Lawson Birthday Conference

Instantons in G_2-geometry

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Why G2 ?

§ G_2 in Math.

$\mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$ $\mathbb{O}$ normed algebras.
division

$$G_2 = \text{Aut}_{\text{alg}}(\mathbb{O}).$$

$G_2 \curvearrowright \mathbb{O}$ fixing   1

$\Rightarrow G_2 \curvearrowright \text{Im}\,\mathbb{O} \simeq \mathbb{R}^7.$

§ G_2 in Physics.

String theory $\mathbb{R}^{3,1} \times Y^6$ Calabi-Yau 3-fold
                                                    SU(3)

                                                    $\cap$
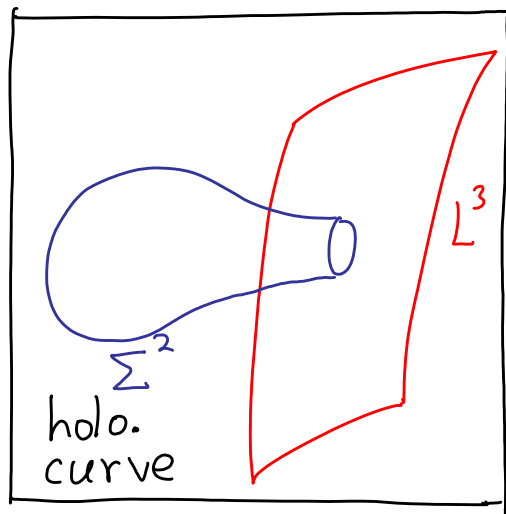
M-theory $\mathbb{R}^{3,1} \times M^7$ G_2 - mfd.

                                                    $\cap$

F-theory $\mathbb{R}^{3,1} \times Z^8$ $\underbrace{\text{Spin(7)}}\text{- mfd.}$

                                                    $\text{Aut}_{twist}(\mathbb{O}).$

Y^6 $\subset Y^3$

L^3

Σ^2

holo. curve

Special Lagrangian submfd.

+

flat bundle

Duality
(eg. Mirror Symmetry)

Y^6 CY^3

M^7 G_2-mfd.

Σ^2

holo. curve

L^3
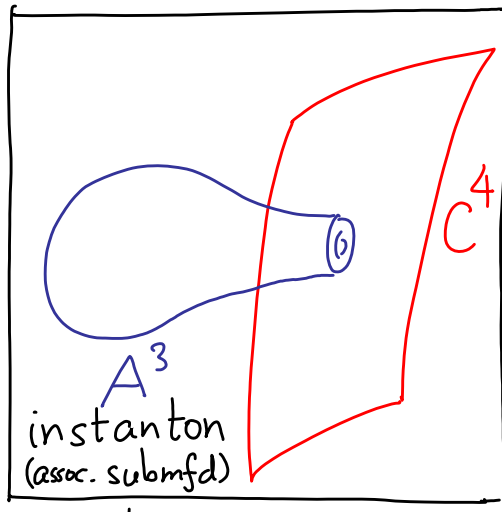
Special Lagrangian submfd.

+

flat bundle

Duality

(eg. Mirror Symmetry)

A^3

instanton

(assoc. submfd)

+

Chern-Simons theory

C^4

brane

(coass. submfd)

+

ASD bundle

Triality ?

Low dim. Geometry

Low: $3, 4 \subseteq 7$

§ G_2- manifolds

Recall $(V \simeq \mathbb{R}^{2n}, g)$ inner product space
$$\simeq T_p X$$

- $\iff$ Hermitian complex structure $J : V \to V$

$$\begin{array}{ccc} J \overset{\vee}{} & \perp & \overset{\vee}{} \\ |J v| & = & |v| \end{array}$$

i.e. 1-VCP
(vector cross product)

$$Aut(V, g, J) = U(n)$$

- $\iff$ (X, g) Kähler
$$J : T_p X \longrightarrow T_p X \quad w/ \quad \nabla J = 0$$

i.e. $Hol(X) \subseteq U(n)$

- $\Sigma \subset X$ cpx submfd
$\iff$ $T_p\Sigma \subset T_pX$ preserved by J
$\iff$ Σ calibrated by $\omega^k/k!$ ($2k = \dim_{\mathbb{R}}\Sigma$)

$\omega(u,v) := g(Ju,v)$ Kähler form.

