Phase Transitions of Random Constraint Satisfaction Problems

Allan Sly, Princeton University

Stony Brook November 2019

Introduction:
random constraint satisfaction problems;

Combinatorics and Theoretical Computer Science

Constraint satisfaction problem (CSP): is it possible to assign values to a set of *variables* to satisfy a given set of *constraints*?

- Scheduling your appointments for the day
- System of linear equations.
- Colouring a graph or finding a large independent set.
- Satisfying a Boolean formula.

Combinatorics and Theoretical Computer Science

Constraint satisfaction problem (CSP): is it possible to assign values to a set of *variables* to satisfy a given set of *constraints*?

- Scheduling your appointments for the day
- System of linear equations.
- Colouring a graph or finding a large independent set.
- Satisfying a Boolean formula.

Random CSPs

Our focus is to investigate properties when the constraints are chosen randomly.

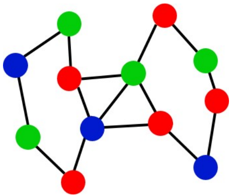
Combinatorial properties of Random Graphs:

- Erdős-Rényi Random Graph: $G(n, \alpha/n)$ with n vertices and edges with probability α/n (average degree α).
- Random α-regular graph: Uniformly chosen from α-regular graphs on n vertices.

Combinatorial properties of Random Graphs:

- Erdős-Rényi Random Graph: $G(n, \alpha/n)$ with n vertices and edges with probability α/n (average degree α).
- Random α-regular graph: Uniformly chosen from α-regular graphs on n vertices.

When is there a proper k-colouring?



K-SAT The random K-SAT problem, a model of a random Boolean formula, is perhaps the canonical random CSP.

K-SAT The random K-SAT problem, a model of a random Boolean formula, is perhaps the canonical random CSP.

Basic Definition:

Variables: $x_1, \dots, x_n \in \{\text{TRUE}, \text{FALSE}\} \equiv \{+, -\}$

Constraints: m clauses taking the OR of k variables uniformly chosen from $\{+x_1, -x_1, \dots, +x_n, -x_n\}$.

K-SAT The random K-SAT problem, a model of a random Boolean formula, is perhaps the canonical random CSP.

Basic Definition:

Variables: $x_1, \ldots, x_n \in \{\text{TRUE}, \text{FALSE}\} \equiv \{+, -\}$

Constraints: m clauses taking the OR of k variables uniformly chosen from $\{+x_1, -x_1, \ldots, +x_n, -x_n\}$.

Example: A 3-SAT formula with 4 clauses:

$$\begin{aligned}\mathscr{G}(\underline{x}) = (+x_1 \text{ OR } +x_2 \text{ OR } -x_3) \text{ AND } &\overbrace{(+x_3 \text{ OR } +x_4 \text{ OR } -x_5)}^{\text{clause}} \\ \text{AND } (-x_1 \text{ OR } -x_4 \text{ OR } +x_5) \text{ AND } &(+x_2 \text{ OR } -x_3 \text{ OR } +x_4)\end{aligned}$$

K-SAT The random K-SAT problem, a model of a random Boolean formula, is perhaps the canonical random CSP.

Basic Definition:

Variables: $x_1, \ldots, x_n \in \{\text{TRUE}, \text{FALSE}\} \equiv \{+, -\}$

Constraints: m clauses taking the OR of k variables uniformly chosen from $\{+x_1, -x_1, \ldots, +x_n, -x_n\}$.

Example: A 3-SAT formula with 4 clauses:

$$\mathscr{G}(\underline{x}) = (+x_1 \text{ OR } +x_2 \text{ OR } -x_3) \text{ AND } \overbrace{(+x_3 \text{ OR } +x_4 \text{ OR } -x_5)}^{\text{clause}}$$
$$\text{AND } (-x_1 \text{ OR } -x_4 \text{ OR } +x_5) \text{ AND } (+x_2 \text{ OR } -x_3 \text{ OR } +x_4)$$

Clause density: The K-SAT model is parameterized the problem by the density of clauses $\alpha = m/n$.

K-SAT The random K-SAT problem, a model of a random Boolean formula, is perhaps the canonical random CSP.

Basic Definition:

Variables: $x_1, \ldots, x_n \in \{\text{TRUE}, \text{FALSE}\} \equiv \{+, -\}$

Constraints: m clauses taking the OR of k variables uniformly chosen from $\{+x_1, -x_1, \ldots, +x_n, -x_n\}$.

Example: A 3-SAT formula with 4 clauses:

$$\begin{aligned}\mathscr{G}(\underline{x}) = (+x_1 \text{ OR } +x_2 \text{ OR } -x_3) \text{ AND } & \overbrace{(+x_3 \text{ OR } +x_4 \text{ OR } -x_5)}^{\text{clause}} \\ \text{AND } (-x_1 \text{ OR } -x_4 \text{ OR } +x_5) \text{ AND } & (+x_2 \text{ OR } -x_3 \text{ OR } +x_4)\end{aligned}$$

Clause density: The K-SAT model is parameterized the problem by the density of clauses $\alpha = m/n$.

Variant NAE-SAT: An assignment $\underline{x}$ is a solution if both $\underline{x}$ *and* $-\underline{x}$ are satisfying.

Graphical description: We can encode a K-SAT formula as a bipartite hyper-graph:

Graphical description: We can encode a K-SAT formula as a bipartite hyper-graph:

Take a 4-SAT formula with 3 clauses: $\mathscr{G}(\underline{x}) =$

$$(+x_1 \text{ OR } +x_3 \text{ OR } -x_5 \text{ OR } -x_7) \text{ AND } (-x_1 \text{ OR } -x_2 \text{ OR } +x_5 \text{ OR } +x_6)$$
$$\text{AND } (-x_3 \text{ OR } +x_4 \text{ OR } -x_6 \text{ OR } +x_7)$$

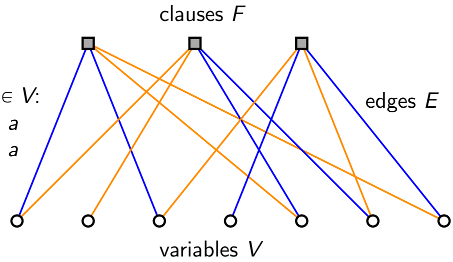
Graphical description: We can encode a K-SAT formula as a bipartite hyper-graph:

Take a 4-SAT formula with 3 clauses: $\mathscr{G}(\underline{x}) =$

$$(+x_1 \text{ OR } +x_3 \text{ OR } -x_5 \text{ OR } -x_7) \text{ AND } (-x_1 \text{ OR } -x_2 \text{ OR } +x_5 \text{ OR } +x_6) \text{ AND } (-x_3 \text{ OR } +x_4 \text{ OR } -x_6 \text{ OR } +x_7)$$

We can encode the formula as a bipartite graph $\mathscr{G} \equiv (V, F, E)$:

clause $a \in F$, variable $v \in V$:
blue edge *(av)* if $+x_v$ in clause a
yellow edge *(av)* if $-x_v$ in clause a

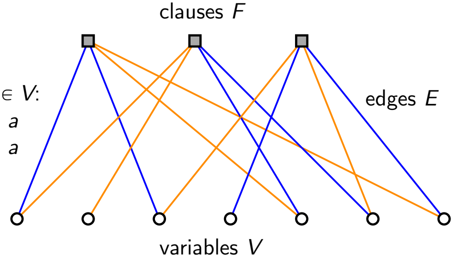


(4-SAT: each clause has degree 4)

Graphical description: We can encode a K-SAT formula as a bipartite hyper-graph:
Take a 4-SAT formula with 3 clauses: $\mathscr{G}(\underline{x}) =$
$(+x_1 \text{ OR } +x_3 \text{ OR } -x_5 \text{ OR } -x_7) \text{ AND } (-x_1 \text{ OR } -x_2 \text{ OR } +x_5 \text{ OR } +x_6)$
$\text{AND } (-x_3 \text{ OR } +x_4 \text{ OR } -x_6 \text{ OR } +x_7)$

We can encode the formula as a bipartite graph $\mathscr{G} \equiv (V, F, E)$:

clause $a \in F$, variable $v \in V$:
blue edge *(av)* if $+x_v$ in clause a
yellow edge *(av)* if $-x_v$ in clause a



(4-SAT: each clause has degree 4)

The resulting random graph is locally tree-like, almost no short cycles and it's local distribution can be described completely.

Main Question:

- *Satisfiability Threshold*: For which α are there satisfying assignments?

