Lecture 25

Radicals as Powers with Rational Exponents

Roots

Let a be a real number. The **n-th root** of a is a number b such that $b^n = a$.

If $n = 2$ then the n-th root is the square root which we studied in the preceding lecture.

Examples. The **2nd** root of 49 is 7, since $7^2 = 49$

The **4th** root of 81 is 3, since $3^4 = 81$.

The **5th** root of -32 is -2, since $(-2)^5 = -32$.

The **4th** root of -81 does **not** exist,

since there is no real number which **4th** power is negative.

Cube root

The **3rd** root has a special name: it is called a **cube root**.

Notation for the cube root: $\sqrt[3]{\ }$. By definition, $b = \sqrt[3]{a} \iff b^3 = a$.

For any number a, there exists a unique cube root of a,

since the equation $x^3 = a$ has a unique solution.

Examples. $\sqrt[3]{1} = 1$ since $1^3 = 1,$

$\sqrt[3]{8} = 2$ since $2^3 = 8,$

$\sqrt[3]{27} = 3$ since $3^3 = 27,$

$\sqrt[3]{64} = 4$ since $4^3 = 64,$

$\sqrt[3]{0} = 0$ since $0^3 = 0,$

$\sqrt[3]{-1} = -1$ since $(-1)^3 = -1,$

$\sqrt[3]{-8} = -2$ since $(-2)^3 = -8.$

Odd-order roots

Let n be a positive **odd** integer, and a be a real number.

Then the equation $x^n = a$ has a unique solution. So there exists a unique n-th root of a.

Notation for the n-th root: $\sqrt[n]{\ }$. By definition, $b = \sqrt[n]{a} \iff b^n = a$.

The number n is called the **index** of the n-th root.

Examples. $\sqrt[5]{1} = 1$ since $1^5 = 1$,

$\sqrt[9]{-1} = -1$ since $(-1)^9 = -1$,

$\sqrt[3]{-125} = -5$ since $(-5)^3 = -125$,

$\sqrt[5]{243} = 3$ since $3^5 = 243$,

$\sqrt[7]{128} = 2$ since $2^7 = 128$,

$\sqrt[7]{-128} = -2$ since $(-2)^7 = -128$.

Even-order roots

Let n be a positive **even** integer, and a be a **non-negative** real number.

Then the equation $x^n = a$ has **two** solutions, which differ by their signs.

So there exist two n-th roots of a.

The positive root is called the **principal** n-th root and denoted by $\sqrt[n]{\ }$.

By definition, $b = \sqrt[n]{a} \iff b^n = a$.

The number n is called the **index** of the n-th root.

It's a custom to omit the index of 2: the second root $\sqrt[2]{a}$ is written as $\sqrt{a}$.

Examples. $\sqrt[4]{1} = 1$ since $1^4 = 1$,

$\sqrt[4]{16} = 2$ since $2^4 = 16$,

$\sqrt[4]{-16}$ is undefined since 4 is even and $-16 < 0$,

$\sqrt[6]{64} = 2$ since $2^6 = 64$,

$\sqrt[4]{81} = 3$ since $3^4 = 81$,

$\sqrt[6]{-81}$ is undefined since 6 is even and $-81 < 0$.

Precautions

Dealing with n-th roots, we have to distinguish two cases: when n is odd and when n is even.

- For **odd** n, $\sqrt[n]{a}$ is defined for **all** a.
  In this case, $\sqrt[n]{a}$ may be positive, negative, or zero (depending on a).
- For **even** n, $\sqrt[n]{a}$ is defined only for **non-negative** a. In this case, $\sqrt[n]{a} \geq 0$.

Operations of taking the n-th power and n-th root are **inverse** to each other:

n-th power

$\sqrt[n]{a} = b \quad \rightleftarrows \quad a = b^n$

n-th root

For even n, we have to restrict ourselves to non-negative a and b.

Then $\sqrt[n]{b^n} = b$ and $(\sqrt[n]{b})^n = b$.

