

Einstein Metrics,

Complex Surfaces, &

Symplectic 4-Manifolds

Claude LeBrun

Stony Brook University

Departmental Colloquium
Thursday, October 17, 2019

Let (M, h) be a Riemannian n -manifold,

Let (M, h) be a Riemannian n -manifold, $p \in M$.

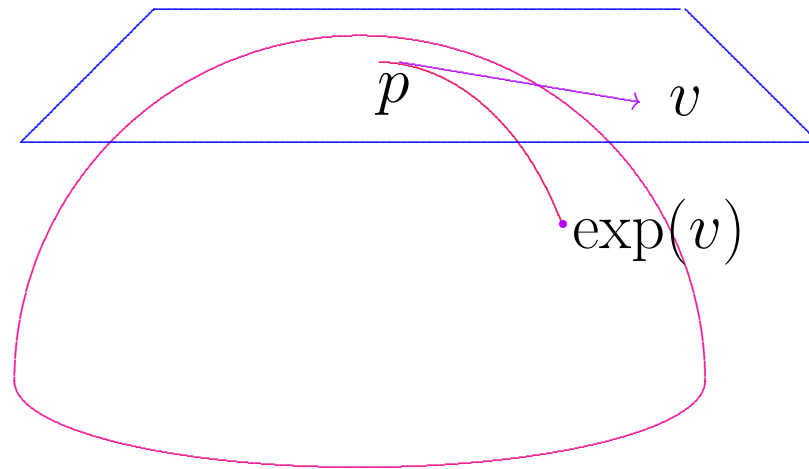
Let (M, h) be a Riemannian n -manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_p M \rightarrow M$$

Let (M, h) be a Riemannian n -manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_p M \rightarrow M$$

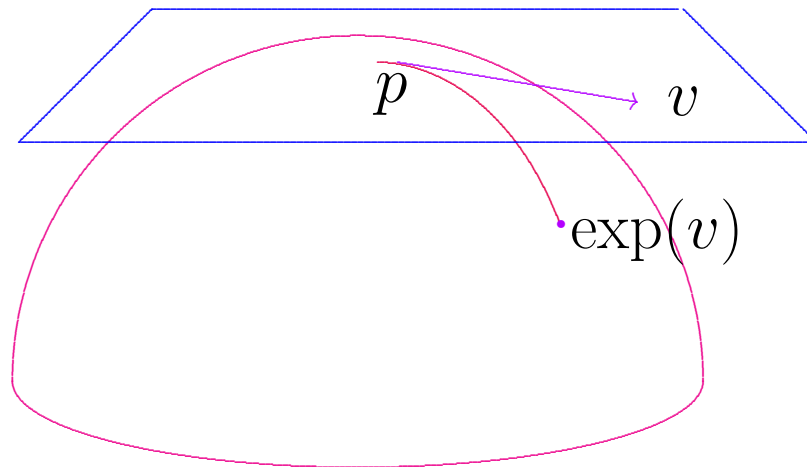
which is a diffeomorphism on a neighborhood of 0:



Let (M, h) be a Riemannian n -manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_p M \rightarrow M$$

which is a diffeomorphism on a neighborhood of 0:



Now choosing $T_p M \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal
basis gives us special coordinates on M .

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = d\mu_{\text{Euclidean}},$$

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = \left[1 - \right] d\mu_{\text{Euclidean}},$$

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = \left[1 - \frac{1}{6} r_{jk} x^j x^k + \dots \right] d\mu_{\text{Euclidean}},$$

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = \left[1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},$$

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = \left[1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor*

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = \left[1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* r_{jk} .

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = \left[1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i_{j ik}$.

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = \left[1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i_{jik}$.

The *Ricci curvature* is by definition the function on the unit tangent bundle

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = \left[1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i_{jik}$.

The *Ricci curvature* is by definition the function on the unit tangent bundle

$$STM = \{v \in TM \mid h(v, v) = 1\}$$

In these geodesic normal coordinates the metric volume form is given by

$$d\mu_h = \left[1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i_{jik}$.

The *Ricci curvature* is by definition the function on the unit tangent bundle

$$STM = \{v \in TM \mid h(v, v) = 1\}$$

given by

$$v \longmapsto r(v, v).$$

Definition.

Definition. A Riemannian metric h is said to be Einstein if it has constant Ricci curvature

Definition. A Riemannian metric h is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda h$$

for some constant $\lambda \in \mathbb{R}$.

Definition. A Riemannian metric h is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda h$$

for some constant $\lambda \in \mathbb{R}$.

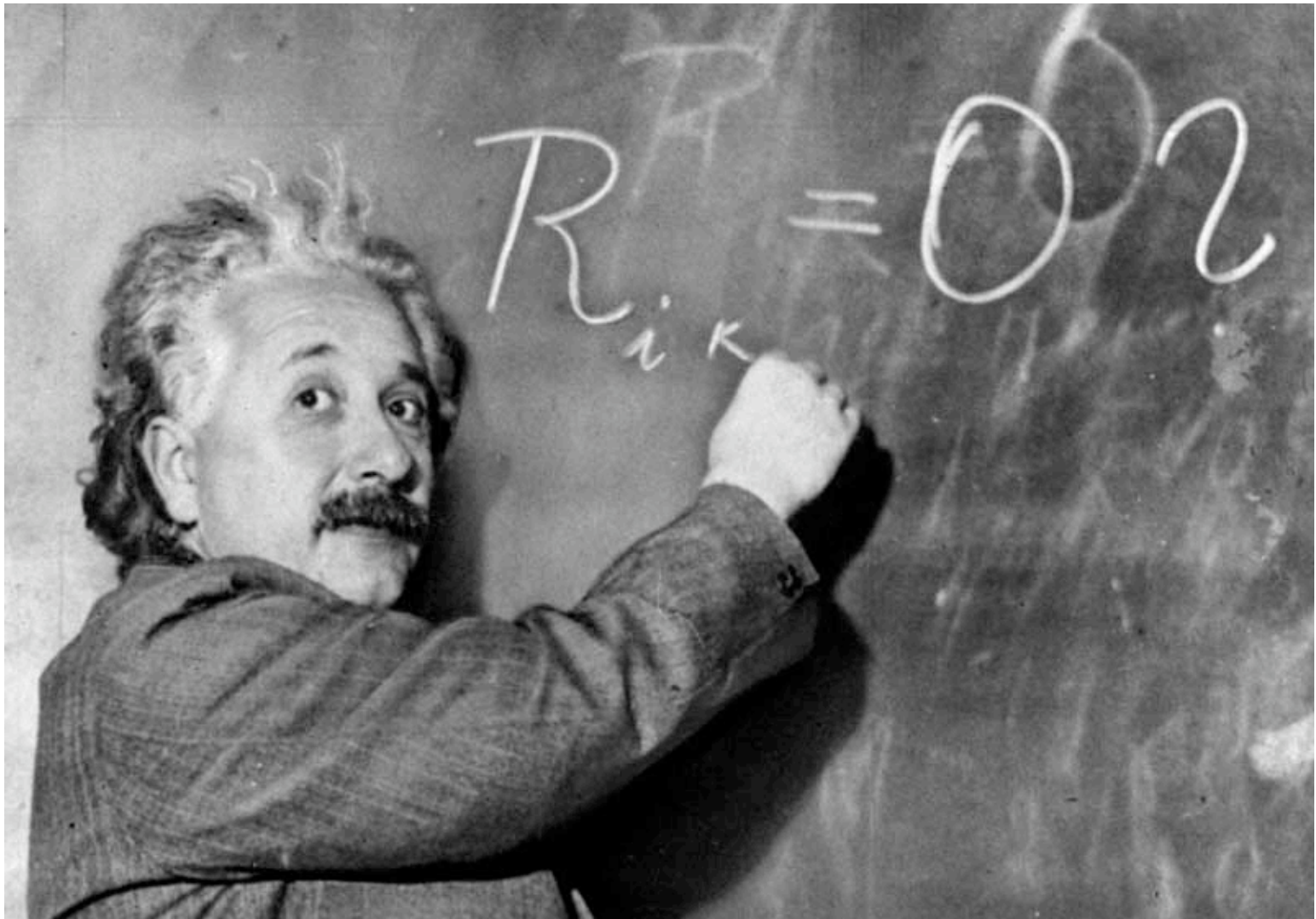
“... the greatest blunder of my life!”

— A. Einstein, to G. Gamow

Definition. A Riemannian metric h is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda h$$

for some constant $\lambda \in \mathbb{R}$.



Definition. A Riemannian metric h is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda h$$

for some constant $\lambda \in \mathbb{R}$.

As punishment ...

Definition. A Riemannian metric h is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda h$$

for some constant $\lambda \in \mathbb{R}$.

λ called Einstein constant.

Definition. A Riemannian metric h is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda h$$

for some constant $\lambda \in \mathbb{R}$.

λ called Einstein constant.

Has same sign as the *scalar curvature*

$$s = r_j^j = \mathcal{R}^{ij}_{ij}.$$

Definition. A Riemannian metric h is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda h$$

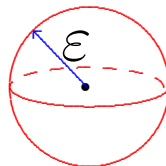
for some constant $\lambda \in \mathbb{R}$.

λ called Einstein constant.

Has same sign as the *scalar curvature*

$$s = r_j^j = \mathcal{R}^{ij}_{ij}.$$

$$\frac{\text{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



Recognition Problem:

Recognition Problem:

Suppose M^n admits Einstein metric h .

Recognition Problem:

Suppose M^n admits Einstein metric h .

What, if anything, does h then tell us about M ?

Recognition Problem:

Suppose M^n admits Einstein metric h .

What, if anything, does h then tell us about M ?

Can we recognize M by looking at h ?

Recognition Problem:

Suppose M^n admits Einstein metric h .

What, if anything, does h then tell us about M ?

Can we recognize M by looking at h ?

When $n = 3$, h has constant sectional curvature!

Recognition Problem:

Suppose M^n admits Einstein metric h .

What, if anything, does h then tell us about M ?

Can we recognize M by looking at h ?

When $n = 3$, h has constant sectional curvature!

So M has universal cover $S^3, \mathbb{R}^3, \mathcal{H}^3 \dots$

Recognition Problem:

Suppose M^n admits Einstein metric h .

What, if anything, does h then tell us about M ?

Can we recognize M by looking at h ?

When $n = 3$, h has constant sectional curvature!

So M has universal cover $S^3, \mathbb{R}^3, \mathcal{H}^3 \dots$

But when $n \geq 5$, situation seems hopeless.

Recognition Problem:

Suppose M^n admits Einstein metric h .

What, if anything, does h then tell us about M ?

Can we recognize M by looking at h ?

When $n = 3$, h has constant sectional curvature!

So M has universal cover $S^3, \mathbb{R}^3, \mathcal{H}^3 \dots$

But when $n \geq 5$, situation seems hopeless.

$\{\text{Einstein metrics on } S^n\} / \sim$ is highly disconnected.

Recognition Problem:

Suppose M^n admits Einstein metric h .

What, if anything, does h then tell us about M ?

Can we recognize M by looking at h ?

When $n = 3$, h has constant sectional curvature!

So M has universal cover $S^3, \mathbb{R}^3, \mathcal{H}^3 \dots$

But when $n \geq 5$, situation seems hopeless.

$\{\text{Einstein metrics on } S^n\} / \sim$ is highly disconnected.

When $n = 4$, situation is more encouraging...

Moduli Spaces of Einstein metrics

Moduli Spaces of Einstein metrics

$$\mathcal{E}(M) = \{\text{Einstein } h\} / (\text{Diffeos} \times \mathbb{R}^+)$$

Moduli Spaces of Einstein metrics

$$\mathcal{E}(M) = \{\text{Einstein } h\} / (\text{Diffeos} \times \mathbb{R}^+)$$

Known to be connected for certain 4-manifolds:

Moduli Spaces of Einstein metrics

$$\mathcal{E}(M) = \{\text{Einstein } h\} / (\text{Diffeos} \times \mathbb{R}^+)$$

Known to be connected for certain 4-manifolds:

$$M =$$

Moduli Spaces of Einstein metrics

$$\mathcal{E}(M) = \{\text{Einstein } h\} / (\text{Diffeos} \times \mathbb{R}^+)$$

Known to be connected for certain 4-manifolds:

$$M = T^4,$$

Berger,

Moduli Spaces of Einstein metrics

$$\mathcal{E}(M) = \{\text{Einstein } h\} / (\text{Diffeos} \times \mathbb{R}^+)$$

Known to be connected for certain 4-manifolds:

$$M = T^4, \quad K3,$$

Berger, Hitchin,

Moduli Spaces of Einstein metrics

$$\mathcal{E}(M) = \{\text{Einstein } h\} / (\text{Diffeos} \times \mathbb{R}^+)$$

Known to be connected for certain 4-manifolds:

$$M = T^4, \quad K3, \quad \mathcal{H}^4/\Gamma,$$

Berger, Hitchin, Besson-Courtois-Gallot,

Moduli Spaces of Einstein metrics

$$\mathcal{E}(M) = \{\text{Einstein } h\} / (\text{Diffeos} \times \mathbb{R}^+)$$

Known to be connected for certain 4-manifolds:

$$M = T^4, \quad K3, \quad \mathcal{H}^4/\Gamma, \quad \mathbb{C}\mathcal{H}_2/\Gamma.$$

Berger, Hitchin, Besson-Courtois-Gallot, L.

