A Tale of Two Series

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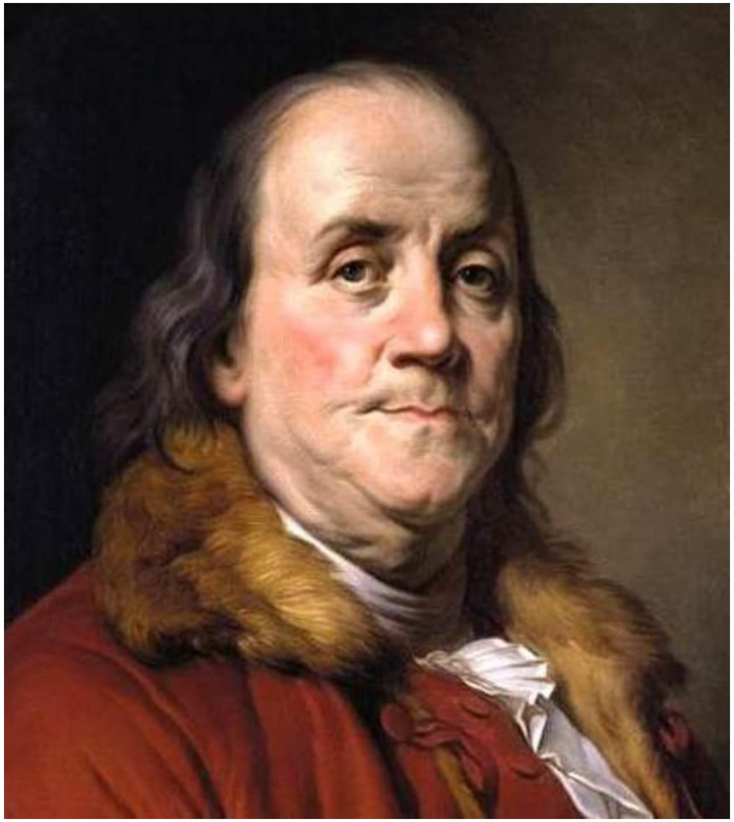
Leonhard Euler
1707 - 1783



Benjamin Franklin
1706 - 1790

"All celebrated mathematicians now alive are his disciples: there is no one who is not guided and sustained by the genius of Euler."

– Condorcet

ACT ONE:

Reciprocals of the Primes

(1737)

Harmonic Series (1689)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

" The sum of the infinite harmonic progression is *infinite*."

– Jakob Bernoulli

Tractatus de seriebus infinitis, 1689

In 1734, Euler had shown that the harmonic series grows like the natural logarithm:

DE PROGRESSIONIBVS HARMONICIS 157

Quae ſeries, cum ſint conuergentes, ſi proxime ſummentur prodibit $1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{i} = l(i+1) + 0,577218$

In modern notation:

$$\sum_{k=1}^{i} \frac{1}{k} \approx \ln(i+1) + \gamma$$

Begin with the harmonic series:

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

$$\frac{1}{2}H = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots$$

$$\frac{1}{2}H = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \ldots$$

"…in which no denominators are even."

$$\frac{1}{2}H = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \ldots$$

$$\frac{1}{3}\left[\frac{1}{2}H\right] = \frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \ldots$$

$$\frac{1 \times 2}{2 \times 3}H = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots$$

"…whose denominators are divisible by neither 2 nor 3."

$$\frac{1\times2}{2\times3}H = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \ldots$$

$$\frac{1}{5}\left[\frac{1\times2}{2\times3}H\right] = \frac{1}{5} + \frac{1}{25} + \frac{1}{35} + \frac{1}{55} + \ldots$$

$$\frac{1\times2\times4}{2\times3\times5}H = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \ldots$$

Continue to get …

$$\frac{1\times 2\times 4\times 6\times 10...}{2\times 3\times 5\times 7\times 11...}H = 1$$

$$\text{So, } H = \frac{2\times 3\times 5\times 7\times 11...}{1\times 2\times 4\times 6\times 10...}$$

$$= \frac{1}{(1/2)\times(2/3)\times(4/5)\times(6/7)...}$$

$$= \frac{1}{(1-1/2)\times(1-1/3)\times(1-1/5)\times(1-1/7)...}$$

Euler's product-sum formula (1737)

$$P = \cfrac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\ \&c.,}$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = \prod_{p \text{ prime}} \frac{1}{1-1/p}$$

