Basic Notions of Geometric Invariant Theory

With an Eye to Hilbert Points

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www.fordham.edu/morrison/pdfs/MorrisonStonyBrookSlides.pdf

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Conventions and disclaimers

A talk aimed at those with little or no background in GIT

- Try to indicate ideas underlying proofs but few details.
- Important doors to moment maps, symplectic quotients, balanced metrics, K-stability are pointed out but not opened.
- Treat most results about algebraic groups as black boxes.

Work over $\mathbb{C}$

- Most results remain have analogues over general algebraically closed fields.
- Proofs require deeper techniques from algebraic groups.

Plan

- Review linear case—representations of reductive groups—in some detail.
- Briefly discuss passage to more general actions.
- Focus on analysis of stability of Hilbert points with hints about Chow points.
- Throughout try to get a feel for the results through key examples.

Quotients of algebraic group actions

- Start from a linear representation $G \curvearrowright W$ of an algebraic group G:
  - To fix notation that I will suppress when possible, a map $\alpha : G \times W \to W$ with each $\overline{\alpha}(g) : W \to W$ in $\mathrm{GL}(W)$ and the map $\overline{\alpha} : G \to \mathrm{GL}(W)$ a homomorphism.
  - View W as an affine space with induced G-action on $S = \mathbb{C}[W]$.
  - Want to form a quotient affine variety $\pi : W \to W /\!/ G$.
  - The inclusion of the invariant subring $i : S^G \subset S$ gives a natural candidate: define $W /\!/ G := \mathrm{Spec}(S^G)$ and use the map π induced by i.
  - This π tautologically meets the first requirement of being constant on G-orbits in W.

- Problem: Goldilocks and the 3 invariants. We need to worry that there are
  - Not too many invariants: is the subring S^G finitely generated?
  - Not too few invariants: do invariants separate orbits?
  - Not just right: by construction, π is dominant, but must it be surjective?

Finite generation of S^G, or Hilbert's 14$^{\text{th}}$ Problem

- Not automatic.
  - Can fail even for simple G like $(\mathbb{G}_a)^r$ with r small.
  - First examples, in positive characteristic, are due to Nagata.
  - Totaro gives examples with $r \geq 3$ over any field.

- Does hold when G is reductive: our main examples are $T = \mathbb{G}_m^n$ and $\mathrm{SL}(n)$.
  - Over $\mathbb{C}$, such G are linearly reductive: any representation V has a canonical splitting $S = S^G \oplus S'$ where S' is the sum of all non-trivial irreducibles.
  - By projecting, we get a Reynolds operator $\rho : S \to S^G$ which is an S^G-module homomorphism.
  - Given $R \subset S$ with S noetherian and an R-module homomorphism $\rho : S \to R$, then R is finitely generated. (Imitate the proof of the Hilbert basis theorem).

Separation by invariants

- Some cautionary examples: not always “just right” even when S^g is f.g.
  - If $\mathbb{G}_m \curvearrowright \mathbb{C}^n$ by homotheties, then any closed invariant set contains the origin $\mathbf{o}$, so all invariants are constant.
  - If $\mathbb{G}_a \curvearrowright \mathbb{C}^2$ by $t \cdot (a, b) = (a, ta + b)$, then $S^G = \mathbb{C}[a]$. All the points $(0, b)$ are closed orbits with image the same image 0 under π.
  - If $\mathbb{G}_a \curvearrowright W = M_2(\mathbb{C})$ by $t \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+tc & b+td \\ c & d \end{smallmatrix}\right)$, then $S^G = \mathbb{C}[c, d, ad - bc]$ so $W /\!\!/ \mathbb{G}_a = \mathbb{A}^3$ but $[0, 0, z]$ is not in $\pi(W)$ if $z \neq 0$.
- Invariants for reductive group actions separate disjoint closed G-invariant subsets.
  - Given X and Y, use Nullstellensatz to write $1 = f + g$ with $f \in I(X)$ and $g \in I(Y)$.
  - Apply ρ to get $1 = \rho(f) + \rho(g)$ with $f \in I(X)^G$ and $g \in I(Y)^G$.
  - That is, $\rho(f)$ is an invariant that is 1 on X and 0 on Y.
- More typical: $\mathbb{G}_m \curvearrowright W = M_2(\mathbb{C})$ by $t \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & t \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & t^2 b \\ t^{-2} c & d \end{smallmatrix}\right)$
  - The closure of the orbit of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ contains $\mathbf{o}$.
  - More generally, the closure of the orbit of $\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$ contains the orbit of $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$.

Stability

- These examples show that not all orbits are created equal.

w is unstable	$\mathbf{o} \in \overline{G \cdot w}$	Any homogeneous invariant not vanishing at w is constant.	$\pi(w) = \mathbf{o}$
w is semistable	$\mathbf{o} \notin \overline{G \cdot w}$	Some non-constant homogeneous invariant vanishes at w.	$\pi(w) \neq \mathbf{o}$
w is polystable	$G \cdot w$ is closed	Invariants separate $G \cdot w$ from other *closed* orbits.	$w' \in \pi^{-1}(w) \iff \overline{G \cdot w'} \cap G \cdot w \neq \emptyset$.
w is stable	$G \cdot w$ is closed and G_w is finite.	Invariants separate $G \cdot w$ from *all* other orbits.	$\pi^{-1}(w) = G \cdot w$.

- A few terminological warnings are in order:
  - Unstable and semistable are antonyms. Unstable and stable are *not*.
  - Many references use stable/properly stable for our polystable/stable.
- Example: Smooth hypersurfaces are semistable for $\mathrm{SL}(n) \curvearrowright \mathrm{Sym}^d(\mathbb{C}^n)$: The discriminant Δ is non-zero at equations of smooth hypersurfaces.

