Guarding Art Galleries, Patroling Prisons, Shoveling Snow, and Surveying Planets

Joe Mitchell

Guarding Polygons

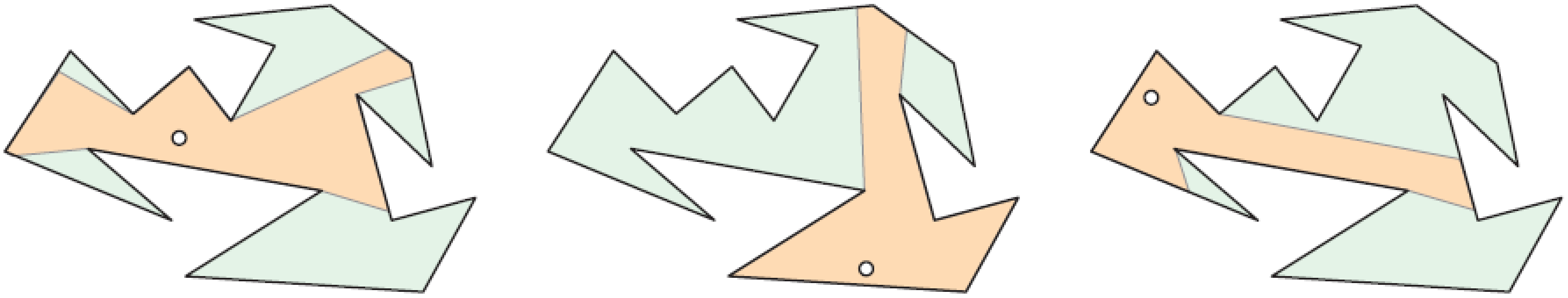


Figure 1.12: Examples of the range of visibility available to certain placement of guards.

V(p) = **visibility polygon** of p inside P
= set of all points q that p sees in P

Guarding Polygons

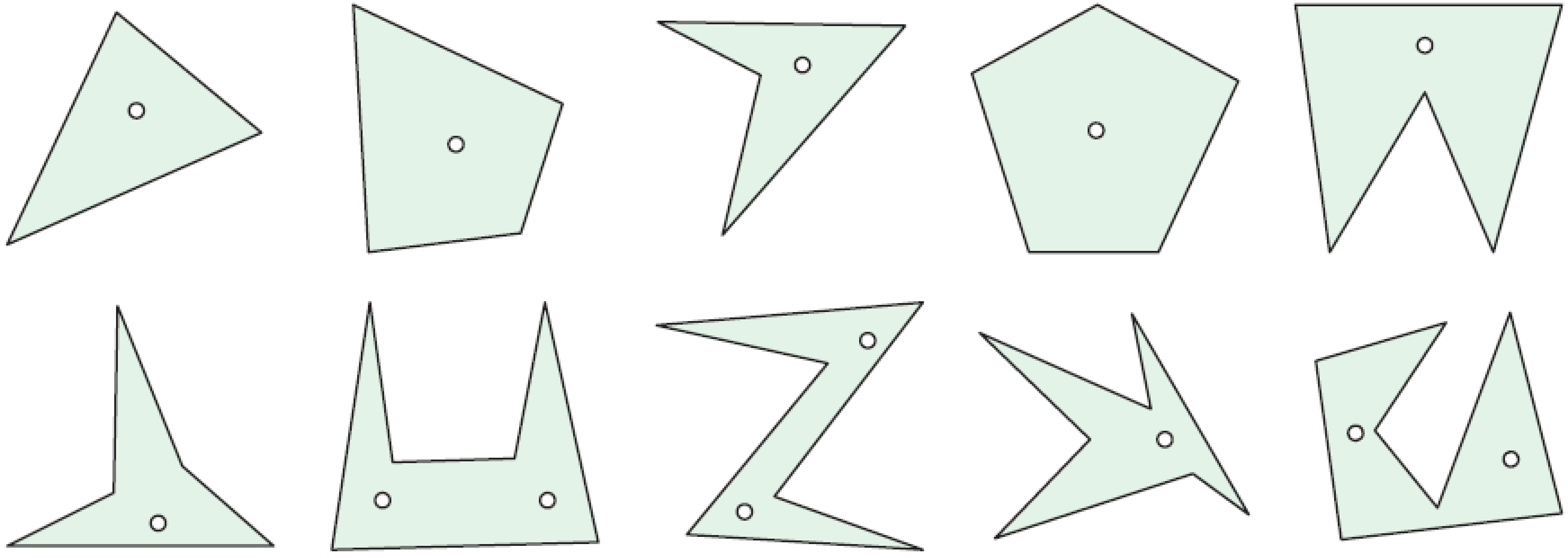
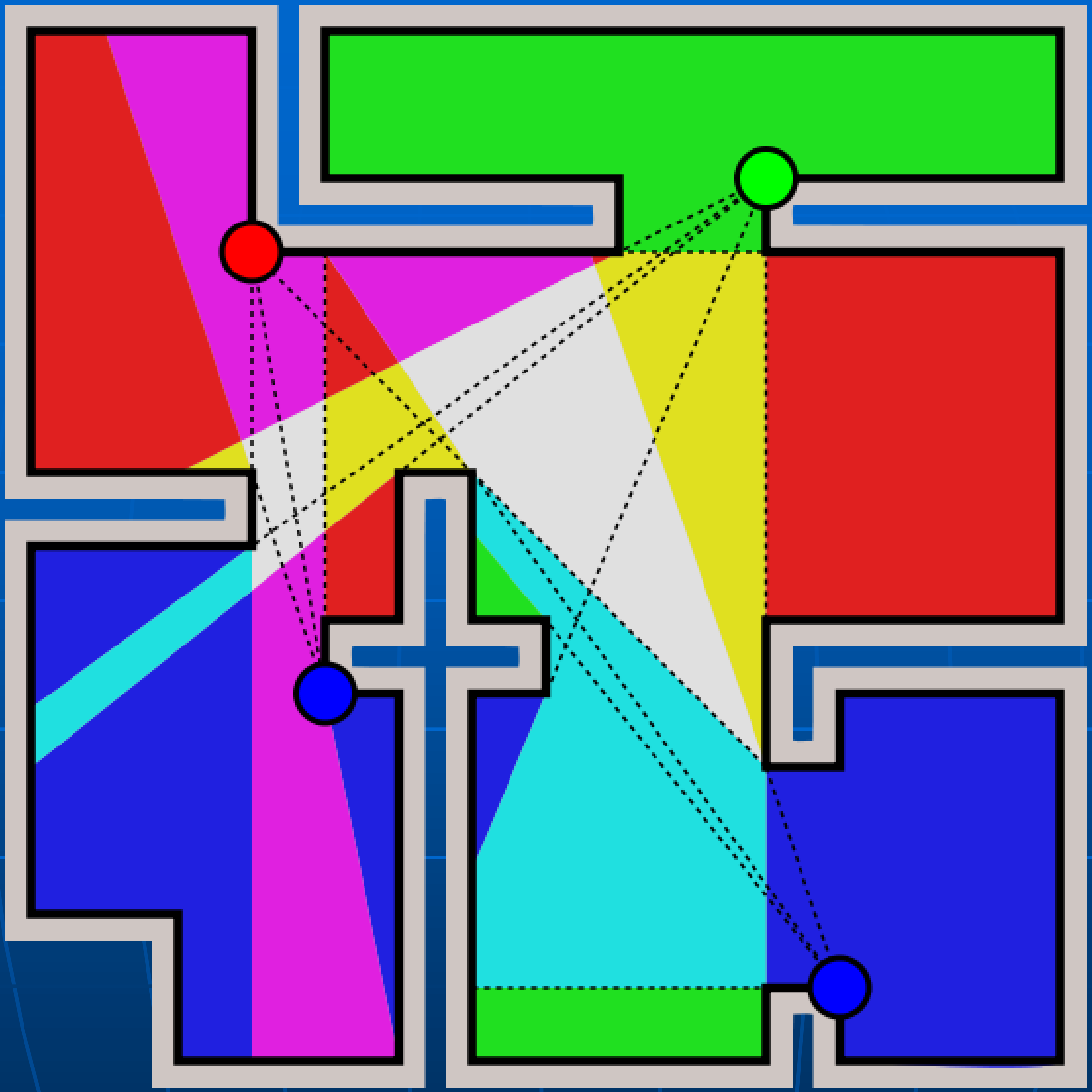


Figure 1.13: Examples of guard placements for different polygons.

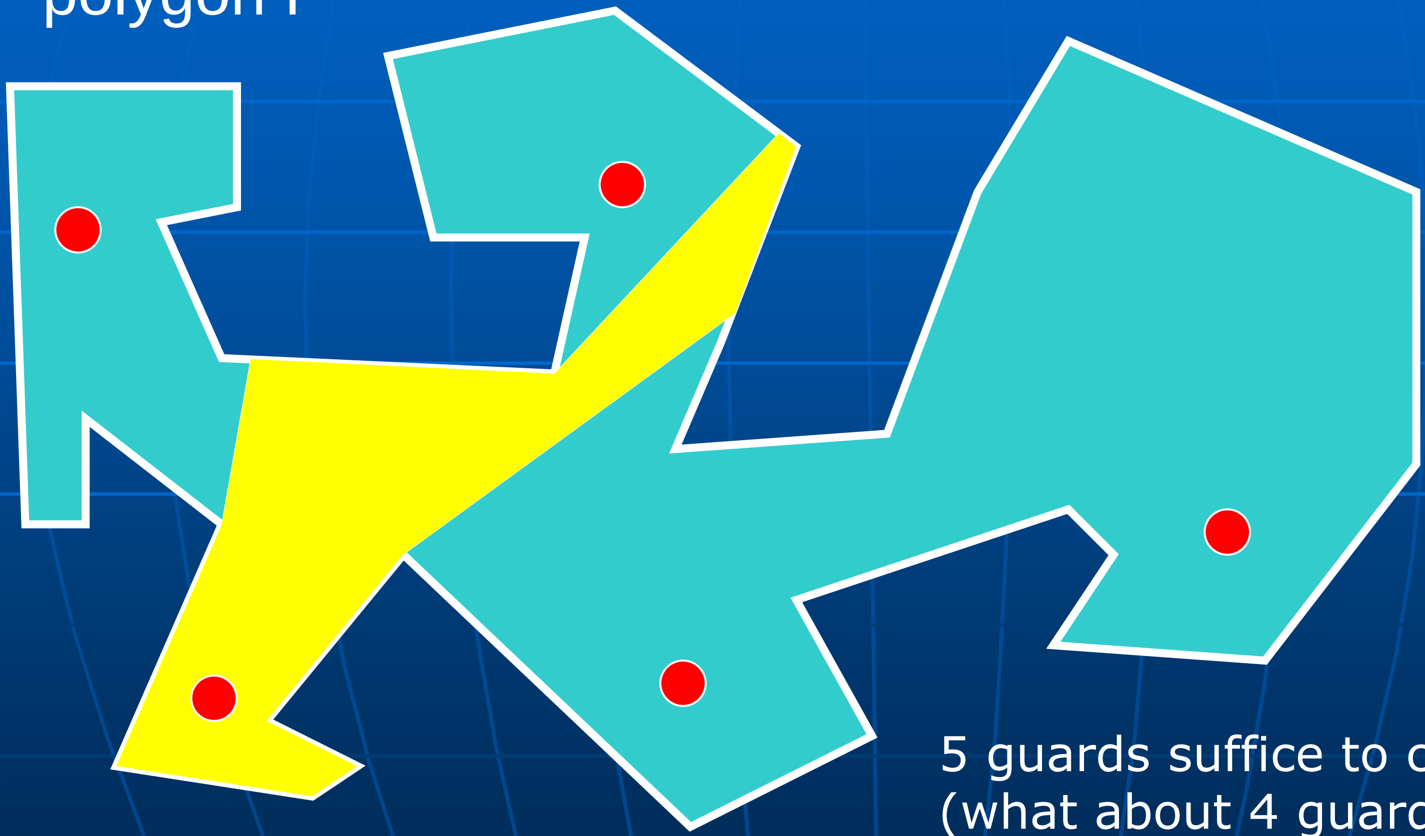
Goal: Find a set of points ("guards") within P so that their VP(p) sets cover P
"Guard cover"
"Point guards" versus "vertex guards"
Regular visibility versus "clear visibility"



A gallery P for which g(P)=4 [wikipedia]

Min-Guard Coverage Problem

- Determine a small set of guards to see all of a given polygon P



5 guards suffice to cover P
(what about 4 guards? 3?)

