Cookie Monster Plays Games

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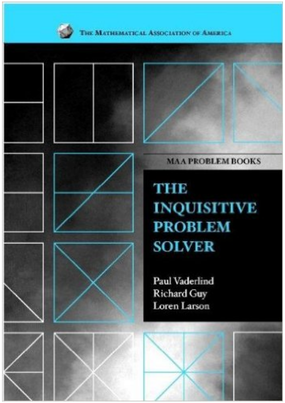
also: PRIMES, Leigh Marie Braswell, Eric Nie, Alok Puranik, Joshua Xiong, Dhroova Aiylam

Stony Brook Math Club, February 17, 2016

The Cookie Monster Problem Origins

First appeared in the book *The inquisitive Problem Solver* by Paul Vaderlind, Richard Guy, Loren Larson.



The Cookie Monster Problem

- Set-up: The Cookie Monster is presented with a set of cookie jars

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- Set-up: The Cookie Monster is presented with a set of cookie jars
- One move: Choose a subset of the jars and take the same number of cookies from each
- Goal: Minimize the number of moves


An Example

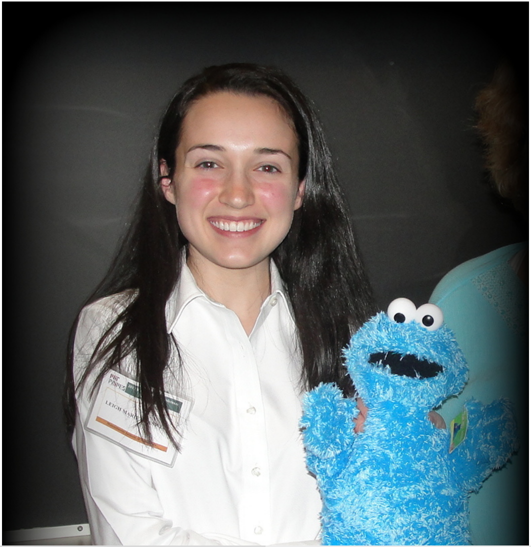
- Start with (1,2,3)

An Example

- After the first move (2,2)

Leigh Marie Braswell

PRIMES project with Leigh Marie Braswell



Trivial bounds

Prior results for n jars.
Assume distinct number of cookies.

- $\leq n$
- $\geq \log_2 n$.

Trivial bounds achieved

The Cookie Monster number is n for sequences that grow at least as fast as powers of 2:
1, 2, 4, 8, 16, . . .

Trivial bounds achieved

The Cookie Monster number is n for sequences that grow at least as fast as powers of 2:
$1, 2, 4, 8, 16, \ldots$
The Cookie Monster number is about $\log_2 n$ for arithemtic progressions:
$1, 2, 3, 4, 5, \ldots$

Fibonacci numbers

- Start with $(1, 2, 3, 5)$

Fibonacci numbers

- Start with $(1, 2, 3, 5)$
- After the move: $(1, 2, 0, 2) = (1, 2)$

Nacci numbers

The best strategy for n jars:

- Fibonacci numbers: $\to \frac{n}{2}$

Nacci numbers

The best strategy for n jars:

- Fibonacci numbers: $\rightarrow \frac{n}{2}$
- Tribonacci numbers: $\rightarrow \frac{2n}{3}$
- Tetranacci numbers: $\rightarrow \frac{3n}{4}$

Theorem

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For any $0 \leq r \leq 1$*, we can build a sequence such that the number of moves tends to* rn*, when the number of jars,* n*, tends to* ∞*.*

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For any $0 \leq r \leq 1$, we can build a sequence such that the number of moves tends to rn, when the number of jars, n, tends to ∞.

Main idea of proof:

- Start with the sequence of powers of 2:
  1, 2, 4, 8, 16, ...
  and add more numbers when needed.