Four Dimensions is Exceptional

Four Dimensions is Exceptional

When $n = 4$, Einstein metrics are genuinely non-trivial: not typically spaces of constant curvature.

Four Dimensions is Exceptional

When $n = 4$, Einstein metrics are genuinely non-trivial: not typically spaces of constant curvature.

There are beautiful and subtle global obstructions to the existence of Einstein metrics on 4-manifolds.

Four Dimensions is Exceptional

When $n = 4$, Einstein metrics are genuinely non-trivial: not typically spaces of constant curvature.

There are beautiful and subtle global obstructions to the existence of Einstein metrics on 4-manifolds.

Some arise from Seiberg-Witten theory, and so are sensitive to the existence of a symplectic structure:

Four Dimensions is Exceptional

When $n = 4$, Einstein metrics are genuinely non-trivial: not typically spaces of constant curvature.

There are beautiful and subtle global obstructions to the existence of Einstein metrics on 4-manifolds.

Some arise from Seiberg-Witten theory, and so are sensitive to the existence of a symplectic structure:

i.e. a closed non-degenerate 2-form ω :

$$d\omega = 0, \quad \omega \wedge \omega > 0.$$

Symplectic 4-manifolds:

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold,*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h (unrelated to ω)?*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω .

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h (unrelated to ω)? What if we also require $\lambda \geq 0$?*

Theorem (L '09).

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold*

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω .*

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h*

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$*

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$M \stackrel{\text{diff}}{\approx}$ 

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \end{array} \right.$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \end{array} \right.$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \end{array} \right.$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \#^k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \end{array} \right.$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \end{array} \right.$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \end{array} \right.$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \end{array} \right.$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{array} \right.$$

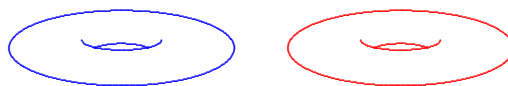
Conventions:

$\overline{\mathbb{C}P}_2$ = reverse oriented $\mathbb{C}P_2$.

Conventions:

$\overline{\mathbb{C}P}_2$ = reverse oriented $\mathbb{C}P_2$.

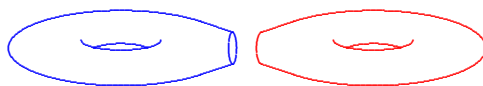
Connected sum #:



Conventions:

$\overline{\mathbb{C}P}_2$ = reverse oriented $\mathbb{C}P_2$.

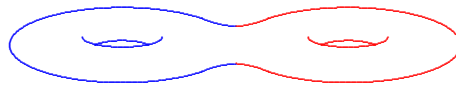
Connected sum #:



Conventions:

$\overline{\mathbb{C}P}_2$ = reverse oriented $\mathbb{C}P_2$.

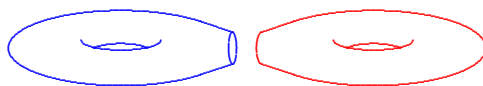
Connected sum #:



Conventions:

$\overline{\mathbb{C}P}_2$ = reverse oriented $\mathbb{C}P_2$.

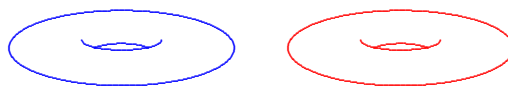
Connected sum #:



Conventions:

$\overline{\mathbb{C}P}_2$ = reverse oriented $\mathbb{C}P_2$.

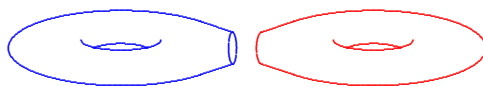
Connected sum #:



Conventions:

$\overline{\mathbb{C}P}_2$ = reverse oriented $\mathbb{C}P_2$.

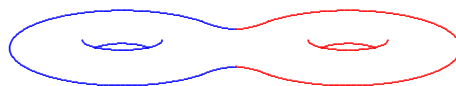
Connected sum #:



Conventions:

$\overline{\mathbb{C}P}_2$ = reverse oriented $\mathbb{C}P_2$.

Connected sum #:



Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{array} \right.$$

K3

$K3 = \text{Kummer}$

$K3 =$ Kummer-Kähler

$K3 =$ Kummer-Kähler-Kodaira

$K3$ = Kummer-Kähler-Kodaira surface.

$K3$ = Kummer-Kähler-Kodaira surface.

—André Weil

K3 = Kummer-Kähler-Kodaira surface.



“...et de la belle montagne K2 au Cachemire.”

—André Weil, 1958

$K3$ = Kummer-Kähler-Kodaira surface.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Only one deformation type.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Only one diffeomorphism type.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{C}P_3$.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{C}P_3$.



$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Only one diffeomorphism type.

$K3$ = Kummer-Kähler-Kodaira manifold.

$K3 =$ Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: T^4/\mathbb{Z}_2 : Singular quartic in $\mathbb{C}P_3$.



$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: T^4/\mathbb{Z}_2 : Singular quartic in $\mathbb{C}P_3$.



T^4 = Picard torus of curve of genus 2.

$K3 =$ Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: T^4/\mathbb{Z}_2 : Singular quartic in $\mathbb{C}P_3$.



Remove singularities by deforming equation.

$K3 =$ Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: T^4/\mathbb{Z}_2 : Singular quartic in $\mathbb{C}P_3$.



Remove singularities by deforming equation.

$K3 =$ Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: T^4/\mathbb{Z}_2 : Singular quartic in $\mathbb{C}P_3$.



Remove singularities by deforming equation.

$K3 =$ Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: T^4/\mathbb{Z}_2 : Singular quartic in $\mathbb{C}P_3$.



Remove singularities by deforming equation.

$K3 =$ Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: T^4/\mathbb{Z}_2 : Singular quartic in $\mathbb{C}P_3$.



Remove singularities by deforming equation.

$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: T^4/\mathbb{Z}_2 : Singular quartic in $\mathbb{C}P_3$.



Generic quartic is a $K3$ surface.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{C}P_3$.



$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{C}P_3$.



Calabi/Yau: Admits $\lambda = 0$ Einstein metrics.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{C}P_3$.



Calabi/Yau: Admits Ricci-flat Kähler metrics.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{C}P_3$.



Calabi/Yau: Admits Ricci-flat Kähler metrics.

Hitchin: Every Einstein h on $K3$ is Calabi-Yau.

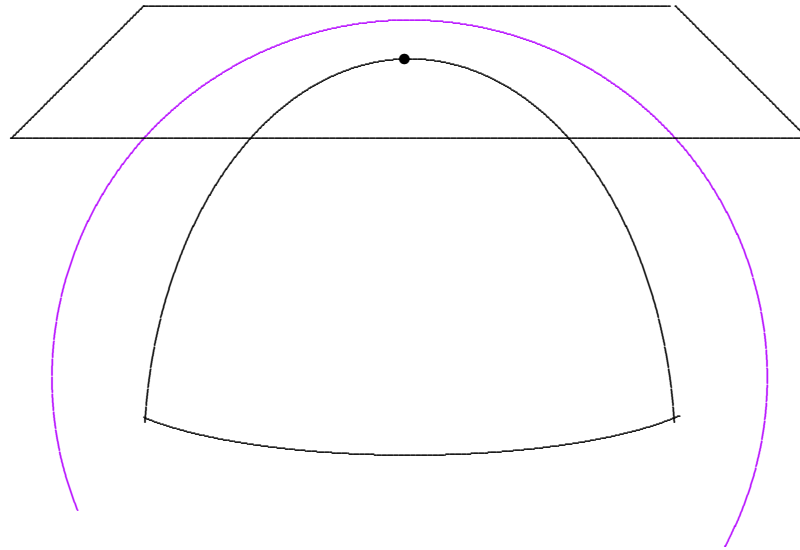
Kähler?

(M^n, g) :

holonomy

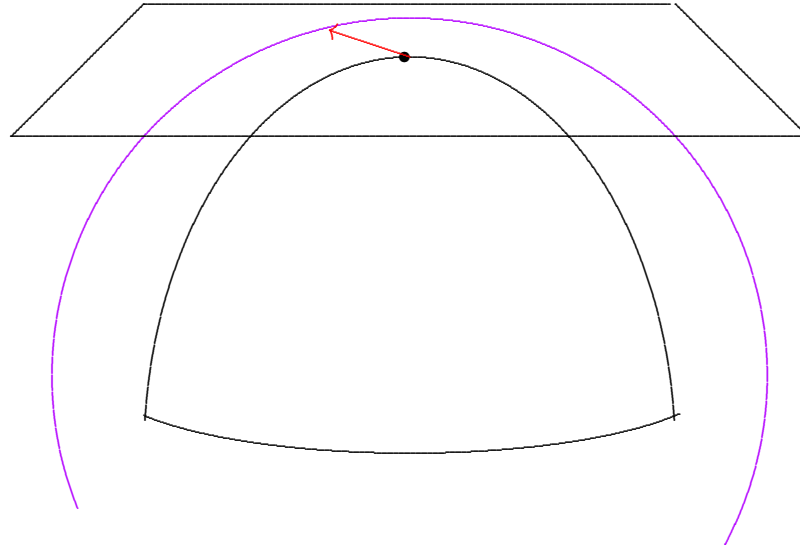
(M^n, g) :

holonomy



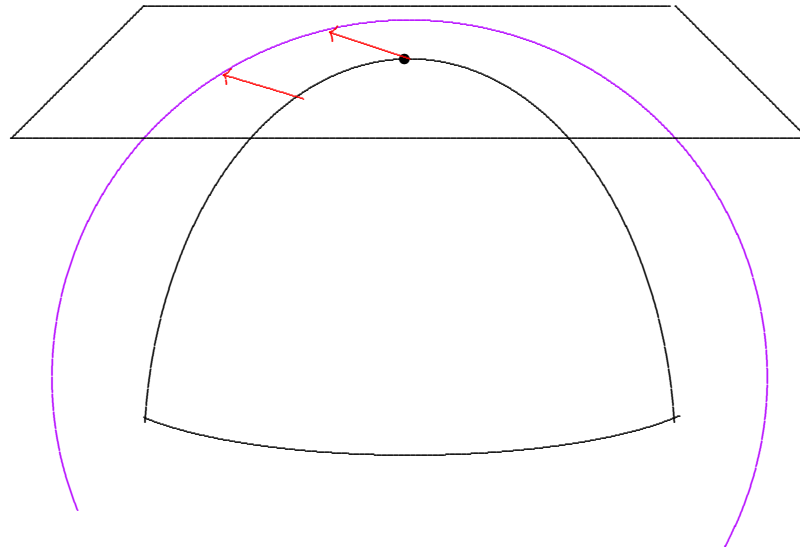
(M^n, g) :

holonomy



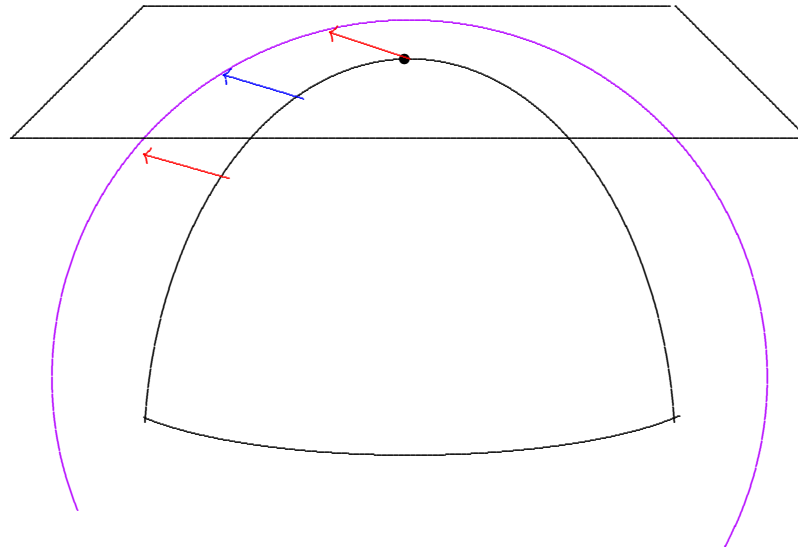
(M^n, g) :

holonomy



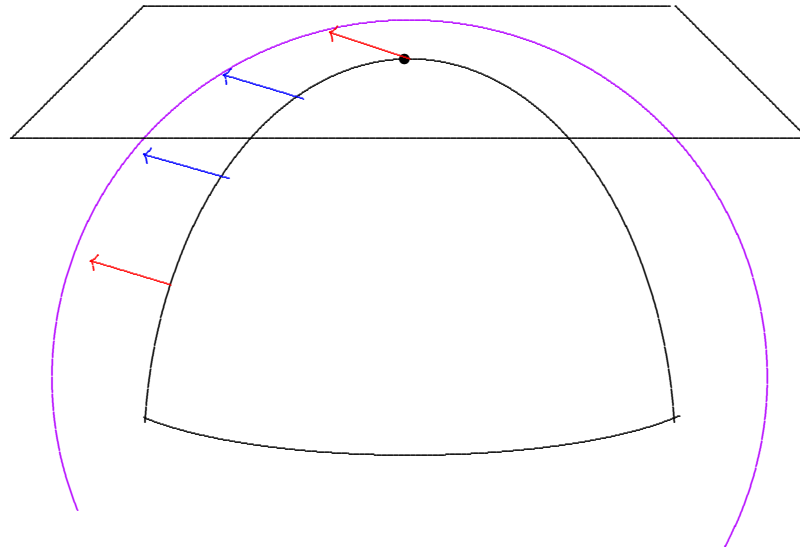
(M^n, g) :