Main Question:

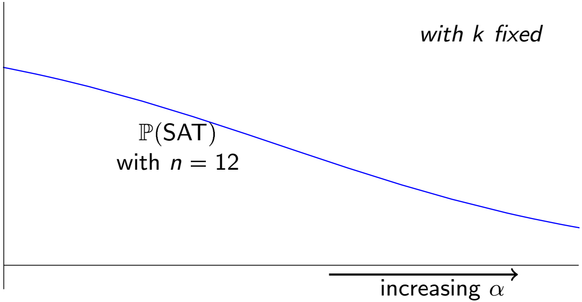
- *Satisfiability Threshold*: For which α are there satisfying assignments?

Other Question:

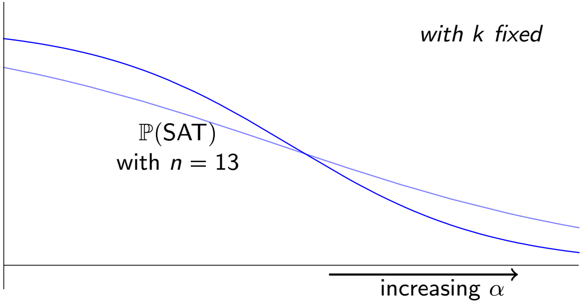
- *Free Energy*: How many solutions are there?
- *Local Statistics*: Properties of solutions such as how many clauses are satisfied only once?
- *Algorithmic*: Can solutions be found efficiently?

The Satisfiability Conjecture. *For each* $k \geqslant 2$, *random* k*-SAT has a sharp satisfiability threshold* α_{sat}.

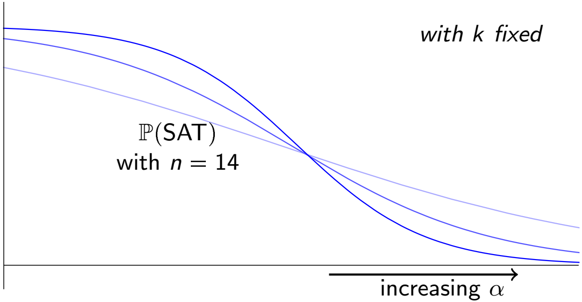
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold* α_{sat}.



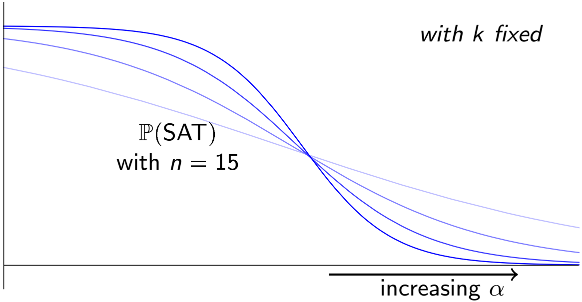
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold* α_{sat}.



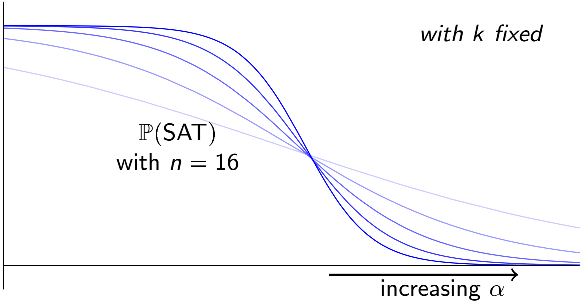
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold* α_{sat}.



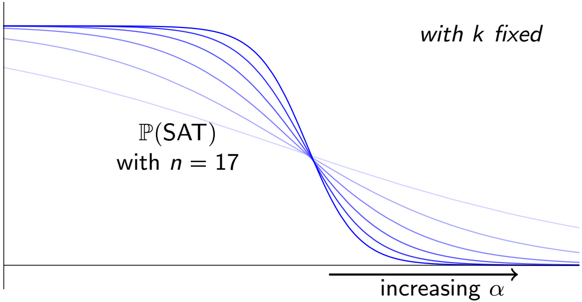
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold α_{sat}.*



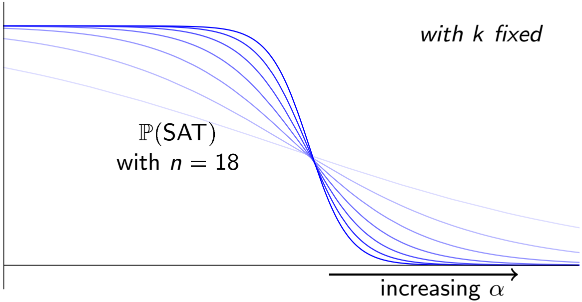
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold* α_{sat}.



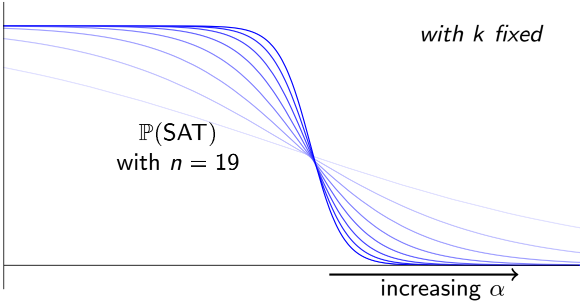
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold* α_{sat}.



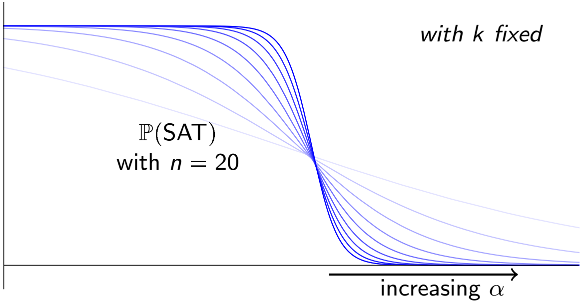
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold* $\color{red}{\alpha_{\mathsf{sat}}}$.



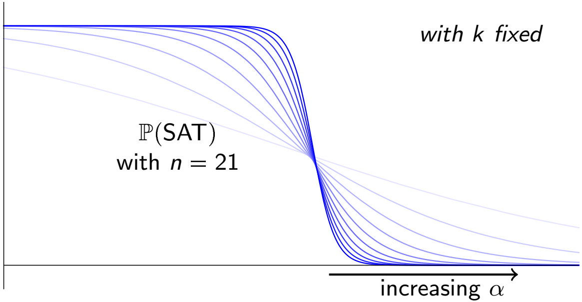
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold α_{sat}.*



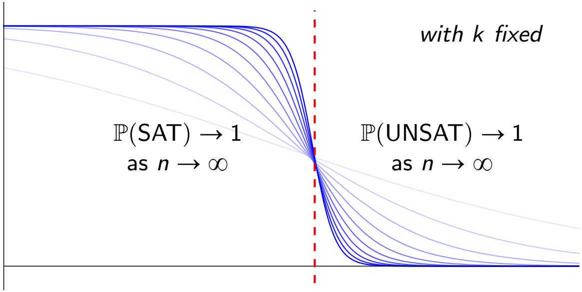
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold* α_{sat}.



The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold* α_{sat}.



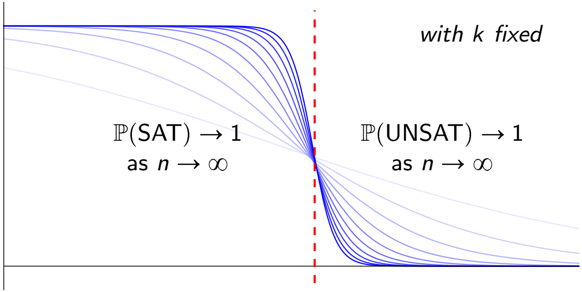
The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold α_{sat}.*



— that is, a single critical value α_{sat} separates SAT|UNSAT

(with high probability in the limit $n \to \infty$; fixed k)

The Satisfiability Conjecture. *For each $k \geqslant 2$, random k-SAT has a sharp satisfiability threshold α_{sat}.*



— that is, a single critical value α_{sat} separates SAT|UNSAT

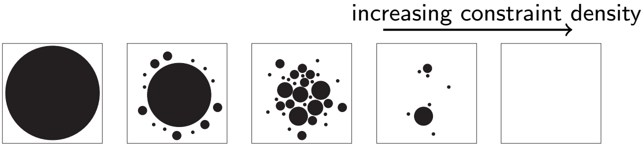
(with high probability in the limit $n \to \infty$; fixed k)

For general k, Friedgut ('99) proved the transition sharpens around a (possibly non-convergent) *threshold sequence* $\alpha_{\mathsf{sat}}(n)$

(whereas conjecture requires $\alpha_{\mathsf{sat}}(n) \to \alpha_{\mathsf{sat}}$ as $n \to \infty$)

Theoretical Physics

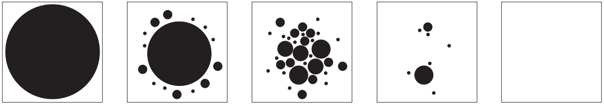
Disordered systems such as *spin glasses* are models of interacting particles/variables with frustrated interactions. Many random constraint satisfaction problems can be recast as dilute mean-field spin glasses.