Computing min # of guards, g(P), for n-gon P is **NP-hard**
Challenge/open: Compute g(P) approximately

Art Gallery Theorem

The Combinatorics of Guarding

- Answers a question of Victor Klee: How many guards are needed to see a simple n-gon?
- Proofs: Chvatal (induction); Fisk (simple coloring argument)

Theorem 1.32 (Art Gallery). *To cover a polygon with n vertices, $\lfloor n/3 \rfloor$ guards are needed for some polygons, and sufficient for all of them.*

g(P) = min number of guards for P
G(n) = max of g(P), over all n-gons P
What is G(n)?
Answer: G(n) = floor(n/3)

In fact, floor(n/3) vertex guards suffice

Chvatal Comb: Necessity of n/3 Guards in Some Cases

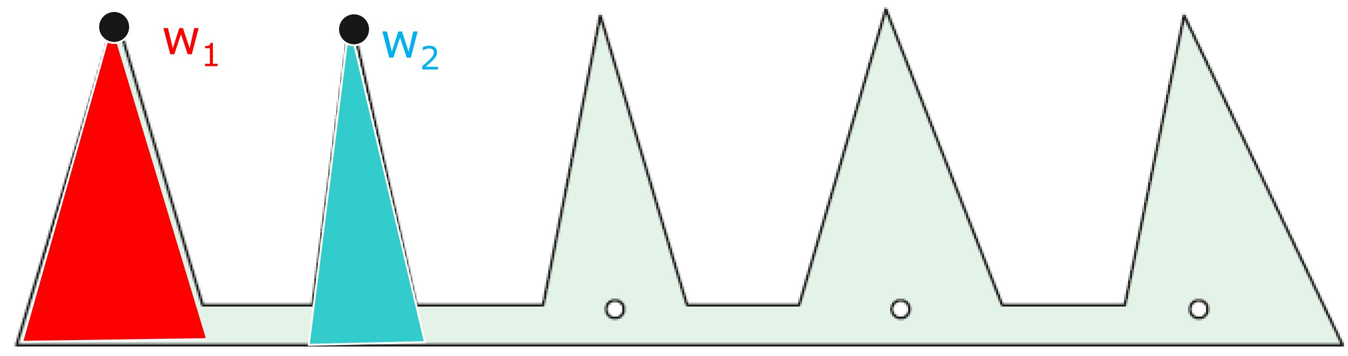


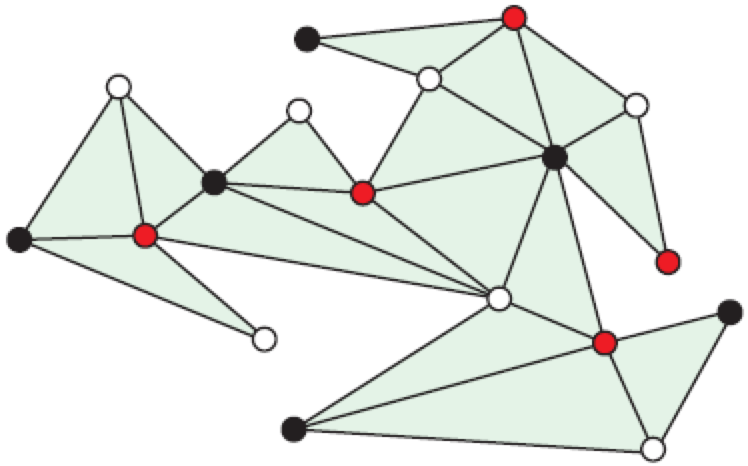
Figure 1.14: A comb-shaped example.

Shows that some n-gons require at least n/3 guards, since we can place "independent witness points", w_i , near each tip, and must have a separate guard in each of their visibility regions (triangles)

Can extend to cases where n is not a multiple of 3, showing lower bound of floor(n/3).

Thus: $G(n) \geq \text{floor}(n/3)$

Fisk Proof: Floor(n/3) Guards Suffice: G(n) ≤ floor(n/3)



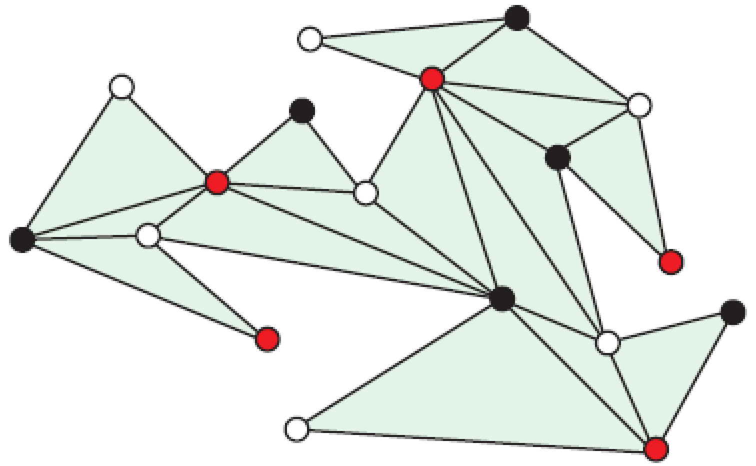


Figure 1.15: Triangulations and colorings of vertices of a polygon with $n = 18$ vertices. In both figures, red is the least frequently used color, occurring five times.

1. Triangulate P (we know a triangulation exists)
2. 3-color the vertices (of triangulation graph)
3. Place guards at vertices in smallest color class

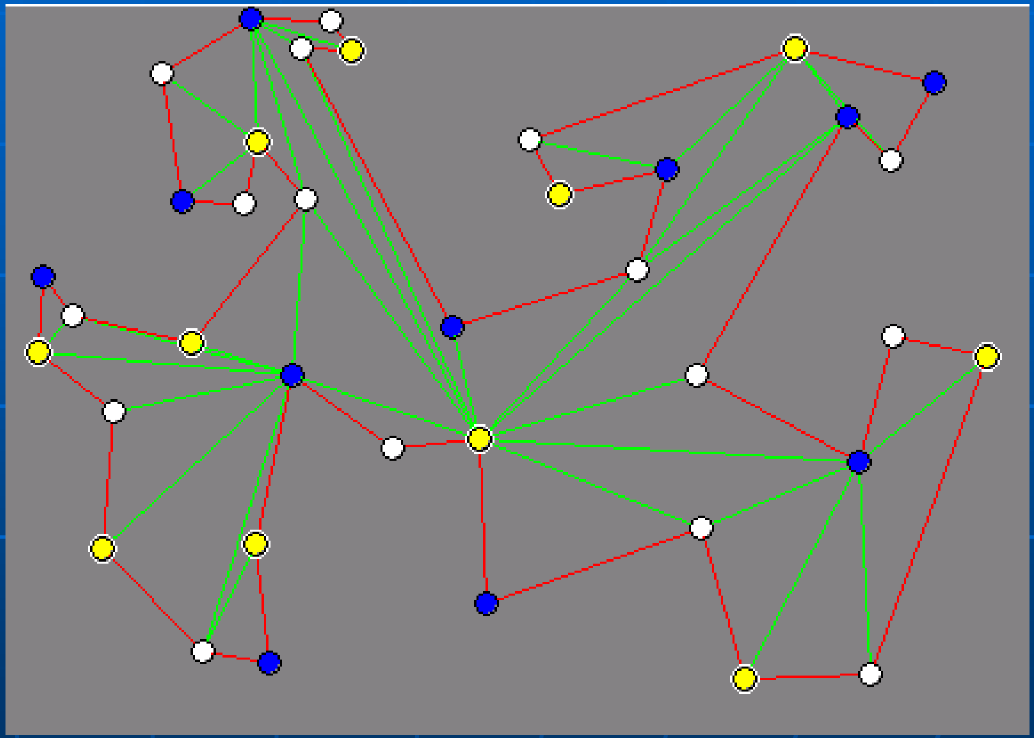
(claim: every point of P is seen, since each triangle has a guard at a corner, and that guard sees all of the (convex) triangle)

Vertex Guarding a Simple Polygon

- Vertex guarding applet

11 yellow vertices
11 blue vertices
16 white vertices

Place guards at yellow (or blue) vertices: at most n/3 vertex guards (here, n=38)

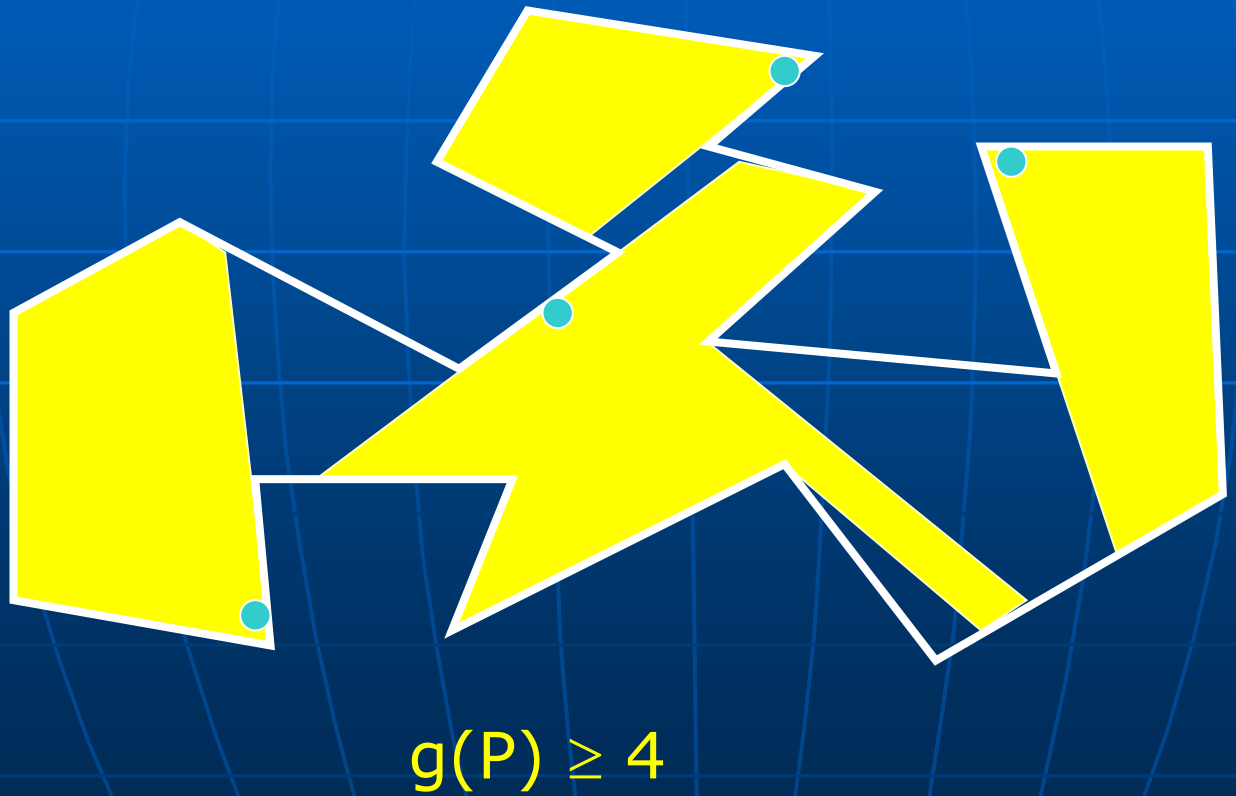


Computing g(P) by Inspection

- By inspection, find a large set of "visibility independent witness points" within P
- If we find w indep witness points, then we know that $g(P) \geq w$
- By inspection, find a small set of m guards that see all of P: $g(P) \leq m$
- If we are lucky, $m=w$; otherwise, more arguments are needed!