holonomy



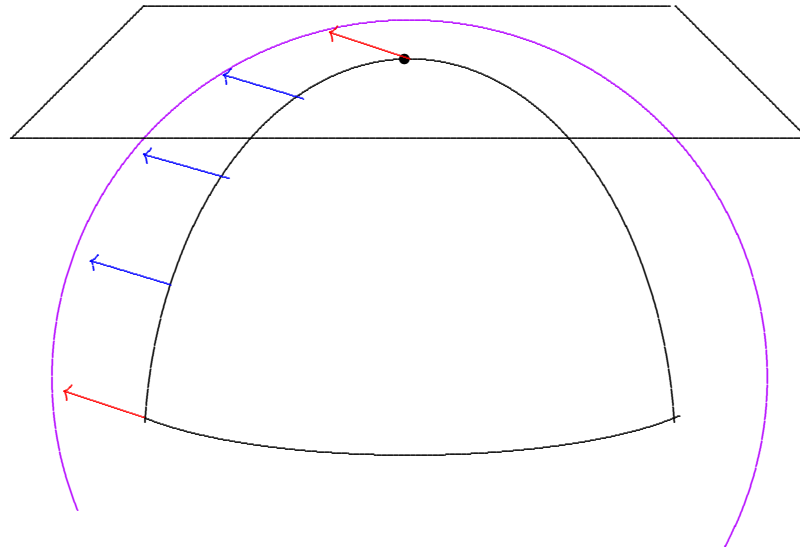
(M^n, g) :

holonomy



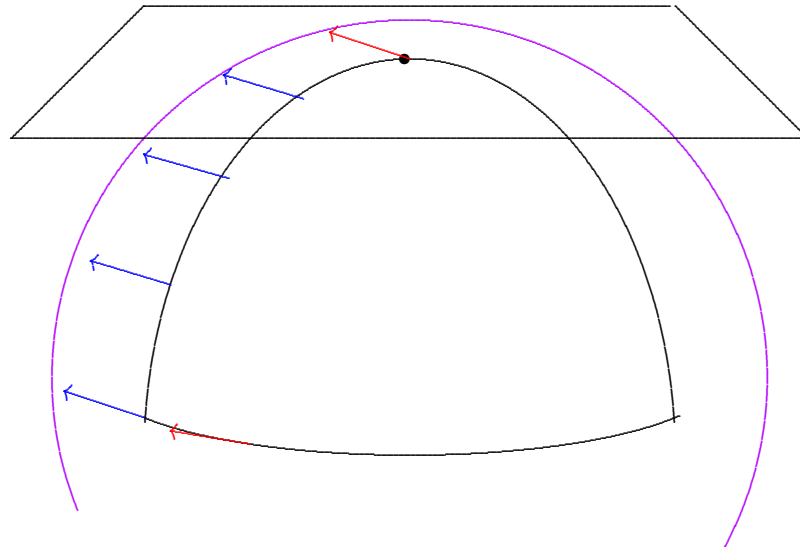
(M^n, g) :

holonomy



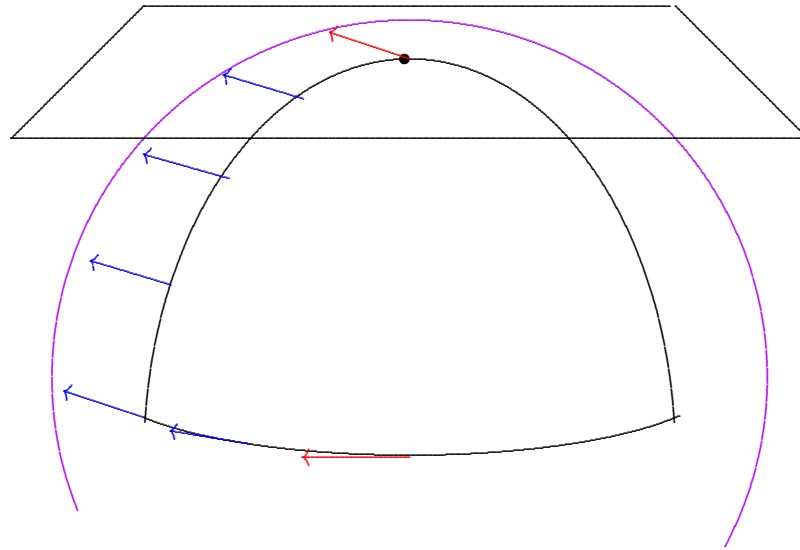
(M^n, g) :

holonomy



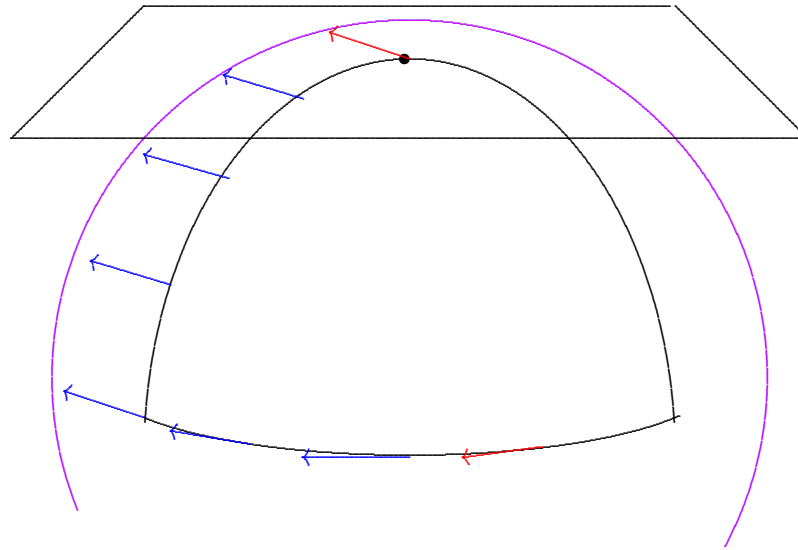
(M^n, g) :

holonomy



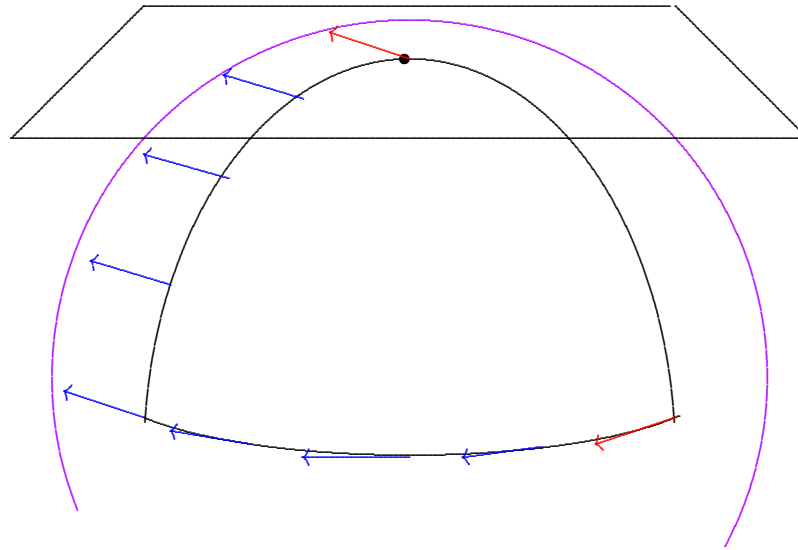
(M^n, g) :

holonomy



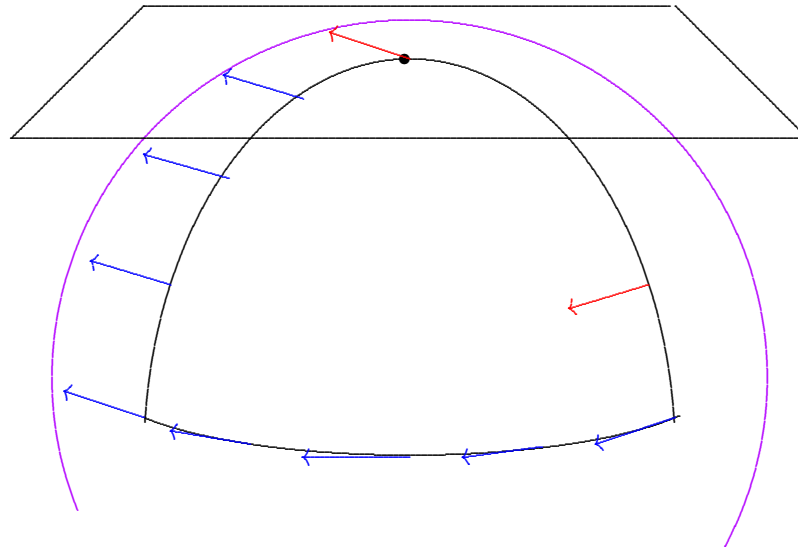
(M^n, g) :

holonomy



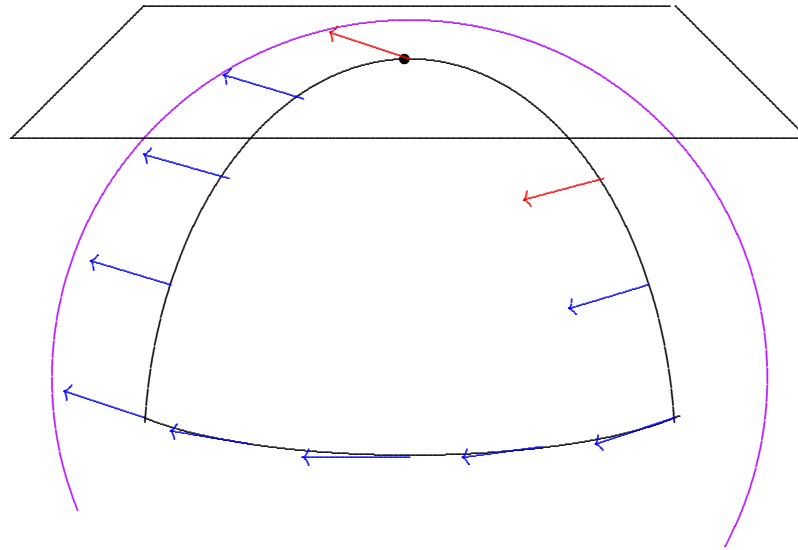
(M^n, g) :

holonomy



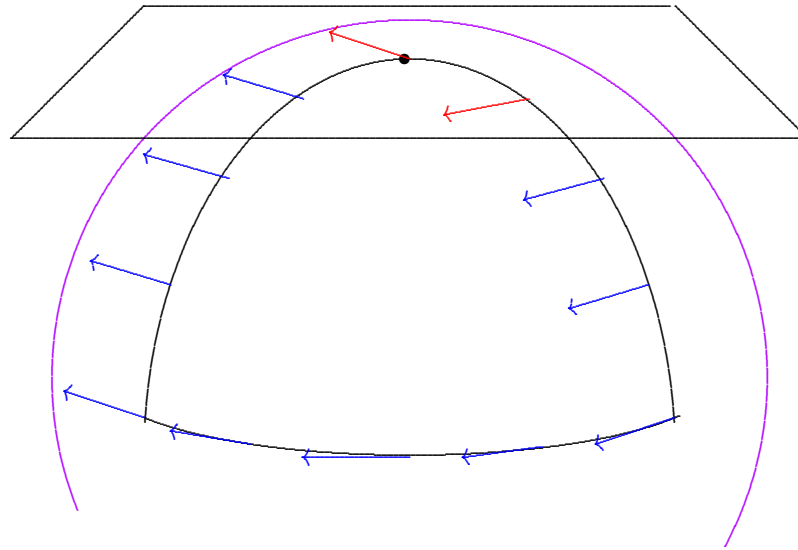
(M^n, g) :

holonomy



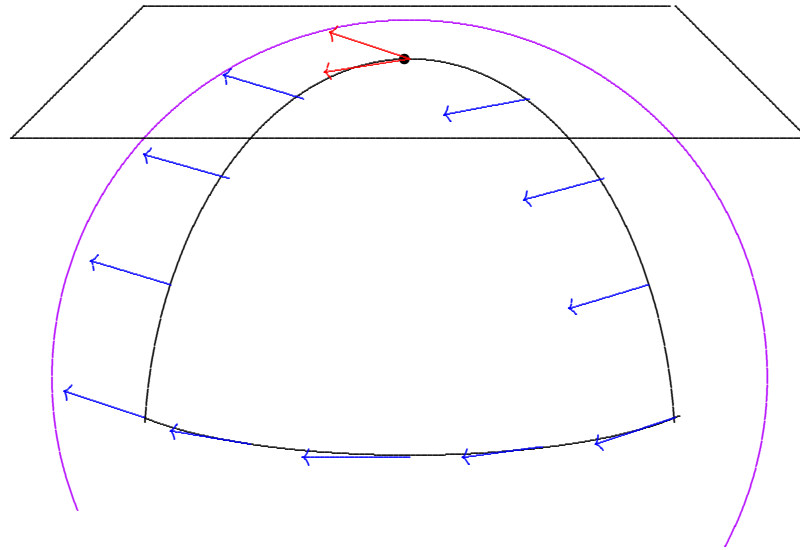
(M^n, g) :