$\implies$ abs. volume min.

Harvey - Lawson,
Calibrated Geometry, Acta 1982.

When $\dim_{\mathbb{C}}\Sigma = 1$
$\rightsquigarrow$ holo. curve/instanton in String Theory.

Def. $\quad \times : \quad V \otimes V \longrightarrow V \qquad$ 2-VCP

if $\left\{ \begin{array}{l} v_1 \times v_2 \quad \perp \quad v_1 \ \& \ v_2 \\ \\ |v_1 \times v_2| = Area\left(\ \overset{v_2}{\underset{v_1}{\diagup\!\!\!\!\diagup}} \ \right) \end{array} \right.$

Eg. $\quad V = \mathbb{R}^3 \qquad = Im \mathbb{H}$

$\quad v_1 \times v_2 \qquad = Im(v_1 \cdot v_2)$

Another eg. : $\qquad Im \mathbb{O}$

$Aut(\mathbb{R}^7, g, \times) = Aut_{alg}(\mathbb{O}) = G_2$

Classification :   No other eg.

Def: (i) $(M^7, g, \times)$ w/ $\nabla \times = 0$ G_2-mfd.

(iii) $A^3 \subset M^7$ preserved by $\times$

associative submfd. / instanton.

(in M-theory).

Harvey-Lawson

($\Leftrightarrow$ calibrated by $\Omega \in \Omega^3(M)$)

$$\Omega(u, v, w) = g(u \times v, w)$$

Remark: Closed Vector Cross Product :

$$\underline{\text{Instanton}}$$

2-VCP	$\Omega \in \Omega^3(M)$	$d\Omega = 0$	Associative submfd.
1-VCP	$\omega \in \Omega^2(M)$	$d\omega = 0$	Holomorphic curve
0-VCP	$\alpha \in \Omega^1(M)$	$d\alpha = 0$ $(\alpha = df)$	Gradient flow line of f (Instantons in Witten's Morse theory)

Boundary valued problem:

$$\Sigma^2 \subset X \text{ Kähler}$$
$$\cup \qquad \cup$$
$$\partial\Sigma \subset L$$
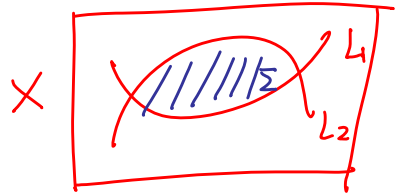
If $\omega|_L = 0$, then

Σ calibrated by $\omega \Rightarrow$ abs. vol. min.

Fredholm $\Rightarrow \quad \dim L = \frac{1}{2} \dim X$

i.e. Lagrangian submfd.

(brane in open string theory).

- Lagrangian intersection theory

X count holo. disks $\Sigma^2 \subset X$

w/ $\partial\Sigma \subset L_1 \cup L_2 \cup \ldots \cup L_k$

$\rightsquigarrow$ Fukaya category $\mathrm{Fuk}(X)$

- Kontsevich HMS

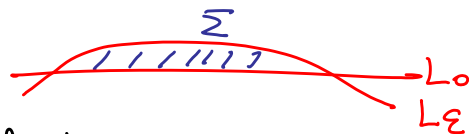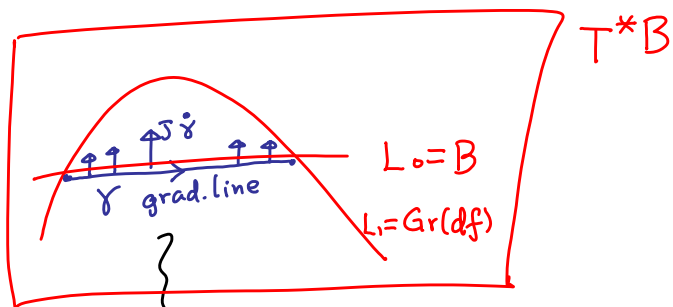$$\mathrm{Fuk}(X) \simeq D^b(X^\vee)$$

sympl.   complex.

mirror mfd.