Theoretical Physics

Disordered systems such as *spin glasses* are models of interacting particles/variables with frustrated interactions. Many random constraint satisfaction problems can be recast as dilute mean-field spin glasses.

increasing constraint density $\longrightarrow$



One-step Replica Symmetry Breaking Predictions:

Developed to study dense spin-glasses such as the Sherrington-Kirkpatrick model.

- **Replica Symmetry Breaking**: Clustering of assignments.
- **Cavity Method**: Heuristic for analyzing adding one variable.

First Moment method on $Z \equiv \left|\{\text{satisfying assignments of } \mathscr{G}\}\right|$:

First Moment method on $Z \equiv |\{\text{satisfying assignments of } \mathscr{G}\}|$:

$$\mathbb{E}Z = 2^n(1-1/2^k)^m$$

First Moment method on $Z \equiv |\{\text{satisfying assignments of } \mathscr{G}\}|$:

$$\mathbb{E}Z = 2^n(1-1/2^k)^m = \exp\{n\,\underbrace{[\ln 2 + \alpha \log(1-1/2^k)]}\}$$

First Moment method on $Z \equiv |\{\text{satisfying assignments of } \mathscr{G}\}|$:

$$\mathbb{E}Z = 2^n(1-1/2^k)^m = \exp\{n \underbrace{[\ln 2 + \alpha \log(1-1/2^k)]}\}$$

exponent decreases in α, crosses zero at $\alpha_1 \approx 2^k \ln 2 + O(1)$

First Moment method on $Z \equiv |\{\text{satisfying assignments of } \mathscr{G}\}|$:

$$\mathbb{E}Z = 2^n(1-1/2^k)^m = \exp\{n \underbrace{[\ln 2 + \alpha \log(1-1/2^k)]}_{}\}$$

exponent decreases in α, crosses zero at $\alpha_1 \approx 2^k \ln 2 + O(1)$

First moment threshold α_1 separates $\mathbb{E}Z \to \infty \mid \mathbb{E}Z \to 0$.

First Moment method on $Z \equiv |\{\text{satisfying assignments of } \mathscr{G}\}|$:

$$\mathbb{E}Z = 2^n(1-1/2^k)^m = \exp\{n \underbrace{[\ln 2 + \alpha \log(1-1/2^k)]}_{\text{exponent decreases in } \alpha, \text{ crosses zero at } \alpha_1 \approx 2^k \ln 2 + O(1)}\}$$

First moment threshold α_1 separates $\mathbb{E}Z \to \infty \mid \mathbb{E}Z \to 0$.

$\alpha_1 \neq \alpha_{\mathsf{sat}}$: At least ϵn unconstrained variables so $Z > 0 \Rightarrow Z \geqslant 2^{\epsilon n}$.

First Moment method on $Z \equiv \left|\{\text{satisfying assignments of } \mathscr{G}\}\right|$:

$$\mathbb{E}Z = 2^n(1-1/2^k)^m = \exp\{n \underbrace{[\ln 2 + \alpha \log(1-1/2^k)]}\}$$

exponent decreases in α, crosses zero at $\alpha_1 \approx 2^k \ln 2 + O(1)$

First moment threshold α_1 separates $\mathbb{E}Z \to \infty \,\big|\, \mathbb{E}Z \to 0$.

$\alpha_1 \neq \alpha_{\text{sat}}$: At least ϵn unconstrained variables so $Z > 0 \Rightarrow Z \geqslant 2^{\epsilon n}$.

Second Moment method:

$$\mathbb{P}[Z > 0] \geqslant \frac{(\mathbb{E}Z)^2}{\mathbb{E}[Z^2]}$$

First Moment method on $Z \equiv |\{\text{satisfying assignments of } \mathscr{G}\}|$:

$$\mathbb{E}Z = 2^n(1-1/2^k)^m = \exp\{n \underbrace{[\ln 2 + \alpha \log(1-1/2^k)]}_{\text{exponent decreases in } \alpha, \text{ crosses zero at } \alpha_1 \approx 2^k \ln 2 + O(1)}\}$$

First moment threshold α_1 separates $\mathbb{E}Z \to \infty \,\big|\, \mathbb{E}Z \to 0$.

$\alpha_1 \neq \alpha_{\mathsf{sat}}$: At least ϵn unconstrained variables so $Z > 0 \Rightarrow Z \geqslant 2^{\epsilon n}$.

Second Moment method:

$$\mathbb{P}[Z > 0] \geqslant \frac{(\mathbb{E}Z)^2}{\mathbb{E}[Z^2]}$$

To be useful, requires always $\mathbb{E}[Z^2] \asymp (\mathbb{E}Z)^2$. Fails, for **all** $\alpha > 0$.

First Moment method on $Z \equiv \big|\{\text{satisfying assignments of } \mathscr{G}\}\big|$:

$$\mathbb{E}Z = 2^n(1 - 1/2^k)^m = \exp\{n \underbrace{[\ln 2 + \alpha \log(1 - 1/2^k)]}\}$$

exponent decreases in α, crosses zero at $\alpha_1 \approx 2^k \ln 2 + O(1)$

First moment threshold α_1 separates $\mathbb{E}Z \to \infty \,\big|\, \mathbb{E}Z \to 0$.

$\alpha_1 \neq \alpha_{\mathsf{sat}}$: At least ϵn unconstrained variables so
$Z > 0 \Rightarrow Z \geqslant 2^{\epsilon n}$.

Second Moment method:

$$\mathbb{P}[Z > 0] \geqslant \frac{(\mathbb{E}Z)^2}{\mathbb{E}[Z^2]}$$

To be useful, requires always $\mathbb{E}[Z^2] \asymp (\mathbb{E}Z)^2$. Fails, for **all** $\alpha > 0$.

For random colourings and NAE-SAT, second moment method succeeds up to $\alpha_2 = \alpha_{\mathsf{sat}} - O(1)$.

Some physics perspective:
condensation and replica symmetry breaking

Spin glasses are marked by a prevalence of *frustrated interactions* — e.g. Sherrington Kirkpatrick spin-glass ('75): sample $(g_{ij})_{i<j}$, standard $N(0,1)$ then use them to define

Spin glasses are marked by a prevalence of *frustrated interactions* — e.g. Sherrington Kirkpatrick spin-glass ('75): sample $(g_{ij})_{i<j}$, standard $N(0,1)$ then use them to define

$$\mathbb{P}(\underline{x}) \cong \frac{1}{Z} \exp\Big\{ \frac{\beta}{\sqrt{n}} \sum_{i<j} g_{ij} x_i x_j \Big\}, \quad \underline{x} \in \{+1, -1\}^n$$

Spin glasses are marked by a prevalence of *frustrated interactions* — e.g. Sherrington Kirkpatrick spin-glass ('75): sample $(g_{ij})_{i<j}$, standard $N(0,1)$ then use them to define

$$\mathbb{P}(\underline{x}) \cong \frac{1}{Z} \exp\left\{ \frac{\beta}{\sqrt{n}} \sum_{i<j} g_{ij} x_i x_j \right\}, \quad \underline{x} \in \{+1, -1\}^n$$

Some remarkable predictions proved for *dense* graphs

— e.g. for the SK spin-glass

Guerra '03, Talagrand '06: Parisi formula (conjecture: Parisi '79, '80)

Panchenko '11: Parisi ultrametricity (conjecture: Parisi '79, '80)

and for optimization on complete graphs with random edge weights:

Aldous '00: random assignment (conjecture: Mézard–Parisi '85, '86, '87)

Frieze '02, Wästlund '10: TSP (conjecture: Mézard–Parisi '86, Krauth–Mézard '89)

Spin glasses are marked by a prevalence of *frustrated interactions* — e.g. Sherrington Kirkpatrick spin-glass ('75): sample $(g_{ij})_{i<j}$, standard $N(0,1)$ then use them to define

$$\mathbb{P}(\underline{x}) \cong \frac{1}{Z} \exp\left\{ \frac{\beta}{\sqrt{n}} \sum_{i<j} g_{ij} x_i x_j \right\}, \quad \underline{x} \in \{+1, -1\}^n$$

Some remarkable predictions proved for *dense* graphs

— e.g. for the SK spin-glass

Guerra '03, Talagrand '06: Parisi formula (conjecture: Parisi '79, '80)

Panchenko '11: Parisi ultrametricity (conjecture: Parisi '79, '80)

and for optimization on complete graphs with random edge weights:

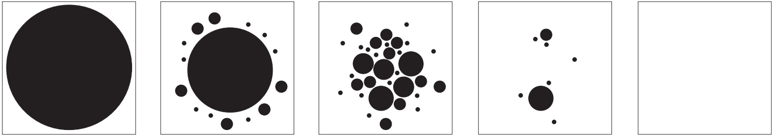
Aldous '00: random assignment (conjecture: Mézard–Parisi '85, '86, '87)

Frieze '02, Wästlund '10: TSP (conjecture: Mézard–Parisi '86, Krauth–Mézard '89)

More recently a set of predictions for *sparse* random systems emerged: Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07, Montanari–Ricci-Tersenghi–Semerjian '08

Phase Diagram

Two solutions are connected if they differ by one bit.