Leigh Marie Braswell PRIMES 2013

Two joint papers:

- *Cookie Monster Devours Naccis*, in The College Mathematics Journal, Vol. 45, No. 2 (March 2014), pp. 129-135.
- *On the Cookie Monster Problem*, in *The Mathematics of Various Entertaining Subjects: Research in Recreational Math*, Princeton University Press, 2015, pp. 231-244.

PRIMES 2014

Not very recreational

- Noah Golowich, *Resolving a Conjecture on Degree of Regularity, with some Novel Structural Results*. Intel Competition, **First Prize**
- Brice Huang, *Monomization of Power Ideals and Generalized Parking Functions*. Intel Competition, **Second Prize**
- Shashwat Kishore, *Multiplicity Space Signatures and Applications in Tensor Products of sl_2 Representations*. Intel Competition, **Third Prize**
- Peter Tian, *Extremal Functions of Forbidden Multidimensional Matrices*. Siemens Competition, **First Prize**
- Joseph Zurier, *Generalizations of the Joints Problem*. Siemens Competition, **Second Prize**

PRIMES 2015

- Siemens: 6 finalists and 6 semifianlists
- Intel: 11 semifinalist and 2 finalists

PRIMES 2015

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Intel finalists:

- Meena Jagadeesan, *The Exchange Graphs of Weakly Separated Collections*
- Rachel Zhang, *Statistics of Intersections of Curves on Surfaces*

Joshua Xiong, PRIMES 2014

The rest of the cookie monster is jointly with Joshua Xiong

Cookie Monster Game

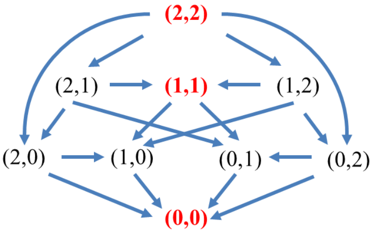
Moves—Game
The last person to move wins.

Nim

- Take at least one cookie from any one pile
- The player who takes the last cookie wins

P-Positions

- $(3, 3)$ is a P-position
- P-positions: **p**revious player wins
- All other positions are N-positions: **n**ext player wins
  - Moves from P-positions can only go to N-positions
  - At least one move from every N-position goes to a P-position
  - The zero position $(0, \ldots, 0)$ is a P-position
- Winning strategy is to move to a P-position



Winning Strategy for Nim

Theorem (Bouton's Theorem)

In Nim, $P = (a_1, \ldots, a_n) \in \mathcal{P}$ *if and only if* $\bigoplus_{i=1}^{n} a_i = 0$.

- The operator $\oplus$ is the bitwise XOR operator, (nim-sum) – represent each of the numbers in binary and add them column-wise modulo 2.

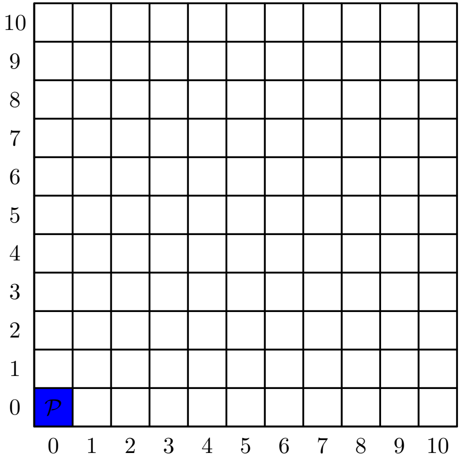
Wythoff's Game

- Take same number of cookies from two piles or any number from one pile

- P-Position $(1, 2)$
- Can only move to $(0, 2)$, $(1, 1)$, $(1, 0)$ and $(0, 1)$:

Calculating P-positions

- P- and N-positions can be calculated from the terminal position



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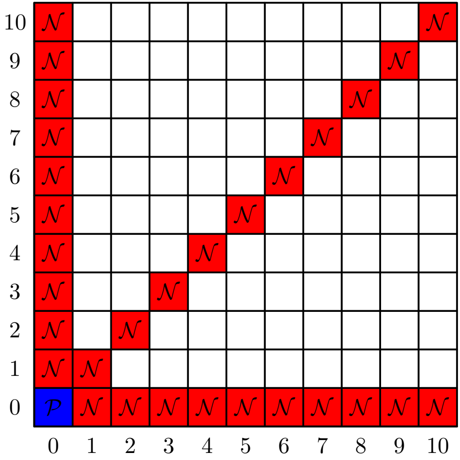

