Conic Kähler-Einstein Metrics

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Let M be a Kähler manifold and $D \subset M$ be a smooth divisor

A conic Kähler metric on M with angle $2\pi\beta$ $(0 < \beta \leq 1)$ along D is a Kähler metric on $M\backslash D$ asymptotically equivalent along D to the model

$$\omega_{0,\beta} = \sqrt{-1}\left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2-2\beta}} + \sum_{j=2}^{n} dz_j \wedge d\bar{z}_j\right),$$

where $z_1, z_2, \cdots, z_n$ are holomorphic coordinates such that $D = \{z_1 = 0\}$ locally.

A conic Kähler metric can be given by its Kähler form ω which represents a cohomology class, i.e., Kähler class,

$$[\omega] \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$$

A conic Kähler-Einstein metric is a conic Kähler metric which are also Einstein outside conic points. It satisfies as currents:

$$\mathrm{Ric}(\omega) = \mu\omega + (1-\beta)[D]$$

Cohomological constraint: $c_1(M) = \mu[\omega] + (1-\beta)[D]$.

If $n = 1$ and $[\omega]$ is the positive generator of $H^2(M, \mathbb{Z}) = \mathbb{Z}$, then $\mu = 2 - 2g - (1 - \beta)d$, where d is the degree of $D = p_1 + \cdots + p_k$.

If $\mu \leq 0$, there is a unique conic metric of constant curvature μ in given class $[\omega]$.

If $\mu > 0$, there is a unique spherical conic metric if and only if

$$\forall i, \ \ (1 - \beta_i) < \sum_{j \neq i} (1 - \beta_j).$$

These are classical (Troyanov, McOwen, Thurston and Luo-Tian etc.). Note that last condition is equivalent to saying: $(M, \sum_i (1 - \beta_i) p_i)$ is K-stable in modern terminology.

• Brendle: Ricci-flat conic Kähler metrics with cone angle $\leq \pi$ along D with restriction;

• Jeffres-Mazzeo-Rubinstein: Existence of conic Kähler-Einstein metrics with non-positive scalar curvature. Also a result in the case of positive scalar curvature;

• Related works by Campana, Guenancia and Paun.

There are at least two motivations to study in higher dimensions:

Conjecture: There are only finitely many rational curves in a complex surface M of general type.

Bogomolov: True if $c_1(M)^2 > c_2(M)$;

In 1994, I observed that if we have a conic version of the Miyaoka-Yau inequality: $c_1(M, D)^2 \le 3c_2(M, D)$, then Bogomolov's theorem still holds even if $c_1(M)^2 = c_2(M)$ and one has an effective bound on the number of rational curves if $c_1(M)^2 > c_2(M)$.

More recently, Donaldson proposed a new continuity method by using conic Kähler-Einstein metrics for solving:

Conjecture: If M is a Fano manifold without holomorphic fields, M admits a Kähler-Einstein metric if and only if M is K-stable.

The necessary part of this conjecture was proved by myself in 1996.

One approach to proving the sufficient part is to use Aubin's continuity method:

Consider $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h-t\varphi}\,\omega^n$, where ω and f are given.

Known: The set of t for which it is solvable is non-empty and open and if it is closed if the K-energy or its Lagrangian is proper.

In general, the K-energy is not proper due to holomorphic obstructions, such as, the K-stability. The K-stability was introduced by myself in 1996 and reformulated algebraically by Donaldson in early 2000s. My definition relates the properness of K-energy to the K-stability.

Assume M is Fano, so K_M^{-1} is ample. By Kodaira, for $\ell >> 1$, any basis of $H^0(M, K_M^{-\ell})$ gives an embedding $\phi : M \mapsto \mathbb{C}P^N$, then we get a family of metrics

$$\mathcal{K}_\ell = \{\frac{1}{\ell}\phi^*\tau^*\omega_{FS}\}.$$

• $\bigcup \mathcal{K}_\ell$ is dense in the space of Kähler metrics on M with Kähler class $c_1(M)$.

• The K-stability is equivalent to the properness of the K-energy restricted to $\mathcal{K}_\ell$ for some sufficiently large ℓ.

hence, the conjecture follows if one can affirm the partial C^0-estimate conjecture (Tian, 1990).