Lower Bound on g(P)

- Fact: If we can place w visibility independent witness points, then g(P) ≥ w.



Witness Number

- Let $w(P)$ = max # of independent witness points possible in a set of visibility independent witness points for P
- Then, $g(P) \geq w(P)$
- Note: It is hard to find $g(P)$, and it is also hard to find $w(P)$

Some polygons have $g(P)=w(P)$; I call these *perfect polygons* – they are very special; most polygons P have a “gap”: $g(P)>w(P)$

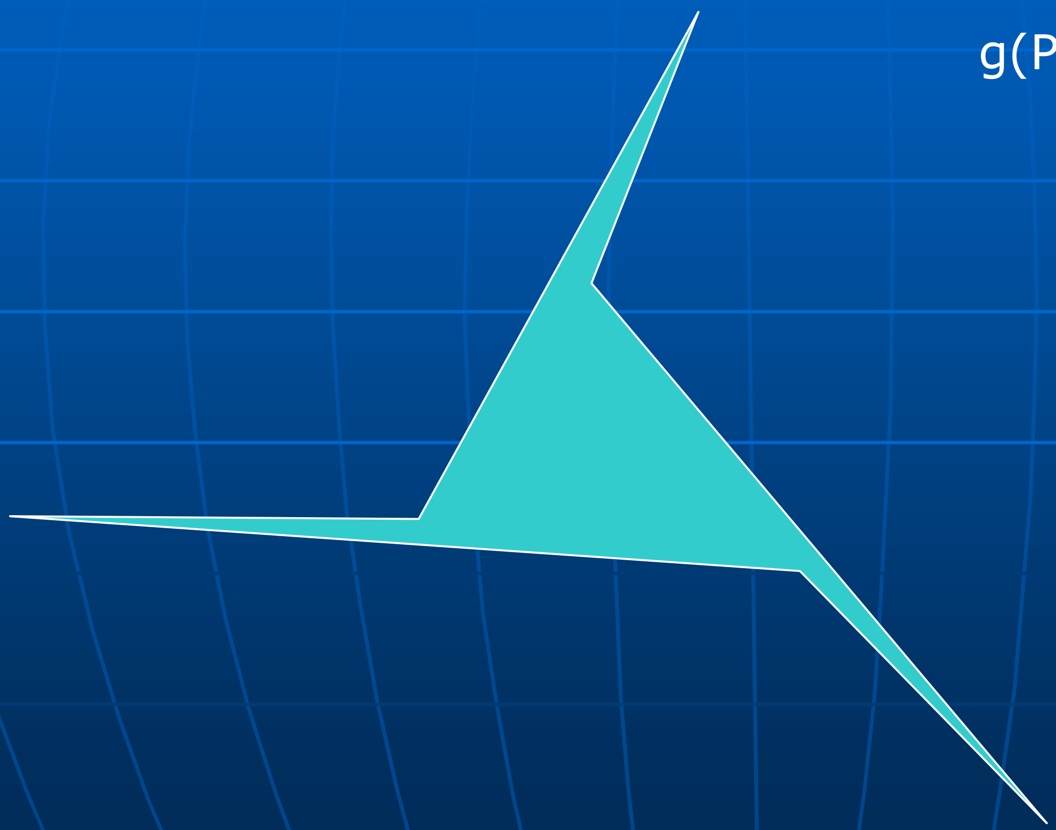
Witness Number: Vertex Guards

- We say that a set, W, of points inside P are *independent with respect to vertex guards* if, for any two points of W, the set of *vertices* of P they see are disjoint
- Let $w_V(P)$ = max # of witness points possible in a set of witness points for P that are indep wrt vertex guards
- Then, $g_V(P) \geq w_V(P)$
- Note: It is hard to find $g_V(P)$, and it is also hard to find $w_V(P)$

Useful Polygon Example

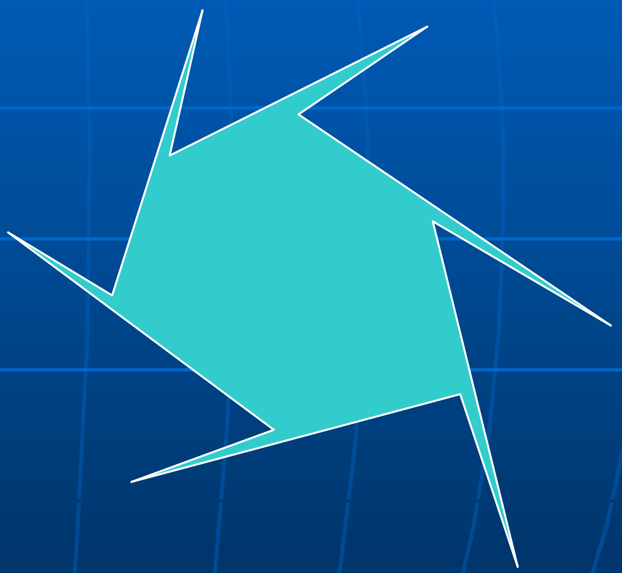
- "Godfried's Favorite Polygon"