Calculating P-positions

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10	$\mathcal{N}$										$\mathcal{N}$
9	$\mathcal{N}$									$\mathcal{N}$	
8	$\mathcal{N}$								$\mathcal{N}$		
7	$\mathcal{N}$							$\mathcal{N}$			
6	$\mathcal{N}$						$\mathcal{N}$				
5	$\mathcal{N}$					$\mathcal{N}$					
4	$\mathcal{N}$				$\mathcal{N}$						
3	$\mathcal{N}$			$\mathcal{N}$							
2	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$								
1	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$								
0	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
	0	1	2	3	4	5	6	7	8	9	10

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2	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
1	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
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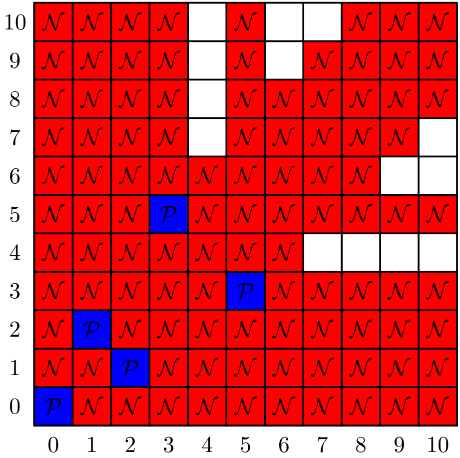

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3	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
2	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
1	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
0	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
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3	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
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0	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
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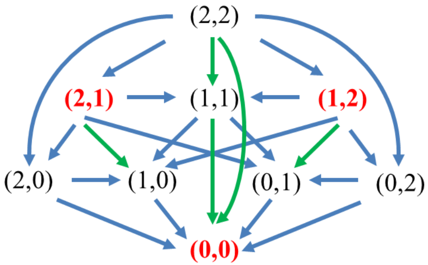
	0	1	2	3	4	5	6	7	8	9	10

Winning Strategy for Wythoff

Theorem (Wythoff's Theorem)

In Wythoff's game, $P = (a_1, a_2) \in \mathcal{P}$ if and only if $\{a_1, a_2\} = \{\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor\}$ for some integer n, where $\phi = \frac{1+\sqrt{5}}{2}$.



Rectangular Games

The Cookie Monster game is too difficult. We generalized it:

- Move consists of taking same number of cookies from specified subsets of piles
- Adjoins rules onto the Nim rule (taking at least one cookie from exactly one of the piles)

Odd Cookie Monster Game

We are only allowed to take from an odd number of piles.

Theorem

The P-positions are the same as the ones in Nim.

Main idea of proof:

- New moves do not allow to get from a P-position in Nim to another P-position in Nim.

Not-from-All Cookie Monster Game

We are allowed to take from any set of piles except from all of them.

Theorem

The position where all jars have the same number of cookies, $(n, n, \ldots, n)$ is a P-position for any n. If the number of cookies have two distinct values, then it is an N-position.

Three piles

All possible games with three piles

1. Nim: no additional sets.
2. Wythoff plus Nim: $\{1,2\}$.
3. One-or-All game = Odd: $\{1,2,3\}$.
4. One-or-Two jars = Not-from-All: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$.
5. Consecutive: $\{1,2\}$, $\{2,3\}$, $\{1,2,3\}$.
6. Consecutive One-or-Two: $\{1,2\}$, $\{2,3\}$.
7. Always include the first jar: $\{1,2\}$, $\{1,2,3\}$.
8. Cookie Monster game: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$.