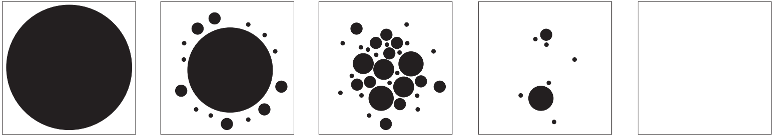


increasing α

KMRSZ '07, MRS '08

Phase Diagram

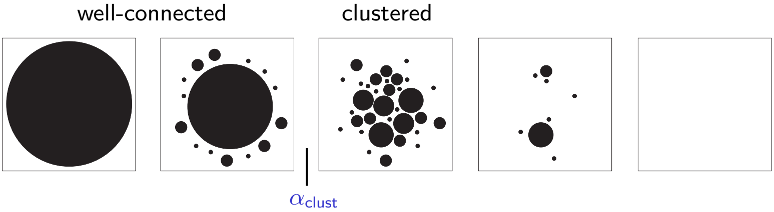
well-connected



KMRSZ '07, MRS '08

The solution space **SOL** starts out as a well-connected cluster.

Phase Diagram



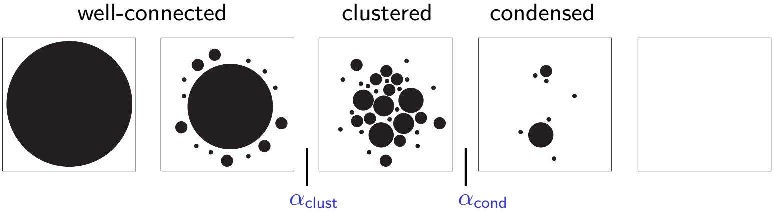
KMRSZ '07, MRS '08

The solution space **SOL** starts out as a well-connected cluster.

After α_{clust}, **SOL** decomposes into exponentially clusters

–Clustering Achlioptas, Coja-Oghlan '10

Phase Diagram



KMRSZ '07, MRS '08

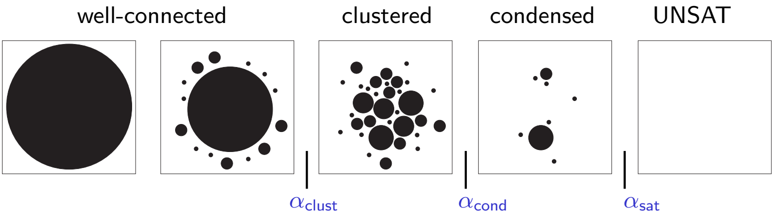
The solution space **SOL** starts out as a well-connected cluster.

After α_{clust}, **SOL** decomposes into exponentially clusters

–Clustering Achlioptas, Coja-Oghlan '10

After α_{cond}, **SOL** is dominated by a few large clusters

Phase Diagram



KMRSZ '07, MRS '08

The solution space **SOL** starts out as a well-connected cluster.

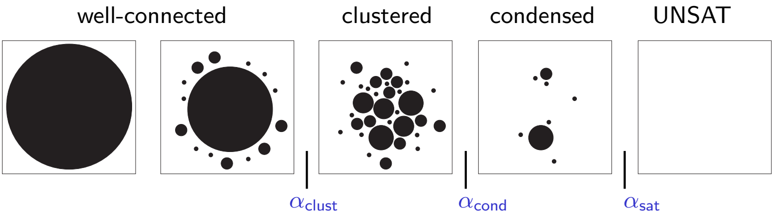
After α_{clust}, **SOL** decomposes into exponentially clusters

–Clustering Achlioptas, Coja-Oghlan '10

After α_{cond}, **SOL** is dominated by a few large clusters

After α_{sat}, no solutions w.h.p.

Phase Diagram



KMRSZ '07, MRS '08

The solution space **SOL** starts out as a well-connected cluster.

After α_{clust}, **SOL** decomposes into exponentially clusters

–Clustering Achlioptas, Coja-Oghlan '10

After α_{cond}, **SOL** is dominated by a few large clusters

After α_{sat}, no solutions w.h.p.

RSB: The one step replica symmetry breaking (1RSB) heuristic roughly says there is no extra structure at the cluster level and decay of correlation.

Cluster Model: We represent clusters as a new spin system on $V(\mathscr{G})$.

Cluster Model: We represent clusters as a new spin system on $V(\mathscr{G})$.

- Start from $\underline{x} \in \{+, -\}^{V(\mathscr{G})}$ and explore the cluster $\mathcal{C}$.

Cluster Model: We represent clusters as a new spin system on $V(\mathscr{G})$.

- Start from $\underline{x} \in \{+, -\}^{V(\mathscr{G})}$ and explore the cluster $\mathcal{C}$.
- If a spin can be flipped between $+$ and $-$ without violating any clauses it is set to $\mathtt{f}$.
- Iterate until done.

Cluster Model: We represent clusters as a new spin system on $V(\mathscr{G})$.

- Start from $\underline{x} \in \{+, -\}^{V(\mathscr{G})}$ and explore the cluster $\mathcal{C}$.
- If a spin can be flipped between $+$ and $-$ without violating any clauses it is set to $\mathtt{f}$.
- Iterate until done.
- Each variable is mapped to a value from $\{+, -, \mathtt{f}\}$.

Cluster Model: We represent clusters as a new spin system on $V(\mathscr{G})$.

- Start from $\underline{x} \in \{+, -\}^{V(\mathscr{G})}$ and explore the cluster $\mathcal{C}$.
- If a spin can be flipped between + and - without violating any clauses it is set to $\mathtt{f}$.
- Iterate until done.
- Each variable is mapped to a value from $\{+, -, \mathtt{f}\}$.

This resulting configuration on $\{+, -, \mathtt{f}\}^{V(\mathscr{G})}$ is our definition of a cluster. It is a spin system satisfying the following conditions:

- $\mathtt{f}$ are not forced by any clause.
- + and - variables must be forced by at least one clause.
- No violated clause.

Cluster Model: We represent clusters as a new spin system on $V(\mathscr{G})$.

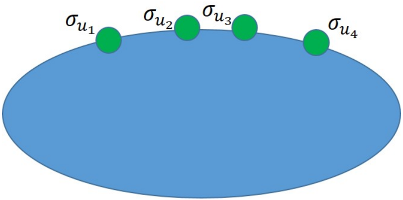
- Start from $\underline{x} \in \{+, -\}^{V(\mathscr{G})}$ and explore the cluster $\mathcal{C}$.
- If a spin can be flipped between + and − without violating any clauses it is set to f.
- Iterate until done.
- Each variable is mapped to a value from $\{+, -, \mathtt{f}\}$.

This resulting configuration on $\{+, -, \mathtt{f}\}^{V(\mathscr{G})}$ is our definition of a cluster. It is a spin system satisfying the following conditions:

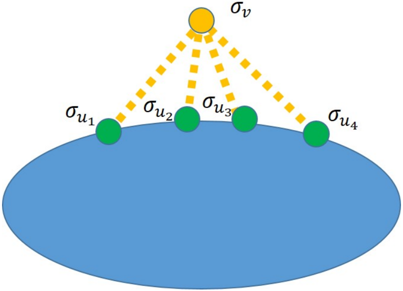
- f are not forced by any clause.
- + and − variables must be forced by at least one clause.
- No violated clause.

We call this the cluster model. Let Ω_n be the number of $\{+, -, \mathtt{f}\}^{V(\mathscr{G})}$ configurations. Locally rigid resulting in no clustering.

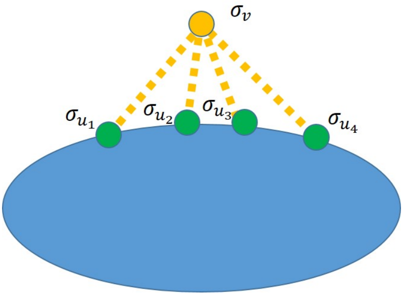
Cavity Method: Adding a new vertex v (or clause).



Cavity Method: Adding a new vertex v (or clause).



Cavity Method: Adding a new vertex v (or clause).

