Lecture 27

Quadratic Formula

Goal: to solve any quadratic equation . . . 2
Quadratic formula . . . 3
Plan . . . 4
Completing the square . . . 5
Completing the square . . . 6
Proving quadratic formula . . . 7
Proving quadratic formula . . . 8
Proving quadratic formula . . . 9
Discriminant . . . 10
How to apply the quadratic formula . . . 11
How to apply the quadratic formula . . . 12
How to apply the quadratic formula . . . 13
When an equation is not in the standard form . . . 14
When the quadratic formula is not the best choice . . . 15
Summary . . . 16

Goal: to solve any quadratic equation

In previous lecture, we learned how to solve some special quadratic equations, namely, **binomial** equations, that is, equations of types $ax^2 + c = 0$ or $ax^2 + bx = 0$.

In this lecture, we will learn how to solve a general quadratic equation $ax^2 + bx + c = 0$ for **arbitrary** coefficients $a \neq 0,\ b$ and c.

This will take some time and efforts,
but we'll get a formula which allows to solve **any** quadratic equation!

Quadratic formula

Theorem. Let $ax^2 + bx + c = 0$ be a quadratic equation
with **arbitrary** coefficients $a \neq 0,\ b$ and c.

Its solution is given by the **quadratic formula**

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\ , \quad \text{provided } b^2 - 4ac \geq 0}$$

If $b^2 - 4ac < 0$, then the equation has **no** solutions.

Remarks. We are going to **prove** and discuss the quadratic formula,
and master it by various numerical examples.

The deduction of the quadratic formula is the most **difficult** part of our course.

It's normal to go over this proof several times until complete understanding.

Important. Quadratic formula will be used throughout all your math studies.
It makes sense to **memorize** it.

Plan

Let $ax^2 + bx + c = 0$ be a quadratic equation,

where x is unknown, a, b, c are given numbers (coefficients) and $a \neq 0$.

We have to **solve** this equation, that is to find the unknown x in terms of the coefficients a, b, c.

For this, we perform a standard trick which turns any quadratic **tri**nomial into a quadratic **bi**nomial.

This trick is called **completing the square**.

Once the quadratic trinomial is converted to a quadratic binomial,

the equation becomes a **binomial equation**, which we know how to solve.

Completing the square

Let $ax^2 + bx + c$ be a quadratic trinomial.

The expression $ax^2 + bx$ may be considered as a "sprout" of a square, an **incomplete** square:

$$ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right) = a\underbrace{\left(x^2 + 2 \cdot x \cdot \frac{b}{2a}\right)}_{\text{incomplete square}}$$

To complete this incomplete square,

we add (and then subtract to keep the balance) the missing term, namely, $\left(\frac{b}{2a}\right)^2$:

$$a\underbrace{\left(x^2 + 2 \cdot x \cdot \frac{b}{2a}\right)}_{\text{incomplete square}} = a\left(\underbrace{x^2 + 2 \cdot x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2}_{\text{complete square}} - \left(\frac{b}{2a}\right)^2\right) = a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right)$$

Completing the square

We have got that

$$ax^2 + bx = a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}.$$

The **tri**nomial may be rewritten as

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c.$$

Note that the resulting expression is a quadratic **binomial**.

Indeed, x is a variable, so is $x + \frac{b}{2a}$. Since a, b, c are constants, so is $-\frac{b^2}{4a} + c$.

If we denote $x + \frac{b}{2a}$ by y and $-\frac{b^2}{4a} + c$ by d,

then the expression $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$ turns to $ay^2 + d$, which is a **bi**nomial.

Proving quadratic formula

By completing the square,

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

Therefore

$$ax^2 + bx + c = 0 \iff a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0.$$

Let us solve the latter **binomial** equation:

$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$ Move $-\frac{b^2}{4a} + c$ to RHS

$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$ Divide both sides by a

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ Combine terms in RHS: $\frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$

Proving quadratic formula

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Notice that the equation has a solution **only if** $\frac{b^2 - 4ac}{4a^2} \geq 0$.

$\frac{b^2 - 4ac}{4a^2} \geq 0 \iff \boxed{b^2 - 4ac \geq 0}$ since $4a^2 > 0$.

Take the square roots from both sides of the equation:

$\left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}}$ Take down the absolute value sign

$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ Simplify the radical

$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$ Move $\frac{b}{2a}$ to RHS

Proving quadratic formula

$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ Combine terms on RHS

$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$ Done!

This is the **quadratic formula** for finding roots (solutions) of a quadratic equation.

It is applicable **only** if $\mathbf{b^2 - 4ac \geq 0}$.

The expression $\mathbf{b^2 - 4ac}$ is of special importance,

it is called the **discriminant** of the quadratic equation.

A quadratic equation has solutions if and only if its discriminant is **non-negative**.

Discriminant

What does the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ give us?