- Key cases: $X = T^*B$ (Fukaya-Oh)

$$Fuk(X) \simeq Morse(B)$$

count holo. disks   count gradient flow line γ

of $f: B \longrightarrow \mathbb{R}$

$\rightsquigarrow$ Lagr. $L_1 = Graph(df) \subset T^*B$

$L_0 = $ o-section

$L_0 \cap L_1 = \{$ crit. pt. of $f\}$

T^*B

$L_0 = B$

γ grad. line

$L_1 = Gr(df)$

Σ

L_0

L_ε

(after perturbation) $\rightsquigarrow$ holo. disk Σ w/ bdy in

$L_0 \cup \underbrace{L_\varepsilon}_{Graph(\varepsilon(df))}$ for small ε

Back to G_2 : $A^3 \subset M^7$

$$\cup \qquad \cup$$

$$\partial A = C$$

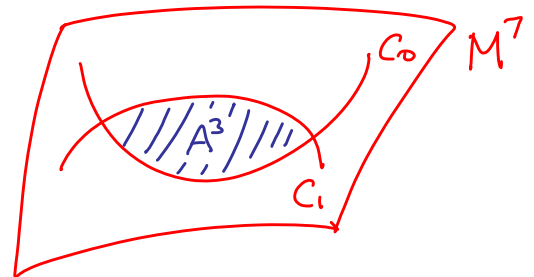- If $\Omega|_C = 0$, then

  A calibrated by $\Omega \Rightarrow A$ abs. vol. min

- Fredholm $\Rightarrow$ $\dim C = [\frac{1}{2} \dim M] = 4$.

  i.e. C : Coassociative submfd./Brane

- Coassociative Intersection

  Theory ?

Prop. Coass. $C^4 \subset (M^7, \times)$ G_2-mfd.

$\Rightarrow$ (i) $N_{C/M} \simeq \Lambda^2_+ T_C^*$

(ii) infinitesimal deformation
$$\longleftrightarrow \eta \in \Omega^2_+(C) \text{ s.t. } d\eta = 0$$
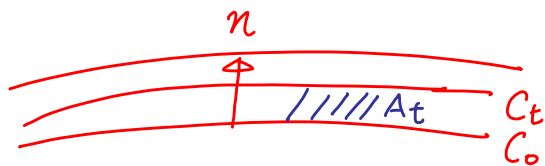
Compare w/. Lagr. $L \subset (X, \omega)$ Sympl. mfd.

(i) $N_{L/X} \simeq T_L^*$

(ii) inf. deform $\longleftrightarrow \alpha \in \Omega^1(L)$ s.t. $d\alpha = 0$

If $\eta(p) \neq 0$ (normalize $\eta/|\eta|$), then

$$J : T_p C \longrightarrow T_p C \quad \text{w/} \quad g(Ju, v) = \frac{\eta}{|\eta|}(u, v)$$

$$\Rightarrow \left(T_p C, g, J, \frac{\eta}{|\eta|} \right) : \text{Herm. vector space.}$$

$$C_t \overset{\text{coass.}}{\subset} (M, \times)$$

$$n = \frac{dC_t}{dt}\Big|_{t=0} \qquad \text{normal v.f.}$$

$$\rightsquigarrow \quad \eta = \imath_n \Omega \in \Omega_+^2(C).$$

Theorem (L.-Wang-Zhu, JDG 2012)

Assume η nonvanishing, then

(1) instantons $A_t \subset M$

s.t. $\partial A_t \subset C_t \cup C_0$

$\Rightarrow \Sigma_0 := \lim_{t \to 0} (A_t \cap C_0)$ is J-holo. curve in C_0

(2) Conversely, every regular J-holo. curve in C_0 is the limit of a family of instantons A_t's as described above.