holonomy



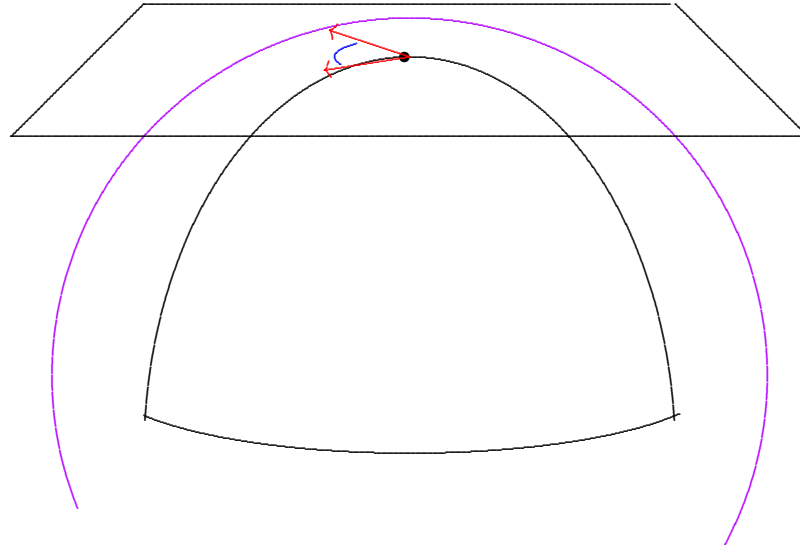
(M^n, g) :

holonomy



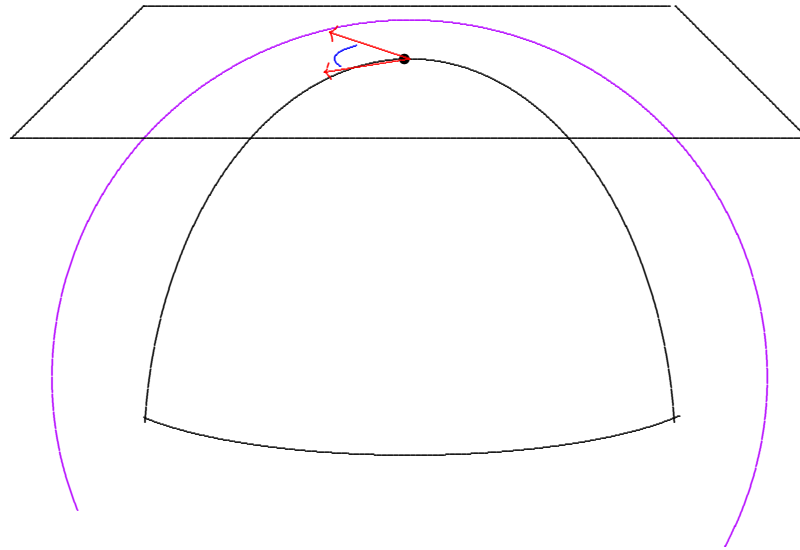
(M^n, g) :

holonomy



(M^n, g) :

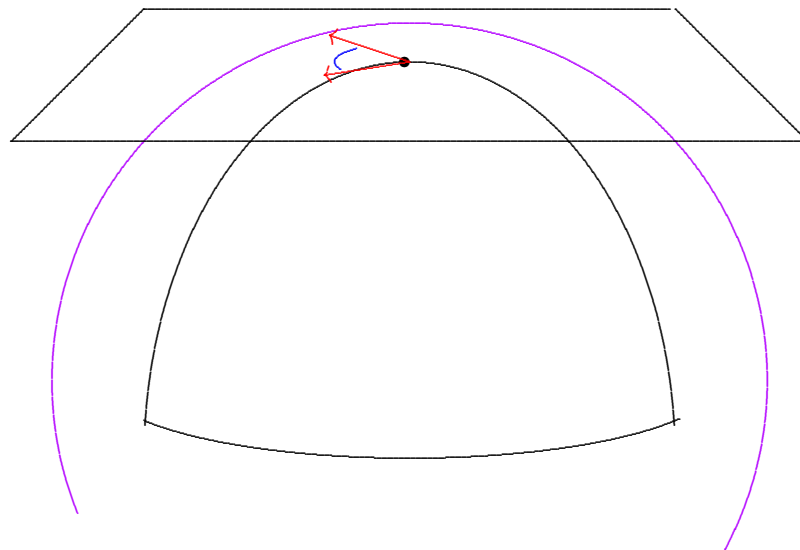
holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

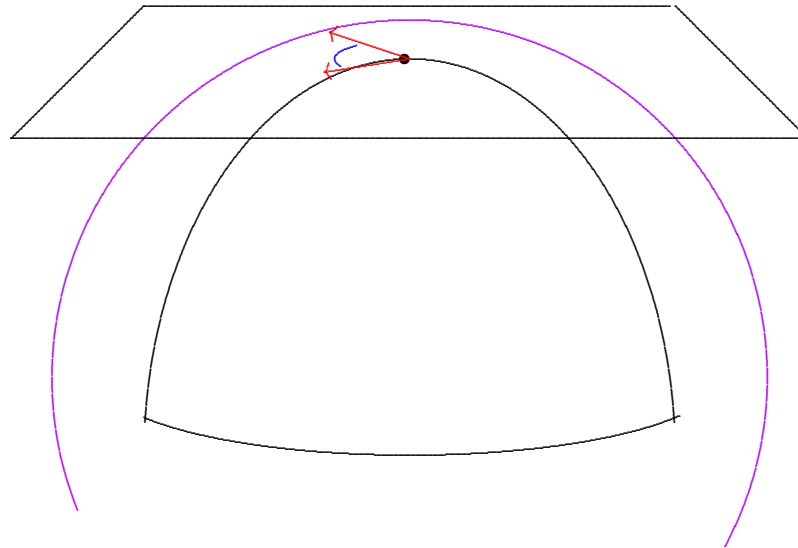
(M^{2m}, g) :

holonomy



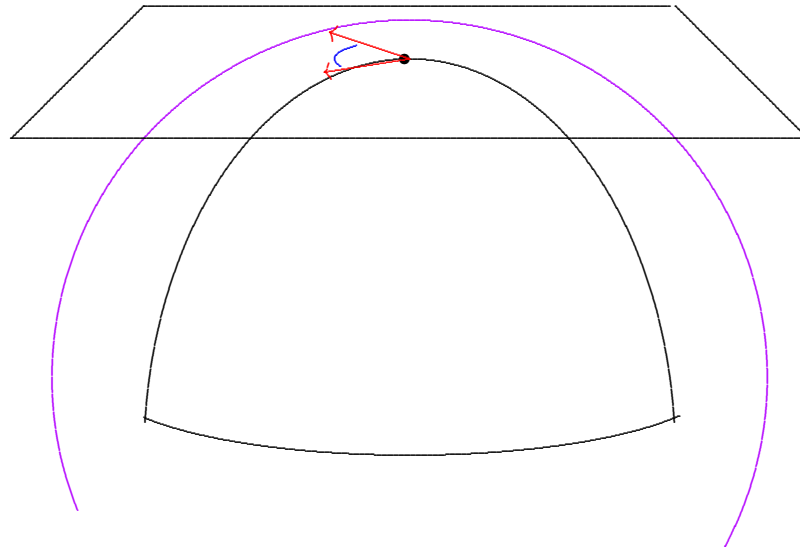
Kähler metrics:

(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$



Kähler metrics:

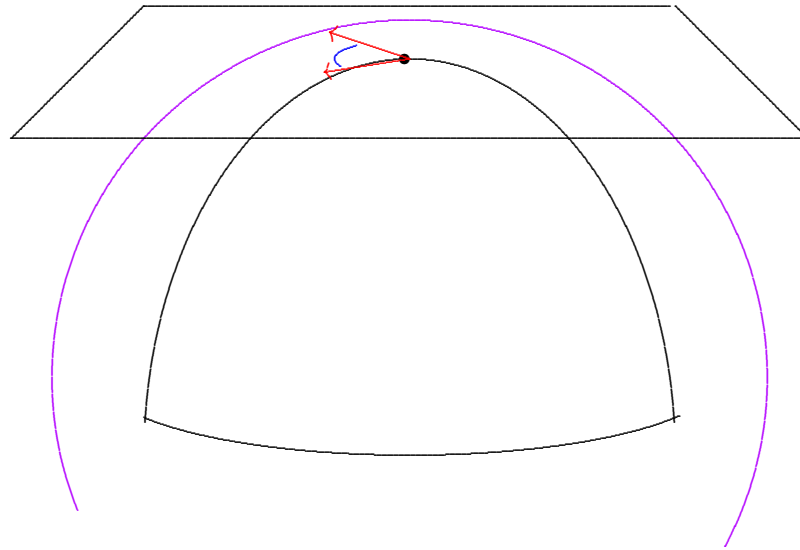
(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$



$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

Kähler metrics:

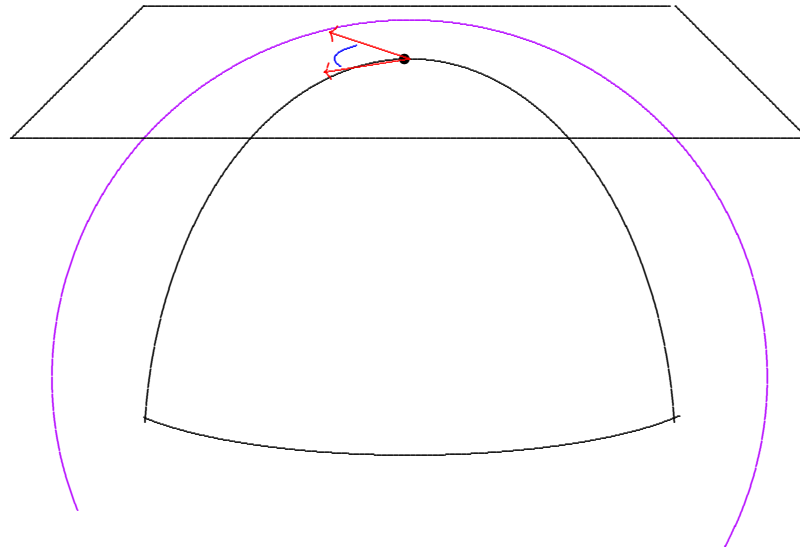
(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$



Makes tangent space a complex vector space!

Kähler metrics:

(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$



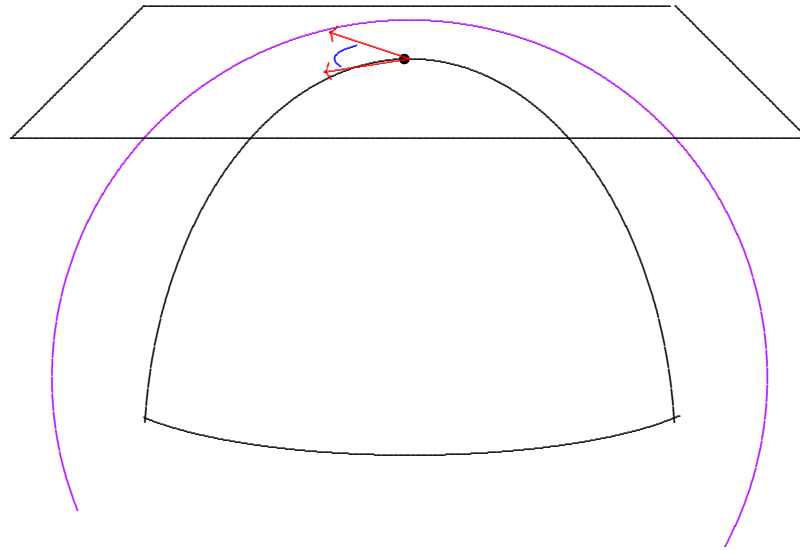
Makes tangent space a complex vector space!

$$J : TM \rightarrow TM, \quad J^2 = -\text{identity}$$

“almost-complex structure”

Kähler metrics:

(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$



Makes tangent space a complex vector space!

Invariant under parallel transport!

Kähler metrics:

(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$

$\iff \exists$ almost complex-structure J with $\nabla J = 0$
and $g(J\cdot, J\cdot) = g$.

Kähler metrics:

(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$

$\iff \exists$ almost complex-structure J with $\nabla J = 0$
and $g(J\cdot, J\cdot) = g$.

$\iff (M^{2m}, g)$ is a complex manifold & $\exists J$ -invariant
closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

Kähler metrics:

(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$

$\iff \exists$ almost complex-structure J with $\nabla J = 0$
and $g(J\cdot, J\cdot) = g$.