"Pruning" the harmonic series

Cut out the odds:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots \qquad \text{diverges}$$

Cut out the non-squares:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots \qquad \text{converges}$$

Cut out the composites:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ldots \quad ?$$

$$H = \frac{1}{(1 - 1/2)\times(1 - 1/3)\times(1 - 1/5)\ldots}$$

$$\ln H = -\ln(1 - 1/2) - \ln(1 - 1/3) - \ln(1 - 1/5) - \ldots$$

Recall: $$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

So $$-\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

$$\ln H = -\ln(1-1/2) - \ln(1-1/3) - \ln(1-1/5) - \ldots$$

$$= (1/2) + \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} + \frac{(1/2)^4}{4} + \ldots$$

$$+ (1/3) + \frac{(1/3)^2}{2} + \frac{(1/3)^3}{3} + \frac{(1/3)^4}{4} + \ldots$$

$$+ (1/5) + \frac{(1/5)^2}{2} + \frac{(1/5)^3}{3} + \frac{(1/5)^4}{4} + \ldots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\ln H = \sum (1/p) + \frac{1}{2}\sum (1/p^2) + \frac{1}{3}\sum (1/p^3) + \ldots$$

$$= A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \ldots$$

where

$$A = \sum 1/p \;,\; B = \sum 1/p^2 \;,\; C = \sum 1/p^3$$

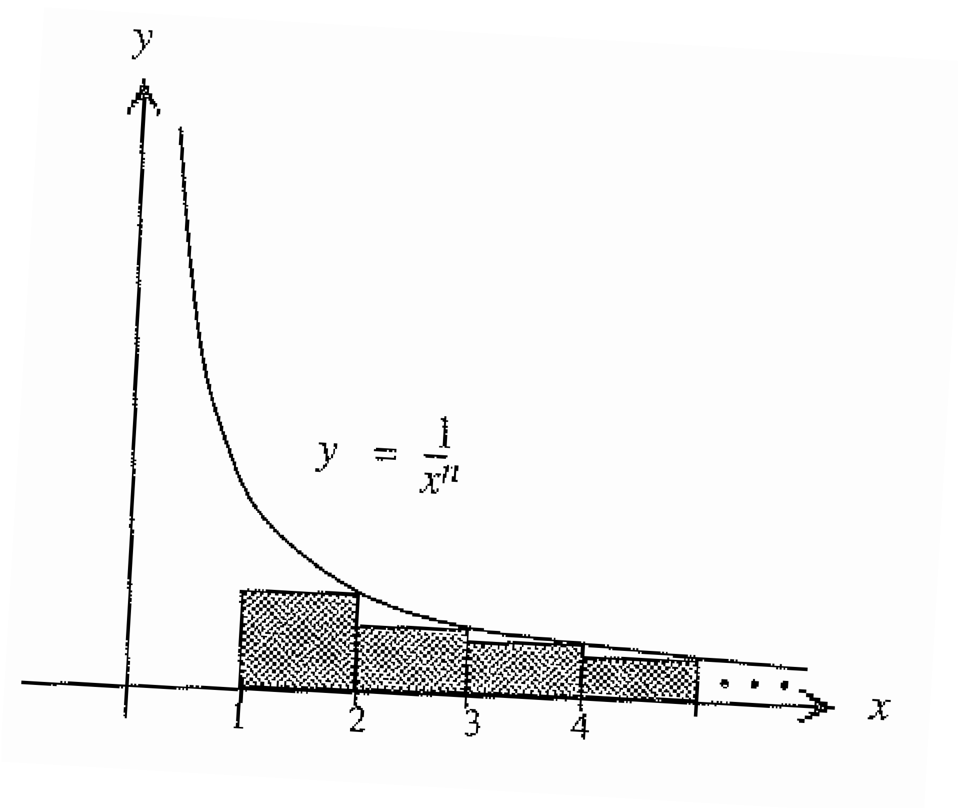
Euler observed, “Not only do *B*, *C*, *D*, etc. have finite values, but

$$\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\ldots$$

has a finite value as well.”

