Spectrum, Renormalization, and 2D Rational Dynamics associated with certain Self-Similar Groups

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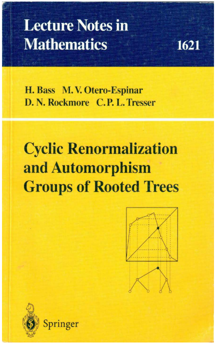
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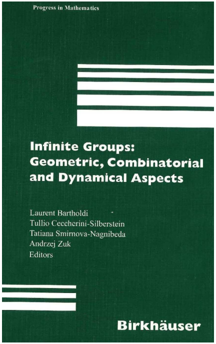
D. N. Rockmore C. P. L. Tresser
Cyclic Renormalization
and Automorphism
Groups of Rooted Trees
Springer


Progress in Mathematics
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Geometric, Combinatorial
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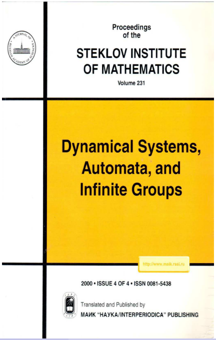
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Volume 231
Dynamical Systems,
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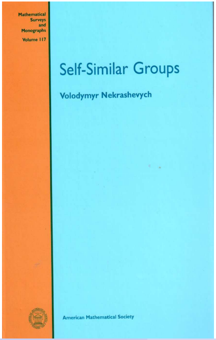
Mathematical
Surveys
and
Monographs
Volume 117
Self-Similar Groups
Volodymyr Nekrashevych
American Mathematical Society

The Goal of the Talk

- Advertise the use of Self-Similar Groups in Dynamics and Spectral Theory of Graphs.
- Explain how renormalization appear in the association with self-similar groups.
- Present some results
- Outline further research.

The first Renormalization maps

N.Sibony 2008: "But this are very special maps."

M. Lyubich 2019: "Intrinsically, they are too degenerate".

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$$F: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{2x^2}{4-y^2} \\ y + \frac{xy^2}{4-y^2} \end{pmatrix}$$

$$G: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{2(4-y^2)}{x^2} \\ -y + \frac{4(4-y^2)}{x^2} \end{pmatrix}$$

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Both maps come from the same group $\mathcal{G} = \langle a, b, c, d \rangle$ called the **First G-group** and we will call them respectively the "**First**" and the "**Second**" **map**.

The group $\mathcal{G}$ has **intermediate growth between polynomial and exponential** and answered in 1984 the question of J. Milnor from 1967. It is also the first example of **amenable but not elementary amenable group**.

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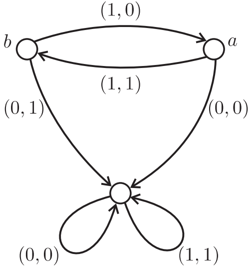
Both maps are "responsible" for the same spectrum (**a joint spectrum of a certain pencil of operators associated with** $\mathcal{G}$.

The Basilica map

$$B\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -2 + \frac{x(x-2)}{y^2} \\ \frac{2-x}{y^2} \end{pmatrix}$$

comes from the **Basilica group** $\mathcal{B}$ generated by automaton



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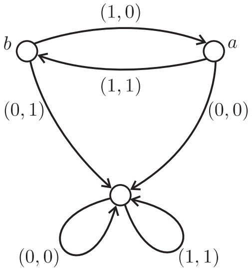

$\mathcal{B} \simeq IMG(z^2 - 1)$ – iterated monodromy group. $\mathcal{B}$ is the first example of **amenable but not subexponentially amenable** group.

The Lamplighter map

$$L\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{x^2-y^2-2}{y-x} \\ \frac{2}{y-x} \end{pmatrix}$$

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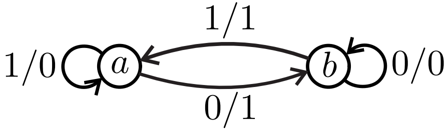
$$L\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{x^2-y^2-2}{y-x} \\ \frac{2}{y-x} \end{pmatrix}$$

comes from the **Lamplighter group** $\mathcal{L} = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} = (\oplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$ realized as a group generated by the automaton



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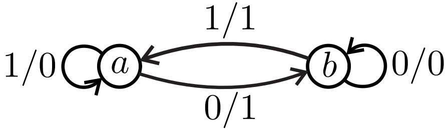

Observe

$$x' + y' = x + y$$

where $(x', y') = F(x, y)$. Hence lines $x + y = c$ are L-invariant.