$\iff (M^{2m}, g)$ is a complex manifold & $\exists J$ -invariant
closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

ω called “Kähler form.”

Kähler metrics:

(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$

$\iff \exists$ almost complex-structure J with $\nabla J = 0$
and $g(J\cdot, J\cdot) = g$.

$\iff (M^{2m}, g)$ is a complex manifold & $\exists J$ -invariant
closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

$$d\omega = 0$$

Kähler metrics:

(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$

$\iff \exists$ almost complex-structure J with $\nabla J = 0$
and $g(J\cdot, J\cdot) = g$.

$\iff (M^{2m}, g)$ is a complex manifold & $\exists J$ -invariant
closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

$$d\omega = 0$$

$$\omega^m \neq 0$$

Kähler metrics:

(M^{2m}, g) Kähler \iff holonomy $\subset \mathbf{U}(m)$

$\iff \exists$ almost complex-structure J with $\nabla J = 0$
and $g(J\cdot, J\cdot) = g$.

$\iff (M^{2m}, g)$ is a complex manifold & $\exists J$ -invariant
closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

$$d\omega = 0$$

$$\omega^m \neq 0$$

So ω is, in particular, a symplectic form.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{C}P_3$.



Calabi/Yau: Admits Ricci-flat Kähler metrics.

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{array} \right.$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{array} \right.$$

del Pezzo surfaces,

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}\mathbb{P}_2 \# k \overline{\mathbb{C}\mathbb{P}_2}, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{array} \right.$$

del Pezzo surfaces,
K3 surface,

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{array} \right.$$

del Pezzo surfaces,
K3 surface, Enriques surface,

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}\mathbb{P}_2 \# k \overline{\mathbb{C}\mathbb{P}_2}, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{array} \right.$$

del Pezzo surfaces,
 K3 surface, Enriques surface,
 Abelian surface,

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω . Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \left\{ \begin{array}{l} \mathbb{C}\mathbb{P}_2 \# k \overline{\mathbb{C}\mathbb{P}_2}, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{array} \right.$$

del Pezzo surfaces,

K3 surface, Enriques surface,

Abelian surface, Hyper-elliptic surfaces.

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$

$$S^2 \times S^2,$$

$$K3,$$

$$K3/\mathbb{Z}_2,$$

$$T^4,$$

$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$

$$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$$

Definitive list ...

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$

$$S^2 \times S^2,$$

$$K3,$$

$$K3/\mathbb{Z}_2,$$

$$T^4,$$

$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$

$$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$$

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$

$$S^2 \times S^2,$$

$$K3,$$

$$K3/\mathbb{Z}_2,$$

$$T^4,$$

$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$

$$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$$

But we understand some cases better than others!

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$

$$S^2 \times S^2,$$

$$K3,$$

$$K3/\mathbb{Z}_2,$$

$$T^4,$$

$$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$$

$$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).$$

But we understand some cases better than others!

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

But we understand some cases better than others!

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

But we understand some cases better than others!

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

But we understand some cases better than others!

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$

But we understand some cases better than others!

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M) = \{\text{Einstein } h\} / (\text{Diffeos} \times \mathbb{R}^+)$

But we understand some cases better than others!

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ completely understood.

But we understand some cases better than others!

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!

Above the line:

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!

Above the line:

Know an Einstein metric on each manifold.

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!

Above the line:

Moduli space $\mathcal{E}(M) \neq \emptyset$.

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!

Above the line:

Moduli space $\mathcal{E}(M) \neq \emptyset$. But is it connected?

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!

Del Pezzo surfaces:

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points,
in general position,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

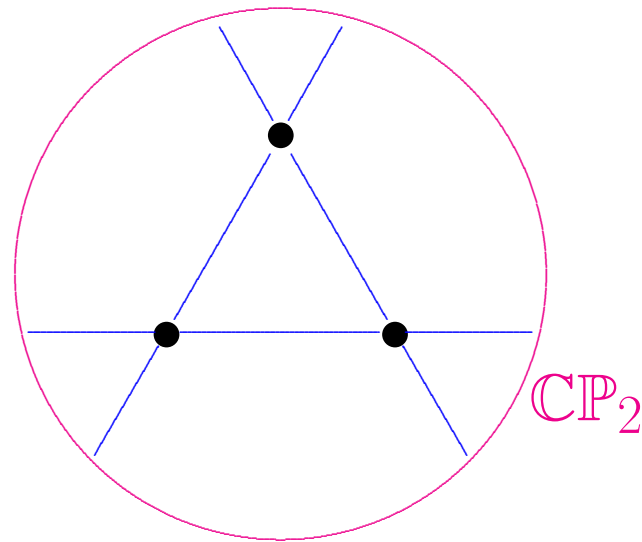
Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.



Blowing up:

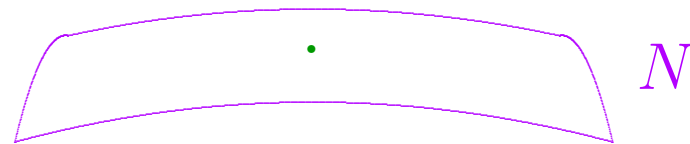
Blowing up:

If N is a complex surface,



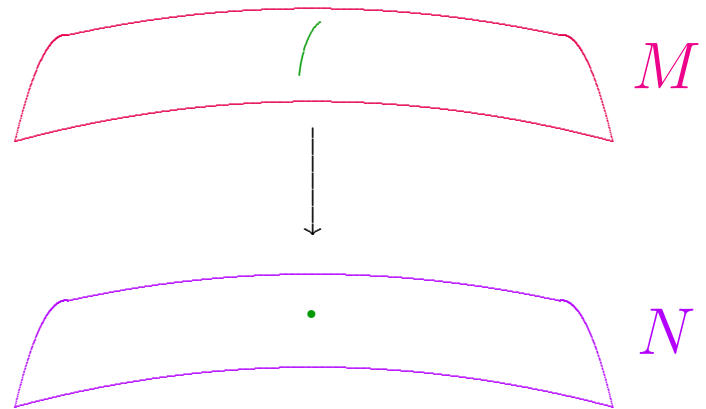
Blowing up:

If N is a complex surface, may replace $p \in N$



Blowing up:

If N is a complex surface, may replace $p \in N$
with $\mathbb{C}P_1$

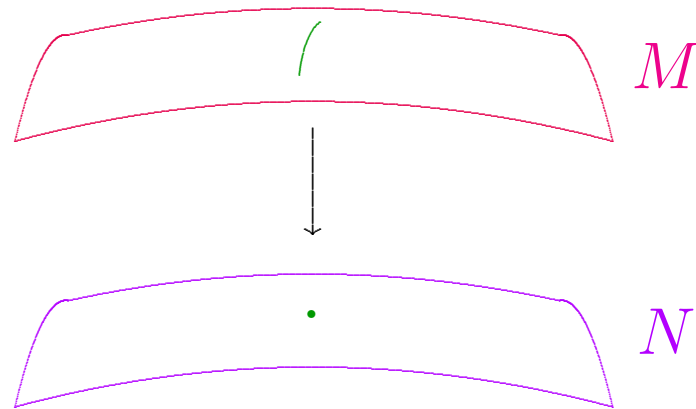


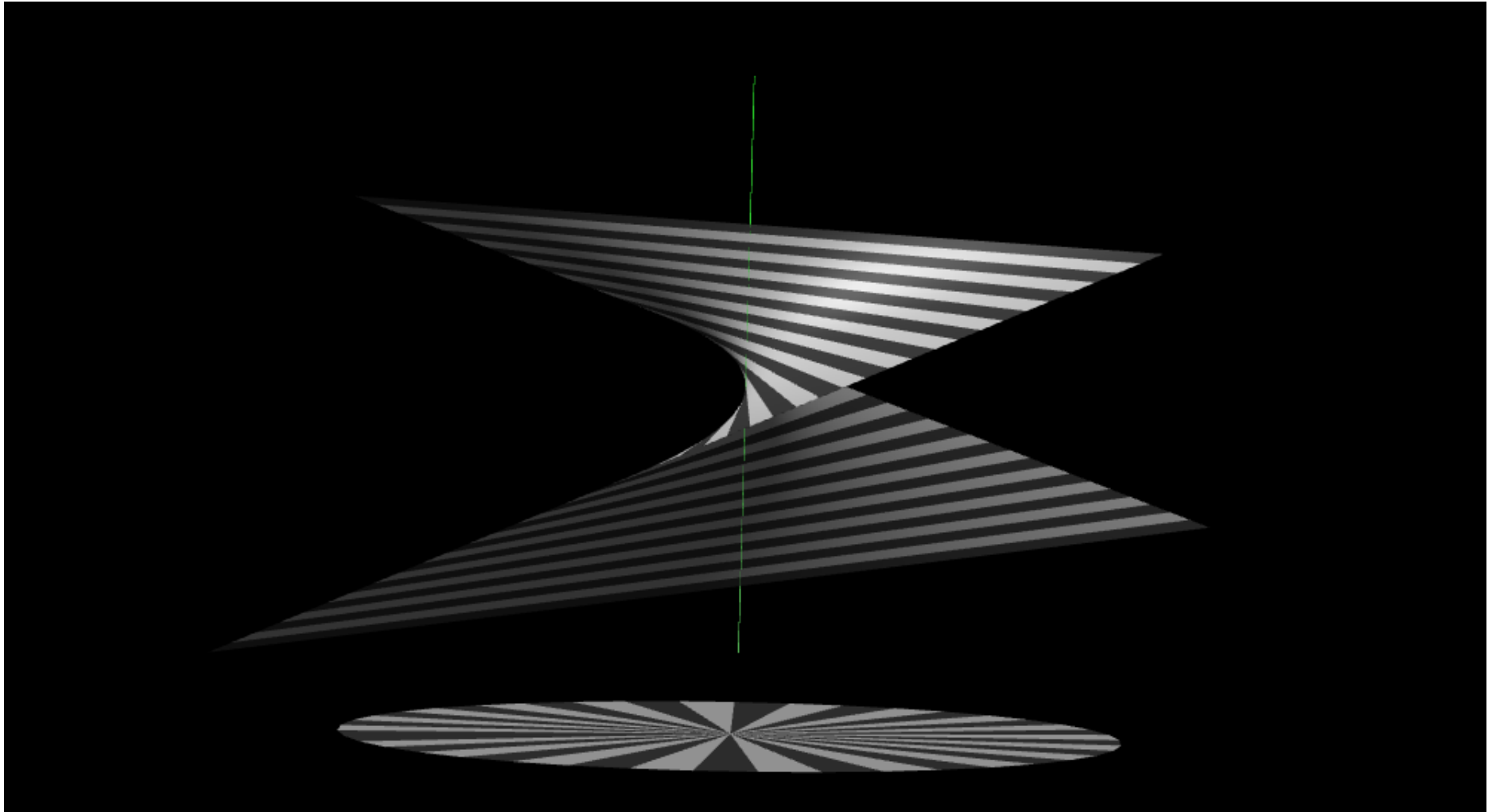
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{C}P_1$ to obtain **blow-up**

$$M \approx N \# \overline{\mathbb{C}P_2}$$

in which added $\mathbb{C}P_1$ has normal bundle $\mathcal{O}(-1)$.



