Lecture #17
MAT 123

Exponential Growth & Decay

Recall:

Date	Deposits
Jan 1	.01
Jan 2	.03 = (.01)(3)
Jan 3	.09 = (.01)(3)(3) = (.01)(3^2)
Jan 4	.27 = (.01)(3^3)
⋮	
Jan 10	.01 (3^9) "10^{th} Day"

← exponential!

"Half life"   Radio Active Elements Decay
"Radio Active Decay"

↑ How long it takes to have half of what you had before

ex: The half life of an element is 5 days. What percent is remaining after 20 days?

Day 0 → 100 grams
Day 5 → 50 grams
Day 10 → 25 grams
Day 15 → 12.5 grams
Day 20 → 6.25 grams

notice since it said % we use 100 grams.

$$\frac{6.25\text{grams}}{100\text{grams}} = 6.25\%$$

$$A = 100\left(\frac{1}{2}\right)^{\frac{t}{5}}$$ ← this formulas will tell us how much is left at any time t

General Formula: $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{k}}$

K = half-life
A_0 = starting amount
$A(t)$ = amount at time t

ex: The half life of an element is 6 hours. If there are 400gms originally, how much will there be after 10 hours?

$A(10) = ?$
$A_0 = 400$
$t = 10$
$K = 6$

$$\underbrace{A(10) = 400\left(\tfrac{1}{2}\right)^{\frac{10}{6}}} = 125.99\text{gms}$$

↑
acceptable answer on an exam b/c you will not have a calculator.

ex: If originally you have 180gms of an element, and after 6 hours you have 40 gms, how many will you have after 20 hours?

2 step problem ⟹ ① figure out the decay rate
② solve for what they ask for

$$\boxed{y = a \cdot b^x}$$ general exponential formula

a = initial amount
b = growth/decay rate

use info they tell us to get these two points < (0, 180)
(6, 40)

$180 = a \cdot \cancel{b^0}$
$a = 180$

$$\frac{40}{180} = \frac{\cancel{180}b^6}{\cancel{180}}$$

$\left(\frac{2}{9}\right)^{1/6} = \left(b^6\right)^{1/6}$

$b = \left(\frac{2}{9}\right)^{1/6}$

↑ step 1

$y = 180\left(\left(\frac{2}{9}\right)^{1/6}\right)^x$

$y = 180\left(\frac{2}{9}\right)^{\frac{x}{6}}$ ← step 2

$y = 180\left(\frac{2}{9}\right)^{\frac{20}{6}}$

ex: If you have 180 grams of an element and 10 hrs later you have 60 grams. How many will you have after 30 hrs?

$(0, 180) \rightarrow$

$(10, 60)$

$y = ab^x$

$180 = a \cdot b^0$

$180 = a(1)$

$a = 180$

$y = 180b^x$

$\frac{60}{180} = \frac{\cancel{180}b^{10}}{\cancel{180}}$

$\left(\frac{1}{3}\right)^{1/10} = \left(b^{10}\right)^{1/10}$

$b = \left(\frac{1}{3}\right)^{1/10}$

notice you need to find "b" before you can solve the given problem!

$y = 180\left(\frac{1}{3}\right)^{\frac{x}{10}}$

$y = 180\left(\frac{1}{3}\right)^{\frac{30}{10}} = 180\left(\frac{1}{3}\right)^3$

ex: If originally you have 24 gms of an element, and 2 hours later you have 20 gms, how much will you have after 7 hours?

$(0, 24)$

$(2, 20)$

$y = ab^x$

$24 = ab^0$

$a = 24$

$y = 24b^x$

$\frac{20}{24} = \frac{\cancel{24}b^2}{\cancel{24}}$

$\left(\frac{5}{6}\right)^{1/2} = \left(b^2\right)^{1/2}$

$b = \left(\frac{5}{6}\right)^{1/2}$

* Find b first then solve!

Now we know: $y = (24)\left(\frac{5}{6}\right)^{\frac{x}{2}}$

x = hours or time

$y = (24)\left(\frac{5}{6}\right)^{\frac{x}{2}}$

ex: If you originally have 60gms of an element and 3 hrs later you have 50 gms, what is the half-life of the element?

