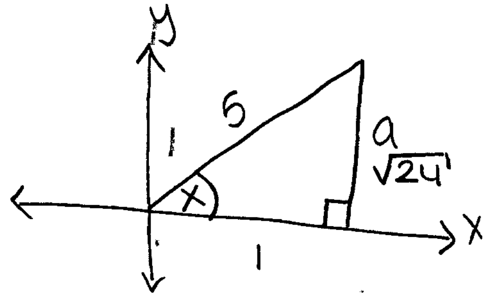
Lecture #23
MAT 123

# Angle sum, double angle, & half-angle formulas

**Recap from last lecture:**

$\sin(\cos^{-1}(\frac{1}{5})) = ?$

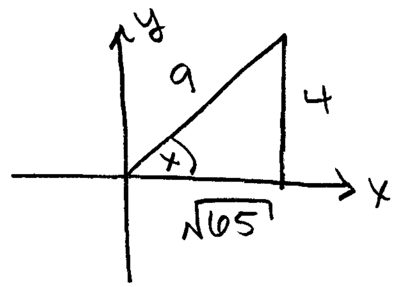
$\sin x = \boxed{\frac{\sqrt{24}}{5}}$




$5^2 = a^2 + 1^2$
$24 = a^2$
$a = \sqrt{24}$

$\tan(\sin^{-1}(\frac{4}{9})) = ?$

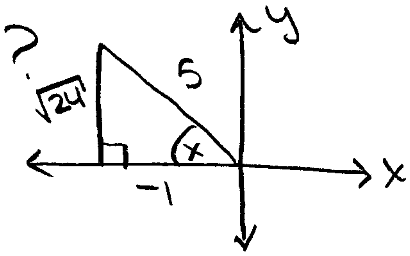
$\tan x = \boxed{\frac{4}{\sqrt{65}}}$




$9^2 = 4^2 + x^2$
$x = \sqrt{65}$

$\sin(\cos^{-1}(-\frac{1}{5})) = ?$

$\sin x = \boxed{\frac{\sqrt{24}}{5}}$

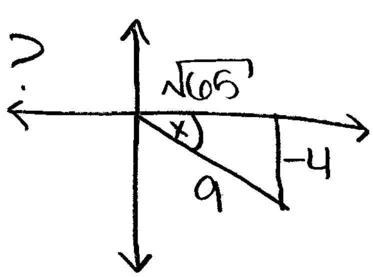


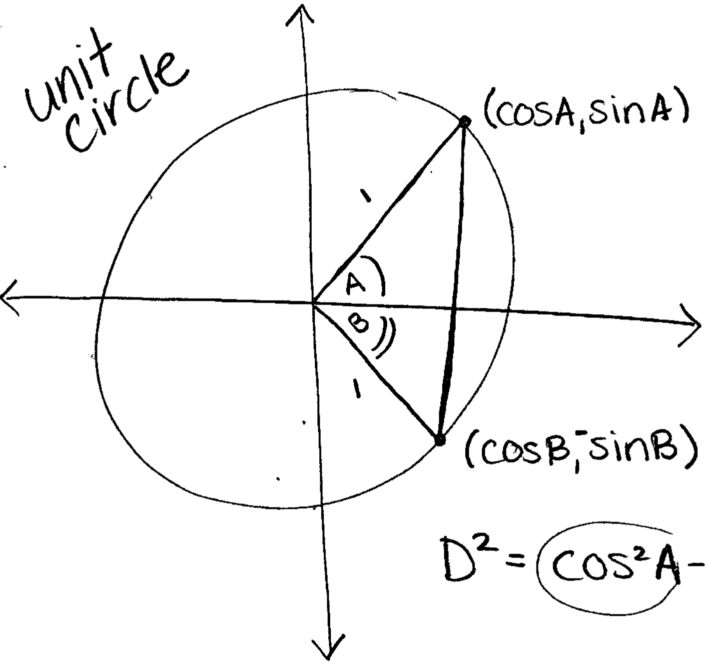

$\tan(\cos^{-1}(-\frac{1}{5})) = ?$ same picture

$\tan(x) = -\frac{\sqrt{24}}{1} = \boxed{-\sqrt{24}}$

$\tan(\sin^{-1}(\frac{-4}{9})) = ?$

$\tan x = \boxed{\frac{-4}{\sqrt{65}}}$






Find the distance between the two points.