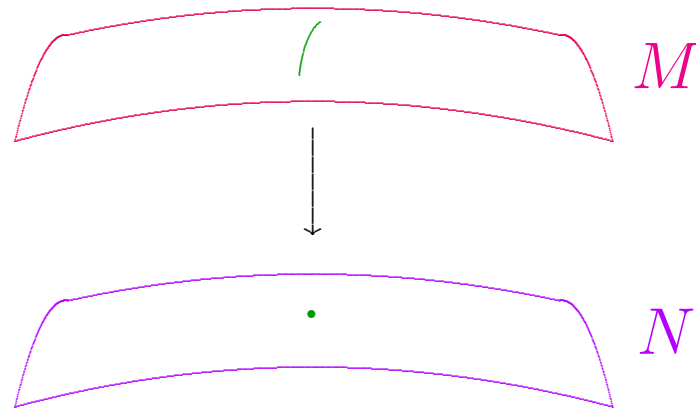


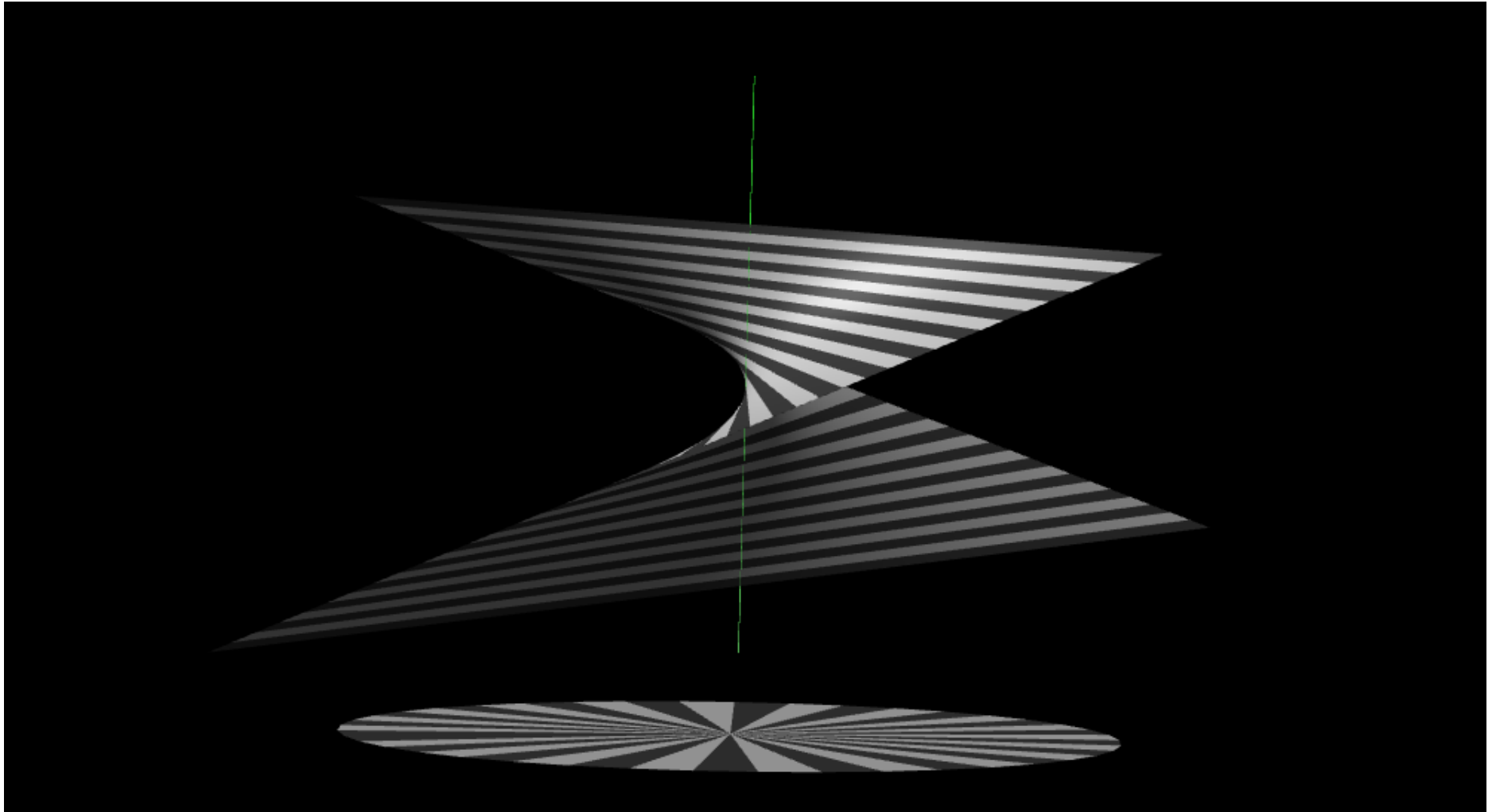
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{C}P_1$ to obtain **blow-up**

$$M \approx N \# \overline{\mathbb{C}P_2}$$

in which added $\mathbb{C}P_1$ has normal bundle $\mathcal{O}(-1)$.



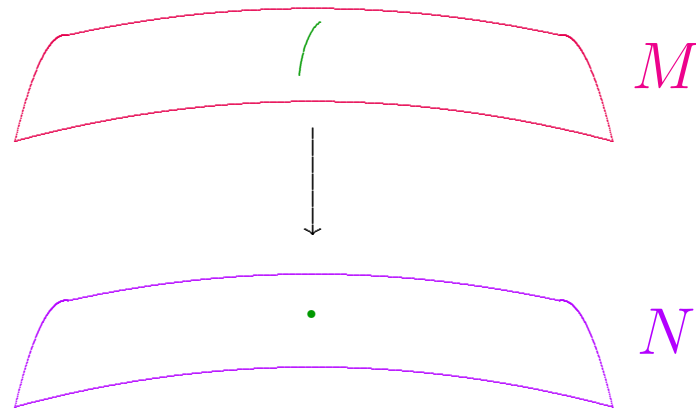


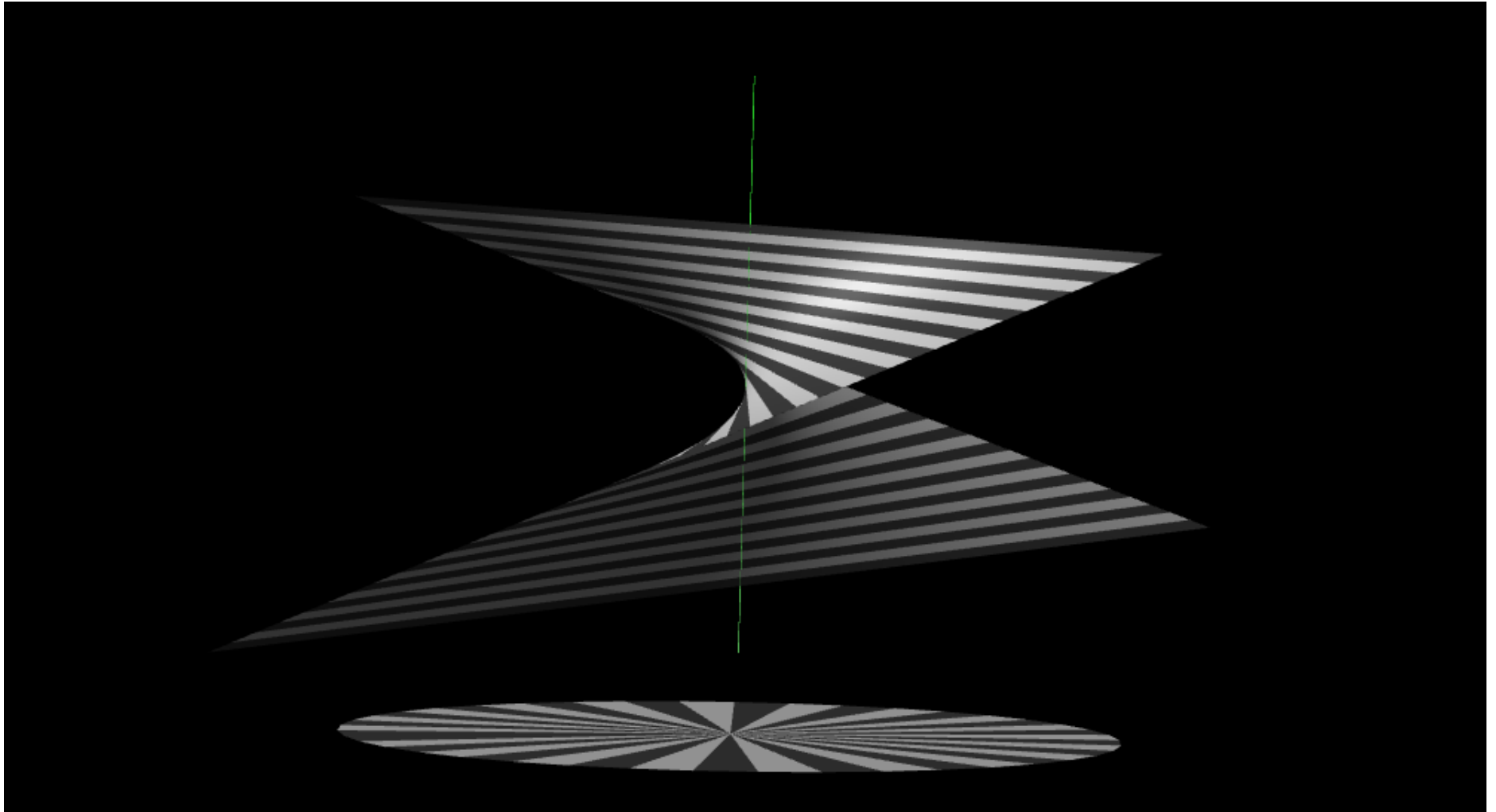
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{C}P_1$ to obtain **blow-up**

$$M \approx N \# \overline{\mathbb{C}P_2}$$

in which added $\mathbb{C}P_1$ has normal bundle $\mathcal{O}(-1)$.



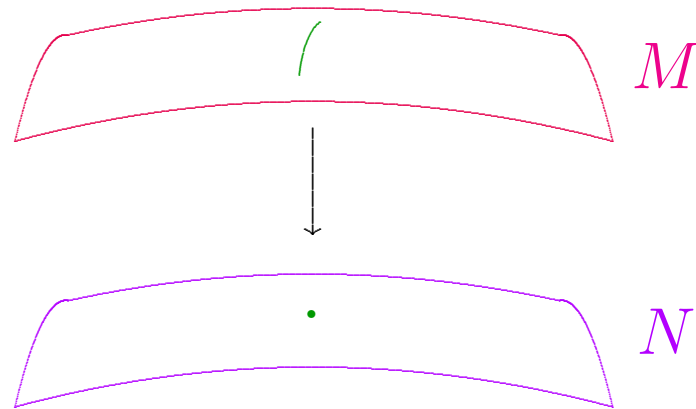


Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{C}P_1$ to obtain **blow-up**

$$M \approx N \# \overline{\mathbb{C}P_2}$$

in which added $\mathbb{C}P_1$ has normal bundle $\mathcal{O}(-1)$.

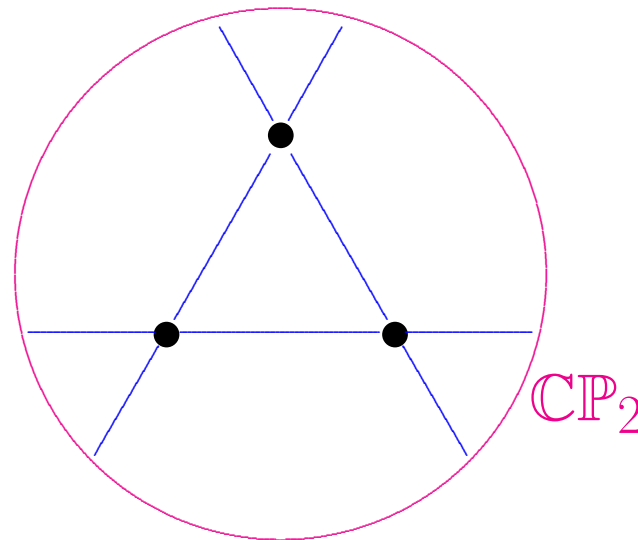


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

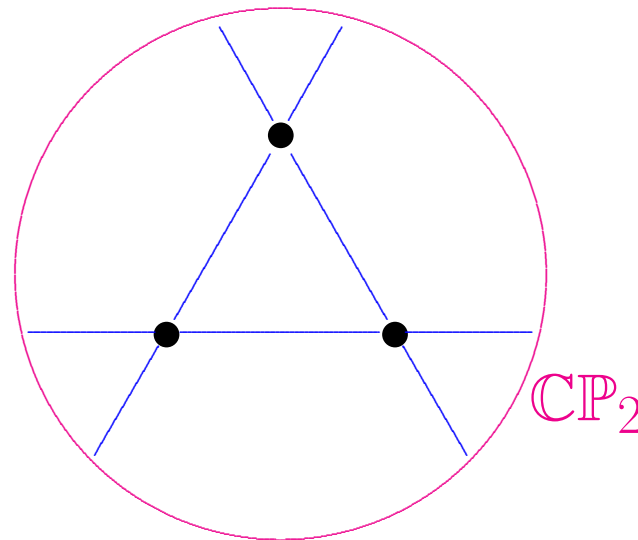
Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.

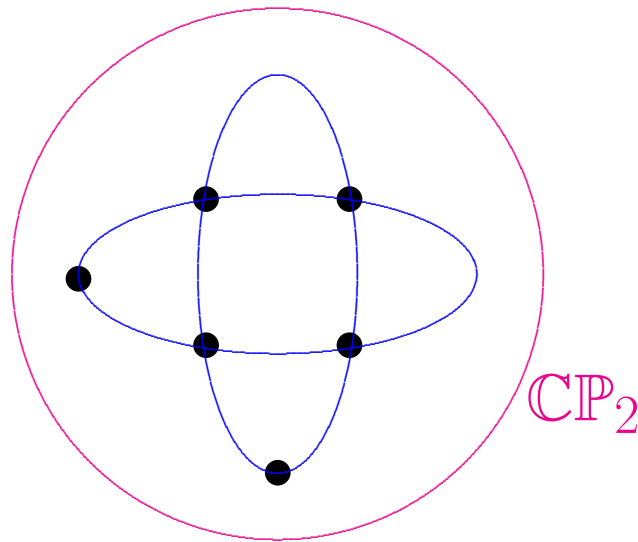


No 3 on a line,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.

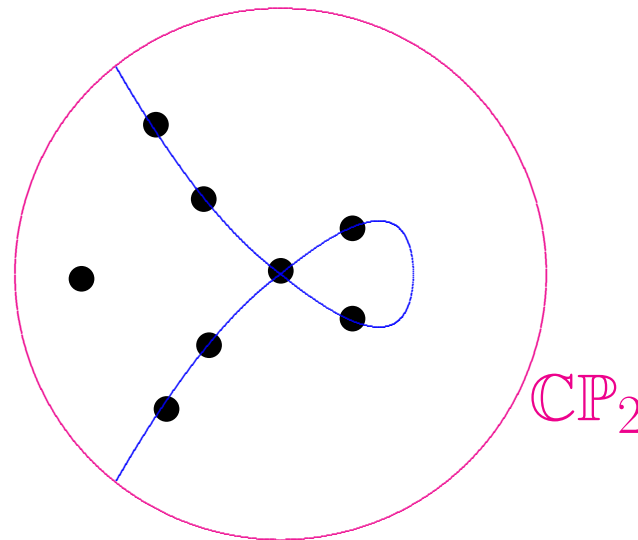


No 3 on a line, no 6 on conic,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.