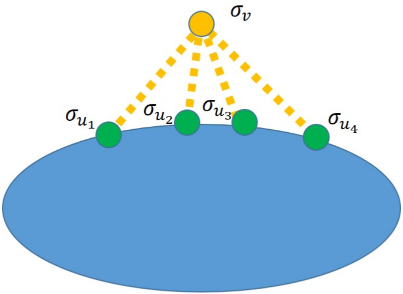


If we know the joint distribution of σ_{u_i} we can:

1. Calculate the law of σ_v
2. Evaluate the change in the partition function from Z_{n+1}/Z_n.

Write $\log Z_n = \sum_{i=1}^{n} \log Z_i/Z_{i-1}$.

Cavity Method: Adding a new vertex v (or clause).



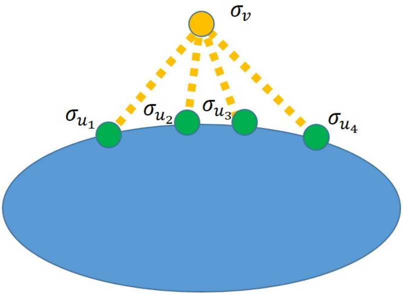
If we know the joint distribution of σ_{u_i} we can:

1. Calculate the law of σ_v
2. Evaluate the change in the partition function from Z_{n+1}/Z_n.

Write $\log Z_n = \sum_{i=1}^{n} \log Z_i/Z_{i-1}$.

The Replica Symmetric heuristic assumes that σ_{u_i} are independent drawn from some law μ.

Cavity Method: Adding a new vertex v (or clause).



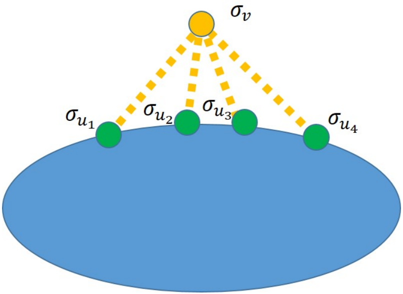
If we know the joint distribution of σ_{u_i} we can:

1. Calculate the law of σ_v
2. Evaluate the change in the partition function from Z_{n+1}/Z_n.

Write $\log Z_n = \sum_{i=1}^{n} \log Z_i/Z_{i-1}$.

The Replica Symmetric heuristic assumes that σ_{u_i} are independent drawn from some law μ.
The 1-RSB heuristic assumes this for the cluster model.

Cavity Method: Adding a new vertex v (or clause).



If we know the joint distribution of σ_{u_i} we can:

1. Calculate the law of σ_v
2. Evaluate the change in the partition function from Z_{n+1}/Z_n.

Write $\log Z_n = \sum_{i=1}^{n} \log Z_i/Z_{i-1}$.

The Replica Symmetric heuristic assumes that σ_{u_i} are independent drawn from some law μ.

The 1-RSB heuristic assumes this for the cluster model.

Self-consistency: The law of σ_v should also be drawn from μ which means μ must satisfy a fixed point equation.

Explicit formula ($k \geqslant 3$)

Explicit formula ($k \geqslant 3$) Let $\mathscr{P} \equiv$ space of probability measures on $[0,1]$. Define the distributional recursion $\boldsymbol{R}_\alpha : \mathscr{P} \to \mathscr{P}$,

$$\boldsymbol{R}_\alpha \mu(B) \equiv \sum_{\underline{d} \equiv (d^+, d^-)} \pi_\alpha(\underline{d}) \int \mathbf{1}\left\{ \frac{(1-\Pi^-)\Pi^+}{\Pi^+ + \Pi^- - \Pi^+\Pi^-} \in B \right\} \prod_{i,j} d\mu(\eta_{ij}^\pm)$$

$$\text{with } \pi_\alpha(\underline{d}) \equiv \frac{e^{-k\alpha}(k\alpha/2)^{d^+ + d^-}}{(d^+)!(d^-)!}, \ \Pi^\pm \equiv \Pi^\pm(\underline{d}, \underline{\eta}) \equiv \prod_{i=1}^{d^\pm} \left(1 - \prod_{j=1}^{k-1} \eta_{ij}^\pm\right)$$

We show $(\boldsymbol{R}_\alpha)^\ell \mathbf{1}_{1/2} \xrightarrow{\ell \to \infty} \mu_\alpha$.
*Distributional equation for the chance of being **+** in a random cluster.*

Explicit formula ($k \geqslant 3$) Let $\mathscr{P} \equiv$ space of probability measures on $[0,1]$. Define the distributional recursion $\boldsymbol{R}_\alpha : \mathscr{P} \to \mathscr{P}$,

$$\boldsymbol{R}_\alpha \mu(B) \equiv \sum_{\underline{d} \equiv (d^+, d^-)} \pi_\alpha(\underline{d}) \int \mathbf{1}\left\{ \frac{(1-\Pi^-)\Pi^+}{\Pi^+ + \Pi^- - \Pi^+\Pi^-} \in B \right\} \prod_{i,j} d\mu(\eta_{ij}^{\pm})$$

$$\text{with } \pi_\alpha(\underline{d}) \equiv \frac{e^{-k\alpha}(k\alpha/2)^{d^+ + d^-}}{(d^+)!(d^-)!}, \ \Pi^{\pm} \equiv \Pi^{\pm}(\underline{d}, \underline{\eta}) \equiv \prod_{i=1}^{d^{\pm}} \left(1 - \prod_{j=1}^{k-1} \eta_{ij}^{\pm}\right)$$

We show $(\boldsymbol{R}_\alpha)^\ell \mathbf{1}_{1/2} \xrightarrow{\ell \to \infty} \mu_\alpha$.

Distributional equation for the chance of being + in a random cluster.

Define

$$\Phi(\alpha) = \sum_{\underline{d}} \pi_\alpha(\underline{d}) \int \ln\left(\Pi^+ + \Pi^- - \Pi^+\Pi^-\right) \prod_j d\mu_\alpha(\eta_j) \prod_{i,j} d\mu_\alpha(\eta_{ij}^{\pm})$$

$$-\alpha(k-1) \int \ln\left(1 - \prod_{j=1}^{k} \eta_j\right) \prod_j d\mu_\alpha(\eta_j) \prod_{i,j} d\mu_\alpha(\eta_{ij}^{\pm})$$

Expected change in $\log \Omega_n$ *to* $\log \Omega_{n+1}$.

Explicit formula ($k \geqslant 3$) Let $\mathscr{P} \equiv$ space of probability measures on $[0,1]$. Define the distributional recursion $\boldsymbol{R}_\alpha : \mathscr{P} \to \mathscr{P}$,

$$\boldsymbol{R}_\alpha \mu(B) \equiv \sum_{\underline{d} \equiv (d^+, d^-)} \pi_\alpha(\underline{d}) \int \mathbf{1}\left\{ \frac{(1-\Pi^-)\Pi^+}{\Pi^+ + \Pi^- - \Pi^+\Pi^-} \in B \right\} \prod_{i,j} d\mu(\eta_{ij}^\pm)$$

$$\text{with } \pi_\alpha(\underline{d}) \equiv \frac{e^{-k\alpha}(k\alpha/2)^{d^+ + d^-}}{(d^+)!(d^-)!}, \quad \Pi^\pm \equiv \Pi^\pm(\underline{d}, \underline{\eta}) \equiv \prod_{i=1}^{d^\pm} \left(1 - \prod_{j=1}^{k-1} \eta_{ij}^\pm\right)$$

We show $(\boldsymbol{R}_\alpha)^\ell \mathbf{1}_{1/2} \xrightarrow{\ell \to \infty} \mu_\alpha$.

*Distributional equation for the chance of being **+** in a random cluster.*

Define

$$\Phi(\alpha) = \sum_{\underline{d}} \pi_\alpha(\underline{d}) \int \ln\left(\Pi^+ + \Pi^- - \Pi^+\Pi^-\right) \prod_j d\mu_\alpha(\eta_j) \prod_{i,j} d\mu_\alpha(\eta_{ij}^\pm)$$
$$-\alpha(k-1) \int \ln\left(1 - \prod_{j=1}^{k} \eta_j\right) \prod_j d\mu_\alpha(\eta_j) \prod_{i,j} d\mu_\alpha(\eta_{ij}^\pm)$$

Expected change in $\log \Omega_n$ *to* $\log \Omega_{n+1}$.

Then the 1RSB prediction $\alpha_{\text{sat}} \approx 2^k \ln 2 - (1 + \ln 2)/2$ is the root of $\Phi(\alpha) = 0$.