Any ω with Kähler class $c_1(M)$ induces an inner product on each $H^0(M, K_M^{-\ell})$, let $\{S_i\}_{0\le i\le N}$ be an orthonormal basis of $H^0(M, K_M^{-\ell})$ w.r.t. this inner product. Put

$$\rho_{\omega,\ell}(x) = \sum_{i=0}^{N} ||S_i||^2(x).$$

Conjecture (Tian, 1990): There are $c_k = c(k,n) > 0$ for $k \geq 1$ such that for any ω with Kähler class $c_1(M)$ and Ricci curvature $\geq t_0 > 0$,

$$\rho_{\omega,\ell}(x) \geq c_\ell$$

for a sufficiently large ℓ.

This is open except for Kähler-Einstein metrics (Donaldson-Sun, Tian).

Donaldson's continuity method: If D is a smooth anti-canonical divisor, then there is a complete CY metric on $M\backslash D$ (Tian-Yau,1990). It was expected that this complete metric is the limit of Kähler-Einstein metrics with conic angle $2\pi\beta \mapsto 0$. If so, the set E of $\beta \in (0,1]$ such that there is a conic Kähler metric with angle $2\pi\beta$ is non-empty.

Not known if every Fano manifold admits a smooth anti-canonical divisor.

Li-Sun's modified version: Let D be a smooth divisor whose Poincare dual is $\lambda c_1(M)$ and E be the set of $\beta \in (1-\lambda^{-1}, 1]$ such that there is a conic Kähler metric with angle $2\pi\beta$ along D.

- $\beta \in E$ if β is close to $1-\lambda^{-1}$, so E is non-empty (Jeffres-Mazzeo-Rubinstein, Li-Sun)

- E is open (Donaldson)

- $\beta \in E$ if the (twisted) K-energy for β is proper (Jeffres-Mazzeo-Rubinstein)

There is a conic version of the partial C^0-estimate:

There are $c_k = c(k, n, \beta_0) > 0$ for $k \geq 1$ such that for any conic Kähler-Einstein metric ω in E with cone angle $2\pi\beta$ along D ($\beta \geq \beta_0 > 0$) and sufficiently large ℓ,

$$\rho_{\omega,\ell}(x) \geq c_\ell.$$

If this partial C^0-estimate is true, then one can prove that the twisted K-energy is proper along E whenever M is K-stable. This follows from a result of Li-Sun etc. that if M is K-stable, $(M,(1-\beta)D)$ is log K-stable (as defined by C. Li).

Therefore, the crucial step is to establish the conic version of the partial C^0-estimate. As for smooth Kähler-Einstein metrics, we need a compactness theorem of Cheeger-Colding-Tian type for conic metrics.

Let ω_i be a sequence of conic Kähler-Einstein metrics with cone angle $2\pi\beta_i$ along D converging to (M_∞, d_∞) in the GH topology. Then

- $\exists S \subset M_\infty$, closed and of codimension 2, s.t. $M_\infty \backslash S$ is smooth

- d_∞ is induced by a Kähler-Einstein metric ω_∞ outside S

- ω_i converges to ω_∞ in the C^∞-topology outside S

Moreover, we can prove that if $\beta_\infty = 1$, M_∞ is smooth outside $S_0 \subset S$ of codimension at least 4.

To prove the above, we use some arguments in the proof of Cheeger-Colding-Tian, but there are some new inputs, especially, in establishing the smooth convergence of ω_i outside S.

First we study the following problem:

Can one approximate a conic Kähler-Einstein metrics by smooth Kähler metrics with Ricci curvature bounded from below?

If $n = 1$, it is clear and a local problem. But the method of smoothing conic metrics in dimension 1 do not extend to higher dimensions. We solve this problem by using complex Monge-Ampere equations.

Let ω be a conic Kähler-Einstein metric on M with cone angle $2\pi\beta$ along D.