Degrees of Freedom of P-positions

Theorem

For a position with $n-1$ numbers known, and one number unknown: $P = (a_1, \ldots, a_{n-1}, x)$, there is a unique value of x such that $P \in \mathcal{P}$.

- For Nim, this function is $f_{\mathrm{NIM}}(a_1, \ldots a_{n-1}) = \bigoplus_{i=1}^{n-1} a_i$.

Bounds on P-Positions

- General bound that holds for all rectangular games

Theorem

If $P = (a_1, \ldots, a_n) \in \mathcal{P}$ *then* $2(\sum_{i=1}^{n} a_i - a_j) \geq a_j$.

Enumeration of P-positions

The number of P-positions in Nim with three piles as a function of the number of tokens n:

Theorem

The number of P-positions is $3^{\mathrm{wt}(n)}$ if n is even, 0 otherwise, where $\mathrm{wt}(n)$ is the number of ones in the binary representation of n.

1, 0, 3, 0, 3, 0, 9, 0, 3, 0, 9, 0, 9, 0, 27, 0, . . .

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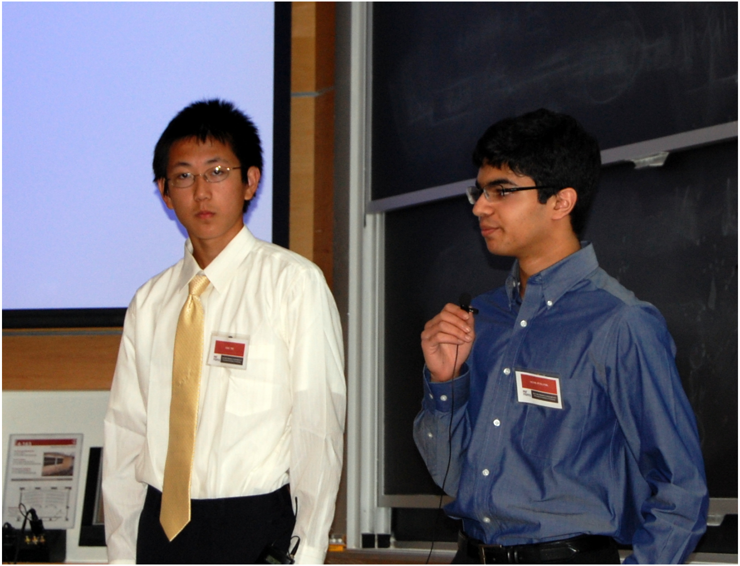
1, 0, 3, 0, 3, 0, 9, 0, 3, 0, 9, 0, 9, 0, 27, 0, ...
SURPRISE!
The same sequence in my other project.

Eric Nie and Alok Puranik



Ulam-Warburton Automaton

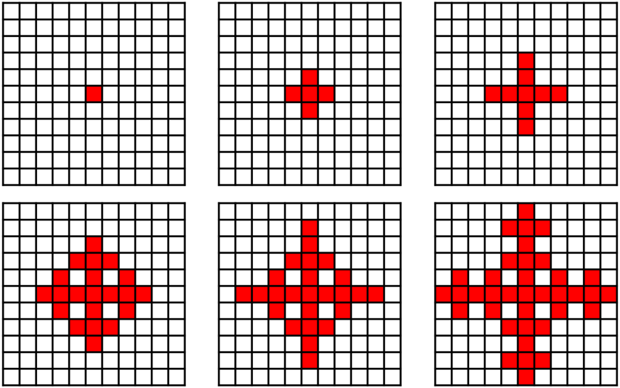


Figure: First generations of the Ulam-Warburton automaton

Enumeration of Cells

The number of cells born at time n:
1, 4, 4, 12, 4, 12, 12, 36, ...

Enumeration of Cells

The number of cells born at time n:

1, 4, 4, 12, 4, 12, 12, 36, . . .

Nim P-positions: 1, 0, 3, 0, 3, 0, 9, 0, 3, 0, 9, 0, 9, 0, 27, . . .