Previous Bounds: Satisfiability conjecture is known in special case $k = 2$, with $\alpha_{\mathsf{sat}} = 1$

Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92

Previous Bounds: Satisfiability conjecture is known in special case $k = 2$, with $\alpha_{\mathsf{sat}} = 1$

Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92

Bounds for $k \geqslant 3$: $(\epsilon_k \to 0$ as $k \to \infty)$

Previous Bounds: Satisfiability conjecture is known in special case $k = 2$, with $\alpha_{\text{sat}} = 1$

Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92

Bounds for $k \geqslant 3$: $(\epsilon_k \to 0$ as $k \to \infty)$

	bound on threshold	gap	
$\alpha_{\text{sat}} \leqslant$	$2^k \ln 2 - (\ln 2)/2 + \epsilon_k$	$O(1)$	trivial

Previous Bounds: Satisfiability conjecture is known in special case $k = 2$, with $\alpha_{\text{sat}} = 1$

Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92

Bounds for $k \geqslant 3$: $(\epsilon_k \to 0 \text{ as } k \to \infty)$

	bound on threshold	gap	
$\alpha_{\text{sat}} \leqslant$	$2^k \ln 2 - (\ln 2)/2 + \epsilon_k$	$O(1)$	trivial
	$2^k \ln 2 - (1 + \ln 2)/2 + \epsilon_k$	ϵ_k	Kirousis et al. '98

Previous Bounds: Satisfiability conjecture is known in special case $k = 2$, with $\alpha_{\text{sat}} = 1$

Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92

Bounds for $k \geqslant 3$: $(\epsilon_k \to 0 \text{ as } k \to \infty)$

	bound on threshold	gap	
$\alpha_{\text{sat}} \leqslant$	$2^k \ln 2 - (\ln 2)/2 + \epsilon_k$	$O(1)$	trivial
	$2^k \ln 2 - (1 + \ln 2)/2 + \epsilon_k$	ϵ_k	Kirousis et al. '98
$\alpha_{\text{sat}} \geqslant$	*(algorithmic)* $1.817 \cdot 2^k/k$	$2^k \ln 2$	Frieze–Suen '96
	(algorithmic) $2^k(\ln k)/k$	$2^k \ln 2$	Coja-Oghlan '10

Previous Bounds: Satisfiability conjecture is known in special case $k = 2$, with $\alpha_{\text{sat}} = 1$

Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92

Bounds for $k \geqslant 3$: $(\epsilon_k \to 0 \text{ as } k \to \infty)$

	bound on threshold	gap	
$\alpha_{\text{sat}} \leqslant$	$2^k \ln 2 - (\ln 2)/2 + \epsilon_k$	$O(1)$	trivial
	$2^k \ln 2 - (1 + \ln 2)/2 + \epsilon_k$	ϵ_k	Kirousis et al. '98
$\alpha_{\text{sat}} \geqslant$	*(algorithmic)* $1.817 \cdot 2^k/k$	$2^k \ln 2$	Frieze–Suen '96
	(algorithmic) $2^k(\ln k)/k$	$2^k \ln 2$	Coja-Oghlan '10
	$2^{k-1} \ln 2 - O(1)$	$2^{k-1} \ln 2$	Achlioptas–Moore '02
	$2^k \ln 2 - O(k)$	$O(k)$	Achlioptas–Peres '03
	$2^k \ln 2 - 3(\ln 2)/2 - \epsilon_k$	$O(1)$	Coja-Oghlan–
	$2^k \ln 2 - (1 + \ln 2)/2 - \epsilon_k$	ϵ_k	–Panagiotou '13, '14

Previous Bounds: Satisfiability conjecture is known in special case $k = 2$, with $\alpha_{\text{sat}} = 1$

Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92

Bounds for $k \geqslant 3$: $(\epsilon_k \to 0$ as $k \to \infty)$

	bound on threshold	gap	
$\alpha_{\text{sat}} \leqslant$	$2^k \ln 2 - (\ln 2)/2 + \epsilon_k$	$O(1)$	trivial
	$2^k \ln 2 - (1 + \ln 2)/2 + \epsilon_k$	ϵ_k	Kirousis et al. '98
$\alpha_{\text{sat}} \geqslant$	*(algorithmic)* $1.817 \cdot 2^k/k$	$2^k \ln 2$	Frieze–Suen '96
	(algorithmic) $2^k(\ln k)/k$	$2^k \ln 2$	Coja-Oghlan '10
	$2^{k-1} \ln 2 - O(1)$	$2^{k-1} \ln 2$	Achlioptas–Moore '02
	$2^k \ln 2 - O(k)$	$O(k)$	Achlioptas–Peres '03
	$2^k \ln 2 - 3(\ln 2)/2 - \epsilon_k$	$O(1)$	Coja-Oghlan–
	$2^k \ln 2 - (1 + \ln 2)/2 - \epsilon_k$	ϵ_k	–Panagiotou '13, '14
$\alpha_{\text{sat}} =$	$\alpha_\star$ $(k \geqslant k_0)$	0 $(k \geqslant k_0)$	exact threshold

Theorem.(Ding, S., Sun) *For $k \geqslant k_0$ (absolute constant), random k-SAT has a sharp satisfiability threshold, with explicit value $\alpha_{\mathsf{sat}} = \alpha_\star$ matching the one-step replica symmetry breaking prediction of Mertens–Mézard–Zecchina '06.*

Beyond the Satisfiability Threshold

Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

$$\mathbb{E}Z = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

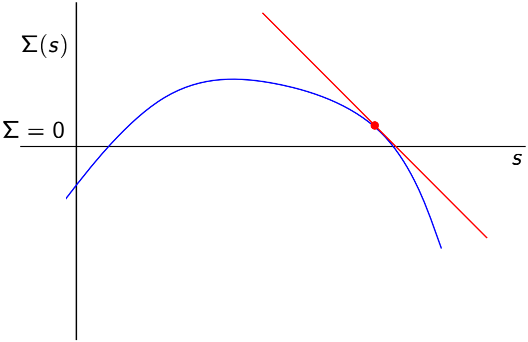
$$\mathbb{E}Z = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

$$\mathbb{E}Z = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

$\mathbb{E}Z$ is dominated by s where $\Sigma'(s) \equiv -1$ (depending on α).

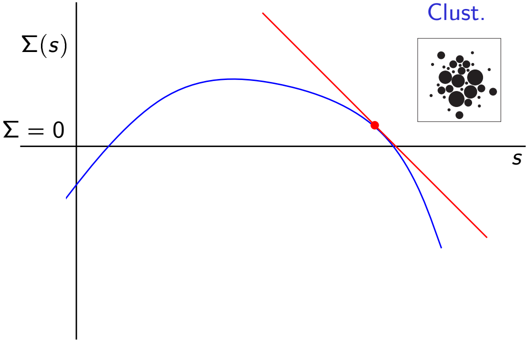


Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

$$\mathbb{E}Z = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

$\mathbb{E}Z$ is dominated by s where $\Sigma'(s) \equiv -1$ (depending on α).

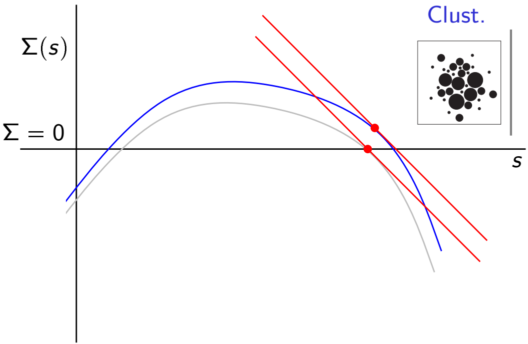


Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

$$\mathbb{E}Z = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

$\mathbb{E}Z$ is dominated by s where $\Sigma'(s) \equiv -1$ (depending on α).

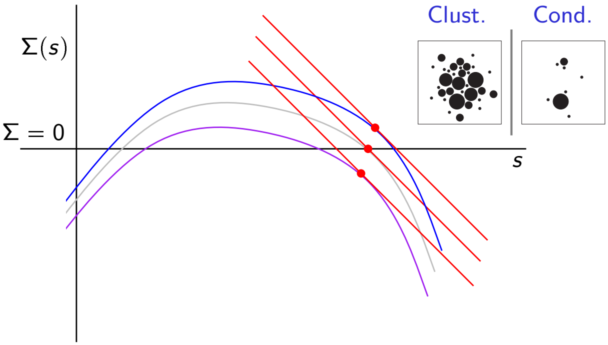


Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

$$\mathbb{E}Z = \sum \underbrace{\text{(cluster size)}}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

$\mathbb{E}Z$ is dominated by s where $\Sigma'(s) \equiv -1$ (depending on α).

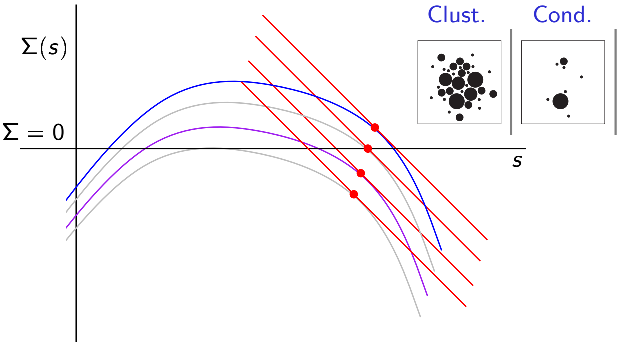


Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

$$\mathbb{E}Z = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

$\mathbb{E}Z$ is dominated by s where $\Sigma'(s) \equiv -1$ (depending on α).