The Hanoi map

Introduced by **Z. Šuníc** and Grigorchuk in 2007

$$\mathcal{H}\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - \frac{2(x^2-x-y^2)y^2}{(x-y-1)(x^2+y-y^2-1)} \\ \frac{(x+y-1)y^2}{(x-y-1)(x^2+y-y^2-1)} \end{pmatrix}$$

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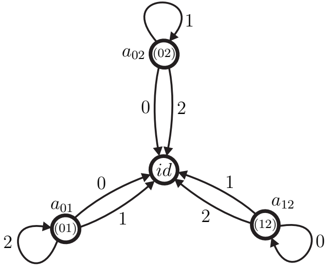
comes from the **Hanoi group** $\mathcal{H}^{(3)}$ associated with the **Hanoi Towers Game on three pegs** and realized as a group generated by the automaton



More about the first two maps

The map $\psi\colon \mathbb{C}^2 \to \mathbb{C}$

$$\psi(x, y) = \frac{4 + x^2 - y^2}{4x}$$

semi-conjugates the First map F to the Chebyshev map $\alpha\colon \mathbb{C} \to \mathbb{C}, \alpha(z) = 2z^2 - 1$.

- The F-preimage of the "horizontal" hyperbola $\mathcal{F}_\theta = 4 + x^2 - y^2 - 4\theta x$ is the union $F_{\theta_1} \cup F_{\theta_2}$ of two hyperbolas, where θ_1, θ_2 are preimages of θ under the Chebyshev map α.
- For values of θ in the interfal $[-1, 1]$ the parts of these hyperbolas fill the domain Ω whose closure $\bar{\Omega}$ is bounded by the lines $x \pm y = \pm 2$.

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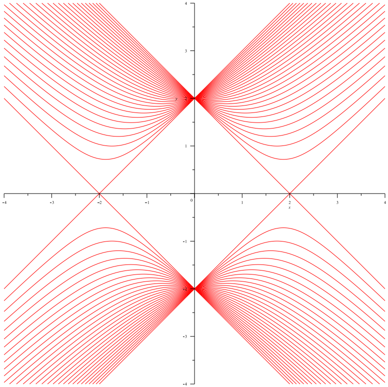

The map

$$\varphi(x,y) = \frac{4 - x^2 + y^2}{4y}$$

semi-conjugates F to the identity map $id\colon \mathbb{C} \to \mathbb{C}$ and the "vertical" hyperbolas $\mathcal{H}_\varphi = \{(x,y)\colon 4 - x^2 + y^2 - 4\varphi y = 0\}$ are F-invariant.

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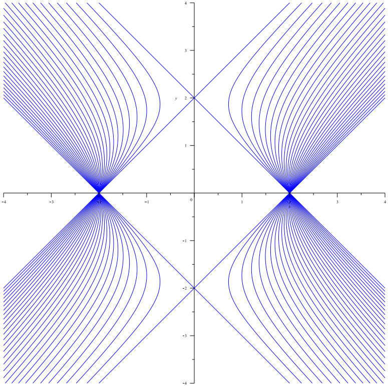

The map $\pi = (\varphi, \psi)\colon \mathbb{C}^2 \to \mathbb{C}^2$ semi-conjugates F with $id \times \alpha$.

This allows to understand the dynamics of F and also to prove

Theorem (Little L-G Equidistribution Theorem)

Let Γ and S be two irreducible algebraic curves in $\mathbb{C}^2$ in coordinates (φ, ψ) such that Γ is not a vertical hyperbola while S is not a horizontal hyperbola. Then

$$\frac{1}{2^n}[(F^n)^*\Gamma \cap S] \to (deg\Gamma) \cdot (degS) \cdot \omega_S,$$

where ω_S is the restriction of the 1-form $\omega = \frac{d\psi}{\pi\sqrt{1-\psi^2}}$ to S.

Here $[S]$ is the **counting measure**.

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This is a starting point for the project initiated few years ago at the Saas-Fee ski resort in Switzerland by Misha Lyubich and speaker, and now accompany by Nguen-Bac Dong. More on this at the end of the talk.

"Extended" version of the first map F

$$\tilde{F}: \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} z+y \\ \frac{x^2(2yzv-u(y^2+z^2-u^2+v^2))}{(y+z+u+v)(y+z-u-v)(y-z+u-v)(-y+z+u-v)} \\ \frac{x^2(2zuv-y(-y^2+z^2+u^2+v^2))}{(y+z+u+v)(y+z-u-v)(y-z+u-v)(-y+z+u-v)} \\ \frac{x^2(2yuv-z(y^2-z^2+u^2+v^2))}{(y+z+u+v)(y+z-u-v)(y-z+u-v)(-y+z+u-v)} \\ u+v+\frac{x^2(2yzu-v(y^2+z^2+u^2-v^2))}{(y+z+u+v)(y+z-u-v)(y-z+u-v)(-y+z+u-v)} \end{pmatrix}$$

"Extended" version of the second map G

$$\tilde{G}: \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \frac{x^2(y+z)}{(u+v+y+z)(u+v-y-z)} \\ u \\ y \\ z \\ v - \frac{x^2(u+v)}{(u+v+y+z)(u+v-y-z)} \end{pmatrix}$$

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$\tilde{F}$ and $\tilde{G}$ viewed as the maps in $\mathbb{R}^5$ have a common invariant set $\tilde{\Omega}$. It is known that sections of $\tilde{\Omega}$ by generic lines are Cantor sets, while sections in some specific directions are unions of two intervals. The set $\tilde{\Omega}$ represents a **joint spectrum** of a certain pencil of operators associated with $\mathcal{G}$.