Closures of orbits

- A good model to keep in mind is $\mathrm{SL}(n) \curvearrowright M_n(\mathbb{C})$ by conjugation.
  - The invariants are generated by the coefficients of the characteristic polynomial which are algebraically independent so $M_n(\mathbb{C})/\!\!/\mathrm{SL}(n) = \mathbb{A}^n$.
  - A is unstable $\iff$ A is nilpotent.
  - A is polystable $\iff$ A is semisimple (is diagonalizable).
  - A is stable $\iff$ A is regular semisimple (has distinct eigenvalues).
  - $B \in \overline{\mathrm{SL}(n) \cdot A}$ $\iff$ The Jordan form of B is obtained from that of A by removing some off-diagonal 1s.

- This illustrates the main properties of closures of orbits.
  - For any w, $\overline{G \cdot w}$ is a finite union of other orbits $G \cdot w'$.
  - If $G \cdot w' \subset \overline{G \cdot w}$, then $\dim(G \cdot w') < \dim(G \cdot w)$ and $\dim(G_{w'}) > \dim(G_w)$.
  - $\overline{G \cdot w}$ will contain a unique orbit $G \cdot w'$ of *minimal* dimension which is closed.
  - But the closures of several orbits can contain the same closed orbit.

Properties of the quotient map

- A categorical quotient $\phi : W \to X$ is a G-equivariant map that has a universal initial property with respect to such maps.
- Such a quotient is good if:
  - It is constant on orbits, surjective and affine.
  - Locally over X, it is given by values of invariants.
  - Closed G-invariant subsets of W have closed images.
  - Disjoint closed G-invariant subsets of W have disjoint images.
- A quotient is geometric if every fiber of ϕ is a single G-orbit (or, more formally, if $W \times_X W$ is isomorphic to the image of the map $(g, w) \xrightarrow{(\alpha,\mathrm{id})} (g \cdot w, w)$).
- The preceding discussion shows that:
  - For G reductive, $\pi : W \to W/\!/G$ is a good categorical quotient.
  - This quotient is only geometric over the stable locus W^s.
  - Remark: If G is not reductive but S^G is finitely generated, then we get a categorical quotient, but, as the $\mathbb{G}_a$ examples above show, not necessarily a good one.

Analysis of one-parameter subgroups

- View an action $\mathbb{G}_m \curvearrowright W$ as a 1-ps, i.e. a homomorphism $\lambda : \mathbb{G}_m \to G$ in $\Lambda(G)$.
  - The irreducibles of $\mathbb{G}_m$ are characters indexed by integers.
  - Decompose $W = \oplus_{i\in\mathbb{Z}} W_i$ where $t \cdot w = t^i w$ for $w \in W_i$.
  - Likewise, if $w \in W$, $w = \sum_{i\in\mathbb{Z}} w_i$.
  - The state $S(W)$ [$S(w)$] is the set of weights: those i s.t. $W_i \neq \{\mathbf{o}\}$ [$w_i \neq \mathbf{o}$].
  - As $t \to 0\,[\infty]$, $t \cdot w \to w_0$ if $\min S_w \geq 0$ [$\max S_w \leq 0$] (and, if not, has no limit).
- If $G \curvearrowright W$ and $\lambda \in \Lambda(G)$, we set $\mu(w, \lambda) = \min S_\lambda(w)$—the least w-weight.

We can translate the discussion above as:

w is unstable	$\Longleftarrow$	For some λ, $\mu(w, \lambda) > 0$.
w is semistable	$\Longrightarrow$	For every λ, $\mu(w, \lambda) \leq 0$.
w is polystable	$\Longrightarrow$	For every λ not fixing w, $\mu(w, \lambda) < 0$.
w is stable	$\Longrightarrow$	For every non-trivial λ, $\mu(w, \lambda) < 0$

- Stable means that every λ has weights of *both* signs (consider λ^{-1}).
- Warning: There is a large Little–Endian school that uses $-\mu$.

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Hilbert-Mumford Numerical Criterion

w is unstable	$\iff$	For some λ, $\mu(w, \lambda) > 0$.
w is semistable	$\iff$	For every λ, $\mu(w, \lambda) \leq 0$.
w is polystable	$\iff$	For every λ not fixing w, $\mu(w, \lambda) < 0$.
w is stable	$\iff$	For every non-trivial λ, $\mu(w, \lambda) < 0$

- Proof is based on the Cartan-Iwahori-Matsumoto decomposition.

Example: Binary quantics

- Consider $\mathrm{SL}(2) \curvearrowright W = \mathrm{Sym}^d(C^2)^\vee$.
- Up to powers and change of coordinates, we can assume $\lambda(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$.
- Then λ acts by $t^i t^{-(d-i)} = t^{2i-d}$ on the monomial $x^i y^{d-i}$.
- If $P(x,y) = \sum_{i=0}^d a_i x^i y^{d-i}$, $\mu(P,\lambda) > [\geq]\ 0 \iff a_i = 0$ whenever $2i < [\leq]\ d$.
- Geometrically, this means $(0,1)$ is a root of P of multiplicity $> [\geq]\ \frac{d}{2}$.
- P is stable [semistable] $\iff$ No root has multiplicity at least [more than] $\frac{d}{2}$.
- The closure of the orbit of a polynomial with a root of multiplicity more than $\frac{d}{2}$ contains the origin.
- The closed orbit in the closure of the orbit of any polynomial with a root of multiplicity exactly $\frac{d}{2}$ is that of $(xy)^{\frac{d}{2}}$.
- Remark: To uniformize notation when taking λ in $\mathrm{SL}(V)$, it is convenient (if non-canonical), to pick coordinates that v_i s.t. $\lambda(t) = \mathrm{diag}(\ldots, t^{\lambda_i} v_i, \ldots)$ allowing repeated weights λ_i.

