Lecture 12

Operations with Rational Expressions

Fractions

A **rational expression** is a quotient of two polynomials.
It is a **fraction** in which numerator and denominator are polynomials.
Therefore rational expressions comply with the same **rules** as fractions:

Cancellation: $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$ for any $c \neq 0$

Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$

Division: $\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$

Addition: $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

These are all the rules that you need to know for operating with fractions,
and with rational expressions.

Multiplying rational expressions

Rational expressions, being fractions, are multiplied as fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a\,c}{b\,d}$$

Example. Simplify the expression $\frac{x}{9-x^2} \cdot \frac{x-3}{x^2+x}$.

Solution. The expression is a product of two rational expressions:

$$\frac{x}{9-x^2} \cdot \frac{x-3}{x^2+x} = \frac{x(x-3)}{(9-x^2)(x^2+x)}\,.$$

To simplify the product, we factor the denominator:

$$\underbrace{(9-x^2)}_{3^2-x^2}\underbrace{(x^2+x)}_{x(x+1)} = (3-x)(3+x)x(x+1)\,.$$

Therefore,

$$\frac{x}{9-x^2} \cdot \frac{x-3}{x^2+x} = \frac{\cancel{x}(x-3)}{(3-x)(3+x)\cancel{x}(x+1)} = \frac{-(3-x)}{(3-x)(3+x)(x+1)} = -\frac{1}{(x+3)(x+1)}\,.$$

Dividing rational expressions

Rational expressions, being fractions, are divided as fractions. To **divide** an expression by a fraction, we multiply the expression by the **reciprocal** of the fraction:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a\,d}{b\,c}\,.$$

Thus,

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a\,d}{b\,c}}$$

Example. Simplify the expression $\dfrac{x^3}{x^2+2x+1} \div \dfrac{x^2}{x+1}$.

Solution. $\dfrac{x^3}{x^2+2x+1} \div \dfrac{x^2}{x+1} = \dfrac{x^3}{x^2+2x+1} \cdot \dfrac{x+1}{x^2} = \dfrac{x(x+1)}{x^2+2x+1}$.

By short multiplication formula, $x^2+2x+1 = (x+1)^2$.

So $\dfrac{x(x+1)}{x^2+2x+1} = \dfrac{x(x+1)}{(x+1)^2} = \dfrac{x\cancel{(x+1)}}{(x+1)\cancel{(x+1)}} = \dfrac{x}{x+1}$.

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Adding rational expressions

Rational expressions, being fractions, are **added** as fractions: if they have a common denominator, then

$$\boxed{\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}}$$

Otherwise the denominators are made coinciding using the relation $\dfrac{a}{b} = \dfrac{ac}{bc}$ and then the same rule applies.

The product of denominators can always serve as a common denominator:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}\,.$$

This gives a formula which always works:

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}}$$

Often fractions have a common denominator simpler than bd.

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Subtracting rational expressions

Subtraction is similar to addition. There are similar formulas:

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

Moreover, subtraction is reduced to addition:

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right)$$

Keep in mind that

$$-\frac{c}{d} = \frac{-c}{d} = \frac{c}{-d}$$

Examples of addition

Example 1. Present $\dfrac{1-x^2}{x} + x$ as a single fraction.

Solution. We have to perform addition of fractions. For this, we need to find a **common denominator** of $\dfrac{1-x^2}{x}$ and $x = \dfrac{x}{1}$. The common denominator is $x \cdot 1 = x$. Therefore,

$$\frac{1-x^2}{x} + x = \frac{1-x^2}{x} + \frac{x}{1} = \frac{(1-x^2)\cdot 1 + x \cdot x}{x \cdot 1} = \frac{1-x^2+x^2}{x} = \frac{1}{x}.$$

Example 2. Present $2 + \dfrac{3}{x+1}$ as a single fraction.

Solution.

$$2 + \frac{3}{x+1} = \frac{2}{1} + \frac{3}{x+1} = \frac{2\cdot(x+1) + 1\cdot 3}{1\cdot(x+1)} = \frac{2x+2+3}{x+1} = \frac{2x+5}{x+1}.$$

Examples of addition

Example 3. Perform the operations and simplify the resulting expression:

$$\frac{1}{x+1} + \frac{1}{x^2-1}.$$

Solution. By the universal formula $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$,

$$\frac{1}{x+1} + \frac{1}{x^2-1} = \frac{(x^2-1)+(x+1)}{(x+1)(x^2-1)} = \frac{x^2+x}{(x+1)(x^2-1)} = \frac{x(x+1)}{(x+1)(x^2-1)} = \frac{x}{x^2-1}.$$

Another solution. Since $x^2-1 = (x-1)(x+1)$, a common denominator is $(x-1)(x+1)$:

$$\frac{1}{x+1} + \frac{1}{x^2-1} = \frac{1}{x+1} + \frac{1}{(x-1)(x+1)} = \frac{(x-1)+1}{(x-1)(x+1)} = \frac{x}{(x-1)(x+1)}.$$

Summary

In this lecture, we have learned

- ☑ how to **multiply** rational expressions
- ☑ how to **divide** rational expressions
- ☑ how to **add** and **subtract** rational expressions