Lecture 28

# Factoring Quadratic Polynomials


Factoring polynomials . . . 2
Irreducible polynomials . . . 3
Factoring quadratic binomials . . . 4
Factorization theorem for quadratic trinomials . . . 5
Proving factorization formula . . . 6
Factoring by finding roots . . . 7
Factoring by finding roots . . . 8
Factoring by finding roots . . . 9
Factoring by finding roots . . . 10
Vieta's theorem . . . 11
Vieta's theorem for finding roots . . . 12
Solving quadratic equations by factoring . . . 13
Summary . . . 14

## Factoring polynomials

To **factor** a polynomial means to present this polynomial as a product of polynomials of degree **less** than the original polynomial.

For example, $x^2 - 1 = (x-1)(x+1)$ is a factoring, but
$3x^2 + 3 = 3(x^2+1)$ is **not** a polinomial factoring,
since the degree of $x^2 + 1$ is **not** less than the degree of $3x^2 + 3$.

Factoring is an important algebraic tool that helps to solve various problems.

The same polynomial can be factored in different ways.

For example, $x^3 - x$ can factored as $x(x^2 - 1)$ or as $x(x-1)(x+1)$ or as $2x(x-1)\left(\frac{1}{2}x + \frac{1}{2}\right)$

**Monomials**, that is, polynomials of type $ax^n$, are easy to factor.

For example, $4x^3 = 4x^2 \cdot x$ or $4x^3 = 4x \cdot x \cdot x$.

In this lecture we will learn how to factor quadratic **binomials** and **trinomials**.

2 / 14

## Irreducible polynomials

If a polynomial can't be factored, it is called **irreducible**.

Polynomials of degree one are **irreducible**, they can't be factored:
we can't present a polynomial of degree one as a product of polynomials of degrees **less** than one.

Some polynomials are easy to factor: $x^2 - 4 = x^2 - 2^2 = (x-2)(x+2)$.
The factors, $x - 2$ and $x + 2$, contain only **integer** coefficients.
Such factoring is called factoring over the integers.

Consider another factoring: $x^2 - 3 = x^2 - (\sqrt{3})^2 = (x - \sqrt{3})(x + \sqrt{3})$.

Here the factors, $x - \sqrt{3}$ and $x + \sqrt{3}$, have **real coefficients**,
Such factoring is called factoring over the reals.

The polynomial $x^2 - 3$ can't be factored over the integers. It is **irreducible** over the integers.

The polynomial $x^2 + 1$ is **irreducible** over the reals,
but can be factored over the complex numbers: $x^2 - 1 = (x - i)(x + i)$.

3 / 14

## Factoring quadratic binomials

Quadratic binomials are expressions of type $ax^2 + bx$ or $ax^2 + c$, they are special types of quadratic polynomials.

It's easy to factor the binomial $ax^2 + bx$: $ax^2 + bx = x(ax + b)$

The binomial $ax^2 + c$ can be factored over the reals only if the coefficients $a$ and $c$ have opposite signs.

If $a$ and $c$ are of the same sign (both positive or both negative) then $ax^2 + c$ is **irreducible**.

**Example.** Factor the following polynomials: $9x^2 - 4$, $9x^2 + 4$.

**Solution.** $9x^2 - 4 = (3x)^2 - 2^2 = (3x - 2)(3x + 2)$.

The polynomial $9x^2 + 4$ is **irreducible**.

For the rest of the course, we will say that a polynomial is irreducible, if it is irreducible over the reals.



## Factorization theorem for quadratic trinomials

**Theorem.** Let $ax^2 + bx + c$ be a quadratic polynomial with non-negative discriminant, that is, $b^2 - 4ac \geq 0$.

Then

$$\boxed{ax^2 + bx + c = a(x - x_1)(x - x_2)}$$

where $x_1$, $x_2$ are the **roots** of the polynomial, that is, the solutions of the equation $ax^2 + bx + c = 0$.

**Remarks.**

**1.** If the discriminant is $0$, then $x_1 = x_2$ is the only root of the equation, and the factoring is

$$ax^2 + bx + c = a(x - x_1)(x - x_1) = a(x - x_1)^2.$$

**2.** Factoring is simple when $a = 1$:

$$x^2 + bx + c = (x - x_1)(x - x_2).$$

**3.** If the discriminant is negative, then the polynomial is **irreducible**.

## Proving factorization formula

By **completing the square**,

$$ax^2+bx+c=a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right)=$$

$$a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right)=$$

$$a\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)=$$

$$a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=a(x-x_1)(x-x_2)\,,$$

as required.



## Factoring by finding roots

**Example 1.** Factor $2x^2-x-1$.

**Solution.** By the factoring theorem,

$2x^2-x-1=2(x-x_1)(x-x_2)$, where

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-1)\pm\sqrt{(-1)^2-4\cdot 2\cdot(-1)}}{2\cdot 2}$$

$$=\frac{1\pm\sqrt{1+8}}{4}=\frac{1\pm\sqrt{9}}{4}=\frac{1\pm 3}{4}$$

So $x_1=\frac{1+3}{4}=1$ and $x_2=\frac{1-3}{4}=-\frac{1}{2}$.