Analysis of torus actions

- Consider an action $T := \mathbb{G}_m^r \curvearrowright W$.
  - Again, we decompose $W = \bigoplus_{\chi \in \mathbf{X}(T)} W_\chi$ where $t \cdot w = \chi(t) w$ for $w \in W_\chi$, but now the character group $\mathbf{X}(T)$ is indexed by $\mathbb{Z}^r$.
  - If $t = (t_1, \ldots, t_r)$ and $z = (z_1, \ldots, z_r)$, then $\chi_z(t) = \prod_{i=1}^r t_i^{z_i}$.
  - The state $\mathcal{S}_T(W)$ $[\mathcal{S}_T(w)]$ is now a set of characters: those χ s.t. $W_\chi \neq \{\mathbf{o}\}$ $[w_\chi \neq \mathbf{o}]$.
  - Define the state polytope $\mathcal{P}_T(w)$ to be the convex hull of $\mathcal{S}_T(w)$ in $\mathbf{X}(T) \otimes \mathbb{R}$.
  - Given a 1-ps $\lambda : \mathbb{G}_m \to T$ and a character $\chi : T \to \mathbb{G}_m$, we get a character $\chi \circ \lambda$ of $\mathbb{G}^m$ which we can write $t \to t^{\langle \lambda, \chi \rangle}$.
  - This defines a non-degenerate bilinear pairing $\langle , \rangle : \Lambda(T) \times \mathbf{X}(T) \to \mathbb{Z}$.
  - Using the standard basis of $\mathbf{X}(T)$, we can identify $\Lambda(T)$ and $\mathbf{X}(T)$ and choose compatible inner products on them.
  - Then $\mu(w, \lambda) = \min\{\langle \lambda, \chi \rangle \mid \chi \in \mathcal{S}_T(w)\}$.
- By Farkas' Lemma, we can restate the Numerical Criterion as:

$$w \text{ is } T\text{-semistable } [T\text{-stable}] \iff \chi_{\mathbf{o}} \in \mathcal{P}_T(w) \ [\ \chi_{\mathbf{o}} \in \mathcal{P}_T(w)^{\circ}\ ]$$

Lengths and parabolic subgroups for 1-ps's

- If w is T-unstable, then we can single out a "worst" 1-ps λ.
  - Let $\|\lambda\| := \left(\sum_i \lambda_i^2\right)^{1/2}$ and "normalize" by setting $\hat{\mu}(w,\lambda) := \dfrac{\mu(w,\lambda)}{\|\lambda\|}$.
  - Note that $\mu(w,\lambda^k) = k\mu(w,\lambda)$ but $\hat{\mu}(w,\lambda^k) = \hat{\mu}(w,\lambda)$.
  - There will be a unique shortest (rational) vector χ_w in $\mathcal{P}_T(w)$.
  - The line dual to this χ contains all λ maximizing $\hat{\mu}(w,\lambda)$ and we pick a primitive integral element.
- If we take any $\|\cdot\|$ invariant under the Weyl group of G w.r.t. T:
  - Get a metric on $\Lambda(G)$ for which $\|g\lambda g^{-1}\| = \|\lambda\|$.
  - For $SL(n)$ (or other semisimple G) there is a canonical choice, $\mathrm{Trace}(\mathrm{ad}(\lambda^* \frac{d}{dt})^2)$.
- Associate to λ the parabolic subgroup $P(\lambda)$
  - $P(\lambda) = \{p \in G \mid \lim_{t\to 0} \lambda(t)p\lambda(t)^{-1}$ exists in $G\}$—such limits centralize λ.
  - For $\mathrm{SL}(n)$, choose a filtration $F(\lambda)$: $\mathbb{C}^n = V_0 \supset V_1 \supset \cdots \supset V_{k-1} \supset V_K = \{\mathbf{o}\}$ with λ acting on V_{i-1}/V_i by t^{λ_i} and the λ_i decreasing. Then $P(\lambda)$ consists of those g preserving $F(\lambda)$.
  - In general, $\hat{\mu}(g\cdot w, g\lambda g^-1) = \hat{\mu}(w,\lambda)$ but $\hat{\mu}(p\cdot w,\lambda) = \hat{\mu}(w,\lambda)$

Worst one-parameter subgroups for unstable w

Kempf-Rousseau Theorem

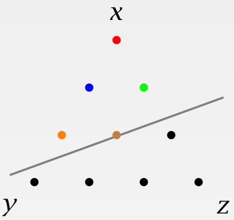
Let $G \curvearrowright W$, w be a G-unstable point and $\|\cdot\|$ be an invariant norm on $\Lambda(G)$. We say that λ is w-worst if $\hat{\mu}(w,\lambda) \geq \hat{\mu}(w,\lambda')$ for any $\lambda' \in \Lambda(G)$.

- The set of w-worst λ is non-empty.
- There is a parabolic subgroup P_w such that any worst λ has $P(\lambda) = P_w$.
- The indivisible worst λ form a single P_w conjugacy class.
- $G_w \subset P_w$.

- The last statement is useful for G semisimple when G_w is "big":
  - If G_w does not lie in any proper parabolic—e.g., if G_w acts irreducibly on W—then w must be semistable.
  - This applies to Chow and Hilbert points of homogeneous spaces and of very ample models of abelian varieties.
  - If W is a multiplicity free G_w-representation and T is chosen compatibly, then T-semistability for w implies G-semistability.

Example: Plane cubics — $\mathrm{SL}(3) \curvearrowright W := \mathrm{Sym}^3(\mathbb{C}^3)^\vee$

- Fix a torus $T \subset \mathrm{SL}(3)$—that is, coordinates (x, y, z).
- Exponents of degree 3 monomials $x^i y^j z^k$ index the character decomposition $W = \oplus W_\chi$ barycentrically (i.e. modulo $i + j + k = 3$).
- Monomials with non-zero coefficient in w—equation of C—give $S_T(w)$.
- Take $\lambda(t) = \mathrm{diag}(t^a, t^b, t^c)$ with $a + b + c = 0$: gray line is $(-5, 1, 4)$.