$g(P)=2$, but $w(P)=1$

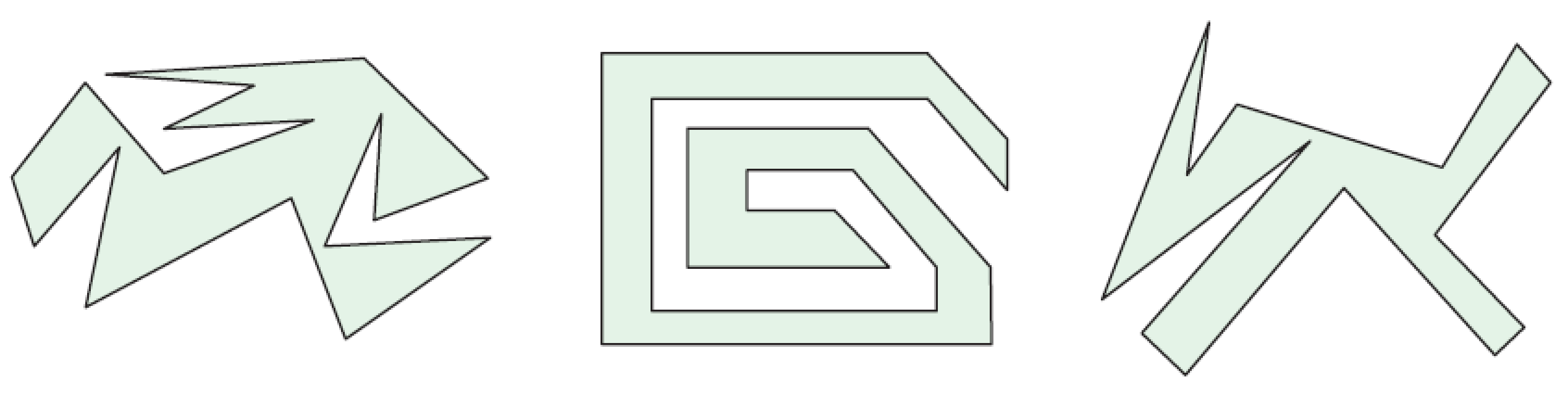


Useful Polygon Examples

- “Godfried’s Favorite Polygon” variations

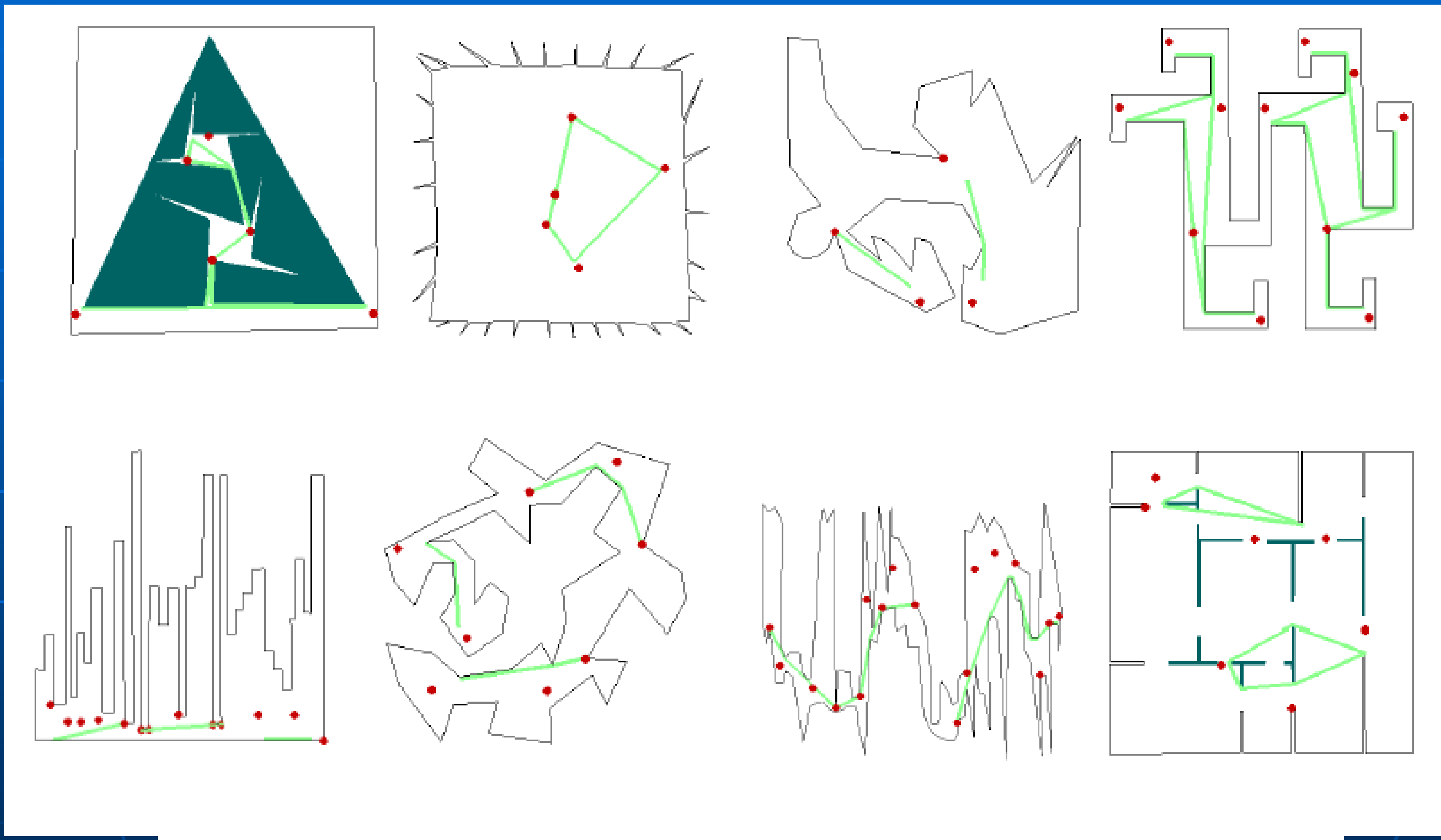


Examples

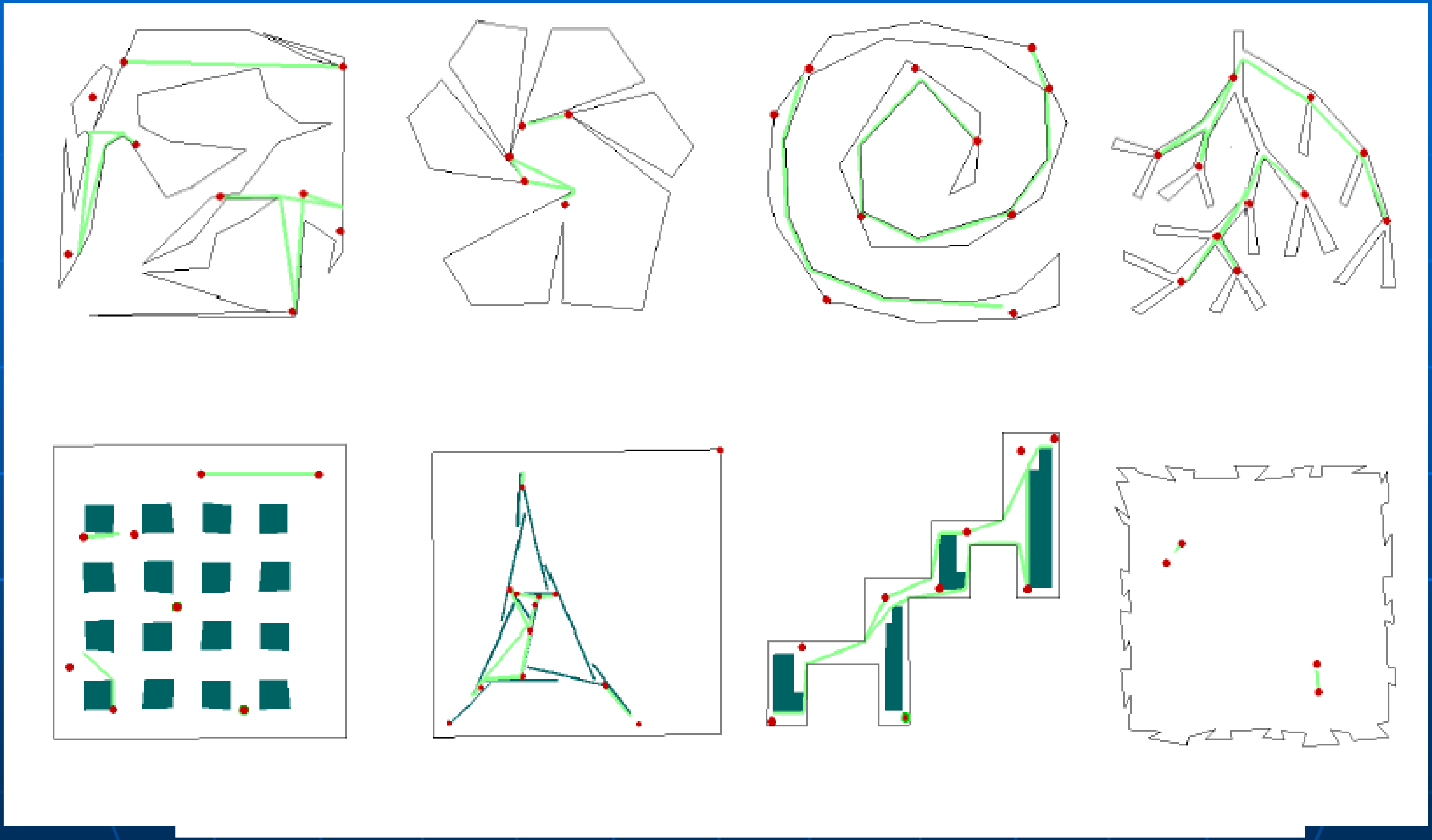


For each of these polygons P, find the point guard number, $g(P)$, and the vertex guard number, $g_V(P)$.
Also, find the witness numbers $w(P)$ and $w_V(P)$

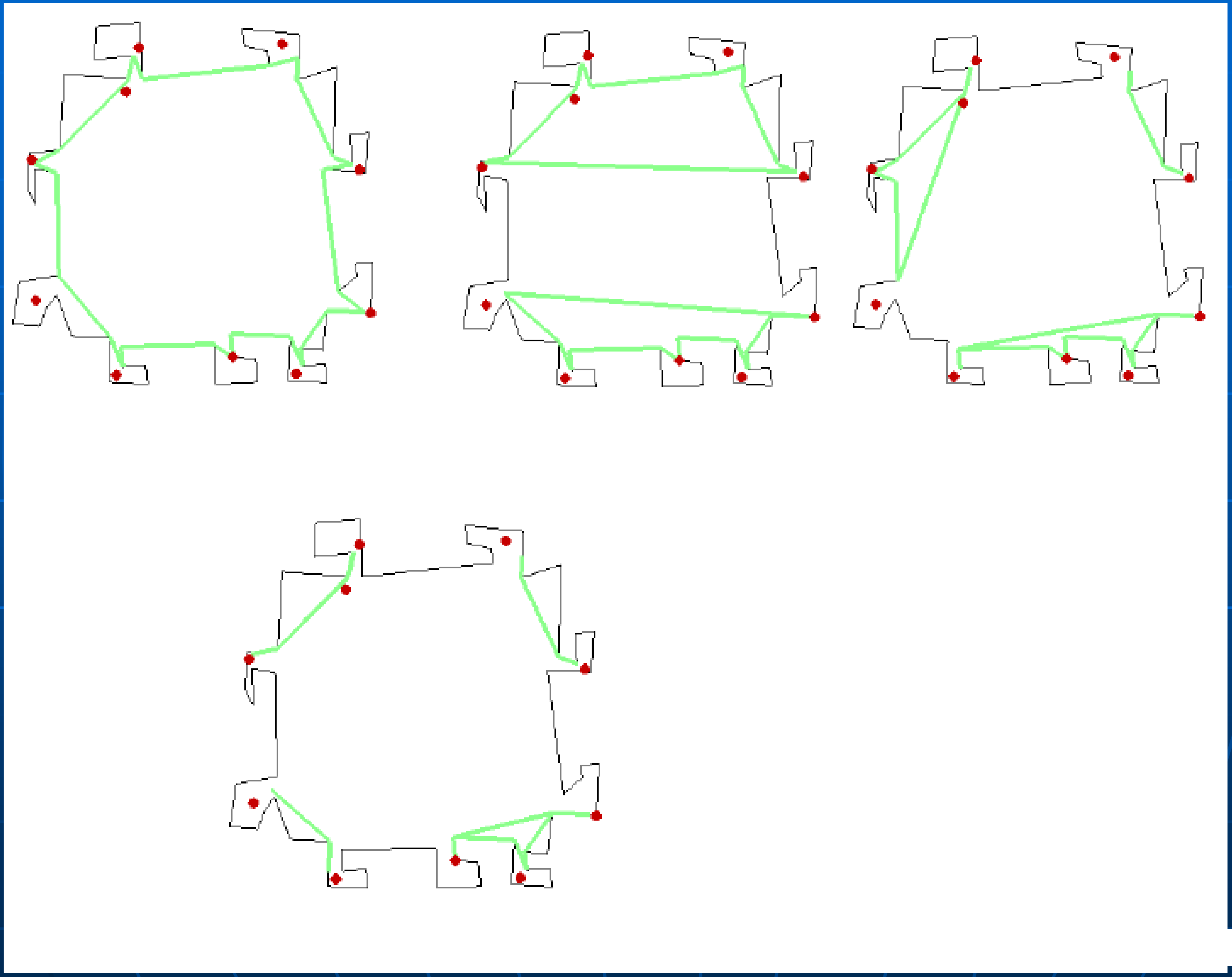
Examples



Examples



Examples



Art Gallery Theorem: Orthogonal (Rectilinear) Polygons

Theorem 1.37 (Orthogonal Gallery). *To cover polygons with n vertices with only right-angled corners, $\lfloor n/4 \rfloor$ guards are needed for some polygons, and sufficient for all of them.*

Polygons with Holes

- Art Galley Theorem: floor((n+h)/3) guards suffice and are sometimes necessary
- (easy: floor((n+2h)/3) suffice – do you see why?)

Exterior Guarding: Fortress Problem

Theorem 1.38 (Fortress). *To cover the* exterior *of polygons with n vertices,* $\lceil n/2 \rceil$ *guards are needed for some polygons, and sufficient for all of them.*

Edge Guards

Unsolved Problem 5 *Edge Guards*

An *edge guard* along edge e of polygon P sees a point y in P if there exists x in e such that x is visible to y. Find the number of edge guards that suffice to cover a polygon with n vertices. Equivalently, how many edges, lit as fluorescent bulbs, suffice to illuminate the polygon? Godfried Toussaint conjectured that $\lfloor n/4 \rfloor$ edge guards suffice except for a few small values of n.

Guarding Polyhedra

- Note: Guards at vertices are NOT enough!

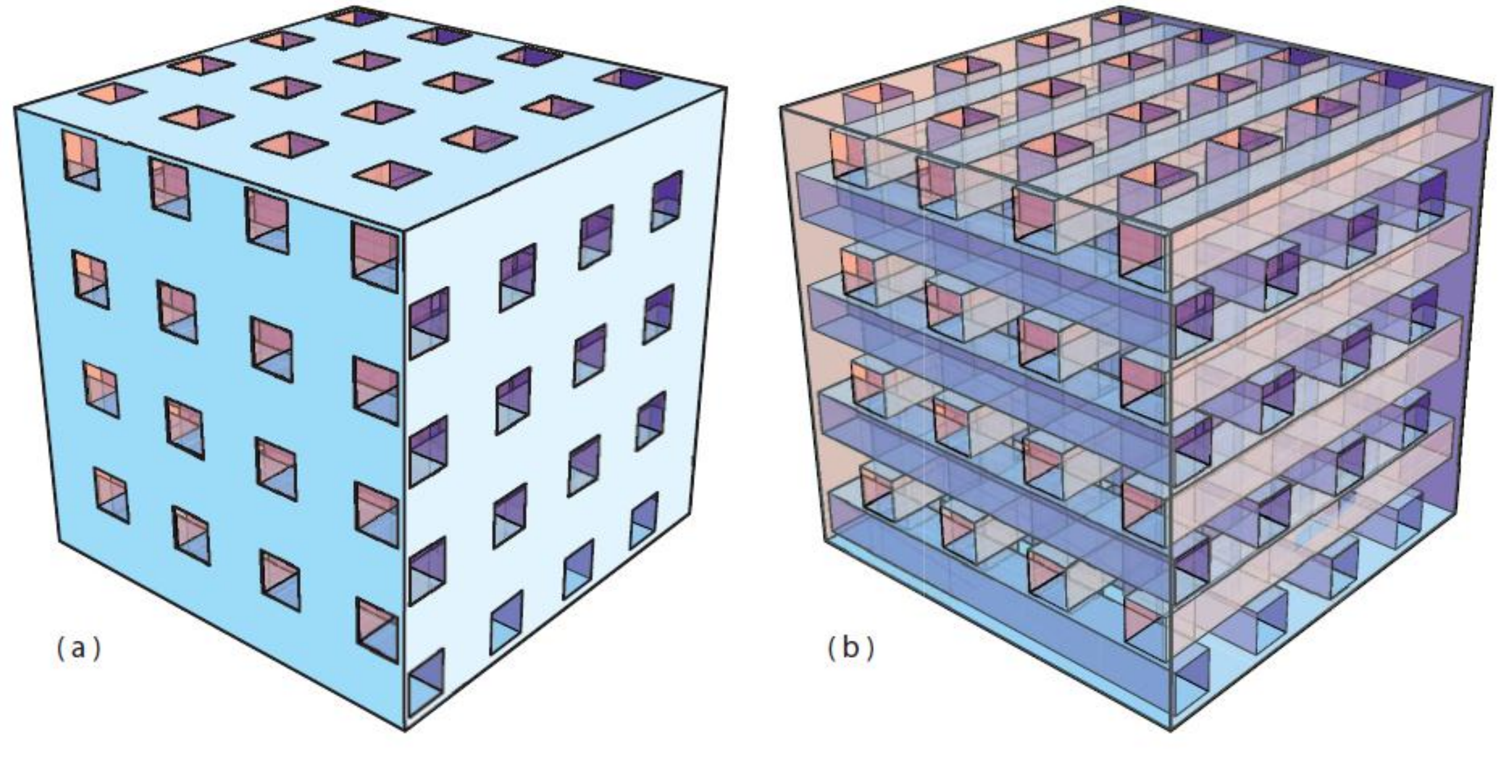


Figure 1.17: (a) The Seidel polyhedron with (b) three faces removed to reveal the interior.

