 was 731 pages long, and Aschbacher's proof of the rank 3 case adds another 159 pages, for a total of 890 pages.
- 1983 Selberg trace formula Hejhal's proof of a general form of the Selberg trace formula consisted of 2 volumes with a total length of 1322 pages.
- Arthur–Selberg trace formula. Arthur's proofs of the various versions of this cover several hundred pages spread over many papers.
- 2000 Almgren's regularity theorem Almgren's proof was 955 pages long.
- 2000 Lafforgue's theorem on the Langlands conjecture for the general linear group over function fields. Laurent Lafforgue's proof of this was about 600 pages long, not counting many pages of background results.
- 2003 Poincaré conjecture, Geometrization theorem, Geometrization conjecture. Perelman's original proofs of the Poincaré conjecture and the Geometrization conjecture were not lengthy, but were rather sketchy. Several other mathematicians have published proofs with the details filled in, which come to several hundred pages.
- 2004 Quasi-thin groups The classification of the simple quasi-thin groups by Aschbacher and Smith was 1221 pages long, one of the longest single papers ever written.
- 2004 Classification of finite simple groups. The proof of this is spread out over hundreds of journal articles which makes it hard to estimate its total length, which is probably around 10000 to 20000 pages.
- 2004 Robertson–Seymour theorem. The proof takes about 500 pages spread over about 20 papers.
- 2005 Kepler conjecture Hales's proof of this involves several hundred pages of published arguments, together with several gigabytes of computer calculations.
- 2006 the strong perfect graph theorem, by Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. 180 pages in the Annals of Mathematics.

WIKIPEDIA
The Free Encyclopedia

Incorrect proofs of correct statements are so abundant that they are impossible to catalogue. Kempe's claimed proof of the four-color theorem stood for more than a decade before Heawood refuted it [Mac01, p. 115]. "More than a thousand false proofs [of Fermat's Last Theorem] were published between 1908 and 1912 alone" [Cor10]. Ralph Boas, former executive editor of Math Reviews, once remarked that proofs are wrong "half the time" [Aus08]. Many published theorems are like the hanging chad

> "Verifying a paper [in mathematics] is becoming just as hard as writing a paper," Voevodsky said. "For writing, you get some reward a promotion, perhaps but to verify someone elses paper, no one gets a reward." (Wired, March 2013)

- The Kepler conjecture asserts that the densest packing of congruent balls in $\mathbb{R}^3$ is achieved by the familiar "cannonball" arrangement.
- The Kepler Conjecture was formulated in the booklet "The six-cornered snowflake," presented as a gift on New Year's day 1611 to Kepler's patron Lord Wacker von Wackenfels.



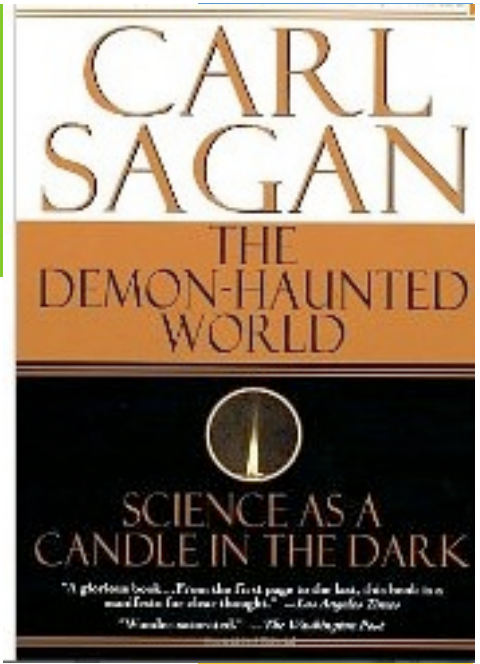
- A proof of the Kepler conjecture was completed in 1998 by Ferguson and H.
- The proof was 300 pages and relied on long computer calculations.
- 12 referees were assigned the task of checking the proof.
- After years of effort, the referees announced they were 99% sure that the proof was essentially correct.
- An editor eventually told me the proof would be published, as soon as I could convince the editors of the proof's correctness.

‘The referees put a level of energy into this that is, in my experience, unprecedented. They ran a seminar on it for a long time. A number of people were involved, and they worked hard. They checked many local statements in the proof, and each time they found that what you claimed was in fact correct. Some of these local checks were highly non-obvious at first, and required weeks to see that they worked out. The fact that some of these worked out is the basis for the 99% statement of Fejes Tóth that you cite.”

“They have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy to devote to the problem.”

How can an editor, who already has the paper in hand, be further convinced that a proof has no blunders?

Baloney Detection Kit



Carl Sagan published a *Baloney Detection Kit* to help readers test the validity of arguments.

- Wherever possible there must be independent confirmation of the facts
- Encourage substantive debate on the evidence by knowledgeable proponents of all points of view.
- Arguments from authority carry little weight (in science there are no “authorities”).

My math baloney detection kit.

- Is the claimed theorem a logical consequence of the axioms of mathematics?
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Formal Proof

A *formal proof* is a style of proof in which every logical inference has been checked all the way back to the fundamental axioms of mathematics.