Vanishing of the indicated monomials means:

$P := (1, 0, 0)$ lies on C.

$z = 0$ is tangent at P.

C has a double point at P,

$z = 0$ is tangent to a branch of C at P.

Tangent cone to C at P is $z^2 = 0$.

Upshot:

Smooth cubics are stable.

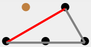
Nodal cubics (even reducible ones) are strictly semistable.

Cubics with cusps (or worse, e.g. multiple line) are unstable.

Example: worst λ's for unstable plane cubics

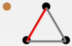
- The invariants are generated by the coefficients g_2 and g_3 of the Weierstrass form $y^2 = 4x^3 - g_2 x - g_3$ with $\Delta = g_2^3 - 27g_3^2$ and $j = \frac{1728 g_2^3}{\Delta}$.
- Below the worst 1-ps's of unstable C are shown in red as a supporting line at the shortest point of a generic state polytope $S(w)$.

$y^3 - xz^2 + z < y, z >^2$ irreducible cuspidal

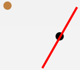


$z(y^2 - xz) + y^2 < y, z >$ conic and tangent line

$< y, z >^3$ three concurrent lines



$z^2 < x, y, z >$ double line



z^3 triple line

- Hesselink showed that this picture generalizes, with the maximum of $\hat{\mu}$ producing a stratification of the unstable locus or nullcone.

Length functions on orbits and moment maps

- Fix $G \curvearrowright W$, a maximal compact $K \subset G$ preserving a hermitian norm $\|\cdot\|$
- For $w \in W$, define $p_w : G \to \mathbb{R}$ by $p_w(g) = \|g \cdot w\|^2$.

Kempf-Ness Theorem

- Any critical point of p_w is a point where it attains its minimum value.
- The function p_w attains its minimum value if an only if $G \cdot w$ is closed. If so:
  - This minimum is taken is a single $K - G_w$-coset M_w.
  - p_w has strictly positive second partials in all directions not tangent to M_w.
- The induced function $\hat{p}_w$ on $K \backslash G$ is a Morse function with a unique minimum if and only if w is stable.

- We pass by, but shall not enter, the door to the symplectic wing here.
  - The moment map $m : \mathbb{P}(W) \to i\mathfrak{k}^\vee$ is defined by $\frac{1}{\|w\|^2} d_e \|g \cdot w\|^2$ so critical points of p_w are zeros of m.
  - Ness and Kirwan study the gradient flow of m on the unstable or nullcone and recover Hesselink's stratification.

Beyond the linear case

- Linearizing actions of reductive groups on affine varieties is straightforward:
  - Any action $G \curvearrowright X$ is rational: i.e., if $V_f := \mathrm{span}\{g \cdot f\}$, then $\dim(V_f) < \infty$.
  - Get an equivariant embedding of $X \subset W$ by taking W to be a sum over generators of V_f.
  - Use linear reductivity to check that $(S/I(X))^G = S^G/(I(X) \cap S^G)$.
  - Thus, the quotient $\pi : W \to W/\!\!/G$ restricts to a quotient $X \to X/\!\!/G$ inheriting many properties, esp. that $X/\!\!/G$ is normal if X is.
- Can we imitate this for $G \curvearrowright X$ with X (quasi-) projective?

$$\begin{array}{ccc} G \times L & \xrightarrow{\hat{\alpha}} & L \\ \downarrow & & \downarrow \\ G \times X & \xrightarrow{\alpha} & X \end{array}$$

  - A G-linearization is a choice of a line bundle L and a lift of the G action to L fixing the 0 section: denote the set of such lifts by $\mathrm{Pic}^G(X)$.
  - The action condition amounts to an isomorphism $\mathrm{pr}_2^*(L) \to \hat{\alpha}^*(L)$.
  - This yields an an exact sequence $0 \to K \to \mathrm{Pic}^G(X) \xrightarrow{\phi} \mathrm{Pic}(X) \to L \to 0$.
  - When X is normal and G is irreducible $L \cong \mathrm{Pic}(G)$ and when X is connected and proper $K \cong \mathbf{X}(G)$.
  - For $G = \mathrm{SL}(n)$ both these groups vanish, so any L has a unique linearization.

Quotients coming from linearizations of ample L

- Fix an action $G \curvearrowright X$ with X normal and projective.
- Fixing an ample line bundle L and a linearization on L yields a good categorical quotient.
  - Get an action $G \curvearrowright R := \oplus_{d \geq 0} \left(H^0(X, L^{\otimes d}) \right)$.
  - Define $X /\!\!/_L G := \mathrm{Proj}(R^G)$—this is a f.g. graded ring because $R(L)$ is.
  - Replacing L by a power if necessary, R^G is generated elements s_j of degree 1.
  - So $X /\!\!/_L G \subset \mathbb{P}(\mathbb{C}^n)$ with ideal $I = \ker(\mathbb{C}[t_i, \ldots, t_n] \to R^G)$ by sending $t_j \mapsto s_j$.
  - This covers the L-semistable locus with affine opens U_j (where $s_j \neq 0$) whose good categorical quotients are the corresponding $(X /\!\!/ G)_j$ (where $t_j \neq 0$).
- Passing by another door we will not open, note that $X /\!\!/_L G$ has a natural ample line bundle, the $\mathcal{O}(1)$ coming from its definition as a Proj.
- Under mild hypotheses, some power of a G-invariant line bundle on X will descend to a bundle on $X /\!\!/_L G$.
- The Grothendieck-Riemann-Roch formula often provides an effective way to relate the two Pic's.

