Lecture 30

Parabolas

Quadratic functions

A **quadratic function** is a function $y = ax^2 + bx + c$, where a, b, c are given numbers and $a \neq 0$.

Examples of quadratic functions:

$$y = x^2$$
$$y = x^2 + x$$
$$y = -3x^2 + 2x - 5$$
$$y = \frac{1}{3}x^2 - \sqrt{2}\,x + 1$$

Functions and, in particular, quadratic functions, are studied in the precalculus and calculus courses.

In this lecture we will learn how to draw the **graph** of a quadratic function.

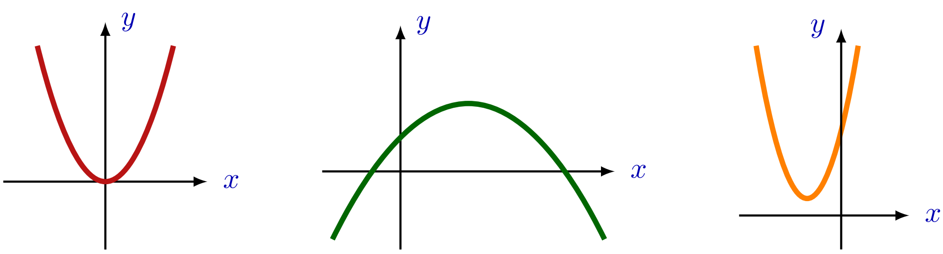
The graph of a function provides a **visualization** of various properties of the function, and helps to understand these properties.

What is the graph

The **graph** of a quadratic function $y = ax^2 + bx + c$ is the set of all points on the plane whose coordinates (x, y) satisfy the equation $y = ax^2 + bx + c$.

The graph of a quadratic function is a plane **curve**, it is called a **parabola**.

Here are a few examples of parabolas:

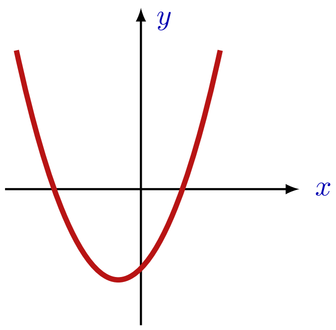


In this lecture, we will learn how to draw a parabola by its equation.

Geometry of a parabola

Any parabola has certain geometric elements which are common for all parabolas.

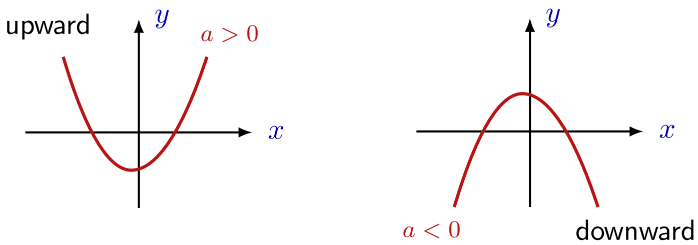
Let us have a look on a typical parabola:



Which geometric elements do we observe on this parabola?

Horns: upward or downward

A parabola has its “horns” turned **upward** or **downward.** (A parabola opens upward or downward.)

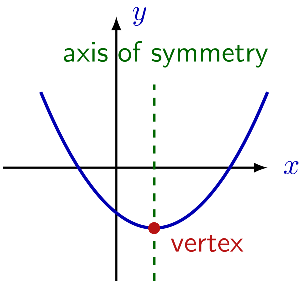


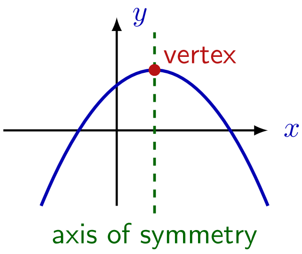
It is the coefficient a (called the **leading coefficient**) which is responsible for this.

- If $a > 0$, then the parabola opens **upward**
- If $a < 0$, then the parabola opens **downward**

Vertex and axis of symmetry

There is a characteristic point on a parabola, where the parabola makes a **turn**.



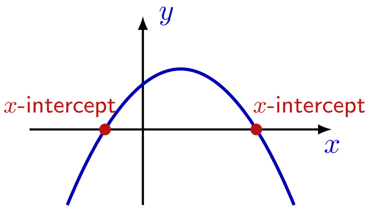


This point is called the **vertex**.
The vertex is the **lowest** point on the parabola if $a > 0$, and the **highest** point if $a < 0$.