Case 1. If the discriminant is positive, that is $b^2 - 4ac > 0$,
then the quadratic formula gives **two** solutions (roots):

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Case 2. If the discriminant equals zero, that is $b^2 - 4ac = 0$,
then the quadratic formula gives **one** solution (root):

$$x = -\frac{b}{2a}.$$

Case 3. If the discriminant is negative, that is $b^2 - 4ac < 0$,
then the quadratic equation has **no** solutions (roots).

How to apply the quadratic formula

Example 1. Solve the equation $x^2 + 2x - 3 = 0$.

Solution. The quadratic equation is written in the standard form

$$ax^2 + bx + c = 0 \text{ with } a = 1,\ b = 2 \text{ and } c = -3.$$

The solution is given by the quadratic formula $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

In our case,

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{-2 \pm \sqrt{4 + 12}}{2} = \frac{-2 \pm \sqrt{16}}{2} = \frac{-2 \pm 4}{2}.$$

From this, $x_1 = \dfrac{-2 + 4}{2} = 1$ and $x_2 = \dfrac{-2 - 4}{2} = -3$.

Answer. $x = 1$ or $x = -3$.

How to apply the quadratic formula

Example 2. Solve the equation $2x^2 - 3x - 1 = 0$.

Solution. In this case, $a = 2,\ b = -3,\ c = -1$. The solution is

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2}$$

$$= \frac{3 \pm \sqrt{9+8}}{4} = \frac{3 \pm \sqrt{17}}{4}.$$

Answer. $x_{1,2} = \dfrac{3 \pm \sqrt{17}}{4}$

Example 3. Solve the equation $x^2 - x + 1 = 0$.

Solution. In this case, $a = 1,\ b = -1,\ c = 1$. The solution is

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{1 \pm \sqrt{-3}}{2}$$

This equation has **no** real solutions.

How to apply the quadratic formula

Example 4. Solve the equation $-x^2 + 6x - 9 = 0$.

Solution. In this case, $a = -1,\ b = 6,\ c = -9$. The solution is

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4 \cdot (-1) \cdot (-9)}}{2 \cdot (-1)}$$

$$= \frac{-6 \pm \sqrt{36 - 36}}{-2} = \frac{-6}{-2} = 3.$$

Answer. $x = 3$.

Remark. Let us have another look on the equation:

$-x^2 + 6x - 9 = 0 \iff x^2 - 6x + 9 = 0$. The left hand side on the latter equation is, actually, a perfect square trinomial: $x^2 - 6x + 9 = (x-3)^2$.

Therefore, $x^2 - 6x + 9 = 0 \iff (x-3)^2 = 0 \iff x - 3 = 0 \iff x = 3$.

When an equation is not in the standard form

Example 5. Solve the equation $5 + x(2 - x) = 4 + x^2$.

Solution. To use the quadratic formula, we have to bring the equation into the standard form:

$5 + x(2 - x) = 4 + x^2 \iff 5 + 2x - x^2 = 4 + x^2 \iff 0 = 2x^2 - 2x - 1$.

The equation is in the standard form with $a = 2,\ b = -2,\ c = -1$.

The solution is

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{2 \pm \sqrt{4+8}}{4}$$

$$= \frac{2 \pm \sqrt{12}}{4} .$$

Let us bring the answer to simplest radical form:

$$\frac{2 \pm \sqrt{12}}{4} = \frac{2 \pm 2\sqrt{3}}{4} = \frac{2(1 \pm \sqrt{3})}{4} = \frac{1 \pm \sqrt{3}}{2} .$$

Answer. $x_{1,2} = \dfrac{1 \pm \sqrt{3}}{2}$

When the quadratic formula is not the best choice

If a quadratic equation is **not** a trinomial, but a binomial,
then the quadratic formula is valid, but is **not** the most efficient tool for solving the equation.

Example. Solve the equation $4x^2 - x = 0$.

Solution. Alternative 1 (using the quadratic formula) $a = 4, b = -1, c = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 4 \cdot 0}}{2 \cdot 4} = \frac{1 \pm \sqrt{1}}{8} = \frac{1 \pm 1}{8} .$$

By this, $x_1 = \dfrac{1+1}{8} = \dfrac{1}{4}$ and $x_2 = \dfrac{1-1}{8} = 0$.

Alternative 2 (by factoring):

$4x^2 - x = 0 \iff x(4x - 1) = 0 \iff x = 0$ or $4x - 1 = 0$

$\iff x = 0$ or $x = \dfrac{1}{4}$.

Answer. $x = 0$ or $x = \dfrac{1}{4}$

Summary

In this lecture, we have learned

- ☑ the **quadratic formula** $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- ☑ how to complete the square
- ☑ how to prove the quadratic formula
- ☑ when the quadratic formula is valid
- ☑ what the **discriminant** of a quadratic equation is
- ☑ how many solutions a quadratic equation has depending on its determinant
- ☑ how to apply the quadratic formula to solving quadratic equations
- ☑ when the quadratic formula is not the best tool to solve a quadratic equation