Automaton as a tree

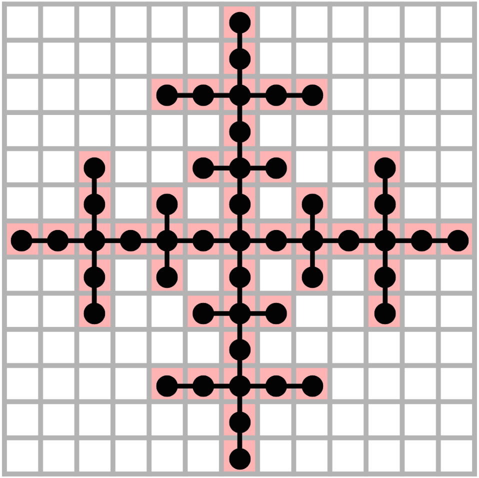
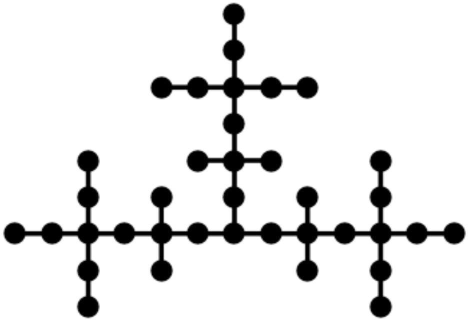
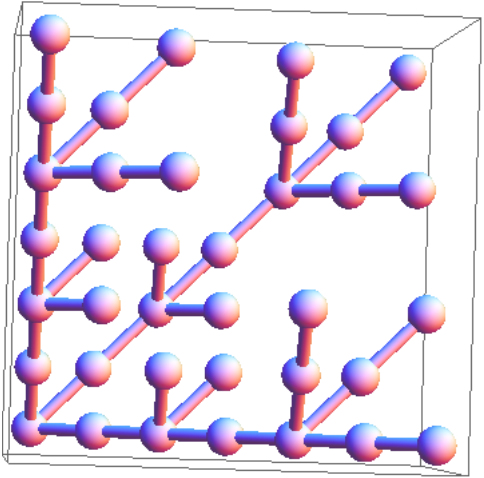


Figure: Picture by Dave Richeson

Automaton corresponding to Nim



P-positions as an automaton

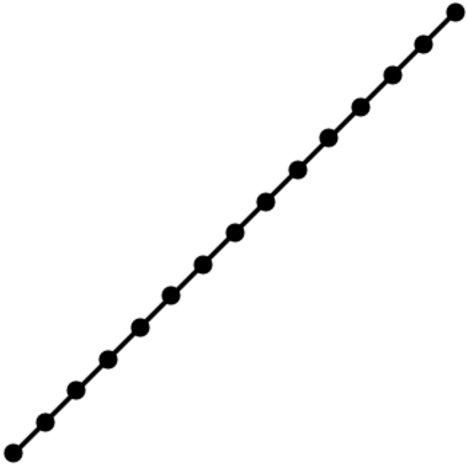


Games as automatons

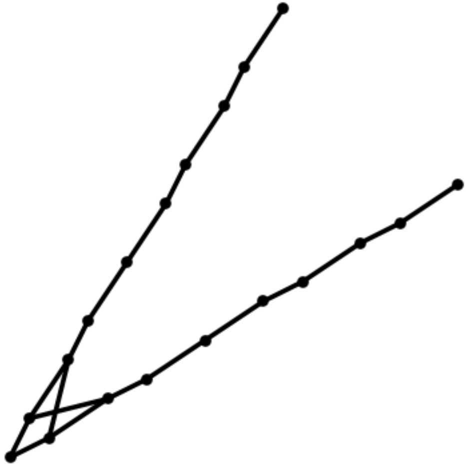
Definition

Two P-position are connected if they are two consecutive P-positions in a longest optimal game.

