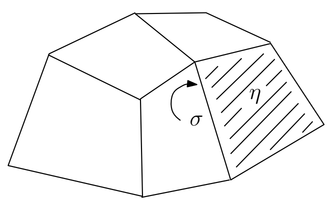
<u>Special Colloquium (4/29/10) : Dennis Sullivan</u>

Leon Takhtajan : *He'll give a talk on continuous and discrete.*
Dennis Sullivan : *Continu***um***!*

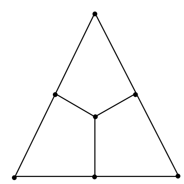
A few references :
Papers of Tony Phillips and David Stone (CMP 80's)
Mahmoud Zeinalian - PhD thesis CUNY 2000
Scott Wilson - PhD thesis SUNY 2005
Nathaniel Rounds - PhD thesis SUNY 2010

Strategy : *To put the usual suspects or structures, like differential forms, d, d^*, wedge, contraction, Lie bracket of vector fields, Hodge star, covariant derivative, of the continuum (both linear and non-linear) on grids or cell decompositions and move these around coherently as these decompositions change.*
Goal : *Apply this to continuum theories obtaining effective theories and descriptions.*

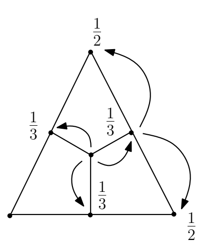
The first is the method due to Poincaré - divide the space into cells and associate chains $(C_\bullet, \partial)$ and cochains $(C^\bullet, \delta)$ to this space. Taking a fine decomposition, this is an obvious approximation to differential forms on the space. I have been thinking about this for quite a few years. One surprise is the interaction of d^* and $[\,,\,]$. We want to have many different decompositions and grids. There is a very nice subdivision called the *pair subdivision.*



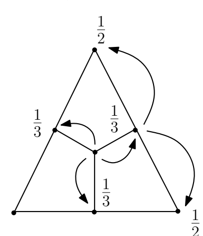
A cell decomposition


The pair subdivision of a triangle

There is a natural inclusion of cells if you give names to the cells; this gives us a partially ordered set. The cells in the pair subdivision are labelled by pairs (σ, η) where $\sigma \subseteq \eta$. This is motivated by the key idea (and a new idea) that any subdivision and the pair subdivision have the same information. There is natural map $f : C_\bullet \to PC_\bullet$. There is a not so obvious map $g : PC_\bullet \to P_\bullet$. You have to fracture your subdivision (like split a point to half of the right vertex and half of the left vertex). Let's see what happens to the edges.



A picture of g

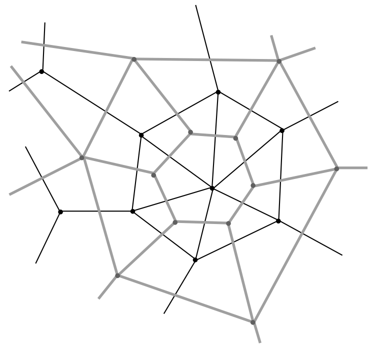


A picture of g

The map f and g both commute with the boundary. We see that $fg = \mathrm{Id}$ but gf is chain homotopic to the identity, i.e., $gf = \mathrm{Id} + \partial S + S\partial$. The operator S has degree 1 and has various pieces like $S_0, S_1, S_2, \cdots$. We inductively solve for $S_0, S_1, \cdots$ and we're working in cells which individually have no homology. Putting an appropriate gauge condition you can solve this uniquely and this leads to Hodge theory. The key new idea is that two chain complexes are quasi-isomorphic and non-linear structures can be moved around via quasi-isomorphisms. This is going to be the main idea or workhorse fact for whatever is presented here.

Remark (1) The pair subdivision of a triangulation is a cubulation. The pair subdivision of a cubulation is

again a cubulation.
(2) Poincaré introduced this idea and used it to show that the Betti numbers in the complementary dimensions are equal.