Hilbert points

- A subvariety $Y^r \subset \mathbb{P}(V)$ is a map $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) \cong V^\vee \to H^0(Y, \mathcal{O}_Y(1))$.
- If we fix the degree r Hilbert polynomial $P(m) := h^0(Y, \mathcal{O}_Y(m))$ for $m \gg 0$, we can choose anb M such that for $m > M$,

  $$0 \to I_m(Y) \to \mathrm{Sym}^m(V^\vee) = H^0\Big(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(m)\Big) \xrightarrow{\mathrm{res}_Y} H^0(Y, \mathcal{O}_Y(m)) \to 0$$

  is exact, and $I_m(Y)$ cuts out Y.
- Thus setting $W := \bigwedge^{P(m)} \mathrm{Sym}^m(V^\vee) \xrightarrow{\bigwedge^{P(m)}(\mathrm{res}_Y)} \bigwedge^{P(m)} H^0(Y, \mathcal{O}_Y(m)) \cong \mathbb{C}$, we get the m^{th}-Hilbert point $[Y]_m \in \mathbb{H}_P(V) \subset \mathrm{Grass}\Big(P(M), \mathrm{Sym}^m(V^\vee)\Big) \subset \mathbb{P}(W)$
- This is the value "at $Y \subset \mathbb{P}(V)$" of a closed embedding of the Hilbert scheme $\mathbb{H}_P(V)$ which represents the functor of "flat families of subschemes of $\mathbb{P}(V)$ with Hilbert polynomial P".
- The representation of $SL(V)$ on W naturally linearizes the bundle $\mathbb{L}_m$ on $\mathbb{H}_p$ obtained by pulling back $\mathcal{O}_{\mathbb{P}(W)}(1)$.
- The $SL(V)$ stability of $[Y]_m$ depends on the embedding in $\mathbb{P}(V)$—that is, on L not just Y. It also depends on m though not, in practice, if we take $m \gg 0$.

The Numerical Criterion for Hilbert points

- Given a non-trivial 1-ps λ of $\mathrm{SL}(V)$, choose coordinates v_i on V w.r.t. which $\lambda(t) = \mathrm{diag}(t^{\lambda_1}, \ldots t^{\lambda_n})$. NB: my dimensions here are affine.
- We think of λ as a choice of a weighted basis $B = (v_i, \lambda_i)$. Likewise:
  - Basis of degree m monomials $M = \prod_i v_i^{m_i}$ with weights $\lambda_M = \sum_i m_i \lambda_i$ on $\mathrm{Sym}^m(V^\vee)$.
  - Basis of Plücker coordinates Z_I obtained by wedging any set I of $P(m)$ distinct monomials with weights $\lambda_{Z_I} := \sum_{M \in I} \lambda_M$ on W.
  - $Z_I([Y]_m) \neq 0 \iff$ The set $\{\mathrm{res}_Y(M) \mid M \in I\}$ is linearly independent in $H^0(Y, \mathcal{O}_Y(m))$. These weights of are the λ-weights of $[Y]_m$.
  - We'll think of Z_I as a monomial basis B_m of $H^0(Y, \mathcal{O}_Y(m))$.

Numerical Criterion for Hilbert Points: First Version

$[Y]_m$ is Hilbert stable [semi-stable] $\iff$ For every weighted basis $B := (v_i, \lambda_i)$ of V with $\sum_i \lambda_i = 0$, there is a monomial basis B_m of $H^0(Y, \mathcal{O}_Y(m))$ of negative [non-positive] weight. I.e., the least such weight is $\mu([Y]_m, \lambda)$.

Example: The Steiner surface

- The Steiner surface $S \subset \mathbb{P}^4$ can be described as
  - the projection of the Veronese surface from a point on it.
  - the image of ruled surface obtained by projectivizing the bundle $E := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ with respect to a linear series $\mathcal{O}(1) + f$ where f is a fiber.
  - the blowup of $\mathbb{P}^2$ at $p = (0,0,1)$ embedded by quadrics passing through p.
- The last viewpoint is perfect for an analysis of stability because it gives us a basis $B_1 = \{x^2, xy, y^2, xz, yz\}$ of $H^0(S, \mathcal{O}_S(1))$.
- If we assign z weight -4 and x and y weight 1, then the basis B_1 has weights $(2,2,2,-3,-3)$ so corresponds to a 1-ps λ.
- Any degree m monomial in the elements of B_1 can be viewed as a monomial of total degree $2m$ in x, y and z having degree *at most* m in z.
- The weight of a monomial in B_m of degree j in z is $(2m-j) - 4j = 2m - 5j$ *regardless of how it arises from* B_1 and there are $2m - j + 1$ such monomials.
- Thus, any (or the only) B_m has weight $\sum_{j=0}^{m}(2m-j+1)(2m-5j)$ which a little algebra gives as $\frac{1}{3}(m-1)m(m+1)$, so $[S_m]$ is λ-unstable for all $m \geq 2$.

A Better Numerical Criterion for Hilbert points

- Bad news: we can (almost) never describe monomials this explicitly.
- Good news: we don't need to; we only used the weight filtration on them.
  - Order B so the weights λ_i decrease, set $V_i = \operatorname{span}\{v_j \mid j > i\}$ to get

$$\begin{array}{ccccccccc} V = V_0 & \supset & V_1 & \supset & \dots & \supset & V_{n-1} & \supset & V_n = \{\mathbf{0}\} \\ & \lambda_1 & \geq & \lambda_2 & \dots & \lambda_{n-1} & \geq & \lambda_n & \end{array}$$

  - Now simply write B for the data of this weight filtration, but note that the flag underlying this filtration is really just that of $P(\lambda)$.
  - We'll get an induced weight filtration on $H^0(Y, \mathcal{O}_Y(m))$—a polynomial's weight is the largest weight of a monomial appearing in it. Now *any* basis B_m has a weight $\lambda(B_m)$.
  - Define $w_B(m) := \min\{\lambda(B_m)\}$—always realized by some monomial basis.
- One advantage: we can shift and scale weights. For this, let $\alpha := \frac{1}{n}\sum_i \lambda_i$. Then $\mu([Y]_m, B) = w_B(m) - mP(m)\alpha$, so

Numerical Criterion for Hilbert Points: Second Version

$[Y]_m$ is Hilbert stable [semi-stable] $\iff$ For all B, $w_B(m) - mP(m)\alpha < [\leq] 0$.