Choose a smooth Kähler metric ω_0 with $[\omega_0] = c_1(M)$. Define h_0 by

$$\mathrm{Ric}(\omega_0) - \omega_0 = \sqrt{-1}\partial\bar{\partial}h_0, \quad \int_M (e^{h_0} - 1)\omega_0^n = 0.$$

This is equivalent to

$$\mathrm{Ric}(\omega_0) = \mu\omega_0 + (1-\beta)[D] + \sqrt{-1}\partial\bar{\partial}(h_0 - (1-\beta)\log ||S||_0^2),$$

where S is a holomorphic section of $K_M^{-\lambda}$ defining D and $||\cdot||_0$ is a Hermitian norm on $K_M^{-\lambda}$ with $\lambda\omega_0$ as its curvature.

Write $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ for some smooth function φ on $M \backslash D$. Then

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_0 - (1-\beta)\log ||S||_0^2 - \mu\varphi}\omega_0^n.$$

Note that φ is Hölder continuous.

To find smooth Kähler metrics which approximate ω, we consider

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_\delta - \mu\varphi}\omega_0^n,$$

where $h_\delta = h_0 - (1-\beta)\log(\delta + ||S||_0^2) + c_\delta$ for some c_δ determined by

$$\int_M \left(e^{h_0 - (1-\beta)\log(\delta + ||S||_0^2) - c_\delta} - 1\right)\omega_0^n = 0.$$

If φ_δ is a smooth solution, then $\omega_\delta = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\delta$ has its Ricci curvatire greater than μ whenever $\beta < 1$ and $\delta > 0$.

We can prove

• There is a smooth solution for the above equation for any $\delta > 0$

• ω_δ converge to ω in the smooth topology on $M\backslash D$.
• ω_δ converge to ω in the GH topology on M. This is proved by a C^2-estimate.

Moreover, we have:

$\forall \ell > 0$, let $< \cdot, \cdot >_\delta$ be the inner product on $H^0(M, K_M^{-\ell})$ induced by ω_δ. Then as $\delta \to 0$, $< \cdot, \cdot >_\delta$ converge to the corresponding inner product by ω. In particular, $\rho_{\omega_\delta, \ell}$ converge to $\rho_{\omega, \ell}$.

Now we have

- For each $p \in M_\infty$, tangent cones T_pM_∞ exist and complex
- The set S of $p \in M_\infty$ for which no tangent cone T_pM_∞ is $\mathbb{R}^{2n}$ is closed and of codimension 2.

It follows from the works of Cheeger-Colding and Cheeger-Colding-Tian. Moreover, if $\lim \beta_i < 1$, then ω_i converge to ω_∞ in the smooth topology outside S.

If $\lim \beta_i = 1$, then the standard arguments do not apply. However, by our approximation theorem, there is a sequence of smooth Kähler metrics $\tilde{\omega}_i$ with Ricci curvature bounded from below by $(1-\beta_i)\lambda$ and converging to (M_∞, d_∞) in the GH topology.

This sequence satisfies the conditions in my work with B. Wang on almost Kähler-Einstein manifolds, so M_∞ is actually smooth outside $S_0 \subset S$ of codimension at least 4.

It remains to prove that if $\lim \beta_i = 1$, ω_i converge to ω_∞ outside S. This is done by studying the limit of defining sections σ_i of D with $||\sigma_i||_i = 1$. We need to prove

- σ_i converge to a holomorphic section σ_∞ of $H^0(M_\infty, K_{M_\infty}^{-\lambda})$ in the suitable sense

- $S = \sigma_\infty^{-1}(0)$

Then one can prove that ω_i converge to ω_∞ in the smooth topology outside S.

Once we have the extension of Cheeger-Colding-Tian, we can use the L^2-estimate to prove the partial C^0-estimate for conic Kähler-Einstein metrics:

There are $c_k = c(k, n, \beta_0) > 0$ for $k \geq 1$ such that for any conic Kähler-Einstein metric ω with positive scalar curvature and cone angle $2\pi\beta$ along D,

$$\rho_{\omega,\ell}(x) \geq c_\ell$$

for sufficiently large ℓ and $\beta \geq \beta_0 > 0$.

With the partial C^0-estimate, by following the arguments described before, we can use conic Kähler-Einstein metrics to prove:

If a Fano manifold M is K-stable, then M admits a Kähler-Einstein metric.

This completes the solution of the folklore conjecture on the existence of Käh;er-Einstein metrics.