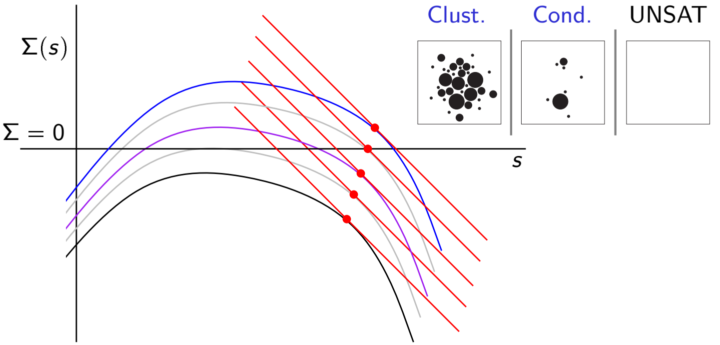


Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

$$\mathbb{E}Z = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

$\mathbb{E}Z$ is dominated by s where $\Sigma'(s) \equiv -1$ (depending on α).

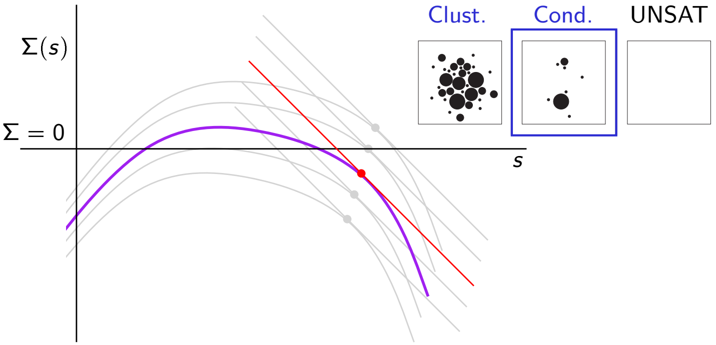


Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

$$\mathbb{E}Z = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

$\mathbb{E}Z$ is dominated by s where $\Sigma'(s) \equiv -1$ (depending on α).

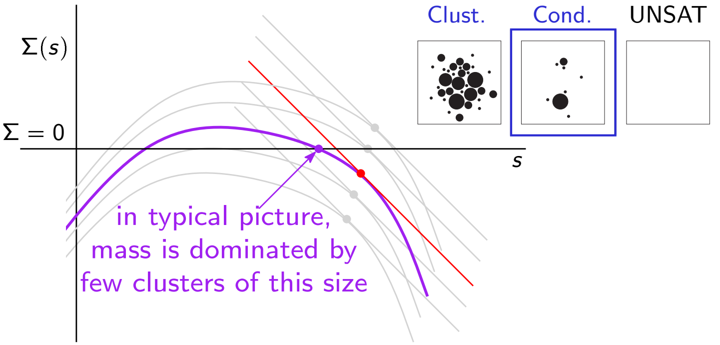


Condensation

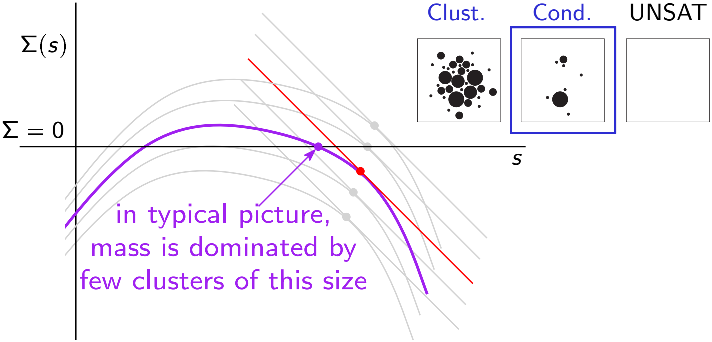
Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

$$\mathbb{E}Z = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\text{number of clusters of that size}]}_{\exp\{n\Sigma(s)\}}$$

$\mathbb{E}Z$ is dominated by s where $\Sigma'(s) \equiv -1$ (depending on α).



Condensation and non-concentration



The 1-RSB prediction:

- Satisfiability Threshold

$$\alpha_{\text{sat}} = \sup\{\alpha : \sup_s \Sigma(s) \geqslant 0\}$$

- Condensation Threshold and free energy

$$\alpha_{\text{cond}} = \sup\left\{\alpha : \sup_s s + \Sigma(s) = \sup_{s:\Sigma(s)\geqslant 0} s + \Sigma(s)\right\}$$

$$\Phi = \lim_{n\to\infty} \frac{1}{n} \log Z = \sup\{s + \Sigma(s) : \Sigma(s) > 0\} = \sup\{s : \Sigma(s) > 0\}$$

Results beyond the condensation threshold:

Results beyond the condensation threshold:

Condensation Threshold:

Random k-Colourings G(n,p) large k [Bapst, Coja-Oghlan, Hetterich, Rassmann, Vilenchik]

Regular k-NAESAT large k [S', Sun, Zhang]

Condensation Regime Free Energy:

Regular k-NAESAT large k [S', Sun, Zhang]

Satisfiability Threshold:

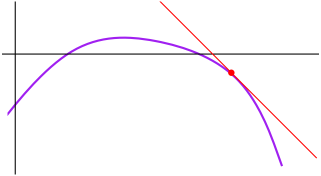
Regular NAESAT large k [Ding, S', Sun]

Maximum Independent Set d-Regular, large d [Ding, S', Sun]

Regular SAT, large k [Coja-Oghlan, Panagiotou]

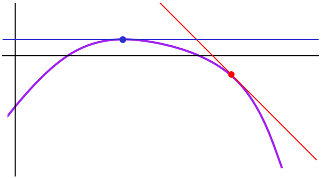
Random k-SAT, large k [Ding, S', Sun]

Free Energy



$$\mathbb{E}Z = \sum\nolimits_s \exp\{n[1 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = -1.$$

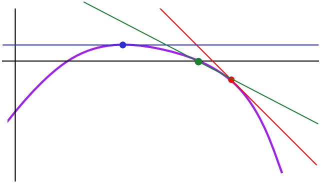
Free Energy



$$\mathbb{E}Z = \sum_s \exp\{n[1 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = -1.$$

$$\mathbb{E}|\Omega| = \sum_s \exp\{n[0 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = 0.$$

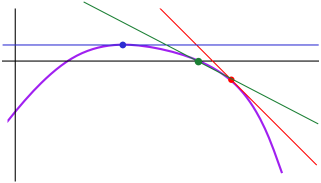
Free Energy



$$\mathbb{E}Z = \sum_s \exp\{n[1 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = -1.$$

$$\mathbb{E}|\Omega| = \sum_s \exp\{n[0 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = 0.$$

Free Energy Weight clusters by (their size)$^{\lambda}$

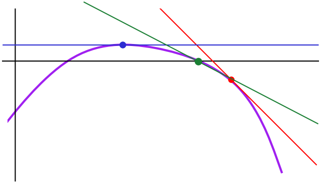


$$\mathbb{E}Z = \sum_s \exp\{n[1 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = -1.$$

$$\mathbb{E}|\Omega| = \sum_s \exp\{n[0 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = 0.$$

$$\mathbb{E}Z_\lambda \equiv \sum_s \exp\{n[\lambda \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = -\lambda$$

Free Energy Weight clusters by (their size)$^\lambda$



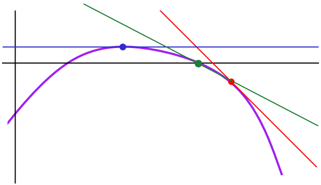
$$\mathbb{E}Z = \sum_s \exp\{n[1 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = -1.$$

$$\mathbb{E}|\Omega| = \sum_s \exp\{n[0 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = 0.$$

$$\mathbb{E}Z_\lambda \equiv \sum_s \exp\{n[\lambda \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = -\lambda$$

In fact, $\frac{1}{n} \log \mathbb{E}Z_\lambda$ is the Legendre transformation of $\Sigma(s)$.

Free Energy Weight clusters by (their size)$^\lambda$



$$\mathbb{E}Z = \sum_s \exp\{n[1 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = -1.$$
$$\mathbb{E}|\Omega| = \sum_s \exp\{n[0 \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = 0.$$
$$\mathbb{E}Z_\lambda \equiv \sum_s \exp\{n[\lambda \cdot s + \Sigma(s)]\}, \qquad \text{maximized at } \Sigma'(s) = -\lambda$$

In fact, $\frac{1}{n}\log \mathbb{E}Z_\lambda$ is the Legendre transformation of $\Sigma(s)$.