Hierarchy of graphs

$\Gamma = (V, E)$ – connected non oriented graph
V – vertices
E – edges.

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Proposition

Each $2k$-regular graph can be realized as a Schreier graph of a free group F_k on k generators

Graphs associated with groups

Cayley graph

$G = \langle S \rangle$ a group with generating set $S \rightsquigarrow \Gamma = \Gamma(G, S)$ – Cayley graph
$V = G$
$E = \{(x, sx) \colon x \in G, s \in S\}$

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H a subgroup of $G \rightsquigarrow \Gamma = \Gamma(G, H, S)$ – Schreier graph
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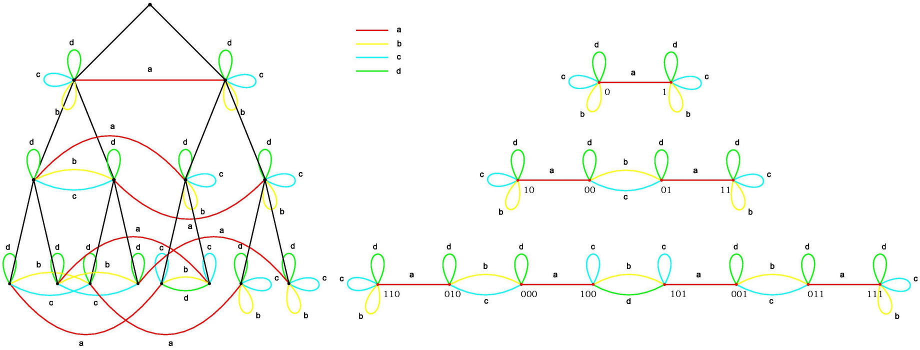
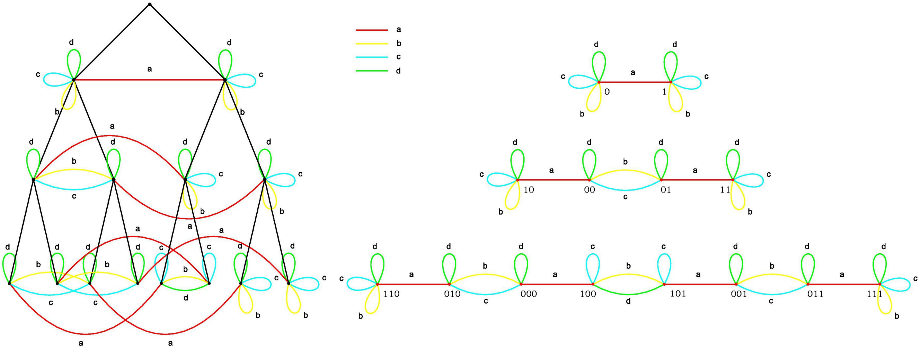
Cayley and Schreier graphs are d-regular, $d = 2|S|$. Schreier graphs are generalization of the Cayley graphs and correspond to the case when $H = \{e\}$ is a trivial subgroup.

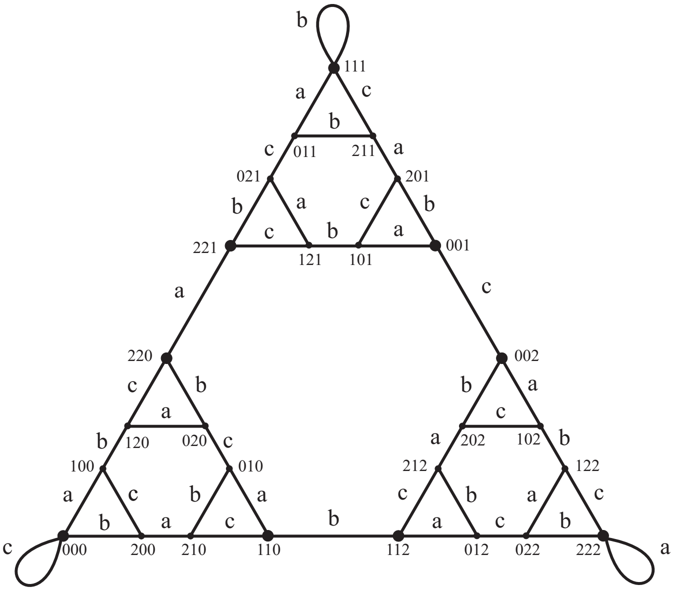
Schreier graphs of group $\mathcal{G}$





This and the next figure contain a hidden information, including the Gray code.