↳ "how long does it take to be left with $\frac{60}{2} = 30$gms"

$y = ab^x$

$a = 60 \rightarrow$ initial amount

(0, 60)
(3, 50)
(x, 30)

$$\frac{50}{60} = \frac{\cancel{60}(b)^3}{\cancel{60}}$$

$$\left(\frac{5}{6}\right)^{1/3} = b$$

$$y = 60\left(\frac{5}{6}\right)^{\frac{x}{3}}$$

$$\frac{30}{60} = \frac{\cancel{60}\left(\frac{5}{6}\right)^{\frac{x}{3}}}{\cancel{60}}$$

✱ notice: $\frac{1}{2}$ appears when we are looking for half-life

$$\frac{1}{2} = \left(\frac{5}{6}\right)^{\frac{x}{3}}$$

$$\frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{5}{6}\right)} = \frac{\frac{x}{3} \cdot \cancel{\log\left(\frac{5}{6}\right)}}{\cancel{\log\left(\frac{5}{6}\right)}}$$

$$3 \cdot \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{5}{6}\right)} = \frac{x}{\cancel{3}} \cdot \cancel{3}$$

$$x = 3 \cdot \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{5}{6}\right)}$$

ex: Originally you have 20 gms of an element and 8 hrs later you have 16gms, what is the half-life?

a.k.a. How long does it take to be left with 10 gms?

$y = a \cdot b^x \qquad a = 20$

$\frac{16}{20} = \frac{\cancel{20} b^8}{\cancel{20}}$

$\left(\frac{4}{5}\right)^{1/8} = (b^8)^{1/8}$

$b = \left(\frac{4}{5}\right)^{1/8}$

$y = 20\left(\frac{4}{5}\right)^{\frac{x}{8}}$

$\frac{10}{20} = \frac{\cancel{20}\left(\frac{4}{5}\right)^{\frac{x}{8}}}{\cancel{20}}$

$\frac{1}{2} = \left(\frac{4}{5}\right)^{\frac{x}{8}}$ ⟸ notice $\frac{1}{2}$ appears!

$\frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{4}{5}\right)} = \frac{\frac{x}{8} \cdot \cancel{\log\left(\frac{4}{5}\right)}}{\cancel{\log\left(\frac{4}{5}\right)}}$

$8 \cdot \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{4}{5}\right)} = \frac{x}{\cancel{8}} \cdot \cancel{8}$

$x = 8 \cdot \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{4}{5}\right)}$

(*) when you have your variable in the power you must use logarithms to solve!

when time is your unknown you will need to use logs to solve.

Log Laws:

$\log(AB) = \log A + \log B$

$\log\left(\frac{A}{B}\right) = \log A - \log B$

we used this in all the previous problems → $\log A^B = B \cdot \log A$

problem: Expand the following:

① $\log(x^2y^3) = \log x^2 + \log y^3$

$= 2 \cdot \log x + 3 \cdot \log y$

② $\log \frac{x^3}{y^5} = \log x^3 - \log y^5$

$= 3 \cdot \log x - 5 \cdot \log y$

③ $\log\left(\frac{x^4\sqrt{y}}{z^2}\right) = \log x^4\sqrt{y} - \log z^2$

$= \log x^4 + \log \sqrt{y} - \log z^2$

$= 4 \cdot \log x + \frac{1}{2} \log y - 2 \log z$

* $\boxed{\sqrt{x} = x^{\frac{1}{2}}}$ *

<u>Problem</u>: Write the following as a single log:

① $3 \log x + 2 \log y - 4 \log z$

$= \log x^3 + \log y^2 - \log z^4$

$= \log\left(\frac{x^3 y^2}{z^4}\right)$

② $4 \log x - 2 \log y + 3 \log z$

$= 4 \log x + 3 \log z - 2 \log y$

$= \log x^4 + \log z^3 - \log y^2$

$= \log\left(\frac{x^4 z^3}{y^2}\right)$

*Be careful!
Rewrite with all "+" first then "–"
This will help you not mess up what goes in the numerator and what goes in the denominator.