2d-Nim as an automaton



Wythoff as an automaton



Other games as automatons

Back to Alok and Eric

A project was suggested by Richard Stanley.
Prior research: a lot was known about the square grid.

Growth on Square Grid (continued)

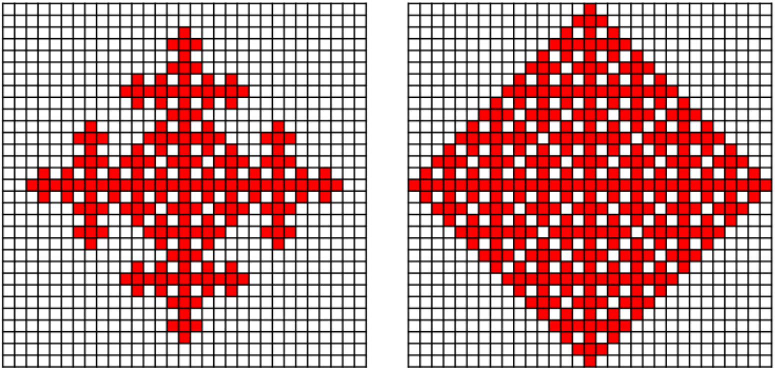


Figure: Generations 13 and 15 of the Ulam-Warburton automaton

Square Grid Known Results

Two major questions:

Square Grid Known Results

Two major questions:

- Which cells are born?

Theorem

A point (x, y) is born if and only if the highest power of 2 dividing x is not equal to the highest power of 2 dividing y.

Square Grid Known Results

Two major questions:

- Which cells are born?

Theorem

A point (x, y) is born if and only if the highest power of 2 dividing x is not equal to the highest power of 2 dividing y.

- In what generation are they born?
  Ugly recursive formula.

Hexagonal Grid Rules

New results.
Rule: A cell is born if it is adjacent to exactly one live cell. A live cell never dies.
Initial conditions: A single live cell at the origin.

Growth on Hexagonal Grid

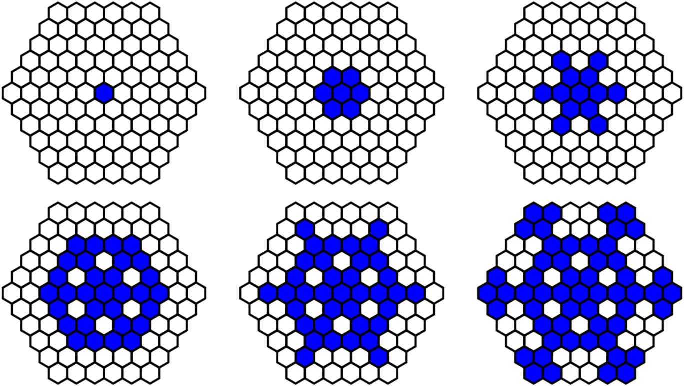


Figure: First generations of Ulam-Warburton-Hex Automaton

Growth on Hexagonal Grid

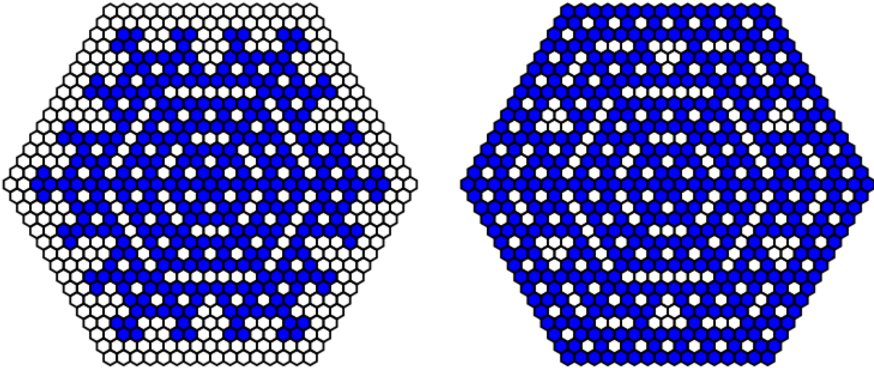


Figure: Generations 13 and 15 of the Hex-UW automaton

Lineage

Definition

Parent: the live cell which caused another cell to be born by being adjacent to it.