Remark : Count such instantons

$$A^3 \subset M^7 \quad \text{w/} \quad \partial A \subset C_0 \cup C_\varepsilon$$

$(C_0 \cap C_\varepsilon = \phi)$ Count holomorphic curves

$$\Sigma^2 \subset C_0$$

i.e. Gromov-Witten inv. of C_0

Taubes $\sim$ $SW(C_0)$. Seiberg-Witten inv.

- Expect this continuous to hold true even if $C_0 \cap C_\varepsilon \neq \phi$.

When $C_0 \cap C_\varepsilon = \phi$

$\Rightarrow \quad (C, \eta = \ln \Omega) \quad$ symplectic.

Otherwise $\overbrace{C_0 \cap C_\varepsilon}^{\{\eta = 0\}} = \coprod S^1 \subset C_0$

and η degenerates along these S^1's.

Taubes analysed holo. curves

$\Sigma \subset C \setminus \{\eta = 0\}$ and they could have

rather nontrivial asym. behaviors near

these S^1's.

§ Applications / Eg. of instantons
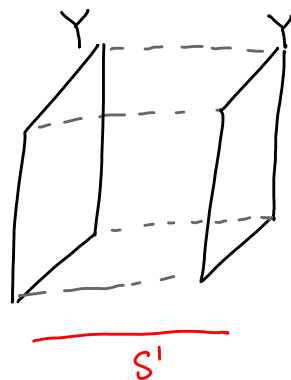
Recall: $Y^{3\mathbb{C}}$ Calabi-Yau 3-fold.

i.e. (Y, g_Y, J_Y, ω_Y) Kähler

w/ $Ricci = 0$
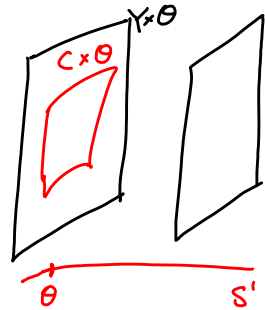
$\leadsto$ $\exists$ parallel holomorphic 3-form

$$\Omega_Y \in \Omega^{3,0}(Y)$$
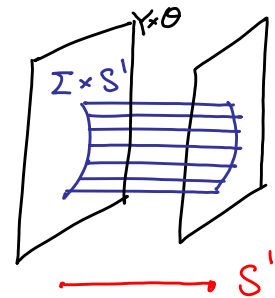
$$\implies M^7 = Y^{3\mathbb{C}} \times S^1 \quad G_2\text{-mfd.}$$

w/ $$\Omega = Re\,\Omega_Y + \omega_Y \wedge d\theta.$$

- Complex surface $C^{2\mathbb{C}} \subset Y^{3\mathbb{C}}$
$\Rightarrow$ coass. submfd. $C \times \{\theta\} \subset M^7$
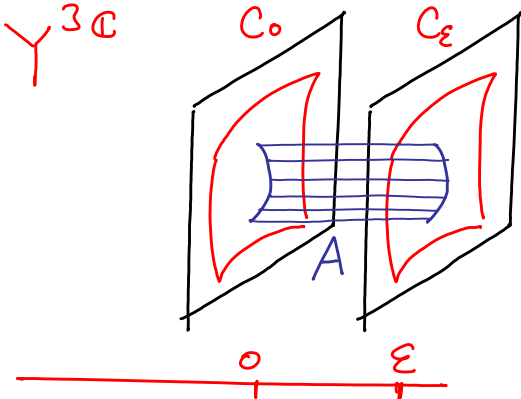(for any $\theta \in S^1$)

- holo. curve $\Sigma^{1\mathbb{C}} \subset Y^{3\mathbb{C}}$
$\Rightarrow$ instanton $\Sigma \times S^1 \subset M$

Thus, if $\Sigma^{1\mathbb{C}} \subset C^{2\mathbb{C}} \subset Y^{3\mathbb{C}}$
$\rightsquigarrow$ trivial example
$A = \Sigma \times [0, \varepsilon] \subset M$
$\partial A \quad \subset C_0 \cup C_\varepsilon$