Schreier graph of Hanoi group $\mathcal{H}^{(3)}$



Markov operators

M – Markov operator. In the case of a d-regular graph

$$(Mf)(x) = \frac{1}{d} \sum_{y \sim x} f(y)$$

where $f \in l^2(V)$, and $x \sim y$ is the adjacency relation.

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- M is self-adjoint and $\|M\| \leq 1 \Rightarrow$ spectrum $sp(M) \subset [-1, 1]$
- Graph Γ is **amenable** if $1 \in sp(M) (\Leftrightarrow \|M\| = 1)$
- The group is amenable if its Cayley graph is amenable.

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Definition

The spectrum of the marked group (G, S) is defined as $sp(\Gamma(G, S))$

Weighted graphs

$w\colon E \to \mathbb{R}$ – weight on edges
M_w – weighted "Markov" operator.

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- Isotropic case when P is a uniform distribution on $S \cup S^{-1}$ (simple random walk case)
- Anisotropic case – P is not uniform

Basic questions about spectra of infinite graphs

Question

What could be the shape of the spectrum of a regular graph or a group?

In particular, can it be a Cantor set or at least have infinitely many gaps?

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Question

Can a **torsion free group** have a gap in the spectrum?

If **YES** then we get a counterexample to the **Kadison-Kaplanski Conjecture on idempotents**.

Define a **spectral measure** by $\mu(B) = \langle E(B)\delta_e, \delta_e\rangle$ where $B \subset \mathbb{R}$ Borel subsets, $\{E(B)\}$ spectral projections associated with M, δ_e – delta function at identity $e \in G$.

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Question

What can be said about μ? In particular what are the components of the decomposition $\mu = \mu_{ac} + \mu_{cs} + \mu_{pp}$?

Some answers

Very little is known.

- Spectra of perturbations of lattices $\mathbb{Z}^d, d \geq 1$ (i.e. crristallographic groups), very classical (Bloch-Floquet theory, Sunada,...)
- Regular trees and their perturbations. In particular Cayley graphs of free products of finite groups.
- H. Kesten 1959, K. Aomoto, D. Cartwright, P. Soardi, Italian School: A. Figa-Talamanca, M. Picardello, W. Woess, T. Steger, G. Kuhn,...
- V. Malozemov and A. Teplyaev 1998, Graph of bounded degree and Cantor spectrum associated with the Sierpinski gasket.

Theorem (L.Bartholdi, R.Grigorchuk 2000)

Spectrum of the Schreier graph can be a Cantor set or a union of a Cantor set and a countable set of isolated points accumulating to it.

- J.-F. Quint, Analyse harmonique sur le graphe de Pascal 2006
- Grigorchuk, Zuk 2002 (Lamplighter group)
- Grigorchuk, Savchuk, Sunik 2005 ($IMG(z^2 + i)$)
- Grigorchuk, Nekrashevych, Self-Similar groups, operator algebras and Schur complement, 2007
- Bajorin, Chen, Dagan, Emmons, Hussein, Khalil, Mody, Steinhurst, Teplyaev, Vibration spectra of finitely ramified, symmetric fractals, 2008

  and many more . . . .
- Grigorchuk, Nekrahsevych, Sunic, 2015, survey.

The spectrum of the Lamplighter group

Theorem (Grigorchuk, A.Zuk, 2001)

The spectrum of the Cayley graph could be a **pure point** *spectrum. Namely the Cayley graph* $\Gamma(\mathcal{L}, \{a, b\})$ *where* a, b *are generators of the lamplighter group* $\mathcal{L}$ *corresponding to the states of the Lamplighter automaton has a pure point spectrum with the eigenvalues of the form* $\cos(\frac{p}{q}\pi), 1 \leq p < q, q = 2, 3, \dots, (p, q) = 1$ *which densely pack the interval* $[-1, 1]$.

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This result was used by P.Linnel, T.Schick, A.Zuk and speaker to construct a closed Riemannian manifold of dimension 7 with noninteger L^2-Betti number $=\frac{1}{3}$ thus answering the question of M.Atiyah and giving a counterexample to the **Strong Atiyah Conjecture**

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Let M_μ be the operator in $l^2(\mathcal{L})$ of convolution with the element $a + a^{-1} + b + b^{-1} + \mu c \in \mathbb{C}[\mathcal{L}]$ where $c = b^{-1}c$.

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Theorem (B. Simanek – Grigorchuk)

For $\mu \in \mathbb{R}$, let M_μ be defined as above. For every $\mu \in \mathbb{R}$, the operator M_μ has pure point spectrum. Moreover

(a) *If $|\mu| \leq 1$, the eigenvalues of M_μ densely pack the interval $[-4-\mu, 4-\mu]$.*

(b) *If $|\mu| > 1$, the eigenvalues of M_μ form a countable set that densely packs the interval $[-4-\mu, 4-\mu]$ and also has an accumulation point $\mu + 2/\mu \notin [-4-\mu, 4-\mu]$.*

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Let M_μ be the operator in $l^2(\mathcal{L})$ of convolution with the element $a + a^{-1} + b + b^{-1} + \mu c \in \mathbb{C}[\mathcal{L}]$ where $c = b^{-1}c$.