The moments of Z_λ may be computed by adding local weights to the free variables in the $\{\texttt{+}, \texttt{-}, \texttt{f}\}$ configurations.

Counting solutions within a cluster

We divide the subgraph of `f`'s into a forest of O(1)-size trees such that assigning values to one tree does not affect the others.

Counting solutions within a cluster

We divide the subgraph of f's into a forest of O(1)-size trees such that assigning values to one tree does not affect the others.

Every edge encodes the 'f-tree' it resides in.

$$\Rightarrow \underline{\tau} \in \{+, -, \mathtt{f}\}^{V(\mathscr{G})} \leftrightarrow \underline{\sigma} \in \{\mathtt{f}\text{-trees}\}^{E(\mathscr{G})}$$

Counting solutions within a cluster

We divide the subgraph of f's into a forest of O(1)-size trees such that assigning values to one tree does not affect the others.
Every edge encodes the 'f-tree' it resides in.

$$\Rightarrow \underline{\tau} \in \{+, -, \mathtt{f}\}^{V(\mathscr{G})} \leftrightarrow \underline{\sigma} \in \{\mathtt{f}\text{-trees}\}^{E(\mathscr{G})}$$

We can use BP algorithm write the number of solutions on trees as a product of weights.

Counting solutions within a cluster

We divide the subgraph of $\texttt{f}$'s into a forest of O(1)-size trees such that assigning values to one tree does not affect the others.
Every edge encodes the '$\texttt{f}$-tree' it resides in.

$$\Rightarrow \underline{\tau} \in \{+, -, \texttt{f}\}^{V(\mathscr{G})} \leftrightarrow \underline{\sigma} \in \{\texttt{f}\text{-trees}\}^{E(\mathscr{G})}$$

We can use BP algorithm write the number of solutions on trees as a product of weights.

Define weight functions Ψ_v, Ψ_a, Ψ_e accordingly s.t. for each $\underline{\sigma} \in \{\texttt{f}\text{-trees}\}^{E(\mathscr{G})}$

$$\begin{aligned} w(\underline{\sigma}) &\equiv \prod_v \Psi_v(\underline{\sigma}_{\delta v}) \prod_a \Psi_a(\underline{\sigma}_{\delta a}) \prod_{e=(av)} \Psi_e(\underline{\sigma}_{(av)}) \\ &= \prod_T (\#\text{ of ways of assigning } \texttt{f}\text{'s. in tree } T) \\ &= (\text{size of cluster}) \end{aligned}$$

Counting solutions within a cluster

We divide the subgraph of $\texttt{f}$'s into a forest of O(1)-size trees such that assigning values to one tree does not affect the others.

Every edge encodes the '$\texttt{f}$-tree' it resides in.

$$\Rightarrow \underline{\tau} \in \{+, -, \texttt{f}\}^{V(\mathscr{G})} \leftrightarrow \underline{\sigma} \in \{\texttt{f}\text{-trees}\}^{E(\mathscr{G})}$$

We can use BP algorithm write the number of solutions on trees as a product of weights.

Define weight functions Ψ_v, Ψ_a, Ψ_e accordingly s.t. for each $\underline{\sigma} \in \{\texttt{f}\text{-trees}\}^{E(\mathscr{G})}$

$$\begin{aligned} w(\underline{\sigma}) &\equiv \prod_v \Psi_v(\underline{\sigma}_{\delta v}) \prod_a \Psi_a(\underline{\sigma}_{\delta a}) \prod_{e=(av)} \Psi_e(\underline{\sigma}_{(av)}) \\ &= \prod_T (\#\text{ of ways of assigning } \texttt{f}\text{'s. in tree } T) \\ &= (\text{size of cluster}) \end{aligned}$$

Then we can define

$$Z_\lambda \equiv \textstyle\sum_{\underline{\sigma}} w^\lambda(\underline{\sigma}).$$

Optimization

We can write

$$\mathbb{E}Z_\lambda = \frac{\sum_{(\mathscr{G},\underline{\sigma})} w^\lambda(\underline{\sigma})}{\#\mathscr{G}}$$

Optimization

We can write

$$\mathbb{E}Z_\lambda = \frac{\sum_{(\mathscr{G},\underline{\sigma})} w^\lambda(\underline{\sigma})}{\#\mathscr{G}}$$

Then partitioning $\underline{\sigma}$ according to its empirical distribution ν,

$$\begin{aligned}\mathbb{E}Z_\lambda[\nu] &= \frac{\binom{n}{n\dot{\nu}}\binom{\alpha n}{\alpha n\hat{\nu}}}{\binom{dn}{dn\bar{\nu}}} {\Psi_v}^{\lambda n\dot{\nu}} {\Psi_a}^{\lambda\alpha n\hat{\nu}} {\Psi_e}^{\lambda dn\bar{\nu}} \\ &\equiv \exp\{n\,[\Sigma(\nu) + \lambda s(\nu)] + o(n)\} \\ &\equiv \exp\{n\,\Phi_\lambda(\nu) + o(n)\}\end{aligned}$$

Optimization

We can write

$$\mathbb{E}Z_\lambda = \frac{\sum_{(\mathscr{G},\underline{\sigma})} w^\lambda(\underline{\sigma})}{\#\mathscr{G}}$$

Then partitioning $\underline{\sigma}$ according to its empirical distribution ν,

$$\begin{aligned}\mathbb{E}Z_\lambda[\nu] &= \frac{\binom{n}{n\dot{\nu}}\binom{\alpha n}{\alpha n\hat{\nu}}}{\binom{dn}{dn\bar{\nu}}}\Psi_v{}^{\lambda n\dot{\nu}}\Psi_a{}^{\lambda\alpha n\hat{\nu}}\Psi_e{}^{\lambda dn\bar{\nu}} \\ &\equiv \exp\{n\,[\Sigma(\nu)+\lambda s(\nu)]+o(n)\} \\ &\equiv \exp\{n\,\Phi_\lambda(\nu)+o(n)\}\end{aligned}$$

Can find optimal $\nu^\star$ by finding fixed points of the Belief Propagation equations (e.g. Dembo–Montanari–Sun '13.)

Optimization

We can write

$$\mathbb{E} Z_\lambda = \frac{\sum_{(\mathscr{G},\underline{\sigma})} w^\lambda(\underline{\sigma})}{\#\mathscr{G}}$$

Then partitioning $\underline{\sigma}$ according to its empirical distribution ν,

$$\begin{aligned}\mathbb{E} Z_\lambda[\nu] &= \frac{\binom{n}{n\dot\nu}\binom{\alpha n}{\alpha n\hat\nu}}{\binom{dn}{dn\bar\nu}} {\Psi_v}^{\lambda n\dot\nu} {\Psi_a}^{\lambda\alpha n\hat\nu} {\Psi_e}^{\lambda dn\bar\nu} \\ &\equiv \exp\{n\,[\Sigma(\nu) + \lambda s(\nu)] + o(n)\} \\ &\equiv \exp\{n\,\Phi_\lambda(\nu) + o(n)\}\end{aligned}$$

Can find optimal $\nu^\star$ by finding fixed points of the Belief Propagation equations (e.g. Dembo–Montanari–Sun '13.)

For regular NAE-SAT and $k \geqslant k_0$, the limit $\Phi(\alpha)$ exists for $\alpha_{\text{cond}} < \alpha < \alpha_{\text{sat}}$, given by an explicit formula matching the 1-RSB prediction from statistical physics. S., Sun, Zhang '16

New Results

Theorem (Nam, S., Sohn 19+) *For $k \geqslant k_0$ (absolute constant), random regular k-NAESAT, WHP the largest and second largest clusters both have a constant fraction of the set total solutions. Two uniformly chosen solutions have normalized hamming distance concentrated on* **two** *points*.

- Requires estimating the partition function up to multiplicative $O(1)$ factor.
- States space of free trees is unbounded.

Future Directions and open problems:

Future Directions and open problems:

Small k?

Future Directions and open problems:

Small k?

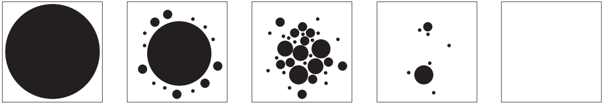
Extension to random graph coloring?

Future Directions and open problems:

Small k?

Extension to random graph coloring?

Other aspects of the 1RSB phase diagram?

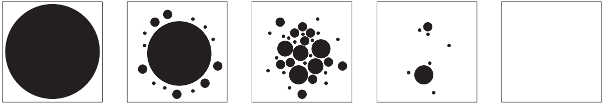


Future Directions and open problems:

Small k?

Extension to random graph coloring?

Other aspects of the 1RSB phase diagram?



Models at finite temperature?

Thanks!