Why?

Note: If $n \geq 2$, then $\sum_{k=2}^{\infty} 1/k^n \leq \frac{1}{n-1}$.



By the integral test,

$$\sum_{k=2}^{\infty} 1/k^n \leq \int_1^{\infty} \frac{1}{x^n}\,dx = \frac{1}{n-1}$$

If $n \geq 2$, then $\displaystyle\sum_{k=2}^{\infty} 1/k^n \leq \frac{1}{n-1}$.

So $$B = \sum 1/p^2 \leq \sum_{k=2}^{\infty} 1/k^2 \leq \frac{1}{2-1} = 1$$

$$C = \sum 1/p^3 \leq \sum_{k=2}^{\infty} 1/k^3 \leq \frac{1}{3-1} = \frac{1}{2}$$

$$D \leq \frac{1}{3} \qquad E \leq \frac{1}{4} \quad \text{etc.}$$

$$\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\ldots$$

$$=\frac{1}{2}\sum 1/p^2+\frac{1}{3}\sum 1/p^3+\frac{1}{4}\sum 1/p^4+\ldots$$

$$\le\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k^2}+\frac{1}{3}\sum_{k=2}^{\infty}\frac{1}{k^3}+\frac{1}{4}\sum_{k=2}^{\infty}\frac{1}{k^4}+\ldots$$

$$\le\frac{1}{2}\cdot 1+\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{3}+\ldots=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots$$

So $$\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\ldots$$

$$\leq \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots$$

$$=1$$

$$\ln H = A + \left[\frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \ldots\right]$$

$\downarrow$ Infinite $\qquad\qquad$ $\downarrow$ Finite

Hence $A = \sum_{p \text{ prime}} 1/p$ is infinite.

Q. E. D.

<u>Corollary</u>: There are infinitely many primes.

$$\ln H = A + \left[\frac{1}{2} B + \frac{1}{3} C + \frac{1}{4} D + \ldots \right]$$

$$\sum \frac{1}{p} = A \sim \ln H$$

Euler had shown in 1734 that

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \ln(n+1) + 0.577218$$

So $H \sim \ln n$

$$\sum \frac{1}{p} \sim \ln H \quad \text{and} \quad H \sim \ln n$$

$$\text{So} \ \sum \frac{1}{p} \sim \ln H \sim \ln(\ln n)$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \text{etc.} = l.\, l\, \infty.$$

“One may well regard these investigations as marking the birth of analytic number theory.”

– André Weil

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \text{etc.} = l.\, l\, \infty.$$

$$\text{"}\, l.\, l\, \infty \,\text{"} \quad \text{means} \quad \lim_{n\to\infty} \Big[\ln(\ln(n)) \Big]$$

“ln(ln(n)) is known to grow to infinity,

although it has never been observed doing so.”

ACT TWO:

Reciprocals of the Squares

(1755)

The Basel Problem

In 1689, Jakob Bernoulli challenged the mathematical community to find the *exact* sum of the infinite series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

Over the course of his career, Euler gave multiple proofs that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots=\frac{\pi^2}{6}$$

1734: used sine series

1741: used arcsine series and integral calculus

1755: used l’Hospital’s rule

INSTITUTIONES CALCULI DIFFERENTIALIS

CUM EIUS VSU

IN ANALYSI FINITORUM

AC

DOCTRINA SERIERUM

AUCTORE

LEONHARDO EULERO

ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROF. HONOR. ACAD. IMP. SCIENT. PETROP. ET ACADEMIARUM
REGIARUM PARISINAE ET LONDINENSIS
SOCIO.

IMPENSIS
ACADEMIAE IMPERIALIS SCIENTIARUM
PETROPOLITANAE
1755.