Interlude: asymptotic-in-d stability

- For $a \geq 2$, set $n = (2a-1)(g-1)$, $V = \mathbb{C}^n$ and $P(m) = (2am-1)(g-1)$.
- Then $\mathbb{H}_P(V)$ contains a-canonical models of smooth curves of genus g.
- Check that the locus $\mathcal{K} \subset \mathbb{H}_P(V)$ of nodal, connected $C \in \mathbb{P}(V)$ such that $\mathcal{O}_C(1) \cong \omega_C^{\otimes a}$ is smooth of (the expected) dimension $(3g-3)+(n^2-1)$.
- As we'll see in a moment, Hilbert points of smooth curves in $\mathbb{H}_P$ are stable.
- Deformation theory of nodes says such points are dense in $\mathcal{K}$.
- Hence, $\overline{\mathcal{K}} /\!/ \mathrm{SL}(V)$ contains M_g as a dense open.

  Question: What happens at the boundary of $\overline{\mathcal{K}} /\!/ \mathrm{SL}(V)$?
- Fact: as $d \to \infty$, stable plane curves of degree d carry singularities with the multiplicities also tending to infinity
- For many years, this led people—even Mumford—to think that the answer above was "all hell breaks loose". In fact, for $a \geq 5$, $\overline{\mathcal{K}} /\!/ \mathrm{SL}(V)$ is $\overline{M}_g$
- Any smoothable Hilbert stable curve of genus g embedded by a complete linear series of degree $d \gg g$ is nodal.

Example: Triple Points

- Consider a reduced, irreducible curve C of arithmetic genus $g \geq 2$ with an ordinary triple point p.
- Embed C in $\mathbb{P}(V)$ by a complete linear series L of degree $d \geq 3(g-1)$.
- Let $\pi : \widetilde{C} \to C$ be the normalization, $Q = q_1 + q_2 + q_3$ be the sum of the preimages of p on $\widetilde{C}$ and $\widetilde{L} = \pi^*(L)$.
- Choose B s.t. $v_i(p) = 0$ for $i \geq 2$ and set $\lambda_1 = 1$ and $\lambda_i = 0$ for $i \geq 0$.
- Claim: For $1 \leq j < m$, we can identify the "weight at most $w - j$" subspace U_j of $H^0(C, L^{\otimes m})$ with $H^0(\widetilde{C}, \widetilde{L}^{\otimes m}(-jQ))$ so $\operatorname{codim}(U_j) = 3j$.
- Any monomial of this weight contains j factors vanishing at p.
- Here we can no longer easily write down a basis of monomials, but we *can* describe the weight filtration geometrically.
- A bit of calculation gives $w_B(m) - mP(m)\alpha = \left(\frac{3}{2} - \frac{d}{n}\right)m^2 + \left(\frac{3}{2} + \frac{g-1}{n}\right)m$.
- Since $n = d - (g-1)$ and $d \geq 3(g-1)$, $\dfrac{d}{n} \leq \dfrac{3}{2}$ so B is destabilizing.

Example: Elliptic tails

- Let $C = C' \cup E \subset P^{n-1}$ where $g(C') = g-1$ and $g(E) = 1$, $C' \cap E = p$ a node.
  - $\mathcal{O}_C(1)$: very ample, non-special, degree d so $n = d - g + 1$, $\mathcal{O}_C(1)\big|_E = \mathcal{O}_E(ap)$.
  - Riemann-Roch then says there is a weighted basis B of the form

$$\begin{array}{ccccccccc} v_1 & v_2 & \dots & v_{\ell-1} & v_\ell & v_{\ell+1} & \dots & v_{n-1} & v_n \\ a & a & \dots & a & a & a-1 & \dots & 2 & 0 \end{array}$$

  with $v_i = 0$ on E for $i < \ell$, $v_{\ell+j} = 0$ on C' and $\lambda_j + \mathrm{ord}_p(v_{l+j}) = a$ for $j > 0$.
  - Claim: the "weight at most w" subspace of $H^0(C, \mathcal{O}_C(m))$ is exactly $H^0((\mathcal{O}_C(m)\big|_E(-(ma-w)p))$ for $w = 0$ and for $2 \le w \le ma - 1$.
  - After some calculation, this yields $w_B(m) - mP(m)\alpha = \frac{1}{2n}(m-1)(m((g-1)a^2 - d(a-2)) + 1)$.
- Taking $a = 4$, we get instability when $\frac{d}{g-1} \le \frac{8}{7}$.
- Elliptic tails destabilize a-canonical curves for $a \le 4$ (but not for $a \ge 5$).
- If $\mathcal{O}_C(1)\big|_E = \mathcal{O}_E(4q)$ for $q \ne p$, unstable range is $\frac{d}{g-1} \le \frac{7}{6}$.
- As $\frac{d}{g}$ gets smaller, stability depends just on extrinsic geometry but in increasingly subtle ways on the embedding.

Instability is geometry, stability is combinatorics.