Examples

Example 1. Find the value of the following expressions:

$$\sqrt[3]{5^3},\ \sqrt[3]{(-5)^3},\ \sqrt[3]{-5^3},\ (\sqrt[3]{5})^3,\ (-\sqrt[3]{5})^3,\ (\sqrt[3]{-5})^3.$$

Solution. $\sqrt[3]{5^3} = 5, \quad \sqrt[3]{(-5)^3} = -5, \quad \sqrt[3]{-5^3} = \sqrt[3]{-125} = -5,$

$(\sqrt[3]{5})^3 = 5, \quad (-\sqrt[3]{5})^3 = -(\sqrt[3]{5})^3 = -5, \quad (\sqrt[3]{-5})^3 = -5.$

Example 2. Find the value of the following expressions:

$$\sqrt[4]{5^4},\ \sqrt[4]{(-5)^4},\ \sqrt[4]{-5^4},\ (\sqrt[4]{5})^4,\ (-\sqrt[4]{5})^4,\ (\sqrt[4]{-5})^4.$$

Solution. Caution! 4 is even and $\sqrt[4]{\ }$ may be not defined.

$\sqrt[4]{5^4} = 5, \quad \sqrt[4]{(-5)^4} = \sqrt[4]{5^4} = 5, \quad \sqrt[4]{-5^4} = \sqrt[4]{-625}$ is undefined.

$(\sqrt[4]{5})^4 = 5, \quad (-\sqrt[4]{5})^4 = (\sqrt[4]{5})^4 = 5, \quad (\sqrt[4]{-5})^4$ is undefined.

Properties of n-th roots

Let a, b be numbers for which n-th roots are defined. Then $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$ and $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.

Indeed, $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$. Therefore, $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$.

$\left(\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n = \dfrac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n} = \dfrac{a}{b}$. Therefore, $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.

Radicals as powers with rational exponents

Reminder:

If n is a **positive** integer, then $x^n = \underbrace{x \cdot x \cdot \dots \cdot x}_{n\,\text{times}}$, and $x^{-n} = \dfrac{1}{x^n}$.

If $n = 0$, then $x^0 = 1$.

What is $x^{\frac{1}{n}}$? Calculate the n-th power of $x^{\frac{1}{n}}$:

$$\left(x^{\frac{1}{n}}\right)^n = \underbrace{x^{\frac{1}{n}} \cdot x^{\frac{1}{n}} \cdot \dots \cdot x^{\frac{1}{n}}}_{n\,\text{times}} = x^{\frac{1}{n}+\frac{1}{n}+\dots+\frac{1}{n}} = x^{n\cdot\frac{1}{n}} = x^1 = x.$$

This means that n-th power of $x^{\frac{1}{n}}$ is x, therefore, $x^{\frac{1}{n}} = \sqrt[n]{x}$.

For positive integers m and n, define a power with **fractional** exponent as follows:

$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.

One can prove that all power rules are valid for fractional exponents.

Operating with fractional exponents

Example. Simplify the following expressions:

$25^{\frac{3}{2}}$, $27^{-\frac{5}{3}}$, $(64)^{\frac{2}{3}}$, $(-64)^{\frac{2}{3}}$, $(64)^{\frac{3}{2}}$, $(-64)^{\frac{3}{2}}$.

Solution.

$25^{\frac{3}{2}} = 25^{\frac{1}{2}\cdot 3} = (25^{\frac{1}{2}})^3 = (\sqrt{25})^3 = 5^3 = 125$

$27^{-\frac{5}{3}} = \dfrac{1}{27^{\frac{5}{3}}} = \dfrac{1}{(\sqrt[3]{27})^5} = \dfrac{1}{3^5} = \dfrac{1}{243}$

$(64)^{\frac{2}{3}} = (\sqrt[3]{64})^2 = 4^2 = 16$

$(-64)^{\frac{2}{3}} = (\sqrt[3]{-64})^2 = (-4)^2 = 16$

$(64)^{\frac{3}{2}} = (\sqrt{64})^3 = 8^3 = 512$

$(-64)^{\frac{3}{2}} = (\sqrt{-64})^3$ is undefined since $-64 < 0$.

Summary

In this lecture, we have learned

- ☑ what the n-th root is
- ☑ what $\sqrt[n]{a}$ is
- ☑ the difference between cases when n is **odd** and **even**
- ☑ defining identities for n-th root: $(\sqrt[n]{x})^n = x$, $\sqrt[n]{x^n} = x$ for $x \geq 0$
- ☑ properties of n-th root
- ☑ that radicals may be written as **powers** with rational exponents:
  $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$
- ☑ how to operate with rational exponents