Write $\sin t$ as an infinite product:

$$\sin t = t\left(1-\frac{t}{\pi}\right)\left(1+\frac{t}{\pi}\right)\left(1-\frac{t}{2\pi}\right)\left(1+\frac{t}{2\pi}\right)\cdots$$

Why?

$$\sin t = 0 \Rightarrow \quad t = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$$

$$t\left(1-\frac{t}{\pi}\right)\left(1+\frac{t}{\pi}\right)\left(1-\frac{t}{2\pi}\right)\left(1+\frac{t}{2\pi}\right)\cdots = 0 \Rightarrow$$

$$t = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$$

Write $\sin t$ as an infinite product:

$$\sin t = t\left(1-\frac{t}{\pi}\right)\left(1+\frac{t}{\pi}\right)\left(1-\frac{t}{2\pi}\right)\left(1+\frac{t}{2\pi}\right)\cdots$$

$$= t\left(1-\frac{t^2}{\pi^2}\right)\left(1-\frac{t^2}{4\pi^2}\right)\left(1-\frac{t^2}{9\pi^2}\right)\cdots$$

Let $t = \pi y$

$$\sin \pi y = \pi y\left(1-y^2\right)\left(\frac{4-y^2}{4}\right)\left(\frac{9-y^2}{9}\right)\cdots$$

$$\sin \pi y = \pi y\left(1 - y^2\right)\left(\frac{4 - y^2}{4}\right)\left(\frac{9 - y^2}{9}\right)\cdots$$

Take logs:

$$\ln(\sin \pi y) = \ln \pi + \ln y + \ln(1 - y^2) + \ln(4 - y^2) - \ln 4$$

$$+ \ln(9 - y^2) - \ln 9 + \cdots$$

Differentiate with respect to y:

$$\frac{\pi \cos \pi y}{\sin \pi y} = \frac{1}{y} - \frac{2y}{1 - y^2} - \frac{2y}{4 - y^2} - \frac{2y}{9 - y^2} - \cdots$$

So, $$\frac{2y}{1-y^2}+\frac{2y}{4-y^2}+\frac{2y}{9-y^2}+\cdots=\frac{1}{y}-\frac{\pi\cos\pi y}{\sin\pi y}$$

$$\Rightarrow \frac{1}{1-y^2}+\frac{1}{4-y^2}+\frac{1}{9-y^2}+\cdots=\frac{1}{2y^2}-\frac{\pi\cos\pi y}{2y\sin\pi y}$$

Let $y=-ix \Rightarrow y^2=(-ix)^2= i^2x^2 =-x^2$

$$\frac{1}{1+x^2}+\frac{1}{4+x^2}+\frac{1}{9+x^2}+\cdots=-\frac{1}{2x^2}+\frac{\pi\cos(-i\pi x)}{2ix\sin(-i\pi x)}$$

Recall: $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

Thus,

$$\frac{\pi \cos(-i\pi x)}{2ix \sin(-i\pi x)} = \frac{\pi \left[\dfrac{e^{\pi x} + e^{-\pi x}}{2}\right]}{2\cancel{i}x \left[\dfrac{e^{\pi x} - e^{-\pi x}}{\cancel{2i}}\right]} = \frac{\pi\left[e^{\pi x} + e^{-\pi x}\right]}{2x\left[e^{\pi x} - e^{-\pi x}\right]}$$

$$= \frac{\pi\left[e^{2\pi x} + 1\right]}{2x\left[e^{2\pi x} - 1\right]}$$

$$\frac{1}{1+x^2}+\frac{1}{4+x^2}+\frac{1}{9+x^2}+\cdots=-\frac{1}{2x^2}+\frac{\pi\cos(-i\pi x)}{2ix\sin(-i\pi x)}$$

$$=-\frac{1}{2x^2}+\frac{\pi\left[e^{2\pi x}+1\right]}{2x\left[e^{2\pi x}-1\right]}$$

$$=\frac{-\left[e^{2\pi x}-1\right]+\pi x\left[e^{2\pi x}+1\right]}{2x^2\left[e^{2\pi x}-1\right]}$$

Hence,

$$\frac{1}{1+x^2}+\frac{1}{4+x^2}+\frac{1}{9+x^2}+\cdots=\frac{1-e^{2\pi x}+\pi x+\pi x e^{2\pi x}}{2x^2e^{2\pi x}-2x^2}$$

$$\frac{1}{1+x^2}+\frac{1}{4+x^2}+\frac{1}{9+x^2}+\cdots=\frac{1-e^{2\pi x}+\pi x+\pi x e^{2\pi x}}{2x^2e^{2\pi x}-2x^2}$$

Let $x = 0$:

LHS: $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$

RHS: $\frac{1-1+0+0}{0-0}=\frac{0}{0}$

Time for l'Hospital !

l’ Hospital

Guillaume François
Antoine de l’Hospital
(1661 – 1704)

Consider P/Q , a fraction whose numerator and denominator simultaneously vanish for $x = a$, thereby giving 0/0.