- In our examples, geometry suggests the choice of 1-ps λ or weighted basis B.
- We described the weight filtration on $H^0(Y, \mathcal{O}_Y(m))$ geometrically in terms of base loci with multiplicity and computed dimensions exactly.
- A general λ or B has no such "exact" geometry: all we can hope to do is to "estimate" the geometry and use this to estimate $w_B(m)$.
  - If $\dim(X) = r$, then by equivariant Riemann-Roch, for any B and $m \gg 0$, $w_B(m)$ is a numerical polynomial of degree $r+1$—a sort of graded Hilbert polynomial:
  $$w_B(m) := \sum_{j=0}^{r+1} e_i(B)m^i = \frac{\epsilon(B)m^{r+1}}{(r+1)!} + \mathrm{O}(m^r) \text{ with } \epsilon(B) \text{ integral.}$$
  - Likewise $P(m) = \frac{dm^r}{r!} + \mathrm{O}(m^{r-1})$. Hence:

Asymptotic-in-m Numerical Criterion for Hilbert Points

$[Y]_m$ is Hilbert stable for $m \gg 0$ if, for all B, $\epsilon(B) < d(r+1)\alpha$.

- For the Chow point $\langle Y \rangle$, $\mu(\langle Y \rangle, \lambda_B) = \epsilon(B)$, hence

  Chow stable $\implies$ Hilbert stable $\implies$ Hilbert semi-stable $\implies$ Chow semi-stable

  and the elliptic tail example with $r = 4$ shows that the converses are false.

K-stability and other asymptotic-in-L notions

- Let's set $M = ms$ with $m \gg s \gg 0$, suppose that $\mathcal{O}_X(1) = L^{\otimes s}$ and write
$$w_{s,M} = w_B(M)sP(s) - w_B(s)MP(M).$$
- We view this as a numerical function associated to a 1-ps λ of $\mathrm{SL}(H^0(X, L^{\otimes s}))$, usually called a *test configuration* of (X, L) in this context.
- We can expand $w_{s,M}$ twice to get $w_{s,M} = \sum_{j=0}^{r+1} e_j(s)M^j = \sum_{j=0}^{r+1}\Big(\sum_{\ell=0}^{r+1} e_{j,\ell}s^\ell\Big)M^j$.
- Our normalization ensures that $e_{r+1,r+1} = 0$ and we define the Donaldson-Futaki invariant to be $\mathrm{DF}(\lambda) = -e_{r,r+1}$.
- The pair (X, L) is K-stable [semistabile] if $\mathrm{DF}(\lambda) \geq [\ >\ ]0$ for all test configurations of $(X, L^{\otimes s})$ for *all* $s \gg 0$.
- Likewise, we say (X, L) is asymptotically Hilbert or Chow (semi)stable if $(X, L^{\otimes s})$ is Hilbert or Chow (semi)-stable for *all* $s \gg 0$.
- The various notions so defined for a pair (X, L) are related by

Asymptotically Chow stable $\implies$ Asymptotically Hilbert stable $\implies$ Asymptotically Hilbert semi-stable $\implies$ Asymptotically Chow semi-stable $\implies$ K-semistable.

Estimating the weight filtration

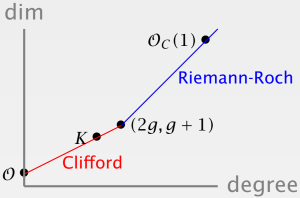
- Given subspaces $U_j \subset H^0(Y, \mathcal{O}_Y(m))$ with $U_j \subset U_k$ if $j \leq k$, $\dim(U_j) \geq d_j$ and all sections in U_j of weight at most w_j, we get an estimate
$$w_B(m) < \sum_j w_j(d_j - d_{j-1}) = \sum_j d_j(w_j - w_{j-1})$$
- Gieseker applies this with $m = m'(p+1)$, taking and the U's of the form
$$\mathrm{Sym}^{m'}\left(V \cdot \mathrm{Sym}^{(p-\ell)}(V_{j_k}) \cdot \mathrm{Sym}^{\ell}(V_{j_{(k+1)}})\right)$$
where $0 = j_0 < j_1 < \cdots < j_h = n$—in effect, we coarsen the filtration on V. Let:
  - Let L_j be the line bundle on Y generated by $|V_j|$, $d_j := \deg(L_j)$ and $e_j = d - d_j$.
  - Projection of Y onto $\mathbb{P}(V_j)$ has image of degree d_j, base locus of degree e_j.
  - $M_{k,\ell} = \mathcal{O}_Y(1) \otimes (L_{j_k})^{(p-\ell)} \otimes (L_{j_{k+1}})^{\ell}$.
  - The blue product is a very ample, base point free sub-linear series of $M_{k,\ell}$ so for $m' >> 0$, each U will be all of $H^0(Y, M_{k,\ell}^{\otimes m'})$.
- Using Riemann-Roch to estimate dimensions gives an estimate for $\epsilon(B)$ that I give only for curves (in general, all the mixed intersection numbers of the L_j appear).

Stability of smooth curves

Gieseker's Criterion for Hilbert Points of Curves

$[C]_m$ is Hilbert stable for $m \gg 0$ if, for any B

$$\epsilon(B) = \max_{\substack{\lambda_1 \geq \cdots \geq \lambda_N = 0 \\ \sum_{i=1}^N \lambda_i = 1}} \left(\min_{1=j_0<\cdots<j_h=N} \left(\sum_{k=0}^{h-1} (e_{j_k} + e_{j_{k+1}})(\lambda_{j_k} - \lambda_{j_{k+1}}) \right) \right) < 2\frac{d}{n}$$



- All the points (d_j, j) associated to the V_j lie below the Riemann-Roch and Clifford lines.
- This gives universal bounds on the e_j and, in terms of these, we are computing the area of the lower convex envelope of the (e_j, λ_j).