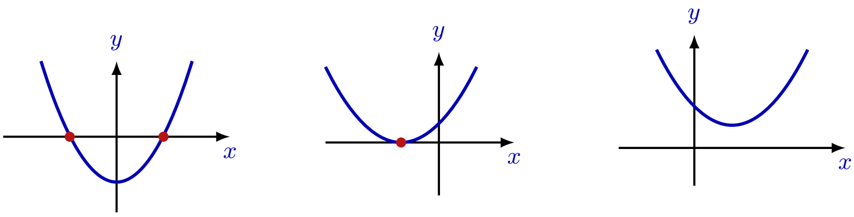
A vertical line passing through the vertex is called the **axis of symmetry**,
because a parabola is symmetric about its axis of symmetry.

The x-intercepts

The points where the parabola intersects the x-axis, are called the **x-intercepts**.

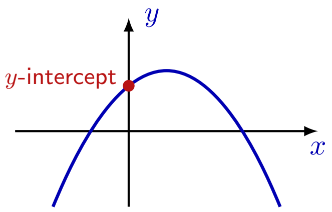


A parabola may have **two**, **one**, or **no** **x-intercepts**.



The y-intercept

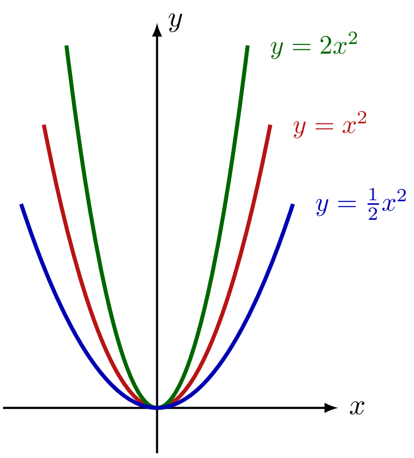
A point where the parabola intersects the y-axis is called the **y-intercept**.



Each parabola has exactly **one** y-intercept.

Wide or narrow?

Some parabolas are wider than others:



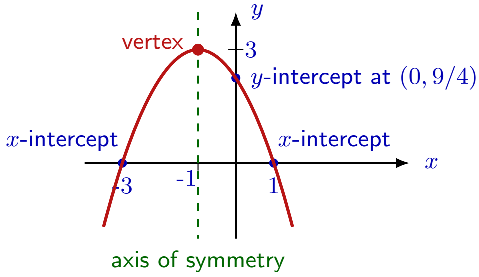
$|a|$ is responsible
for the **width** of the parabola
.
The smaller $|a|$,
the wider the parabola.

What do we need to sketch a parabola?

- the vertex
- the axis of symmetry
- the sign of a (upward or downward)
- the y-intercept
- the x-intercepts (if any)

Example. Sketch a parabola which opens downward, has the vertex at $(-1, 3)$, the y-intercept at $(0, 9/4)$, and the x-intercepts at $(-3, 0)$ and $(1, 0)$.

Solution.



How to find the vertex

The vertex of the parabola $y = ax^2 + bx + c$ is located at the point with coordinates $\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$.

Why is this so? Rewrite the equation of the parabola using **completing the square**:

$$y = ax^2 + bx + c \iff y = a\left(x + \frac{b}{2a}\right)^2 + \left(-\frac{b^2}{4a} + c\right)$$

If $a > 0$, then the vertex is located at the **lowest** point on the parabola, that is at the point, where y takes the **minimal** value.

Since $a\left(x + \frac{b}{2a}\right)^2 \geq 0$ for all x, the minimal value of $y = a\left(x + \frac{b}{2a}\right)^2 + \left(-\frac{b^2}{4a} + c\right)$ occurs exactly when $\left(x + \frac{b}{2a}\right)^2 = 0$, that is when $x = -\frac{b}{2a}$.

Therefore, the vertex is located at $\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$.

How to find the vertex and the axis of symmetry

If $a < 0$, then the vertex is located at the **highest** point on the parabola,
that is at the point, where y takes the **maximal** value.

Since $a\left(x+\frac{b}{2a}\right)^2 \le 0$ for all x, the maximal value of $y = a\left(x+\frac{b}{2a}\right)^2 + \left(-\frac{b^2}{4a}+c\right)$ occurs exactly when $\left(x+\frac{b}{2a}\right)^2 = 0$, that is when $x = -\frac{b}{2a}$.

Therefore, the vertex is located at $\left(-\frac{b}{2a}, -\frac{b^2}{4a}+c\right)$.
Remember that

> The **vertex** of the parabola $y = ax^2+bx+c$
>
> is located at the point where $x = \frac{-b}{2a}$.

The **axis of symmetry** is the vertical line passing through the vertex.

Its equation is $x = -\frac{b}{2a}$

How to find the vertex and the axis of symmetry

Example. Find the vertex and the axis of symmetry of the parabola $y = x^2 - 4x + 1$.