Mobile Guards

- Find shortest route (path or tour) for a mobile guard within P: Watchman route problem
- Efficient algorithms for simple polygons P
- NP-hard for polygons with holes (as hard as the TSP)

Motivations from Robotics, etc

Exploration Strategies for a Robot with a Continously Rotating 3D Scanner

Elena Digor, Andreas Birk, and Andreas Nüchter

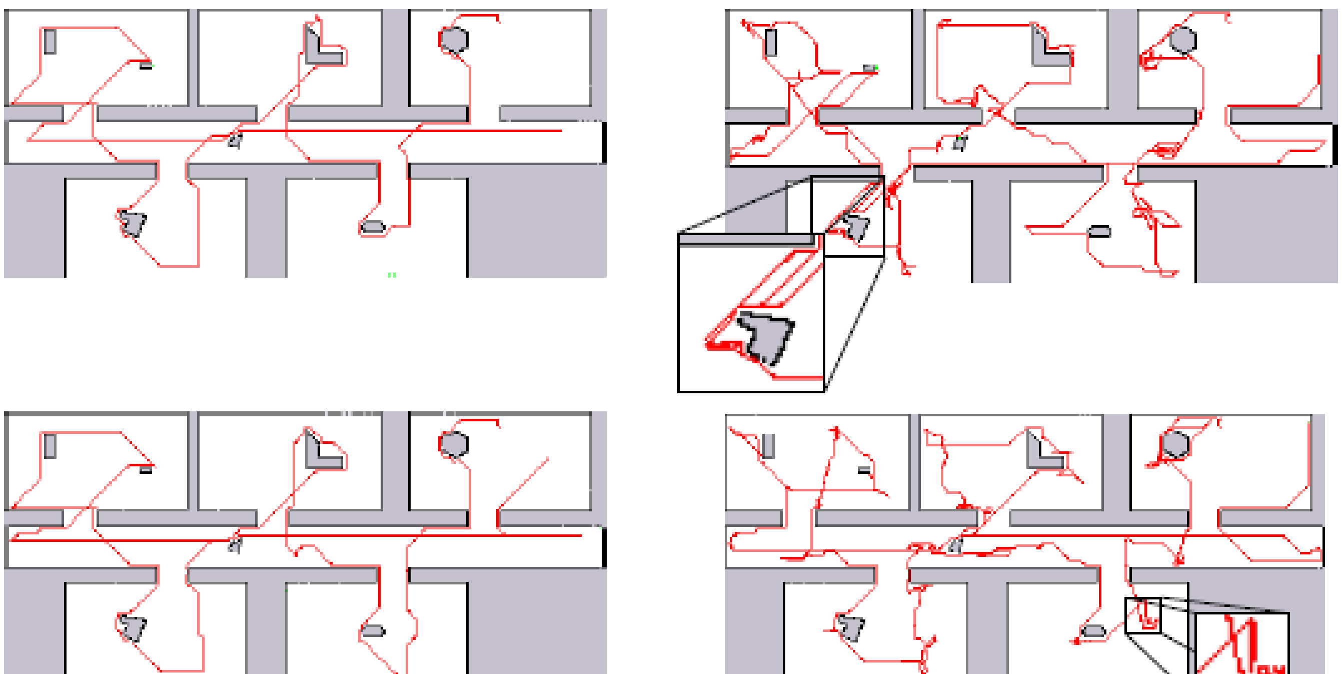
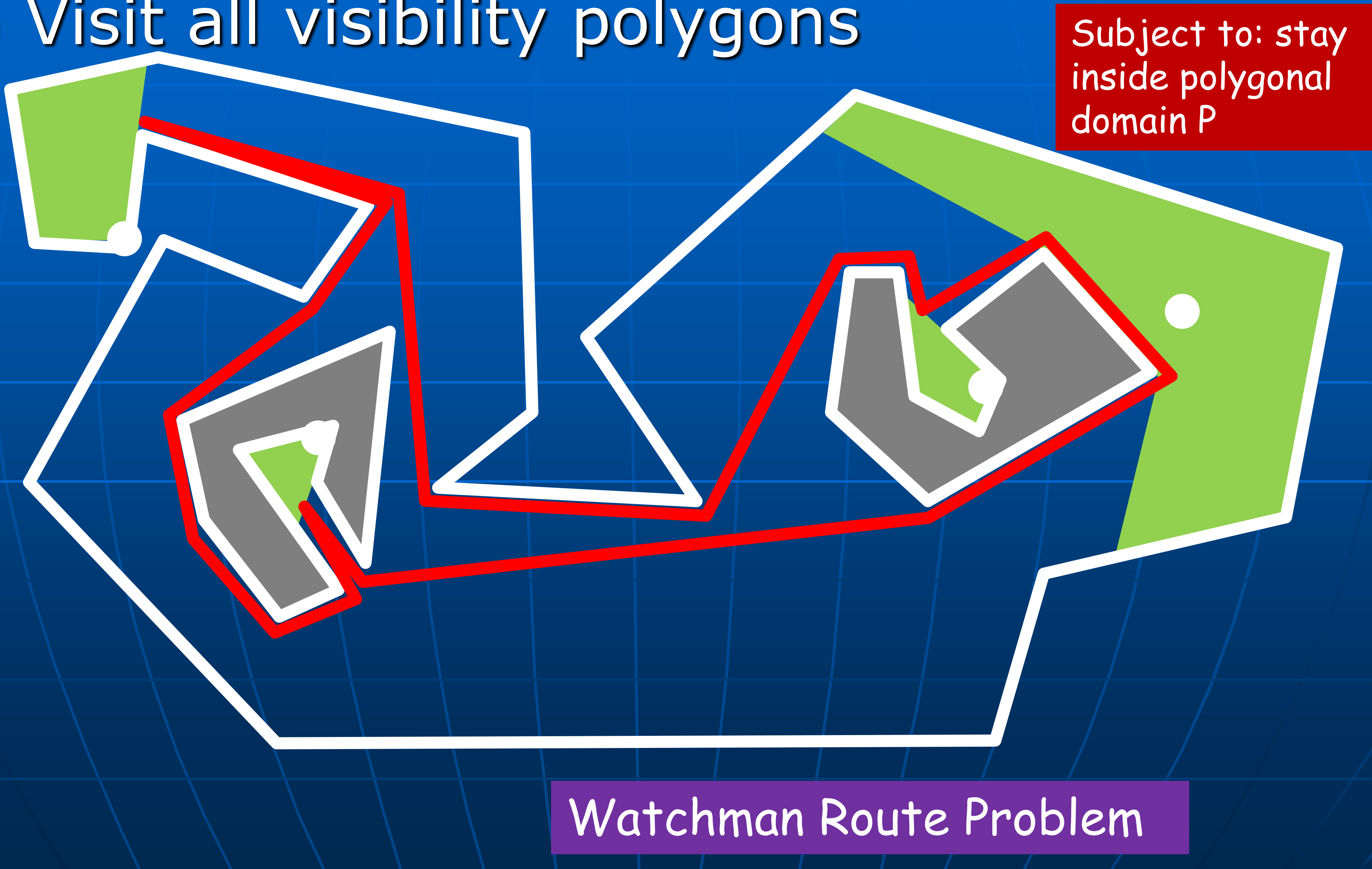


Fig. 7. Results in the office map with clutter. From left to right: Stop-scan-replanning-go, Scan-replanning-go, Continuously-replanning-with-stopping, and Continuously-replanning-go strategy.