Asymptotic Stability of Smooth Curves

For $g \geq 2$ and $d \geq 2g+1$, there is an $M_{g,d}$ such that any smooth curve C of genus g embedded by a complete linear series of degree at least d has stable m^{th} Hilbert points $[C]_m$ for all $m \geq M_{g,d}$.

Truth in advertising

- Disclaimer: the story of smooth curves is totally misleading.
  - Gieseker's motivation was to prove asymptotic stability of pluricanonical models of surfaces of general type. His proof is a tar-baby.
  - His approximate weight filtrations on $H^0(C, \mathcal{O}_C(m))$ are not sharp enough to prove asymptotic stability of nodal curves, or even smooth pointed curves.
  - The span of *all* the U's of fixed weight can be much larger than any one.
  - Using these, get criteria involving a sum, indexed by points p of C, of Gieseker-like sums with e_j is replaced by the multiplicity of p in $\mathrm{Bs}(V_j)$.
  - Using this approach, Li and Wang prove stability of nodal curves and Swinarski of pointed smooth curves, in both cases, only for $d \gg g$.
  - I know of no attempts to do this for varieties of higher dimension.
- You're here today because a very different approach suggested by Mumford and popularized by Yau *does* work: Y is Chow stable if and only if:

(Luo) Y is Chow stable $\iff$ For some $g \in \mathrm{SL}(V)$, $g \cdot Y$ is balanced:

$$\frac{1}{\mathrm{vol}(Y)} \int_{g \cdot Y} \left(\frac{v_i \cdot \overline{v}_j}{|v_1|^2 + \cdots + |v_n|^2} \right) \omega_{\mathrm{FS}}^r = \frac{1}{n} \delta_{ij}.$$

Semistable replacement

- The a-canonical quotients $\overline{\mathcal{K}}/\!\!/\mathrm{SL}(V)$ for *small* a play a key role in the log minimal model program for $\overline{M}_g$.
- But for such a, we can only check the Numerical Criterion for smooth X.
- The substitute uses a procedure called semistable replacement that works quite generally when we have an action on a projective X.
- Fix a DVR R with quotient field K and residue field k and set $S = \mathrm{Spec}(R)$ and $S^* = \mathrm{Spec}(K)$, where $o = \mathrm{Spec}(k)$.
- Given a map $\phi : S^* \to X^s/\!\!/G$, then after a finite base change $S' \to S$ ramified only at o, there exists a lift $\phi' : S' \to X^{ss}$ of f as in the diagram:

$$\begin{array}{ccccccc} & & S' & \xrightarrow{\phi'} & X^{ss} & & \\ & \swarrow & & & & \searrow & \\ S & \hookleftarrow & S^* & \xrightarrow{\phi} & X^s/\!\!/G & \hookrightarrow & X/\!\!/G \end{array}$$

with the orbit of $\phi'(o)$ closed.

Constructing compact quotients

- Given a singular C with $[C]_m \in \overline{\mathcal{K}}$, we smooth it getting a family whose general fiber has stable Hilbert point.
- Base change gives family with projectively equivalent general fiber and m-semistable special fiber.
- Moreover, the universal property of the Hilbert scheme gives us line bundles on the two families that are both generically a-canonical.
- If we can extend this isomorphism of line bundles over the closed fiber, then $[C]_m$ must be the a-canonical Hilbert point of $[C']_m$.
- Gieseker does by proving that only abstractly stable curves have m-semistable a^{th} Hilbert points for $a \geq 5$ and applying semistable reduction.
- For reducible curves, need to rule out possibility of twisting by components of the special fiber: prove a “balanced degree inequality”.
- Extended by Schubert ($a = 3, 4$), Hassett and Hyeon ($a = 2$) to construct log canonical models of $\overline{M}_g$ as moduli spaces for variant moduli problems.
- Open problem: next stages require doing this for fixed values of m.

General surveys of invariant—especially, geometric—invariant theory.

[1] Igor Dolgachev, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series, vol. 296, Cambridge University Press, Cambridge, 2003. MR2004511 (2004g:14051)
A gentle introduction.

[2] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 (95m:14012)
Not the place to start, but one you'll have to visit eventually. This edition contains an 8$^{\text{th}}$ chapter that is a nice introduction to moment maps in GIT.

Worst one-parameter subgroups

[3] George R. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), no. 2, 299–316, DOI 10.2307/1971168. MR506989 (80c:20057)
In addition to the analysis of instability, this gives the definitive treatment of the numerical criterion.

Moment maps and the symplectic viewpoint

[4] Chris Woodward, *Moment maps and geometric invariant theory* (2011), available at `http://arxiv.org/abs/0912.1132`.
A very complete survey of the Kempf-Ness theorem, Hesselink-Kirwan-Ness stratifications, moment maps and much more.

Stability of Hilbert and Chow points

[5] D. Gieseker, *Global moduli for surfaces of general type*, Invent. Math. **43** (1977), no. 3, 233–282. MR0498596 (58 #16687)
"The Feit-Thompson of algebraic geometry", David Mumford

[6] Huazhang Luo, *Geometric criterion for Gieseker-Mumford stability of polarized manifolds*, J. Differential Geom. **49** (1998), no. 3, 577–599. MR1669716 (2001b:32035)
Stability of Hilbert points via a balanced metric.

[7] Ian Morrison, *GIT constructions of moduli spaces of stable curves and maps*, Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces, Surv. Differ. Geom., vol. 14, Int. Press, Somerville, MA, 2009, pp. 315–369. MR2655332 (2011m:14077)
Focuses on curves with lots of details.

[8] David Mumford, *Stability of projective varieties*, Enseignement Math. (2) **23** (1977), no. 1-2, 39–110. MR0450272 (56 #8568)
Still a great reference many of the basic ideas about Chow and Hilbert stability, especially a algebraic geometer's view of test configurations