Solution. The x-coordinate of the vertex is

$$x = \frac{-b}{2a} = \frac{-(-4)}{2\cdot 1} = \frac{4}{2} = 2.$$

To find the y-coordinate of the vertex, we plug in $x = 2$ into the equation of the parabola:

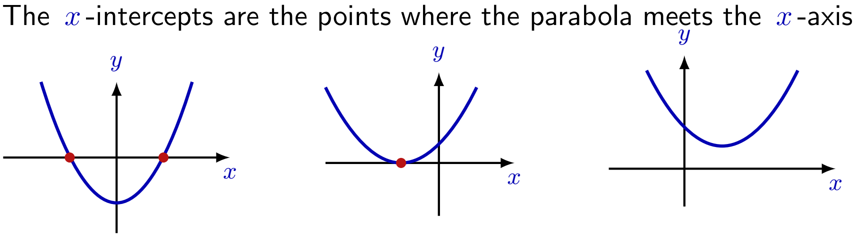
$$y = 2^2 - 4\cdot 2 + 1 = 4 - 8 + 1 = -3.$$

Therefore, the **vertex** of the parabola is at the point with coordinates $(2, -3)$.

The **axis of symmetry** is the vertical line $x = 2$.

How to find the x-intercepts

The x-intercepts are the points where the parabola meets the x-axis.



A parabola may have two, one or no x-intercepts.
At x-intercept, the y-value is equal to 0. Therefore,

> To find the x-intercepts of the parabola $y = ax^2 + bx + c$,
> solve the equation $ax^2 + bx + c = 0$.

If the quadratic equation $ax^2 + bx + c = 0$ has **two** roots,
then the parabola intersects the x-axis at **two** points.

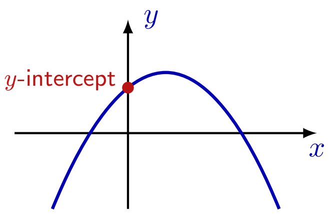
If the equation has **one** root, then the parabola touches the x-axis at **one** point.

If the equation has **no** roots, then the parabola does **not** intersect the x-axis.

14 / 24

How to find the y-intercept

The y-intercept is easy to find.



This is the point where the parabola intersects the y axis.

At this point, the x-coordinate equals 0.

When we plug $x = 0$ into the equation of the parabola $ax^2 + bx + c$, we get

$y = a \cdot 0^2 + b \cdot 0 + c = c$.

Therefore,

> The y-intercept of the parabola $y = ax^2 + bx + c$
> is located at the point $(0, c)$

15 / 24

Step-by-step instruction for drawing a parabola

To draw the parabola $y = ax^2 + bx + c$,

- Determine the **vertex**. It's located at the point where $x = \dfrac{-b}{2a}$.
- Draw the **axis of symmetry**. It's the vertical line $x = \dfrac{-b}{2a}$.
- Determine if the parabola opens **upward** ($a > 0$) or **downward** ($a < 0$).
- Determine the ***y*-intercept**. It's located at the point $(0, c)$.
- Determine the ***x*-intercepts** (if any). They are located at the points $(x_{1,2}, 0)$,

  where $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Draw the parabola, using the information above.
  Make sure that your parabola is smooth and symmetric.

Example 1

Example 1. For the parabola $y = x^2 - x - 2$, determine the vertex, the axis of symmetry, the intercepts, and draw the graph.

Solution.

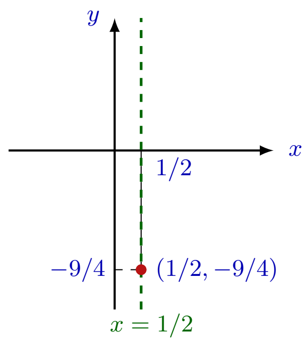
- The **vertex** is at $x = \dfrac{-b}{2a} = \dfrac{-(-1)}{2} = \dfrac{1}{2}$. The y-coordinate of the vertex is

$y = \left(\dfrac{1}{2}\right)^2 - \dfrac{1}{2} - 2 = \dfrac{1}{4} - \dfrac{1}{2} - 2 = -9/4$. So the vertex is located at $(1/2, -9/4)$.

Draw the vertex.

- The **axis of symmetry** is the vertical line $x = 1/2$.

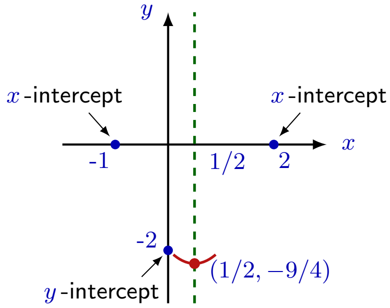
Draw the axis of symmetry.



Example 1

- $a = 1 > 0$, therefore, the parabola opens **upward**.

Draw a small **sprout** of a parabola at the vertex.