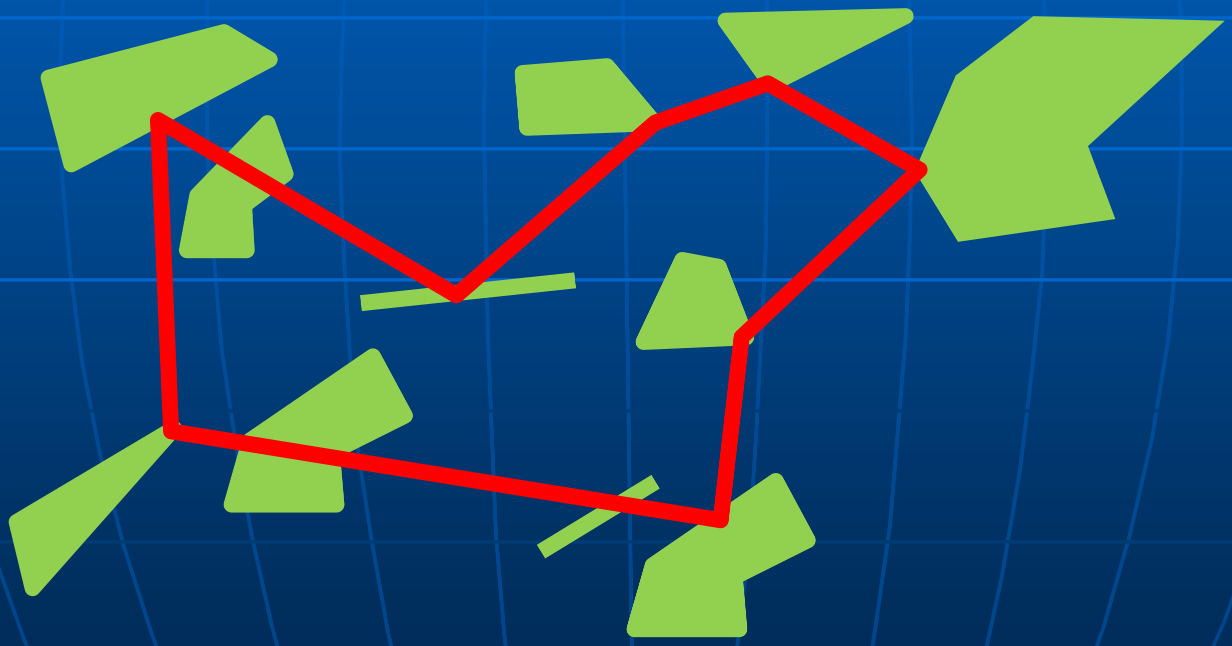
Mobile Robotic Guard

- Visit all visibility polygons



Related Geometric Problems

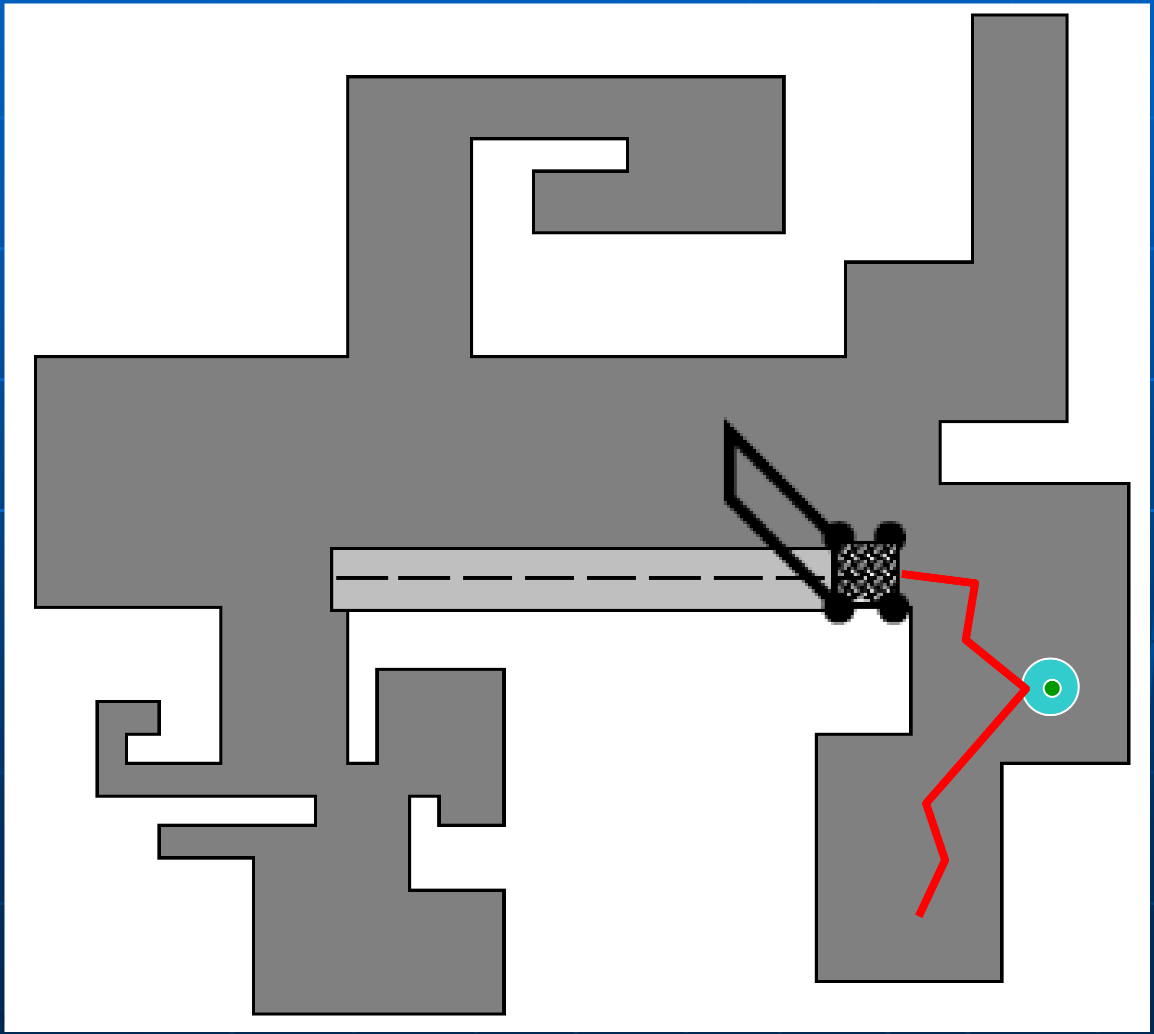
- **TSP with Neighborhoods** (TSPN)



(AND obstacles)

Related Geometric Problems

- Lawnmower/Milling



Best method of **mowing** the lawn?

NC-machining: **milling** a pocket.

[Ntafos, CGTA 1992]: d-sweeper: must be within distance d to see a point

TSPN: Visit the disk centered at each blade of grass

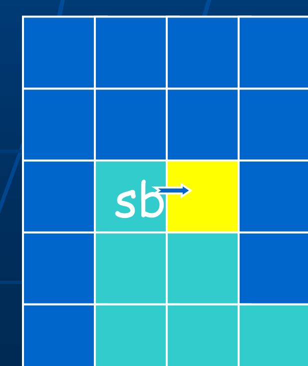
Snowblower: Material-Shifting Machine

- **lifts** snow from one location
- **piles** it on an adjacent location

SB moves from pixel to adjacent pixel

- picks up all snow
- throws to a neighbor pixel
- or over the boundary of region
- max depth of snow *D*

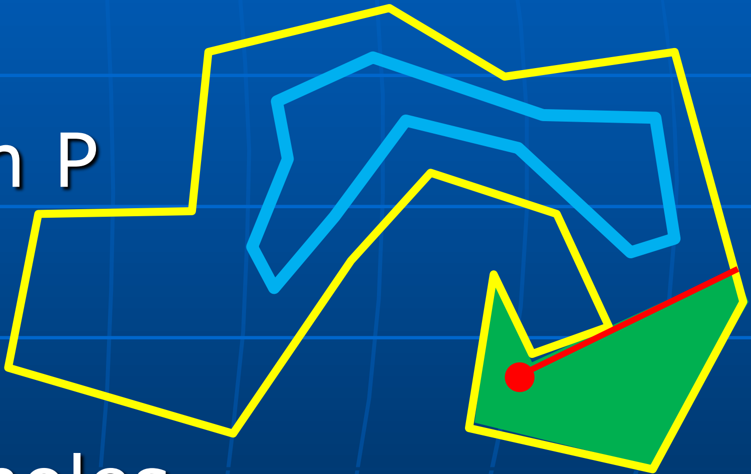
Objective: minimize the length of the tour of the snowblower



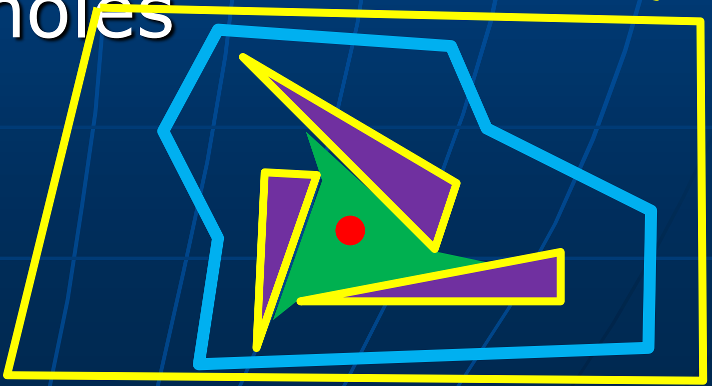
Results: O(1)-approx, in several models [ABMP, WAFR'06]

How Much Needs to be Covered?

- Must visit VP(p) for ***all*** p in P
- **Q**: Is it enough for the tree/tour to see all vertices of P?
  - **YES**, in simple polgyon P



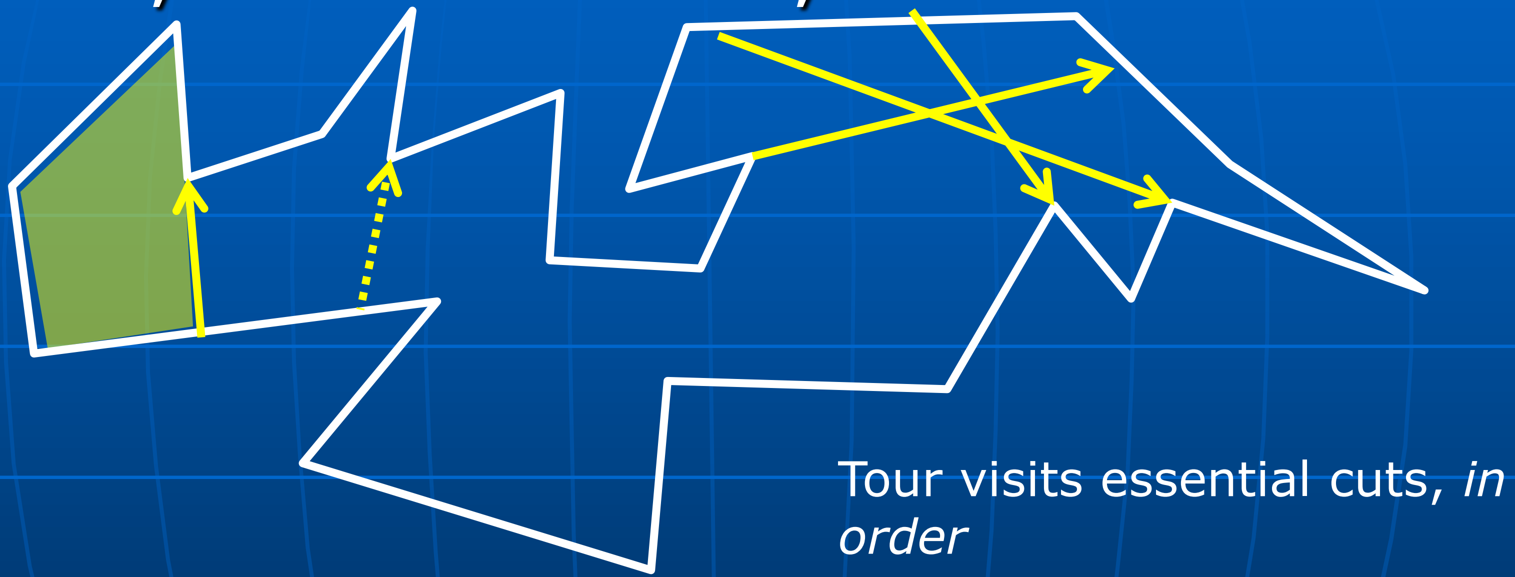
  - **NO,** in polygons with holes



Not even enough to see all
of the boundary of P

WRP Structure in Simple Polygons

- Cuts, essential cuts, corner



- One can compute all essential cuts: O(n)

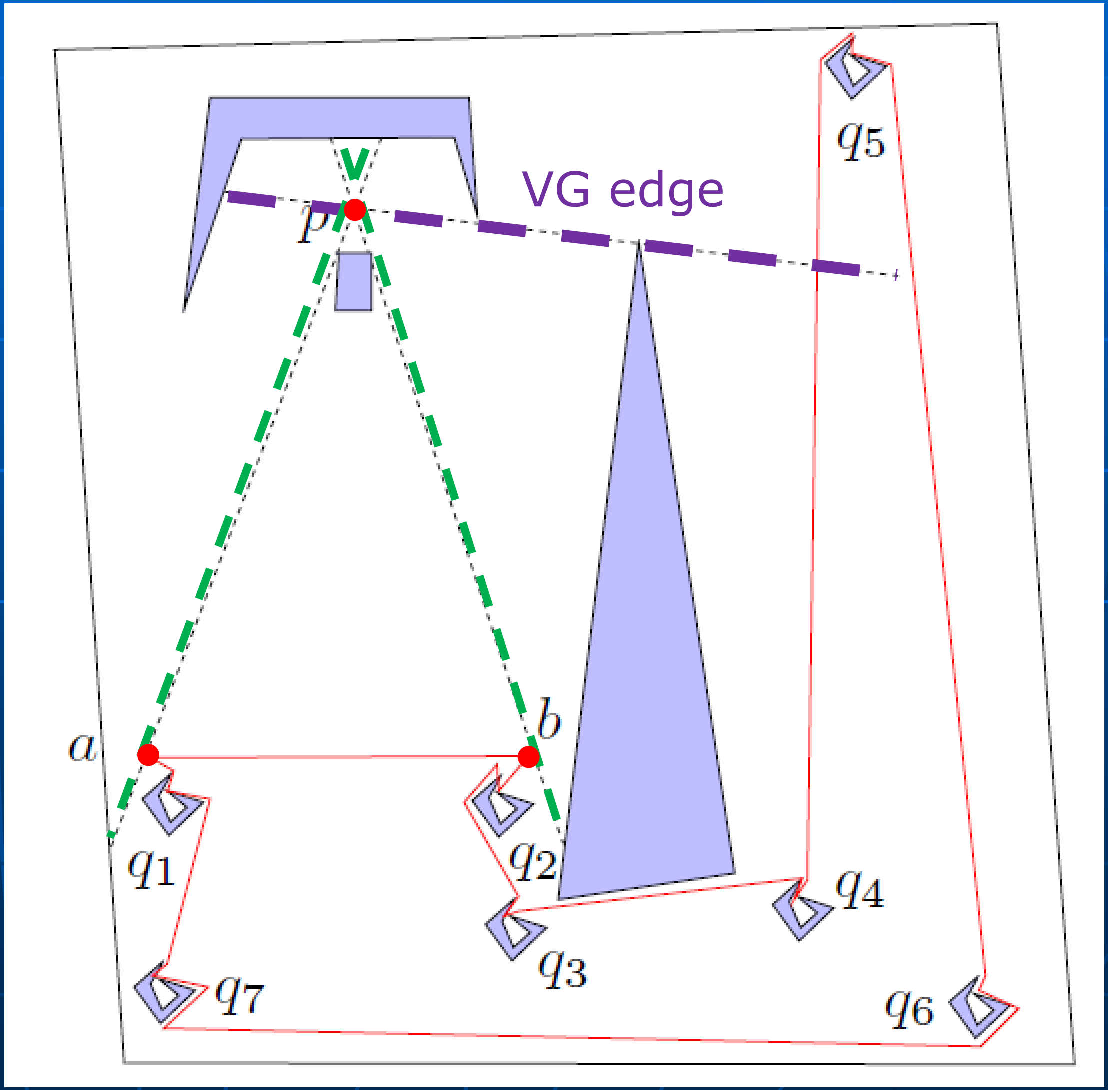
[Tan, 2007]