Theorem (B. Simanek – Grigorchuk)

For $\mu \in \mathbb{R}$, let M_μ be defined as above. For every $\mu \in \mathbb{R}$, the operator M_μ has pure point spectrum. Moreover

(a) *If $|\mu| \leq 1$, the eigenvalues of M_μ densely pack the interval $[-4-\mu, 4-\mu]$.*

(b) *If $|\mu| > 1$, the eigenvalues of M_μ form a countable set that densely packs the interval $[-4-\mu, 4-\mu]$ and also has an accumulation point $\mu + 2/\mu \notin [-4-\mu, 4-\mu]$.*

Corollary

The spectrum of a Cayley graph can have infinitely many gaps.

Theorem (B. Simanek–Grigorchuk)

The spectral measure ν_μ of the operator M_μ is discrete and is given by

$$\nu_\mu = \frac{1}{4}\delta_\mu + \sum_{k=2}^{\infty} \left[\frac{1}{2^{k+1}} \sum_{\{s: G_k(s,\mu)=0\}} \delta_s \right],$$

where

$$G_k(z,\mu) = 2^k \left[U_k\left(\frac{-z-\mu}{4}\right) + \mu U_{k-1}\left(\frac{-z-\mu}{4}\right) \right],$$

and U_k is the degree k Chebyshev polynomial of the second kind.

On the question “Can one hear the shape of a group?”

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Two papers with the same title **“Can one hear the shape of a group?”**: A.Valette 1993 and K.Fujiwara 2016.
The question asks: “Does the spectrum of the Cayley graph determine it up to isometry”?

The answer is **NO** in a very strong sense.

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Theorem (Artem Dudko–Grigorchuk 2018)

(i) Let $\mathcal{G}_\omega = \langle S_\omega \rangle, \omega \in \Omega = \{0,1,2\}^{\mathbb{N}}, S_\omega = \{a, b_\omega c_\omega, d_\omega\}$ *be a family of groups of* **of intermediate growth** *between polynomial and exponential. Then for each* $\omega \in \Omega$ *the spectrum of the Cayley graph* $\Gamma_\omega = \Gamma(\mathcal{G}_\omega, S_\omega)$ *is the union*

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(ii) Moreover, for each $\omega \in \Omega$ *that is not eventually constant sequence the group* $\mathcal{G}_\omega$ *has uncountably many covering amenable groups* $\tilde{G} = \langle \tilde{S} \rangle$ *generated by* $\tilde{S} = \{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ *such that the map* $\tilde{a} \to a, \tilde{b} \to b_\omega, \tilde{c} \to c_\omega, \tilde{d} \to d_\omega\}$ *extends to a surjective homomorphism* $\tilde{G} \twoheadrightarrow \mathcal{G}_\omega$ *and the spectrum of the Cayley graphs* $\Gamma(\tilde{G}, \tilde{S})$ *is the same set* $\Sigma = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$.

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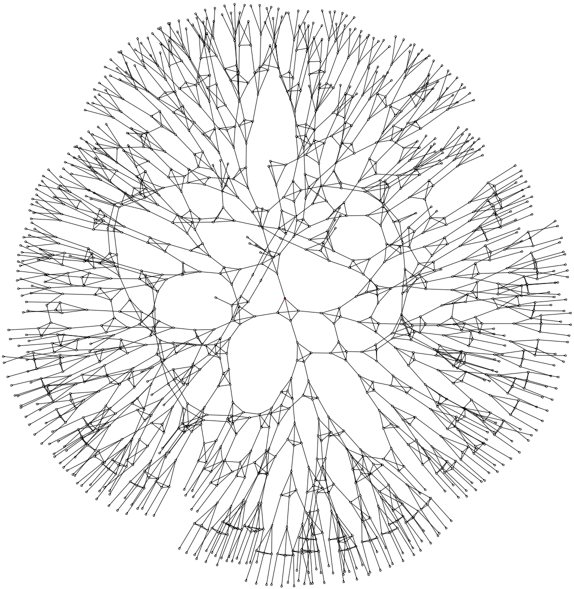
This result is generalized in various directions by T. Nagnibeda, A. Peres and R. Grigorchuk (work in progress).

Cayley graph of $\mathcal{G}$



- The proof uses the Hulanicki theorem on characterization of amenable groups in terms of the **weak containment** of **trivial representation** into **regular representation**, and A.Dudko-Grigorchuk **Weak Hulanicki type theorem for covering graphs**. The above theorem is for the **isotropic case**.