To obtain nontrivial example:

take $\Sigma^{1\mathbb{C}} \subset C^{2\mathbb{C}} \subset Y^{3\mathbb{C}}$ CY^3

satisfying Σ as unique canonical curve of C

i.e. $p_g(C) = 1.$

$\Rightarrow$ (1) $\Sigma \subset C$ is regular

i.e. $H^1(\Sigma, N_{\Sigma/C}) = 0$

(2) C can be deformed inside Y

$\rightsquigarrow$ cpx. surface $C(s) \subset Y$

for small $s \in \mathbb{C}.$

Even though $C \cap C(\Delta) \neq \emptyset$ in Y,

$$\underbrace{C \times \{0\}}_{C_0} \cap \underbrace{C(\Delta) \times \{\varepsilon\}}_{C_\varepsilon} = \emptyset \quad \text{in} \quad M = Y \times S^1$$

$\qquad\qquad\qquad\qquad$ if $\varepsilon \neq 0$

$\rightsquigarrow$ almost complex structure J_Δ on C_0.

For generic Δ, then Σ can be perturbed to regular J_Δ-holo. curve $\Sigma_\Delta \subset C_0$. ($\because$ $\Sigma \subset C$ is regular wrt original $J \sim J_\Delta$)

Thm $\Rightarrow$ $\exists$ instanton $A_\varepsilon \subset M = Y \times S^1$

$\qquad\qquad\qquad\qquad\quad \partial A_\varepsilon \subset (C \times 0) \cup (C(\Delta) \times \varepsilon)$
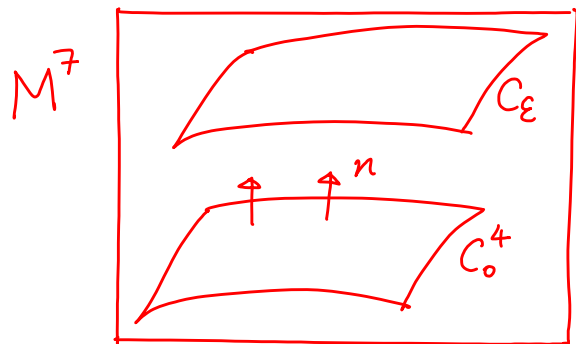
§ Ideas of proof of theorem:

Recall :

Theorem (LWZ): (1) instantons $A_t \subset M$

s.t. $\partial A_t \subset C_t \cup C_0$

$\Rightarrow \Sigma_0 := \lim_{t \to 0} (A_t \cap C_0)$ is J-holo. curve in C_0

(2) Conversely, every regular J-holo. curve in C_0 is the limit of a family of instantons A_t's as described above.
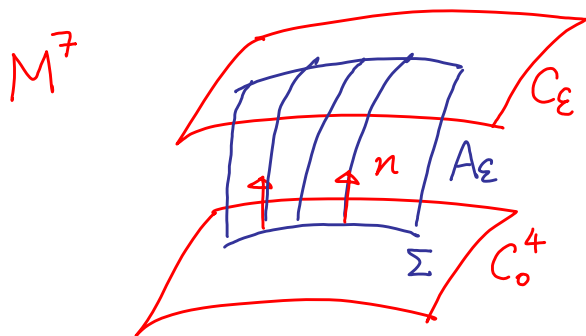
M^7

C_ε

n

C_0^4

$n = \dfrac{dC_t}{dt}\Big|_{t=0}$ normal vector field
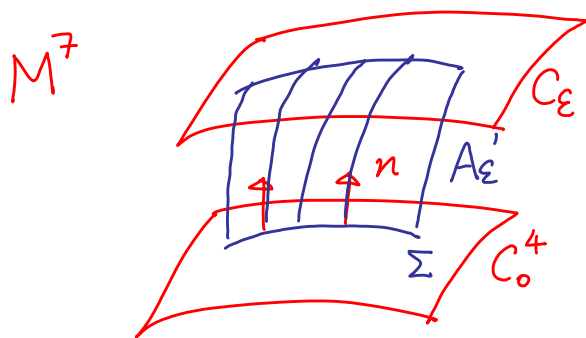
$\eta = \iota_n \Omega \in \Omega_+^2(C_0)$

$\dfrac{\eta}{|\eta|} \xrightarrow{g|_{C_0}} J$ almost complex structure on C_0

Given $\quad \Sigma^2 \subset C_0^4 \quad$ J-holo. curve,

M^7

C_ε

n A_ε

Σ C_0^4

$\rightsquigarrow A_\varepsilon^3 \subset M^7$ obtained by 'flowing' Σ from C_0 to C_ε.