WRP Example: Effect of Holes

Complicating Issue:

Tour reflects off of segments that are not readily known (e.g., edges of P, VG edges)

Reminiscent of art gallery problem



Bounds on WRP Tour Length

- Upper bound on length of tour, in terms of h (# holes), per(P) and dian $O(per(P) + \sqrt{h} \cdot diam(P))$

tight for polygons P with $per(P) > c \cdot diam(P)$, for any fixed $c > 2$

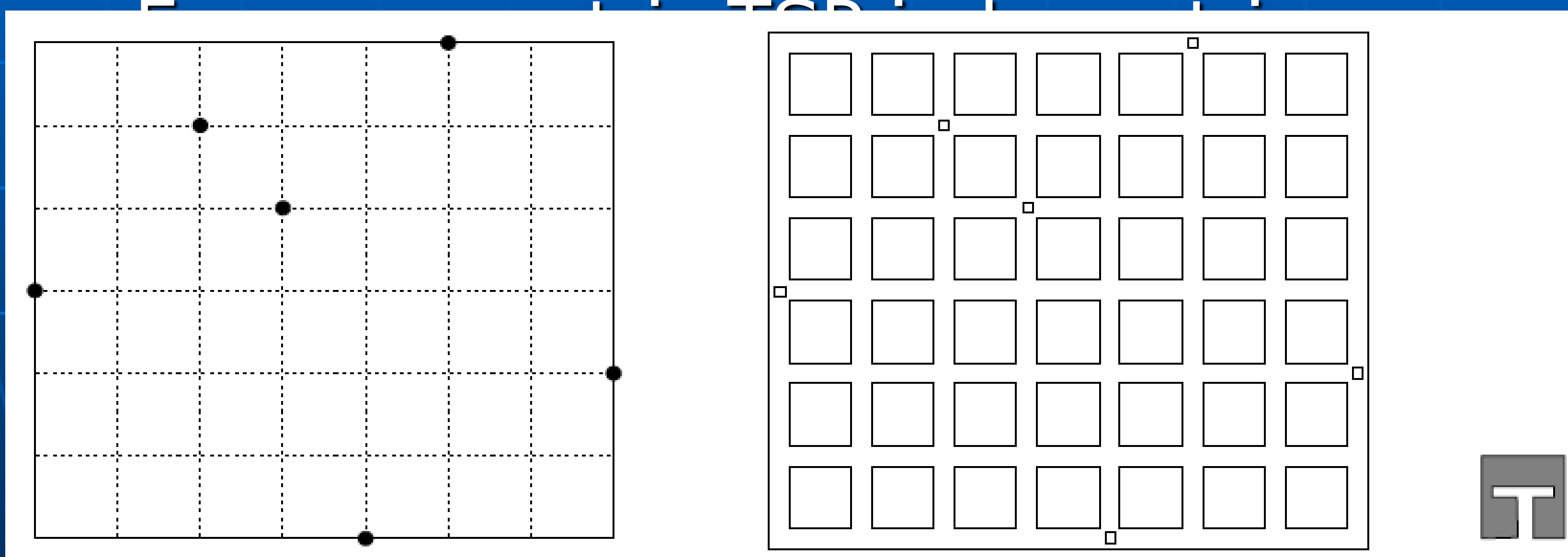
[Dumitrescu, Toth, CCCG 2010, CGTA 2012] Also bounds in 3D

[Czyzowicz,Ilcinkas,Labourel,Pelc, SWAT 2010] Exploring an *unknown* domain. Also bounds in terms of area(P) in limited visibility model

- Given P, can compute in O(n log n) time

WRP in Polygons with Holes

- Rectilinear polygon with holes: **NP-hard**



[Dumitrescu, Toth]

WRP in Simple Polygons

- Best time bounds based on modelling as "Touring Polygons Problem" (TPP)

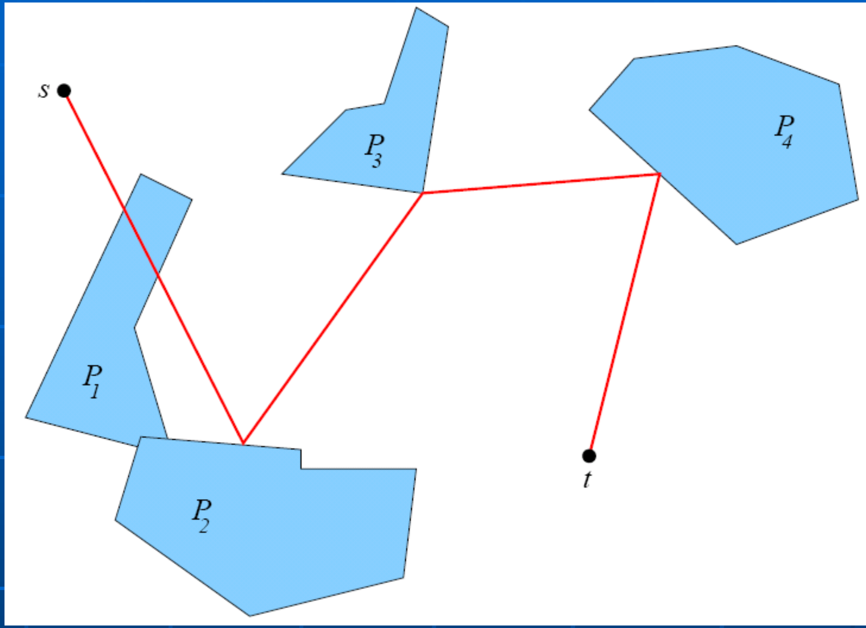
[Dror,Efrat,Lubiw,M, STOC 2003]

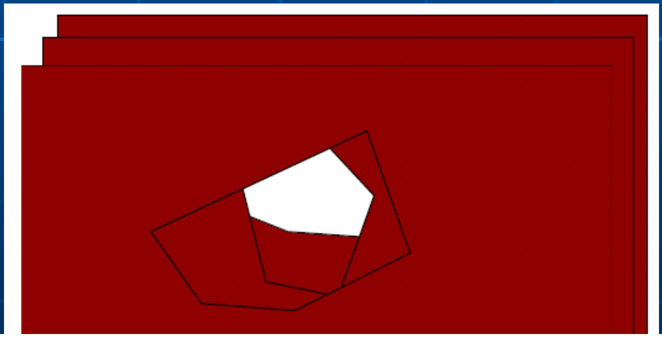
Ordered Covering Tours/Paths

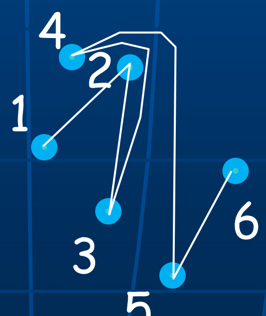
- Order given [DELM, 2003]

  Convex: poly-time

  Non-convex, overlapping: NP-hard
- Related to 3D shortest paths





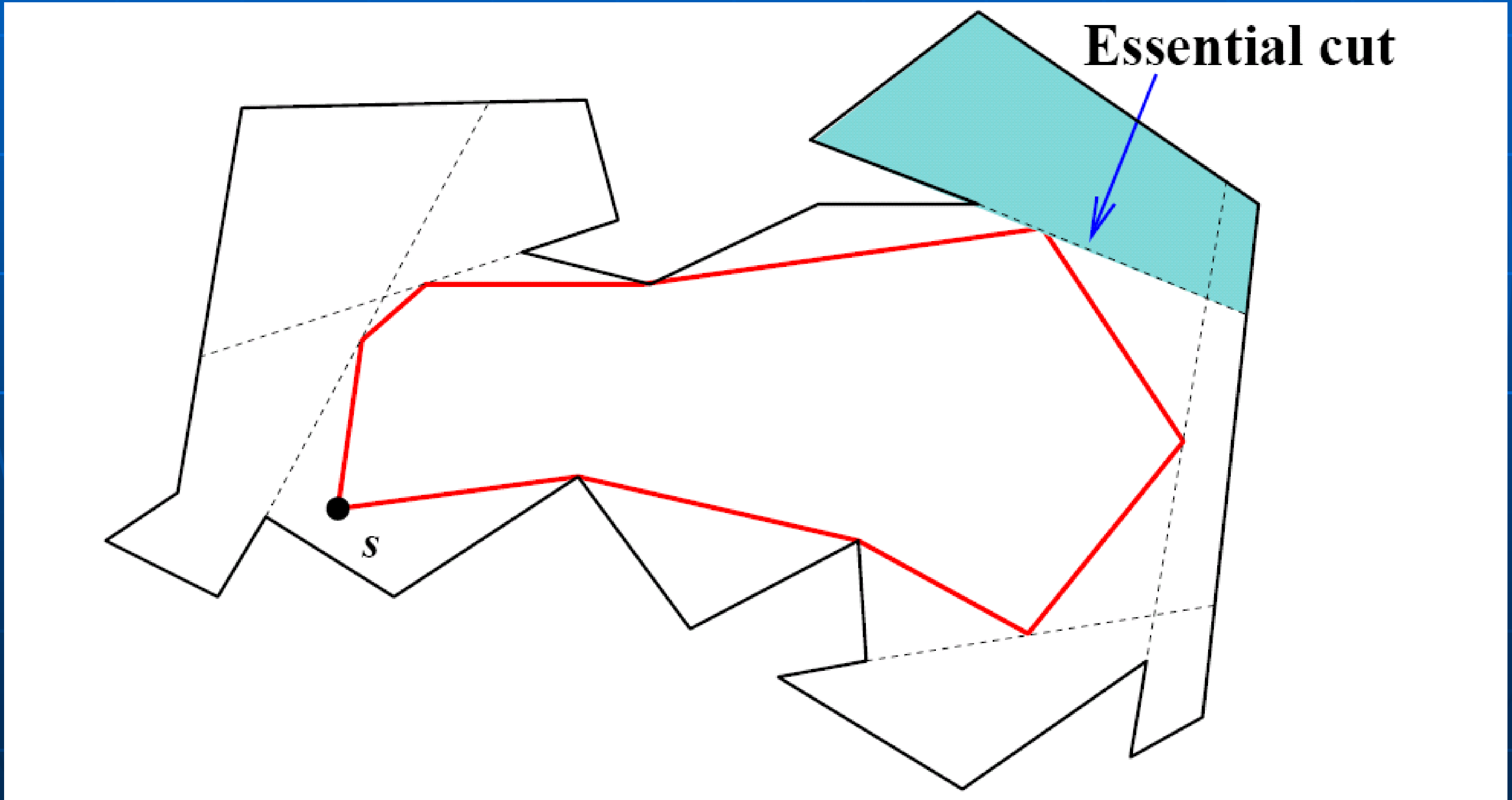


Q: Disjoint non-convex?

Q: Shortest **simple** tour, even for points?