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- In the **anisotropic case** by the result of D. Lenz, T. Nagnibeda and Grigorchuk we only know that $sp(M_p)$ contains a Cantor subset which is a spectrum of **random Schrödinger operator** whose potential is ruled by the **substitutional dynamical system** generated by the substitution

$$\sigma\colon a \to aca, b \to d, c \to b, d \to c$$

Corollary

There are uncountably many groups with pairwise not quasi-isometric Cayley graphs but with the same spectrum.

This is because the family $\mathcal{G}_\omega, \omega \in \Omega$ has uncountably many groups with pairwise different rates of growth and rate of growth is a quasi-isometry invariant.

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Question

Does the spectral measure μ determines Cayley graph up to isometry?

Remark

μ determines the spectrum of M, probabilities $P_{e,e}^{(n)}$ of return, the Ihara zeta function, ... Perhaps the answer could be affirmative.

Self-similar groups

Given invertible Mealy automaton $\mathcal{A}$ with the input and output alphabets X and set of states Q one defines a group $G = G(\mathcal{A})$ generated by initial automata $\mathcal{A}_q, q \in Q$ (the operation is the composition of automata).

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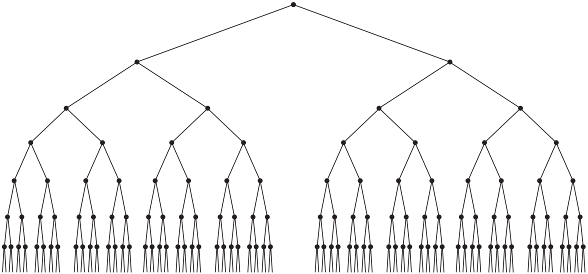
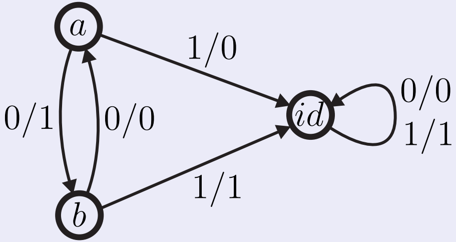


The set Q of states serves as a **generating set**.

Action on T given by finite initial automaton

Definition (By Example)

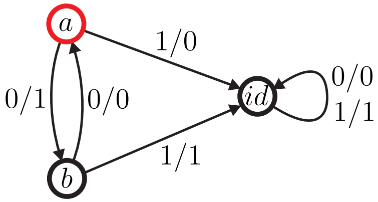
$S_2 = \{\varepsilon, \sigma\}$ acts on $X = \{0, 1\}$.



$\mathcal{A}$ — noninitial automaton,
$\mathcal{A}_q$ — initial automaton, $q \in \{a, b, id\}$.

$\mathcal{A}_q$ acts on X^* (and on T)

Input:	0	0	0	0	1	0	1	1
	↓							
States:	a	b	a	b	a	id	id	id
	↓							
Output:	1	0	1	0	0	0	1	1



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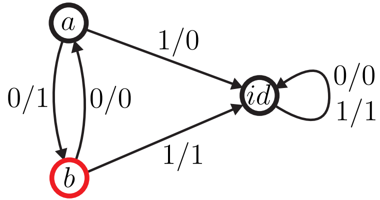

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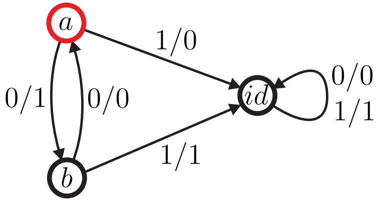

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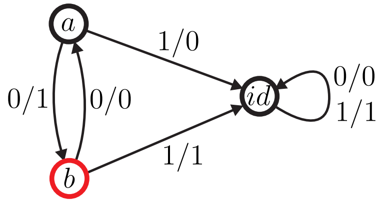

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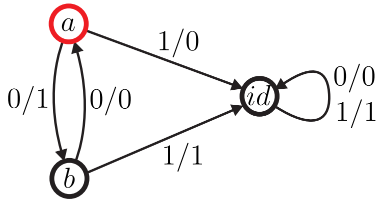

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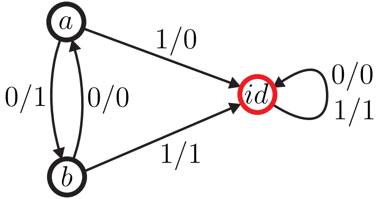

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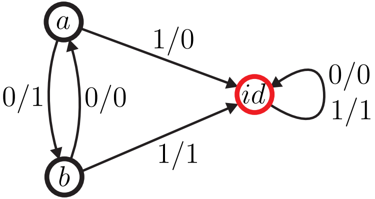