Given $\Sigma^2 \subset C_0^4$ J-holo. curve,

$\rightsquigarrow$ $A'^3_\varepsilon \subset M^7$

"almost" instanton.

HOPE : If ε small enough, then A'_ε can be perturbed to an instanton.

Remark : For $\Sigma^2 \underset{\text{holo.}}{\subseteq} (X, \omega)$

$\partial\Sigma \subset L_o \cup L_\varepsilon$

$L_\varepsilon = \text{Graph}(\varepsilon \, df)$

Fukaya - Oh related holo. curves in X w/ gradient path of f.

Key differences :

- Conformal Geometry
useful in dim. 2 but NOT in dim 3

- $\bar{\partial} \, u = 0$ w/ $u : [0, 1] \times \mathbb{R} \to M$ <u>linear</u> , <u>uniform elliptic</u>.
Not true for associative submanifolds.

For holomorphic disks in Fukaya-Oh,
L^p-type estimate is enough.

For instantons in G_2-setting
We need Schauder estimates for
Dirac operator $\not{D}$ on $A'_\varepsilon = \Sigma \times [0, \varepsilon)$,
a degenerating domain as $\varepsilon \to 0$.

Setup: First, find diff. equation
which characterize instanton $A^3 \subset (M^7, x)$
G_2-mfd.

Prop.: A instanton $\iff$ $\tau|_A = 0$

where $\tau \in \Omega^3(M, T_M)$

$$g(\tau(u,v,w), \nu) = \underbrace{(*\Omega)}_{\in \Omega^4(M)}(u,v,w,\nu)$$

· $\tau|_A \in \Omega^3(A, T_M|_A)$
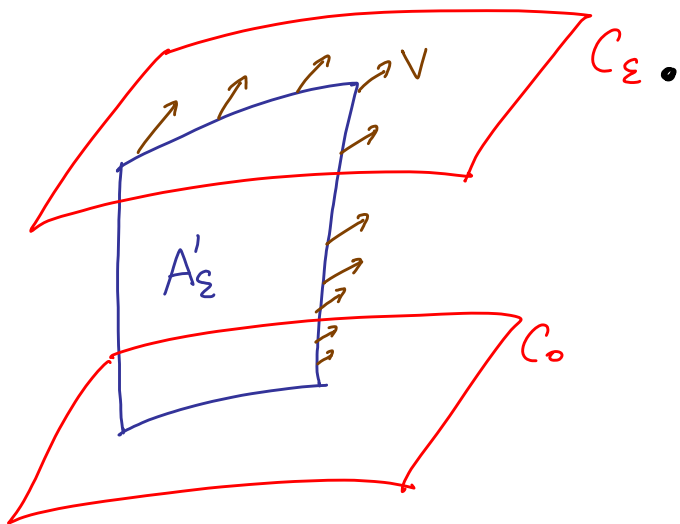
but indeed

$\tau|_A \in \Omega^3(A, N_{A/M})$.

Prop. (McLean) instanton $A^3 \subset (M^7, \times)$ G_2-mfd.