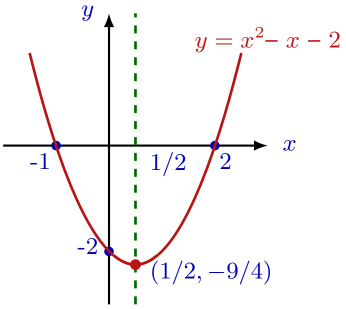
- The $\boldsymbol{y}$**-intercept** is at $(0, c) = (0, -2)$.
- The $\boldsymbol{x}$**-intercepts** are the roots of $x^2 - x - 2 = 0$.

$x^2 - x - 2 = 0 \iff (x+1)(x-2) = 0 \iff x = -1,\ x = 2$.

So the x-intercepts are $(-1, 0)$ and $(2, 0)$.

Example 1

Now we are ready to draw the parabola:



Be neat: the parabola should be smooth and symmetric.

Example 2.

Example 2. For the parabola $y = -x^2 - 2x - 2$, determine the vertex, the axis of symmetry, the intercepts, and draw the graph.

Solution. The vertex is at $x = \frac{-b}{2a} = \frac{-(-2)}{2 \cdot (-1)} = -1$.

The y-coordinate of the vertex is $y = -(-1)^2 - 2 \cdot (-1) - 2 = -1 + 2 - 2 = -1$. By this, the vertex is $\boxed{(-1, -1)}$.

The axis of symmetry is $\boxed{x = -1}$.

$a = -1 < 0$, so the parabola opens $\boxed{\text{downward}}$.

The y-intercept is $(0, c) = (0, -2)$.

For the x-intercepts, solve the equation $-x^2 - 2x - 2 = 0$:

$$-x^2 - 2x - 2 = 0 \iff x^2 + 2x + 2 = 0.$$

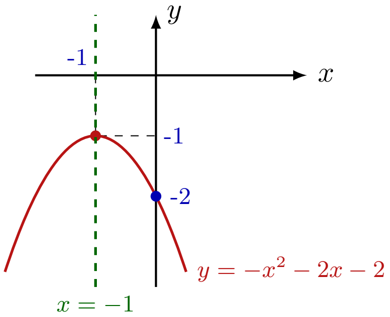
The discriminant is $b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot 2 = -4 < 0$.

Therefore, there are no solutions, and the parabola doesn't meet the x-axis.

20 / 24

Example 2.

Now put all the information on the graph.



21 / 24

The graph of a quadratic monomial

What do we know about the graph of the parabola $y = ax^2$?

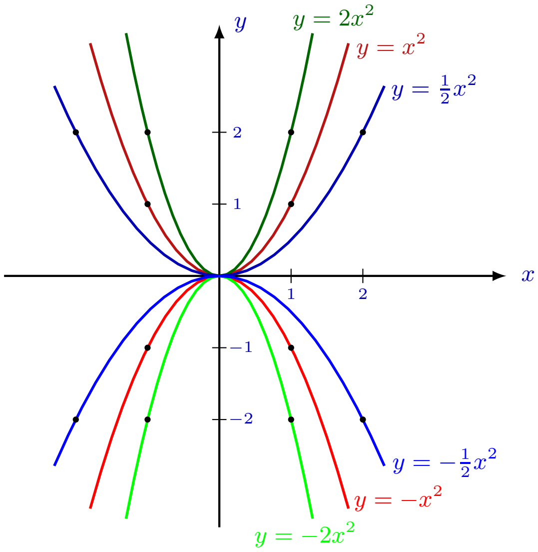
- The vertex at the origin $(0, 0)$, since $\dfrac{-b}{2a} = 0$.
- The axis of symmetry is the line $x = 0$, that is, the y-axis.
- The parabola opens upward if $a > 0$, and downward if $a < 0$.
- The y-intercept is $(0, 0)$.
- The only x-intercept is $(0, 0)$.

This information is not sufficient for a drawing.
We may need to plot a support point, say, $(x, y) = (1, a)$ belonging to the parabola.

By symmetry, we get another point $(x, y) = (-1, a)$ on the parabola.

The graph of a quadratic monomial

Let us draw several parabolas $y = ax^2$ with different coefficients a.



Summary

In this lecture, we have learned

- ☑ what the **graph** of a quadratic function is
- ☑ what a **parabola** looks like
- ☑ what the essential **geometric elements** of the parabola are (vertex, axis of symmetry, intercepts)
- ☑ when a parabola opens **upward** ($a > 0$) or **downward** ($a < 0$)
- ☑ how to find the **vertex** and the **axis of symmetry** of a parabola
- ☑ how to find the x-**intercepts** (if any) and the y-**intercept** of a parabola
- ☑ how to **draw** the parabola from its equation
- ☑ how to draw the graph of a **quadratic monomial**