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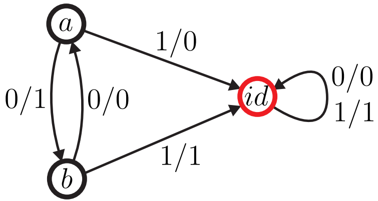

Definition of automaton group

Given an automaton $\mathcal{A}$ every state q defines an automorphism $\mathcal{A}_q$ of X^*

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An **automaton** or **self-similar** group generated by automaton $\mathcal{A}$ is the group

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The examples of automaton groups:

$$\mathbb{Z} = \langle \text{odometer} \rangle$$

$$D_\infty = \langle a, t : a^2 = t^2 = 1 \rangle = \text{infinite dihedral group} = \mathrm{IMG}(z^2 - 2)$$

$$\mathcal{G} = \langle a, b, c, d \rangle \text{ the first group of intermediate growth}$$

Basilica $\mathcal{B}$, Hanoi $\mathcal{H}^{(3)}$, $IMG(z^2 + i)$ and many more important groups.

Random Schreier graphs

$G \curvearrowright \partial T$
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The family $\{\Gamma_x, x \in \partial T\}$ is a **random graph** with respect to the **uniform Bernoulli measure** $\nu = \{1/d, \dots, 1/d\}^{\mathbb{N}}$ on ∂T which is G-invariant.

Let $x = \{v_n\}_{n=1}^{\infty}$ where v_n is vertex of level n on the path $x \in \partial T$.
We have

$$(\Gamma, x) = \lim_{n \to \infty} (\Gamma_n, v_n)$$

(convergence of marked graphs).

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M_n – Markov operator on Γ_n
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$$\mu_n = \frac{1}{d^n} \sum_{\lambda \in sp(M_n)} \delta_\lambda$$

(eigenvalues are presented in the sum according to the multiplicities).

Theorem (Bartholdi-Grigorchuk 2000, extended version by A. Dudko-Grigorchuk 2018)

a) The spectrum of graph Γ_x *does not depend on the point* $x \in \partial T$ *and in amenable case coincide with the set*

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- μ is the analogue of the **density of states** for the **random Schrödinger operator**.
- There is more relation of spectra of graphs with the random Schrödinger operator via the works of B. Saimon, L. Grabowski and B. Virag, D.Lenz, T. Nagnibeda and Grigorchuk, B. Simanek and Grigorchuk.

Schur Complement, Renormalization, and Self-Similar Groups

H Hilbert space, $H = H_1 \oplus H_2$
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(ii) Assume $A \in B(H_1)$ is invertible. Then the *second Schur complement*

$$S_2(M) = D - CA^{-1}B.$$

Theorem

Suppose that D is invertible. Then M is invertible if and only if $S_1(M)$ is invertible. Similarly, if A is invertible, then M is invertible if and only if $S_2(M)$ is invertible.

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where $S_1 = S_1(M)$.

Similarly one defines Schur maps $S_i(M), i = 1, \ldots, d$ for the decomposition $H = H \oplus H \oplus \cdots \oplus H$ (d summands).

Assume $\dim H = \infty$. Any isomorphism

$$\theta : H \to H \oplus H \oplus \cdots \oplus H$$

($d \geq 2$ summands) is called d-similarity (d-similarities are in bijection with $*$-representations of the Cuntz C^*-algebra $\mathcal{O}_d$).

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If $M(z), z \in \mathbb{C}^k, M(z) \in B(H)$ is an operator valued function and we are interested in a **"joint spectrum"** $sp(M(z))$ of $M(z)$ i.e. in

$$sp(M(z)) = \{z : M(z) \text{ is not invertible}\}$$

then it may happen that

there are:
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$$S_i(M(z)) = M(F(z))$$

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In this case the spectral problem usually reduces to the finding of a suitable F-invariant subset $\Omega \subset \mathbb{C}^k$.

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Renormalization Map associated with the given spectral problem.

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The above approach is applicable in many cases related to self-similar groups and their Schreier graphs, and in all tested cases F is a rational map.

If $d = 2$ and F is semi-conjugate to a one-dimensional map f then joint spectrum can be described completely.

The example

Z.Sunic and Grigorchuk 2007

Example

Let $\mathcal{H} = \mathcal{H}^{(3)}$ be the Hanoi Towers group (on three pegs). It is a self-similar group acting on X^* for $X = \{0, 1, 2\}$ with generators a, b, c satisfying the following matrix recursions