In general, "…the fraction dP/dQ takes the value of the fraction P/Q in question."

– Euler

$$\frac{1}{1+x^2}+\frac{1}{4+x^2}+\frac{1}{9+x^2}+\cdots=\frac{1-e^{2\pi x}+\pi x+\pi x e^{2\pi x}}{2x^2e^{2\pi x}-2x^2}$$

Apply l'Hospital to RHS: $\dfrac{1 - e^{2\rho x} + \rho x + \rho x e^{2\rho x}}{2x^2 e^{2\rho x} - 2x^2}$

$$\frac{-2\rho e^{2\rho x} + \rho + 2\rho^2 x e^{2\rho x} + \rho e^{2\rho x}}{4\rho x^2 e^{2\rho x} + 4x e^{2\rho x} - 4x}$$

$$= \frac{-\rho e^{2\rho x} + \rho + 2\rho^2 x e^{2\rho x}}{4\rho x^2 e^{2\rho x} + 4x e^{2\rho x} - 4x}$$

Let $x = 0$: $\dfrac{-\rho + \rho + 0}{0 + 0 - 0} = \dfrac{0}{0}$

Apply l' Hospital again: $\dfrac{-\pi e^{2\pi x}+\pi+2\pi^2 xe^{2\pi x}}{4\pi x^2 e^{2\pi x}+4xe^{2\pi x}-4x}$

$$\frac{-2\pi^2 e^{2\pi x}+4\pi^3 xe^{2\pi x}+2\pi^2 e^{2\pi x}}{8\pi^2 x^2 e^{2\pi x}+8\pi xe^{2\pi x}+8\pi xe^{2\pi x}+4e^{2\pi x}-4}$$

$$=\frac{\pi^3 xe^{2\pi x}}{2\pi^2 x^2 e^{2\pi x}+4\pi xe^{2\pi x}+e^{2\pi x}-1}$$

$$=\frac{\pi^3 x}{2\pi^2 x^2+4\pi x+1-e^{-2\pi x}}$$

$$= \frac{\rho^3 x}{2\rho^2 x^2 + 4\rho x + 1 - e^{-2\rho x}}$$

Let $x = 0$: $\quad \dfrac{0}{0 + 0 + 1 - 1} = \dfrac{0}{0}$

Apply l' Hospital to: $\quad \dfrac{\rho^3 x}{2\rho^2 x^2 + 4\rho x + 1 - e^{-2\rho x}}$

$$\frac{\rho^3}{4\rho^2 x + 4\rho + 2\rho e^{-2\rho x}}$$

Let $x = 0$: $\quad \dfrac{\rho^3}{0 + 4\rho + 2\rho} = \dfrac{\rho^3}{6\rho} = \dfrac{\rho^2}{6}$

$$\frac{1}{1+x^2}+\frac{1}{4+x^2}+\frac{1}{9+x^2}+\cdots=\frac{1-e^{2\pi x}+\pi x+\pi x e^{2\pi x}}{2x^2e^{2\pi x}-2x^2}$$

So, for $x = 0$, LHS is $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$

RHS is $\frac{p^2}{6}$

Thus, $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}$

Q.E.D. ! !

Note that this argument …

- had an all-star cast of transcendental functions: sines, cosines, logs, exponentials

- moved from real to complex and back again

- featured l'Hospital's rule in a starring role

“Euler was the high priest of sum worship,
for he was cleverer than anyone else at
inventing unorthodox methods of summation.”

– Ivor Grattan-Guinness

Way to Go, Uncle Leonhard!