No 3 on a line, no 6 on conic, no 8 on nodal cubic.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.

Theorem. *Each del Pezzo (M^4, J) admits a compatible conformally Kähler, Einstein metric, and this metric is unique up to automorphisms.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.

Theorem. *Each del Pezzo (M^4, J) admits a compatible conformally Kähler, Einstein metric, and this metric is unique up to automorphisms.*

Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: “ $c_1 > 0$.”

Blow-up of $\mathbb{C}P_2$ at k distinct points, $0 \leq k \leq 8$,
in general position, or $\mathbb{C}P_1 \times \mathbb{C}P_1$.

Theorem. *Each del Pezzo (M^4, J) admits a compatible conformally Kähler, Einstein metric, and this metric is unique up to automorphisms.*

Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber.

Uniqueness: Bando-Mabuchi '87, L '12.

Above the line:

Moduli space $\mathcal{E}(M) \neq \emptyset$. But is it connected?

$$\mathbb{C}P_2 \# k \overline{\mathbb{C}P_2}, \quad 0 \leq k \leq 8,$$
$$S^2 \times S^2,$$

$K3$,

$K3/\mathbb{Z}_2$,

T^4 ,

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6$,

$T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$, or $T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!

Basic problem:

Basic problem:

Understand all Einstein metrics on del Pezzos.

Basic problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Basic problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Nice characterizations of known Einstein metrics.

Basic problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Nice characterizations of known Einstein metrics.

Exactly one connected component of moduli space!

This all depends on ...

Special character of dimension 4:

Special character of dimension 4:

On oriented (M^4, h) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Special character of dimension 4:

On oriented (M^4, h) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where Λ^\pm are (± 1) -eigenspaces of

$$\star : \Lambda^2 \rightarrow \Lambda^2,$$

$$\star^2 = 1.$$

Special character of dimension 4:

On oriented (M^4, h) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where Λ^\pm are (± 1) -eigenspaces of

$$\star : \Lambda^2 \rightarrow \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Special character of dimension 4:

On oriented (M^4, h) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where Λ^\pm are (± 1) -eigenspaces of

$$\star : \Lambda^2 \rightarrow \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Moreover, this is conformally invariant!

Special character of dimension 4:

On oriented (M^4, h) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where Λ^\pm are (± 1) -eigenspaces of

$$\star : \Lambda^2 \rightarrow \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Moreover, this is conformally invariant!

$$h \rightsquigarrow u^2 h$$

Special character of dimension 4:

On oriented (M^4, h) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where Λ^\pm are (± 1) -eigenspaces of

$$\star : \Lambda^2 \rightarrow \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Because of this ...

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W^+ + \frac{s}{12}$	$\overset{\circ}{r}$
Λ^-	$\overset{\circ}{r}$	$W^- + \frac{s}{12}$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W^+ + \frac{s}{12}$	$\overset{\circ}{r}$
Λ^-	$\overset{\circ}{r}$	$W^- + \frac{s}{12}$

where

s = scalar curvature

$\overset{\circ}{r}$ = trace-free Ricci curvature

W^+ = self-dual Weyl curvature

W^- = anti-self-dual Weyl curvature

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W^+ + \frac{s}{12}$	$\overset{\circ}{r}$
Λ^-	$\overset{\circ}{r}$	$W^- + \frac{s}{12}$

where

s = scalar curvature

$\overset{\circ}{r}$ = trace-free Ricci curvature

W^+ = self-dual Weyl curvature (*conformally invariant*)

W^- = anti-self-dual Weyl curvature //

Special character of dimension 4:

On oriented (M^4, h) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where Λ^\pm are (± 1) -eigenspaces of

$$\star : \Lambda^2 \rightarrow \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Special character of dimension 4:

On oriented (M^4, h) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where Λ^\pm are (± 1) -eigenspaces of

$$\star : \Lambda^2 \rightarrow \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Also because of this ...

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, d\star\varphi = 0\}.$$

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, d\star\varphi = 0\}.$$

Since \star is involution of RHS, \implies

$$H^2(M, \mathbb{R}) = \mathcal{H}_h^+ \oplus \mathcal{H}_h^-,$$

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, d\star\varphi = 0\}.$$

Since \star is involution of RHS, \implies

$$H^2(M, \mathbb{R}) = \mathcal{H}_h^+ \oplus \mathcal{H}_h^-,$$

where

$$\mathcal{H}_h^\pm = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms.

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, d\star\varphi = 0\}.$$

Since \star is involution of RHS, \implies

$$H^2(M, \mathbb{R}) = \mathcal{H}_h^+ \oplus \mathcal{H}_h^-,$$

where

$$\mathcal{H}_h^\pm = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms.

Notice these spaces are conformally invariant.

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, d\star\varphi = 0\}.$$

Since \star is involution of RHS, \implies

$$H^2(M, \mathbb{R}) = \mathcal{H}_h^+ \oplus \mathcal{H}_h^-,$$

where

$$\mathcal{H}_h^\pm = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms.

Notice these spaces are conformally invariant.

More generally, their dimensions

$$b_\pm(M) = \dim \mathcal{H}_h^\pm$$

are completely metric-independent, and are oriented homotopy invariants of M .

One Riemannian characterization:

Theorem (L '15).

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold*

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω*

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M .

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface,

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler,

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Indeed, the conformally Kähler, Einstein metrics

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Indeed, the conformally Kähler, Einstein metrics on del Pezzo surfaces all satisfy $W^+(\omega, \omega) > 0$.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

$$\text{Kähler} \implies \Lambda^+ = \mathbb{R}\omega \oplus \Re\Lambda^{2,0}$$

$$W^+ = \begin{bmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{bmatrix}$$

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

$$\text{Kähler} \implies \Lambda^+ = \mathbb{R}\omega \oplus \Re\Lambda^{2,0}.$$

$$W^+(\omega, \omega) = \frac{s}{3}$$

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

$$\text{Kähler} \implies \Lambda^+ = \mathbb{R}\omega \oplus \Re\Lambda^{2,0}.$$

$$W^+(\omega, \omega) = \frac{s}{3} > 0$$

for relevant Kähler metrics g .

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

$$\text{Kähler} \implies \Lambda^+ = \mathbb{R}\omega \oplus \Re\Lambda^{2,0}.$$

$$W^+(\omega, \omega) = \frac{s}{3} > 0$$

for relevant Kähler metrics g . Indeed, $h = s^{-2}g$.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

$$\text{Kähler} \implies \Lambda^+ = \mathbb{R}\omega \oplus \Re\Lambda^{2,0}.$$

$$W^+(\omega, \omega) = \frac{s}{3} > 0$$

for relevant Kähler metrics g . Indeed, $h = s^{-2}g$.

Moreover, g Bach-flat & Kähler \implies extremal.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Indeed, the conformally Kähler, Einstein metrics on del Pezzo surfaces all satisfy $W^+(\omega, \omega) > 0$.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Indeed, the conformally Kähler, Einstein metrics on del Pezzo surfaces are completely classified:

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Indeed, the conformally Kähler, Einstein metrics on del Pezzo surfaces are completely classified:

- the Kähler-Einstein metrics with $\lambda > 0$;

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Indeed, the conformally Kähler, Einstein metrics on del Pezzo surfaces are completely classified:

- the Kähler-Einstein metrics with $\lambda > 0$;
- the Page metric on $\mathbb{C}P_2 \# \overline{\mathbb{C}P_2}$; and

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Indeed, the conformally Kähler, Einstein metrics on del Pezzo surfaces are completely classified:

- the Kähler-Einstein metrics with $\lambda > 0$;
- the Page metric on $\mathbb{C}P_2 \# \overline{\mathbb{C}P_2}$; and
- the CLW metric on $\mathbb{C}P_2 \# 2\overline{\mathbb{C}P_2}$.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Every del Pezzo surface has $b_+ = 1$.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Every del Pezzo surface has $b_+ = 1$. \iff

Up to sign, $\forall h, \exists!$ self-dual harmonic 2-form ω :

$$d\omega = 0, \quad \star\omega = \omega, \quad \int_M \omega^2 = 1.$$

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Corollary. *Let M^4 be the underlying smooth manifold of any del Pezzo surface.*

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Corollary. *Let M^4 be the underlying smooth manifold of any del Pezzo surface. Then the conformally Kähler, Einstein metrics*

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Corollary. *Let M^4 be the underlying smooth manifold of any del Pezzo surface. Then the conformally Kähler, Einstein metrics sweep out exactly one connected component*

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Corollary. *Let M^4 be the underlying smooth manifold of any del Pezzo surface. Then the conformally Kähler, Einstein metrics sweep out exactly one connected component of the moduli space $\mathcal{E}(M)$*

Theorem (L '15). *Let (M, h) be a smooth compact oriented Einstein 4-manifold that carries a self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) > 0$$

everywhere on M . Then M is diffeomorphic to a del Pezzo surface, and h is conformally Kähler, with Einstein constant $\lambda > 0$.

Conversely, every del Pezzo surface admits Einstein metrics with $W^+(\omega, \omega) > 0$.

Corollary. *Let M^4 be the underlying smooth manifold of any del Pezzo surface. Then the conformally Kähler, Einstein metrics sweep out exactly one connected component of the moduli space $\mathcal{E}(M)$ of Einstein metrics on M .*

Reasonably satisfying result.

Reasonably satisfying result.

$W^+(\omega, \omega)$ is non-trivially related

Reasonably satisfying result.

$W^+(\omega, \omega)$ is non-trivially related to scalar curv s ,

Reasonably satisfying result.

$W^+(\omega, \omega)$ is non-trivially related to scalar curv s ,
via Weitzenböck for self-dual harmonic 2-form ω :

Reasonably satisfying result.

$W^+(\omega, \omega)$ is non-trivially related to scalar curv s ,
via Weitzenböck for self-dual harmonic 2-form ω :

$$0 = \nabla^* \nabla \omega - 2W^+(\omega, \cdot) + \frac{s}{3} \omega$$

Reasonably satisfying result.

$W^+(\omega, \omega)$ is non-trivially related to scalar curv s ,
via Weitzenböck for self-dual harmonic 2-form ω :

$$0 = \nabla^* \nabla \omega - 2W^+(\omega, \cdot) + \frac{s}{3} \omega$$

Taking inner product with ω and integrating:

Reasonably satisfying result.

$W^+(\omega, \omega)$ is non-trivially related to scalar curv s ,
via Weitzenböck for self-dual harmonic 2-form ω :

$$0 = \nabla^* \nabla \omega - 2W^+(\omega, \cdot) + \frac{s}{3} \omega$$

Taking inner product with ω and integrating:

$$\int_M W^+(\omega, \omega) d\mu \geq \int_M \frac{s}{6} |\omega|^2 d\mu$$

Reasonably satisfying result.

$W^+(\omega, \omega)$ is non-trivially related to scalar curv s ,
via Weitzenböck for self-dual harmonic 2-form ω :

$$0 = \nabla^* \nabla \omega - 2W^+(\omega, \cdot) + \frac{s}{3} \omega$$

Taking inner product with ω and integrating:

$$\int_M W^+(\omega, \omega) d\mu \geq \int_M \frac{s}{6} |\omega|^2 d\mu$$

In particular, an Einstein metric with $\lambda > 0$ has

Reasonably satisfying result.

$W^+(\omega, \omega)$ is non-trivially related to scalar curv s ,
via Weitzenböck for self-dual harmonic 2-form ω :

$$0 = \nabla^* \nabla \omega - 2W^+(\omega, \cdot) + \frac{s}{3} \omega$$

Taking inner product with ω and integrating:

$$\int_M W^+(\omega, \omega) d\mu \geq \int_M \frac{s}{6} |\omega|^2 d\mu$$

In particular, an Einstein metric with $\lambda > 0$ has

$$W^+(\omega, \omega) > 0$$

on average.

Reasonably satisfying result.

$W^+(\omega, \omega)$ is non-trivially related to scalar curv s ,
via Weitzenböck for self-dual harmonic 2-form ω :

$$0 = \nabla^* \nabla \omega - 2W^+(\omega, \cdot) + \frac{s}{3} \omega$$

Taking inner product with ω and integrating:

$$\int_M W^+(\omega, \omega) d\mu \geq \int_M \frac{s}{6} |\omega|^2 d\mu$$

In particular, an Einstein metric with $\lambda > 0$ has

$$W^+(\omega, \omega) > 0$$

on average. But result requires this everywhere.

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Peng Wu proposed an alternate characterization

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Peng Wu proposed an alternate characterization using only a purely local condition on W^+ .

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Peng Wu proposed an alternate characterization using only a purely local condition on W^+ .