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{pmatrix},$$

$$b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$c = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Consider the two-parametric family matrices

$$\Delta(x,y) = \begin{pmatrix} c-x & y & y \\ y & b-x & y \\ y & y & a-x \end{pmatrix} =$$

$$\left(\begin{array}{ccc|ccc|ccc} c-x & 0 & 0 & y & 0 & 0 & y & 0 & 0 \\ 0 & -x & 1 & 0 & y & 0 & 0 & y & 0 \\ 0 & 1 & -x & 0 & 0 & y & 0 & 0 & y \\ \hline y & 0 & 0 & -x & 0 & 1 & y & 0 & 0 \\ 0 & y & 0 & 0 & b-x & 0 & 0 & y & 0 \\ 0 & 0 & y & 1 & 0 & -x & 0 & 0 & y \\ \hline y & 0 & 0 & y & 0 & 0 & -x & 1 & 0 \\ 0 & y & 0 & 0 & y & 0 & 1 & -x & 0 \\ 0 & 0 & y & 0 & 0 & y & 0 & 0 & a-x \end{array}\right).$$

Permuting rows and columns and dividing them into blocks we get the matrix

$$\left(\begin{array}{ccc|cccccc}
c-x & 0 & 0 & y & 0 & 0 & y & 0 & 0 \\
0 & b-x & 0 & 0 & y & 0 & 0 & y & 0 \\
0 & 0 & a-x & 0 & 0 & y & 0 & 0 & y \\
\hline
y & 0 & 0 & -x & 0 & 1 & y & 0 & 0 \\
0 & y & 0 & 0 & -x & 0 & 0 & y & 1 \\
0 & 0 & y & 1 & 0 & -x & 0 & 0 & y \\
y & 0 & 0 & y & 0 & 0 & -x & 1 & 0 \\
0 & y & 0 & 0 & y & 0 & 1 & -x & 0 \\
0 & 0 & y & 0 & 1 & y & 0 & 0 & -x
\end{array}\right).$$

Computation of Schur complement with respect to the given partition of the matrix yields

$$\widehat{S}_1(\Delta(x,y)) = \Delta(x',y'),$$

where

$$x' = x - \frac{2(x^2 - x - y^2)y^2}{(x-y-1)(x^2-1+y-y^2)}$$

and

$$y' = \frac{(x+y-1)y^2}{(x-y-1)(x^2-1+y-y^2)}.$$

The map $F\colon (x,y) \mapsto (x', y')$ is semi-conjugate to the map $f\colon \mathbb{R} \longrightarrow \mathbb{R}\colon x \mapsto x^2 - x - 3$,

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{F} & \mathbb{R}^2 \\ \psi\big\downarrow & & \big\downarrow\psi \\ \mathbb{R} & \xrightarrow{f} & \mathbb{R} \end{array},$$

$$\psi(x,y) = \frac{x^2 - 1 - xy - 2y^2}{y}.$$

The spectrum of $\Delta(x,y)$ is the closure of the union $\bigcup_{\theta \in \bigcup f^{-n}(S)} \mathcal{H}_\theta \cup L_0 \cup L_1 \cup L_2$, where $S = \{-2, 0\}$, $\mathcal{H}_\theta$ is the hyperbola $x^2 - xy - 2y^2 - \theta y = 1$, and L_0, L_1, L_2 are the lines given by the equations

$$x - 1 - 2y = 0,$$
$$x + 1 + y = 0,$$
$$x - 1 + y = 0.$$

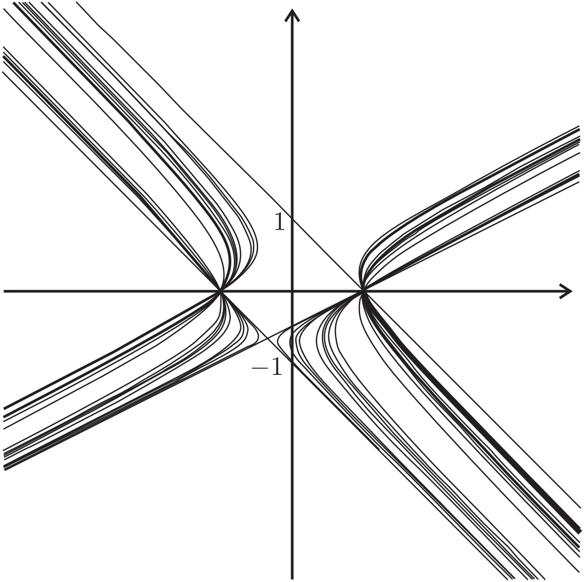

Figure: Joint Spectrum of Schreier graphs of $\mathcal{H}^{(3)}$

The "Saas-Fee Project" N-B. Dang, M. Lyubich, and R. Grigorchuk

The project aims at a detailed dynamical exploration of the spectral renormalization transformations arising in the theory of self-similar groups. This involves:

- Revealing algebro-geometric and dynamical nature of the integrability (i.e. semi-conjugacy to lower dimensional maps, ...) observed for some of these transformations.
- Interpretation of suitable spectral sets and the correponding spectral measures as slices of Julia sets and corresponding Green currents (for multidimensional maps).
- Characterizing the dichotomy "discrete vs continuous spec" in terms of combinatorial and dynamical degrees.
- Looking for a generalization of the Little L-G equidistribution theorem that would serve for a broader class of self-similar groups.

Thank you