$\Rightarrow$ (i) $N_{A/M} \simeq \not{S} \underset{\mathbb{H}}{\otimes} E$

(ii) $\left\{ \begin{array}{c} 1^{st} \text{ order deformations} \\ \text{of instantons} \end{array} \right\} \simeq \text{Ker } \not{D}$

$$F : \Gamma(N_{A/M}) \longrightarrow \Gamma(N_{A/M})$$

$$V \longmapsto \underbrace{T_v \circ ((\exp V)^* \tau)}_{\in \Omega^3(A, T_M)} \longmapsto_{\perp_{N_{A/M}}} \longmapsto_{*} F(V)$$

C_ε •

$F_\varepsilon(v) = 0$

$\rightsquigarrow A_\varepsilon(v) \subset M$ instanton

w/ $\partial A_\varepsilon(v) \subset C_0 \cup C_\varepsilon$

where $F_\varepsilon : C_-^{m,\alpha}(A'_\varepsilon, N_{A'_\varepsilon/M}) \longrightarrow C^{m-1,\alpha}(A'_\varepsilon, N_{A'_\varepsilon/M})$

Linear analysis: $\not{D}$ on $\$_E$ over A_ε

$$A_\varepsilon = \Sigma \times [0, \varepsilon]$$

w/ $g_{A_\varepsilon} = g_\Sigma + |n|^2 dx^2$ warped product

Need control of $\not{D}$ as $\varepsilon \to 0^+$.

LEMMA: $\|\not{D}^{-1}\| \leq C_\delta \, \varepsilon^{-(2d + \delta)}$ $\forall \delta > 0$

Pf: • $\lambda_1(\not{D}$ on $L^{1,2}_-(A_\varepsilon, \$)) \geq \lambda_1(\partial^+$ on $L^{1,2}(\Sigma, \$^+))$

    • L^2-estimate.

    • L^P-estimate   (reflection from A_ε to $A_{k\varepsilon}$)

    • C^0-estimate   Bigger.

    • $C^{1,\alpha}$-estimate   (Schauder). [weaker as no uniform ellipticity.]

After establishing linear estimates, STILL need delicate identification between $\cancel{S} \longrightarrow A_\varepsilon$ and $N_{A'_{\varepsilon,1M}} \longrightarrow A'_\varepsilon$ (expontial along base + // transport along fibers) Then attempt to use a qualitative version of Implicit Function Theorem to solve $F_\varepsilon(v) = 0$.

Complications: Cubic nonlinearity......

Q.E.D.

Remarks on r-fold VCP on $(V, \langle \ \rangle)$

- $\times : \Lambda^r V \longrightarrow V$; $\Omega \in \Lambda^{r+1} V^*$

$$v_1 \times \cdots \times v_r \perp v_1, \ldots, v_r$$

$$|v_1 \times \cdots \times v_r| = \text{Vol}\left(\begin{array}{c} \end{array} \right)$$

- Instanton: $A^{r+1} \subset (M^n, \times, \Omega)$

$\overset{\Delta}{\Longleftrightarrow}$ preserved by $\times$

$\Longleftrightarrow$ calibrated by Ω

$\Longleftrightarrow \tau|_A = 0$ $\qquad \tau \in \Omega^{r+1}(M, \mathfrak{J}_M^\perp)$

$\qquad\qquad\qquad \mathfrak{J}_M \leqslant \Lambda^2 T_M^*$ symmetry of $\times$.

- Brane: $C^{(n+r-1)/2} \subset (M, \times, \Omega)$, $\Omega|_C = 0$

- $\partial A \subset C$

r	Manifold	Instanton	Brane
$n-1$	Oriented	domain	hypersurface
1	Kähler	holo. curve	Lagrangian
2	G_2		
3	Spin(7)	Cayley submfd.	$\not\exists$

- <u>Complex</u> VCP on $(V_{\mathbb{C}}, \langle \ \rangle)$

  $\Rightarrow$ only $r = 1$ or $n-1$

r	Manifold	$\mathbb{C}$-instanton	Neumann Brane	Dirichlet Brane
$n-1$	Calabi-Yau. (complex oriented)	SLag $_{\theta=0}$	$\mathbb{C}$-hyper-surface	SLag $_{\theta=\pi/2}$
1	HyperKähler (complex Lagrangian)	holo. curve	$\mathbb{C}$-Lagr.	$\mathbb{I}-\mathbb{C}$-Lagr

$\rightsquigarrow$ Berger classification of holonomy groups.

$\rightsquigarrow$ Calibrated cycles in Harvey-Lawson.