The factoring is

$$2x^2-x-1=2(x-1)\left(x-\left(-\frac{1}{2}\right)\right)=2(x-1)\left(x+\frac{1}{2}\right)=(x-1)(2x+1)\,.$$

## Factoring by finding roots

**Example 2.** Factor $x^2 - x - 1$ .

**Solution.** By the factoring theorem,

$x^2 - x - 1 = (x - x_1)(x - x_2)$ , where

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} =$$

$$= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

So $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$ .

The factoring is

$$x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right).$$



## Factoring by finding roots

**Example 3.** Factor $x^2 - 4x + 4$ .

**Solution.** By the factoring theorem,

$x^2 - 4x + 4 = (x - x_1)(x - x_2)$ , where

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} =$$

$$= \frac{4 \pm \sqrt{16-16}}{2} = \frac{4 \pm 0}{2} = 2$$

So $x_1 = x_2 = 2$ .

The factoring is $x^2 - 4x + 4 = (x-2)(x-2) = (x-2)^2$ .

**Remark.** If you recognize a perfect square trinomial formula in the expression $x^2 - 4x + 4$ , then the factoring can be achieved faster:

$$x^2 - 4x + 4 = x^2 - 2 \cdot x \cdot 2 + (2)^2 = (x-2)^2 .$$

## Factoring by finding roots

**Example 4.** Factor $3x^2 - x + 1$.

**Solution.** The **discriminant** is

$$b^2 - 4ac = (-1)^2 - 4 \cdot 3 \cdot 1 = 1 - 12 = -11 < 0,$$

therefore, the polynomial has **no** roots and is **irreducible**.



## Vieta's theorem

**Theorem.** If $x_1$, $x_2$ are the roots of the equation $ax^2 + bx + c = 0$,

then $x_1 + x_2 = -\frac{b}{a}$ and $x_1 \cdot x_2 = \frac{c}{a}$.

**Proof.** By the Factorization theorem,

$ax^2 + bx + c = a(x - x_1)(x - x_2)$.

Let us expand RHS of this identity:

$a(x - x_1)(x - x_2) = a(x^2 - x_1 x - x_2 x + x_1 x_2) = ax^2 - a(x_1 + x_2)x + ax_1 x_2$.

Therefore, $ax^2 + \mathbf{b}x + \mathbf{c} = ax^2 \mathbf{-a(x_1 + x_2)}\, x + \mathbf{ax_1x_2}$.

By comparison of the coefficients of these two polynomials, we get

$\mathbf{b = -a(x_1 + x_2)}$ and $\mathbf{c = ax_1x_2}$. From this,

$x_1 + x_2 = -\frac{b}{a}$ and $x_1 x_2 = \frac{c}{a}$, as required.

Vieta's theorem relates the **roots** and the **coefficients** of a quadratic equation.

## Vieta's theorem for finding roots

Vieta's theorem is especially simple if $a = 1$. In this case,

- the roots $x_1$, $x_2$ of the equation $x^2 + bx + c = 0$ satisfy

$$x_1 + x_2 = -b \text{ and } x_1 x_2 = c.$$

Vieta's theorem may be used for finding the roots of a quadratic equation, provided that the coefficients of the equation and the roots are **integers**.

**Example.** Solve the equation $x^2 + x - 6 = 0$.
**Solution.** If $x_1$ and $x_2$ are the roots of $x^2 + x - 6 = 0$, then
$x_1 + x_2 = -b = -1$ and $x_1 x_2 = c = -6$.
Let us guess two numbers, whose sum equals $-1$ and the product equals $-6$. The numbers are $2$ and $-3$. **Answer.** $x = 2$ or $x = -3$.
⚠ **Warning.** Guessing out the roots may be **not** a good idea.
It may happen that the equation has **irrational** roots or **no roots** at all.
Although Vieta's theorem is valid, it can't be used to find the roots in these cases.

Don't waste your time guessing!



## Solving quadratic equations by factoring

**Example.** Solve the equation $x^2 - 2x - 15 = 0$.

**Solution.** For this equation, $a = 1,\ b = -2,\ c = -15$.

The discriminant of the equation is $b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-15) = 64$, which a perfect square.

It means that the roots are **rational** numbers, and we may guess them out.

The factoring is

$x^2 - 2x - 15 = (x - \ ?\ )(x - \ ?\ )$

By guessing, we get

$x^2 - 2x - 15 = (x - 5)(x + 3)$.

So $x^2 - 2x - 15 = 0 \iff (x - 5)(x + 3) = 0$

$$\iff x - 5 = 0 \text{ or } x + 3 = 0 \iff x = 5 \text{ or } x = -3.$$

**Answer.** $x = 5$ or $x = -3$

## Summary

In this lecture, we have learned

- ☑ what it means to factor a polynomial
- ☑ what an irreducible polynomial is
- ☑ how to factor quadratic binomials
- ☑ how to factor quadratic trinomials $ax^2 + bx + c = a(x - x_1)(x - x_2)$
- ☑ how to prove the factorization formula
- ☑ Vieta's theorem
- ☑ how to use Vieta's theorem for solving quadratic equations
- ☑ how to solve quadratic equations by factoring