Watchman Route Problem

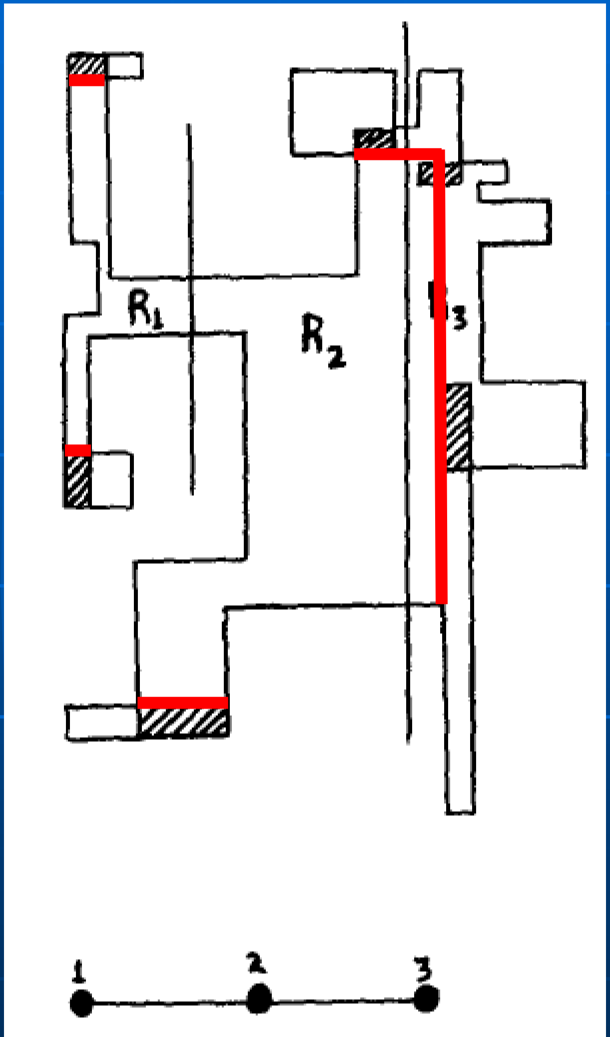
- Find a shortest tour for a guard to be able to see all of the domain



Fact: The optimal path visits the essential cuts in the order they appear along ∂P.

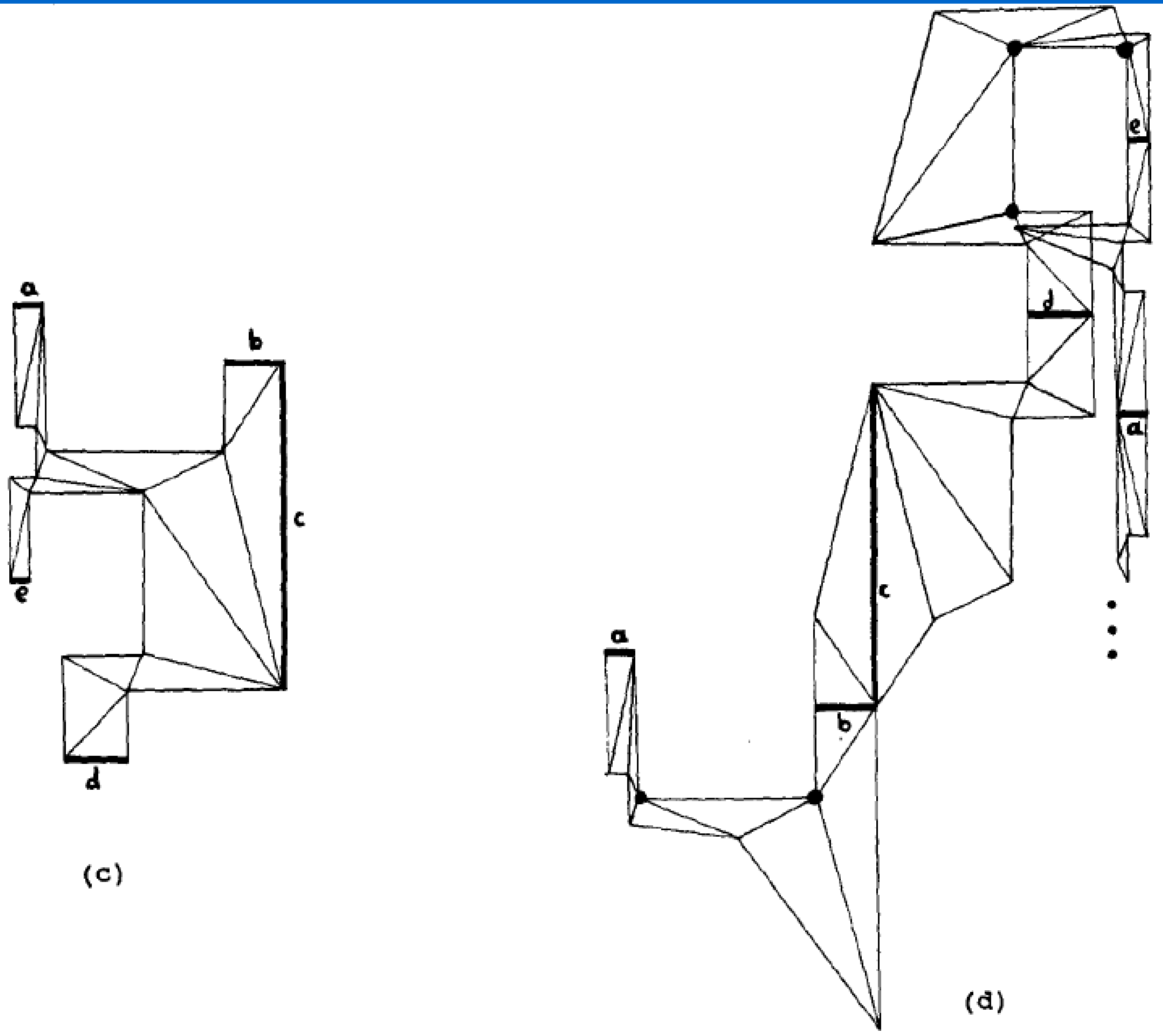
Special Cases of WRP

- (1) Simple, rectilinear polygons:
  O(n) time



Essential Cuts

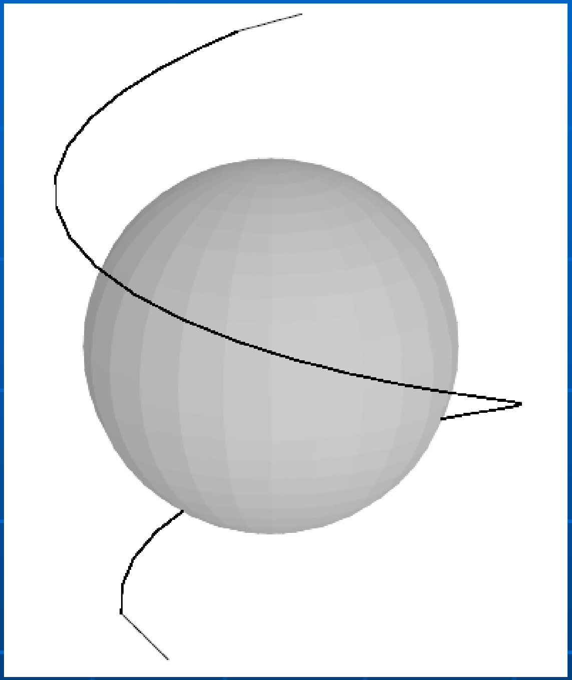
[Chin, Ntafos, SoCG 1986]


a
b
c
e
d
(c)
a
b
c
d
e
a
(d)

WRP in 3D

External Watchman **Path** for a Sphere

- Short **Path**
  Length 11.08



Two segments and a spiral:

$$\{((1 - at^2)\sin(b\pi t),\ (1 - at^2)\cos(b\pi t),\ ct) \mid -1 \le t \le 1\}$$

Fatten spiral near middle

$$a = 0.4,\ b = 1.18,\ c = 1.12,\ x_0 = -0.37,\ y_0 = -0.199,\ z_0 = 1.24$$

By computer search

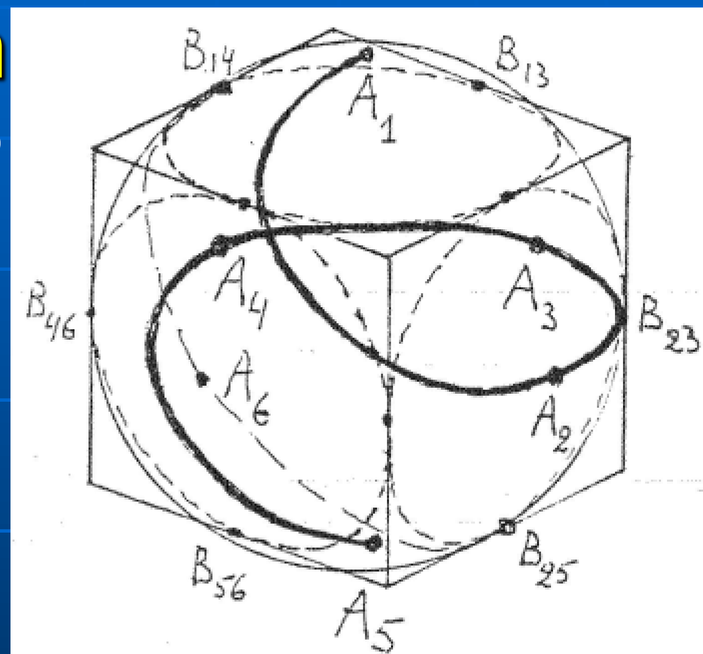
The Asteroid Surveying Problem and Other Puzzles

Timothy M. Chan   Alexander Golynski   Alejandro López-Ortiz   Claude-Guy Quimper

[SoCG'03 video]

External Watchman **Path** for a Sphere

- Short **Path**
  Length 10.726



a "rather short" inspection curve that lies at the constant altitude of $\sqrt{2} - 1$

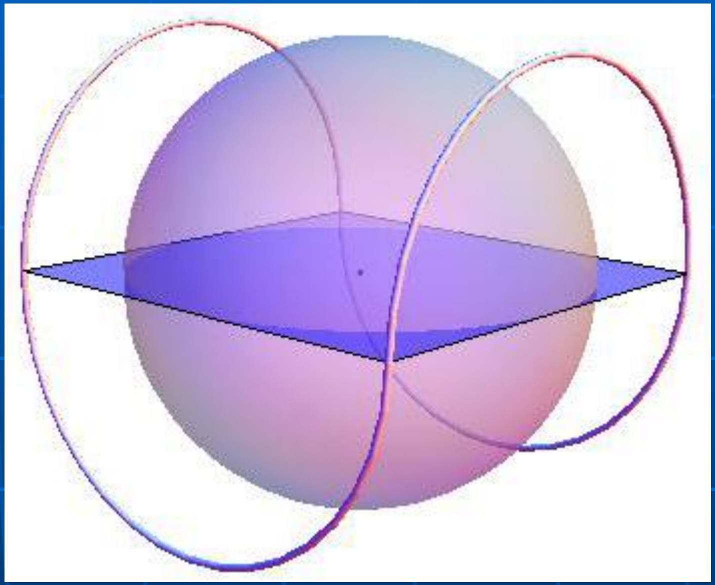
$$L = \pi(2 + \sqrt{2}) \approx 10.726$$

SHORTEST INSPECTION CURVES FOR THE SPHERE

V. A. Zalgaller* *Journal of Mathematical Sciences, Vol. 131, No. 1, 2005*

External Watchman Cycle for a Sphere

Shortest **Cycle** ?



"Shortest Inspection Curves for the Sphere"
V. A. Zalgaller

"baseball stitch curve"

[discussions: Jin-ichi Itoh, Joe O'Rourke, Anton Petrunin, Y. Tanoue, Costin Vilcu]

108 double stitches