No step of the proof is left unchecked, no matter how trivial. NO EXCEPTIONS!

It is not allowed to say a step is “obvious,” even when it is obvious. It is not allowed to say that the “other arguments follow in a similar fashion” even if they do.

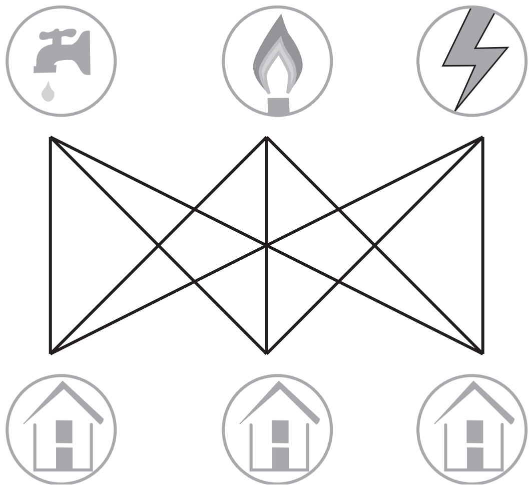
When a proof is expanded in this fashion, it is generally done by computer, because the number of logical steps can run into the millions, even for ordinary mathematical theorems.

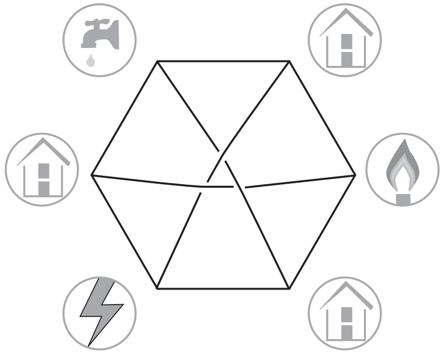
Table 1. Examples of Formal Proofs

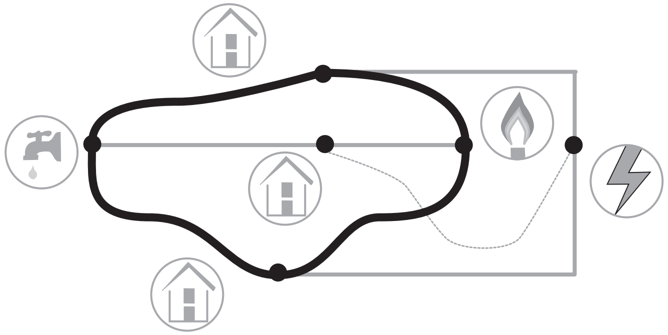
Year	Theorem	Proof System	Formalizer	Traditional Proof
1986	First Incompleteness	Boyer-Moore	Shankar	Gödel
1990	Quadratic Reciprocity	Boyer-Moore	Russinoff	Eisenstein
1996	Fundamental - of Calculus	HOL Light	Harrison	Henstock
2000	Fundamental - of Algebra	Mizar	Milewski	Brynski
2000	Fundamental - of Algebra	Coq	Geuvers et al.	Kneser
2004	Four Color	Coq	Gonthier	Robertson et al.
2004	Prime Number	Isabelle	Avigad et al.	Selberg-Erdös
2005	Jordan Curve	HOL Light	Hales	Thomassen
2005	Brouwer Fixed Point	HOL Light	Harrison	Kuhn
2006	Flyspeck I	Isabelle	Bauer-Nipkow	Hales
2007	Cauchy Residue	HOL Light	Harrison	classical
2008	Prime Number	HOL Light	Harrison	analytic proof

Formal Proof of the Jordan Curve Thm

Math Blunders







```
let JORDAN_CURVE_THEOREM = prove_by_refinement(
  `!C. simple_closed_curve top2 C ==>
     (?A B.  top2 A /\ top2 B /\
       connected top2 A /\ connected top2 B /\
     ~(A = EMPTY) /\ ~(B = EMPTY) /\
      (A INTER B = EMPTY) /\ (A INTER C = EMPTY) /\
          (B INTER C = EMPTY) /\
         (A UNION B UNION C = euclid 2))`,
  (* {{{ proof *)
  [
  REP_BASIC_TAC;
  THM_INTRO_TAC[`C`] jordan_curve_not_one_sided;
  ASM_REWRITE_TAC[];
  FULL_REWRITE_TAC[one_sided_jordan_curve];
```

The formal proof of the Kepler conjecture

- The first proof was presented (by Ferguson and H. in 1998) and published in 2006.
- A project called Flyspeck seeks to give a formal proof of the theorem, which involves a computer verification of every single logical inference in the proof, all the way back to the fundamental axioms of mathematics.
- FLYSPECK comes from F.*P.*K, for the Formal Proof of the Kepler Conjecture.
- The Flyspeck project is about ~~80%~~ complete.

94%

There is a great need to improve the technology of formal proofs so that someday this becomes the standard way for researchers to check that they have not blundered.

We need logicians, computer scientists, and mathematicians to turn to this area of research!