Lineage

Definition

Parent: the live cell which caused another cell to be born by being adjacent to it.

Definition

Lineage: the sequence of live cells from the origin to any live cell such that each cell is the parent of the next one.

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Definition

Pioneer: a point (x, y) which is born in generation $x + y$.

Sierpinski Sieve in Hex Grid

Lemma

The set of all pioneers is equal to the Sierpinski sieve

Sierpinski Sieve in Square Grid

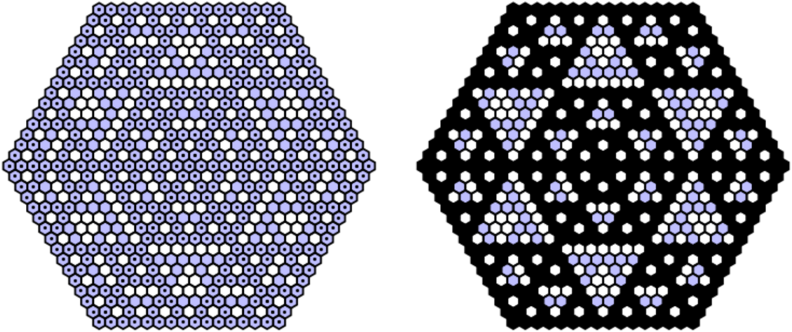
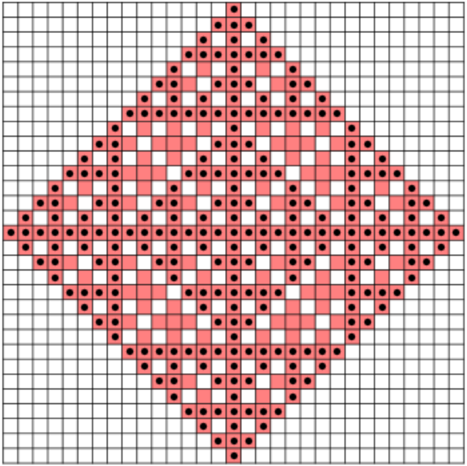


Figure: The Sierpinski gasket in the Hex-UW automaton

Sierpinski



Sierpinski

Theorem

The cells that correspond to the Sierpinki gasket are the ones where you never turn back.

Sierpinski in Nim

Googled Sierpinski and Nim and found two papers:

- Aviezri Fraenkel and Alex Kontorovich, *The Sierpinski Sieve of Nim-varieties and Binomial Coefficients*, 2006. Implies: The Sierpinski triangles are P-positions such that one of the coordinates is the sum of the others.

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- Kevin Gibbons, *The Geometry of Nim*, 2011. Claim: The set of P-positions in d-pile Nim is the full discrete Sierpinski d-demihypercube.

Joshua Xiong PRIMES 2014

Two joint papers:

- *Nim Fractals*, in the Journal of Integer Sequences, Vol. 17 (2014), Article 14.7.8.
- *Cookie Monster Plays Games*, in The College Mathematics Journal, v.46(4) pp.283–293 (2015).

Eric Nie and Alok Puranik PRIMES 2014

One joint paper:

- *The Pioneering Role of the Sierpinski Gasket*, in Math Horizons, Sep 2015, pp. 5–9.

References

1. M. H. Albert, R. J. Nowakowski, D. Wolfe, *Lessons in Play*, A K Peters, Wellesley, MA, 2007
2. E. R. Berlekamp, J. H. Conway and R. K. Guy, *Winning Ways for your Mathematical Plays*, vol. 1, second edition, A. K. Peters, Natick, MA, 2001.
3. C. L. Bouton. *The Annals of Mathematics*, 2nd Ser., Vol. 3, No. 1/4. (1901 - 1902), pp. 35–39.