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$D = \sqrt{(\cos A - \cos B)^2 + (\sin A + \sin B)^2}$$

$$D^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$

$$D^2 = \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A + 2\sin A\sin B + \sin^2 B$$

note: $\sin^2 A + \cos^2 A = 1$
$\sin^2 B + \cos^2 B = 1$ $>$ $1+1=2$

$$\boxed{D^2 = 2 - 2\cos A\cos B + 2\sin A\sin B}$$

Now use law of cosines to find D.

$$D^2 = 1^2 + 1^2 - 2(1)(1)\cos(A+B)$$

$$\boxed{D^2 = 2 - 2\cdot\cos(A+B)}$$

$$\cancel{2} - 2\cos A\cos B + 2\sin A\sin B = \cancel{2} - 2\cos(A+B)$$

$$\frac{-2\cos A\cos B + 2\sin A\sin B}{-2} = \frac{-\cancel{2}\cos(A+B)}{\cancel{-2}}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

Problem:

① $\cos 75° = ?$

$75 = 45 + 30$

$\cos 75° = \cos(45+30)$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(45+30) = \cos 45\cos 30 - \sin 45\sin 30$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\right) = \frac{\sqrt{2}\cdot\sqrt{3}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\boxed{\cos 75° = \frac{\sqrt{6}-\sqrt{2}}{4}}$$

$\cos(15°) = ?$

$15 = 45 - 30$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

- $\cos(15°) = \cos(45° - 30°)$

$$\cos(45° - 30°) = \cos 45°\cos 30° + \sin 45°\sin 30°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\boxed{\cos(15°) = \frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

ex: $\sin 75° = ?$

- $\sin 75° = \sin(45° + 30°)$

$$\sin(45° + 30°) = \sin 45°\cos 30° + \cos 45°\sin 30°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\boxed{\sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}}$$

ex:

$$\sin 15° = \sin(45° - 30°) = \sin 45°\cos 30° - \cos 45°\sin 30°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$\boxed{\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}}$$

ex: $\sin(105°) = \sin(60° + 45°) = \sin 60°\cos 45° + \cos 60°\sin 45°$

- $= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$$\boxed{\sin(105°) = \frac{\sqrt{6}+\sqrt{2}}{4}}$$

ex: $\cos(105°) = \cos(45° + 60°) = \cos 45° \cos 60° - \sin 45° \sin 60°$
$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$
$= \frac{\sqrt{2} - \sqrt{6}}{4}$

$$\boxed{\cos(105°) = \frac{\sqrt{2} - \sqrt{6}}{4}}$$

What about when we are dealing with radians?

$\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$

$\frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$

$\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

if you don't like radians, just convert to degrees!

what about $\tan(A+B)$?

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

Double Angle Formulas:

$\sin(2A) = \sin(A+A) = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A$

$$\sin(2A) = 2 \sin A \cos A$$

$\cos(2A) = \cos(A+A) = \cos A \cdot \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

these are all equivalent!

$\cos(2A) = \cos^2 A - \sin^2 A$
$\cos^2 A = 1 - \sin^2 A$
$= 1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A$

$\cos(2A) = \cos^2 A - \sin^2 A$
$\sin^2 A = 1 - \cos^2 A$
$= \cos^2 A - (1 - \cos^2 A) = \cos^2 A - 1 + \cos^2 A = 2\cos^2 A - 1$

# Half-Angle Formulas

$$\cos(2A) = 2\cos^2 A - 1$$

$$\frac{1+\cos(2A)}{2} = \frac{2\cos^2 A}{2}$$

$$\sqrt{\frac{1+\cos(2A)}{2}} = \sqrt{\cos^2 A}$$

$$\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}} \implies \cos\left(\tfrac{1}{2}A\right) = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\cos(2A) = 1 - 2\sin^2 A$$

$$\frac{\cancel{2}\sin^2 A}{\cancel{2}} = \frac{1-\cos(2A)}{2}$$

$$\sqrt{\sin^2 A} = \sqrt{\frac{1-\cos(2A)}{2}}$$

$$\sin A = \pm\sqrt{\frac{1-\cos(2A)}{2}} \implies \sin\left(\tfrac{1}{2}A\right) = \pm\sqrt{\frac{1-\cos A}{2}}$$

ex: $\sin\left(\frac{\pi}{8}\right) = ?$

$$\sin\left(\frac{\pi}{8}\right) = \sin\left(\frac{\frac{\pi}{4}}{2}\right) = \sin\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) = +\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}$$

$$= +\sqrt{\frac{1-\left(\frac{\sqrt{2}}{2}\right)}{2}} = \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}}$$

$$\boxed{\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{2-\sqrt{2}}{4}}}$$

ex: $\cos\left(\frac{5\pi}{8}\right) = \cos\left(\frac{1}{2}\cdot\frac{5\pi}{4}\right) = -\sqrt{\frac{1+\cos\left(\frac{5\pi}{4}\right)}{2}}$

$\frac{5\pi}{4} = 225°$

$$= -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$\frac{5\pi}{8} = 112.5°$

$$\cos\left(\frac{5\pi}{8}\right) = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

*note since 225° is in Quad II we take the $-\sqrt{\phantom{x}}$ since cos is (−) in Quad II

# Summary of Formulas

Sum/Difference Formulas:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Double Angle Formulas:

$$\sin(2A) = 2\sin A \cos A$$
$$\cos(2A) = \cos^2 A - \sin^2 A$$
$$= 1 - 2\sin^2 A$$
$$= 2\cos^2 A - 1$$

Half Angle Formulas:

$$\sin(A/2) = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos(A/2) = \pm\sqrt{\frac{1+\cos A}{2}}$$

(*) Remember: $\tan X = \dfrac{\sin X}{\cos X}$