A picture proof of Poincaré duality

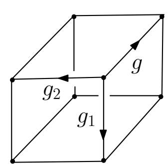
So the pair subdivision is the intersection of any decomposition and its dual decomposition.
(3) If you had a sequence of effective theories and wanted to compute the limit you want them to have good shapes as you go down the limit. The barycentric subdivision is no good as the angle gets smaller and smaller. But the pair subdivision is good.
(4) The pair subdivision has a natural map $cp : P_\bullet \to P_\bullet \otimes P_\bullet$

$$cp((\sigma,\eta)) := \sum_{\sigma\subseteq\tau\subseteq\eta} \pm(\sigma,\tau)\otimes(\tau,\eta)$$

which makes this a coalgebra. It's coassociative but not cocommutative. Now let $\overline{P}_\bullet = P_\bullet/T_\bullet$ for a tree in the 1-skeleton hitting all the vertices. This is called the *reduced coalgebra*. It also has an *eventually zero property.*

So now I'm going to discuss something that's a variant of what's in these papers. Let's do an example.

Example 1. (Lattice field connection)
Let G be a Lie group, the gauge group. If you actually had a continuum, i.e., a bundle with a connection, then you can identify a fibre over a vertex to another fibre by parallel translation. The idea of using a maximal tree removes the ambiguity of choosing an isomorphism of the fibre over a point, with the group. Let A be a dg algebra, thought of as modelling the system of chains on G.



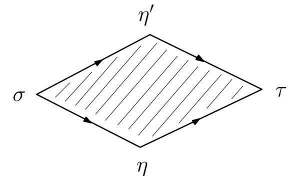
Definition 2. A *completed lattice field* is a sequence of operators $\tau_i : \overline{C}_i \to A_{i-1}$ such that τ_1 assigns to each edge an element of the group and if we form $\tau := \tau_1 + \tau_2 + \cdots \in \mathrm{Hom}_{-1}(\overline{C}_\bullet, A_\bullet)$, this satisfies the Maurer-Cartan equation, i.e.,

$$\not\partial\tau + \tau * \tau = 0, \tag{1}$$

where $*$ is defined by

$$\overline{C}_\bullet \xrightarrow{cp} \overline{C}_\bullet \otimes \overline{C}_\bullet \xrightarrow{\tau\otimes\tau} A_\bullet \otimes A_\bullet \xrightarrow{m} A.$$

The first equation says $\not\partial\tau_2 + \tau_1 * \tau_1 = 0$ and the next says $\not\partial\tau_3 + \tau_1 * \tau_2 + \tau_2 * \tau_1 = 0$. The first deforms a monodromy around a loop to the identity.



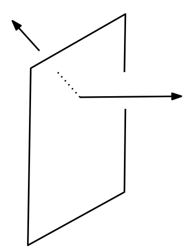
The second gives you a deformation of deformations. Suppose I had an actual connection and I formed the monodromies then these will be small. I fill it in canonically near the identity. Then I fill in with little squares and so on.

If you have a vector space you can form the tensor coalgebra (or algebra if you start with an algebra)

$$\mathcal{B}_A := A \oplus A^{\otimes 2} \oplus \cdots .$$

Given a map $\overline{C} \to A$ from a coalgebra it factors through a map $\overline{C} \to \mathcal{B}_A$. This construction is due to Cartan-Eilenberg. The question is when is this map a map of coalgebras? This happens exactly when (1) has a solution.

Example 3. We can put together a lot of piecewise flat things and approximate any Riemannian manifold like this. You have to parallel translate through a wall



and after a lot of steps one possibly can (after assuming some hypothesis) construct the characteristic classes.

So we have $d, \wedge, d^*, [\,,\,], i, \nabla_A, *$. The fact that d^* is a derivation of $[\,,\,]$ is known to very few people. A lot of what I'll say now is motivated by Scott's thesis. In the dual triangulation picture, boundary corresponds to coboundary. Let $(D_\bullet, \delta_{n-\cdot})$ be the dual decomposition. This is a trivial observation but to appreciate this takes something. The combinatorial Hodge star takes chains to cochains and vice-versa. We also have a commutative diagram

$$\begin{tikzcd}
(\Omega^*, d^*) \arrow[r, "*"] & (\Omega, d_{n-\cdot}) \\
(C_\bullet, \partial) \arrow[u, "\cong"] \arrow[r, "*_c"] & (D^\bullet, \delta_{n-\cdot}). \arrow[u, "\cong"']
\end{tikzcd}$$

You have two complexes here that are quasi-isomorphic. I'll end with two things :
(1) If you transfer this structure to another place, because of the chain homotopies, you don't get associativity but you do get one up to homotopy etc.
(2) If you work over $\mathbb{Z}$ (refer Nathaniel Rounds's thesis) we can use this to determine the homeomorphism type of a manifold (at least in the simply connected case).