Kähler $\implies \Lambda^+ = \mathbb{R}\omega \oplus \Re\Lambda^{2,0}$

$$W^+ = \begin{bmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{bmatrix}$$

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Peng Wu proposed an alternate characterization using only a purely local condition on W^+ .

Kähler $\implies \Lambda^+ = \mathbb{R}\omega \oplus \Re\Lambda^{2,0}$

$$\det(W^+) = \det \begin{bmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{bmatrix} = \frac{s^3}{864} > 0$$

for these metrics

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Peng Wu proposed an alternate characterization using only a purely local condition on W^+ .

Kähler $\implies \Lambda^+ = \mathbb{R}\omega \oplus \Re\Lambda^{2,0}$

$$\det(W^+) = \det \begin{bmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{bmatrix} = \frac{s^3}{864} > 0$$

for these metrics & conformal rescalings:

$$g \rightsquigarrow h = f^2 g \implies \det(W^+) \rightsquigarrow f^{-6} \det(W^+).$$

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Peng Wu proposed an alternate characterization using only a purely local condition on W^+ .

Wu's criterion:

$$\det(W^+) > 0.$$

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Peng Wu proposed an alternate characterization using only a purely local condition on W^+ .

Wu's criterion:

$$\det(W^+) > 0.$$

Wu (2019): cryptic, opaque proof that \iff .

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Peng Wu proposed an alternate characterization using only a purely local condition on W^+ .

Wu's criterion:

$$\det(W^+) > 0.$$

Wu (2019): cryptic, opaque proof that \iff .

L (2019): completely different proof;

Reasonably satisfying result.

But $W^+(\omega, \omega) > 0$ is not purely local condition!

Involves global harmonic 2-form ω .

Peng Wu proposed an alternate characterization using only a purely local condition on W^+ .

Wu's criterion:

$$\det(W^+) > 0.$$

Wu (2019): cryptic, opaque proof that \iff .

L (2019): completely different proof;

method also proves more general results.

Theorem A.

Theorem A. *Let (M, h) be a compact oriented Einstein 4-manifold,*

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold,*

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M .

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

Corollary. *Every simply-connected compact oriented Einstein (M^4, h) with $\det(W^+) > 0$ is diffeomorphic to a del Pezzo surface.*

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

Corollary. *Every simply-connected compact oriented Einstein (M^4, h) with $\det(W^+) > 0$ is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo M^4 carries Einstein h with $\det(W^+) > 0$, and these sweep out exactly one connected component of moduli space $\mathcal{E}(M)$.*

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

One can also show that

$$\int_M \det(W^+) d\mu > 0$$

for any $\lambda \geq 0$ Einstein metric on these M^4 .

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

One can also show that

$$\int_M \det(W^+) d\mu > 0$$

for any $\lambda \geq 0$ Einstein metric on these M^4 .

But result requires pointwise assumption.

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

Simply connected hypothesis is essential!

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

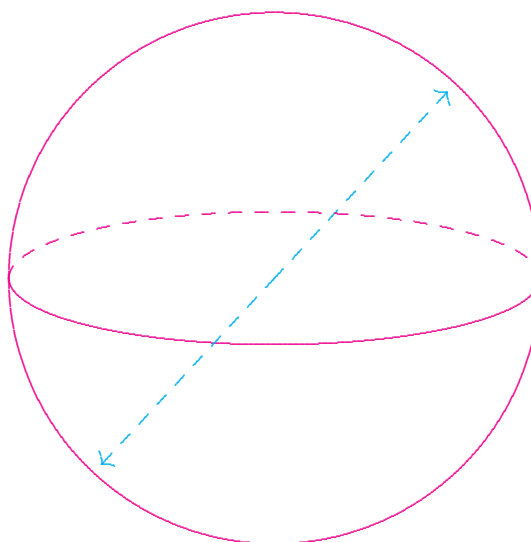
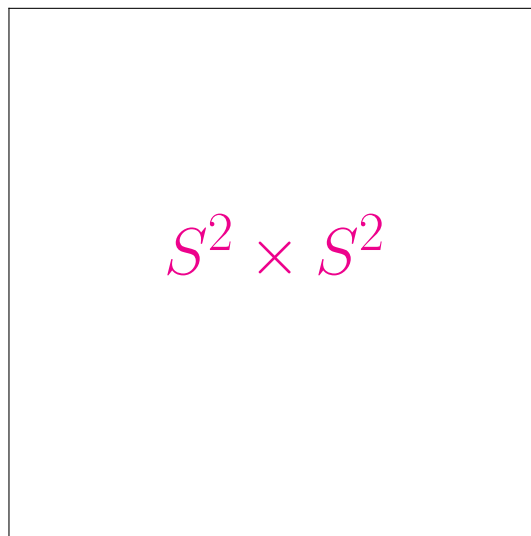
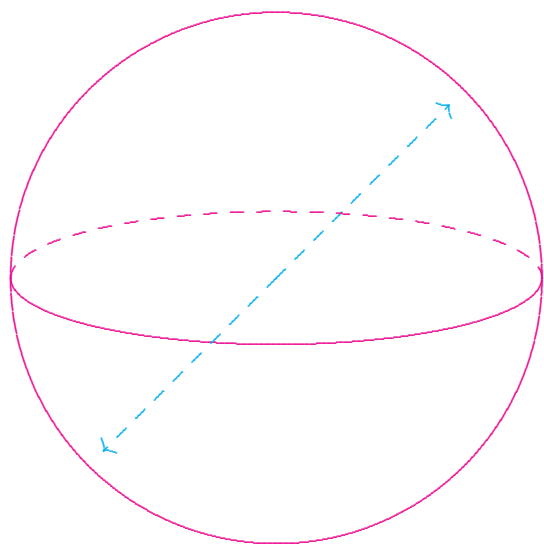
satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

Simply connected hypothesis is essential!

Otherwise, $(S^2 \times S^2)/\mathbb{Z}_2$ would be counter-example,



Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

Simply connected hypothesis is essential!

Otherwise, $(S^2 \times S^2)/\mathbb{Z}_2$ would be counter-example, where antipodal \times antipodal generates \mathbb{Z}_2 -action.

Theorem A. *Let (M, h) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \rightarrow \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M . Then h is conformal to an orientation-compatible Bach-flat extremal Kähler metric g with scalar curvature $s > 0$ on M .

Simply connected hypothesis is essential!

Otherwise, $(S^2 \times S^2)/\mathbb{Z}_2$ would be counter-example, where antipodal \times antipodal generates \mathbb{Z}_2 -action.

However, this example is as bad as it gets...

Proposition.

Proposition. *Let (M, h) be a compact oriented Einstein 4-manifold*

Proposition. *Let (M, h) be a compact oriented Einstein 4-manifold with*

$$\det(W^+) > 0.$$

Proposition. *Let (M, h) be a compact oriented Einstein 4-manifold with*

$$\det(W^+) > 0.$$

Then either

Proposition. *Let (M, h) be a compact oriented Einstein 4-manifold with*

$$\det(W^+) > 0.$$

Then either

(i) $\pi_1(M) = 0,$

Proposition. *Let (M, h) be a compact oriented Einstein 4-manifold with*

$$\det(W^+) > 0.$$

Then either

- (i) $\pi_1(M) = 0$, and M admits an orientation-compatible complex structure J that makes (M, J) into a del Pezzo surface, and relative to which the Einstein metric h becomes conformally Kähler;

Proposition. *Let (M, h) be a compact oriented Einstein 4-manifold with*

$$\det(W^+) > 0.$$

Then either

- (i) $\pi_1(M) = 0$, and M admits an orientation-compatible complex structure J that makes (M, J) into a del Pezzo surface, and relative to which the Einstein metric h becomes conformally Kähler; or else,

Proposition. *Let (M, h) be a compact oriented Einstein 4-manifold with*

$$\det(W^+) > 0.$$

Then either

- (i) $\pi_1(M) = 0$, and M admits an orientation-compatible complex structure J that makes (M, J) into a del Pezzo surface, and relative to which the Einstein metric h becomes conformally Kähler; or else,
- (ii) $\pi_1(M) = \mathbb{Z}_2$,

Proposition. *Let (M, h) be a compact oriented Einstein 4-manifold with*

$$\det(W^+) > 0.$$

Then either

- (i) $\pi_1(M) = 0$, and M admits an orientation-compatible complex structure J that makes (M, J) into a del Pezzo surface, and relative to which the Einstein metric h becomes conformally Kähler; or else,
- (ii) $\pi_1(M) = \mathbb{Z}_2$, and M is doubly covered by a del Pezzo surface (\hat{M}, J) of even signature on which the pull-back of the Einstein metric h becomes conformally Kähler.

One key idea underlying the proof:

One key idea underlying the proof:

By second Bianchi identity,

One key idea underlying the proof:

By second Bianchi identity,

h Einstein \implies

One key idea underlying the proof:

By second Bianchi identity,

h Einstein \implies

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

One key idea underlying the proof:

By second Bianchi identity,

h Einstein \implies

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Our strategy:

One key idea underlying the proof:

By second Bianchi identity,

h Einstein \implies

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Our strategy:

study weaker equation

One key idea underlying the proof:

By second Bianchi identity,

h Einstein \implies

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Our strategy:

study weaker equation

$$\delta W^+ = 0$$

One key idea underlying the proof:

By second Bianchi identity,

h Einstein \implies

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Our strategy:

study weaker equation

$$\delta W^+ = 0$$

as proxy for Einstein equation.

One key idea underlying the proof:

By second Bianchi identity,

h Einstein \implies

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Our strategy:

study weaker equation

$$\delta W^+ = 0$$

as proxy for Einstein equation.

But actually more widely applicable!

Theorem B.

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold*

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$,

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M .

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M . Then M admits an orientation-compatible Kähler metric g

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M . Then M admits an orientation-compatible Kähler metric g of scalar curvature $s > 0$

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M . Then M admits an orientation-compatible Kähler metric g of scalar curvature $s > 0$ such that $h = s^{-2}g$.

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M . Then M admits an orientation-compatible Kähler metric g of scalar curvature $s > 0$ such that $h = s^{-2}g$.

Derdziński: Conversely, if (M^4, g) is Kähler, with scalar curvature $s > 0$, then $h = s^{-2}g$ satisfies $\delta W^+ = 0$

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M . Then M admits an orientation-compatible Kähler metric g of scalar curvature $s > 0$ such that $h = s^{-2}g$.

Derdziński: *Conversely, if (M^4, g) is Kähler, with scalar curvature $s > 0$, then $h = s^{-2}g$ satisfies $\delta W^+ = 0$ and $\det(W^+) > 0$.*

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M . Then M admits an orientation-compatible Kähler metric g of scalar curvature $s > 0$ such that $h = s^{-2}g$.

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M . Then M admits an orientation-compatible Kähler metric g of scalar curvature $s > 0$ such that $h = s^{-2}g$.

Corollary. *A smooth compact oriented M^4 with $b_+(M) \neq 0$ admits metrics with $\delta W^+ := 0$ and $\det(W^+) > 0$ if and only if it is diffeomorphic to a rational or ruled complex surface.*

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M . Then M admits an orientation-compatible Kähler metric g of scalar curvature $s > 0$ such that $h = s^{-2}g$.

Corollary. *A smooth compact oriented M^4 with $b_+(M) \neq 0$ admits metrics with $\delta W^+ := 0$ and $\det(W^+) > 0$ if and only if*

$$M \stackrel{\text{diff}}{\approx} \begin{cases} (\Sigma^2 \times S^2) \# k \overline{\mathbb{C}P}_2, & k \geq 0 \\ \Sigma^2 \# S^2, \text{ or} \\ \mathbb{C}P_2. \end{cases}$$

Theorem B. *Let (M, h) be a compact oriented Riemannian 4-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, and that h satisfies $\det(W^+) > 0$ at every point of M . Then M admits an orientation-compatible Kähler metric g of scalar curvature $s > 0$ such that $h = s^{-2}g$.

Corollary. *A smooth compact oriented M^4 with $b_+(M) \neq 0$ admits metrics with $\delta W^+ := 0$ and $\det(W^+) > 0$ if and only if it is diffeomorphic to a rational or ruled complex surface. When such metrics exist, their moduli space is always infinite dimensional.*

Theorem C.

Theorem C. *Let (M, h) be a compact oriented Riemannian 4-manifold*

Theorem C. *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$.*

Theorem C. *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$. If*

$$W^+ \neq 0$$

Theorem C. *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$. If*

$$W^+ \neq 0 \quad \text{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

Theorem C. *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$. If*

$$W^+ \neq 0 \quad \text{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M ,

Theorem C. *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$. If*

$$W^+ \neq 0 \quad \text{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M , then actually $\det(W^+) > 0$.

Theorem C. *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$. If*

$$W^+ \neq 0 \quad \text{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M , then actually $\det(W^+) > 0$. Thus, after at worst passing to a double cover $\hat{M} \rightarrow M$,

Theorem C. *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$. If*

$$W^+ \neq 0 \quad \text{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

*everywhere on M , then actually $\det(W^+) > 0$. Thus, after at worst passing to a double cover $\hat{M} \rightarrow M$, h becomes conformally Kähler, in the manner described by **Theorem B**.*

Theorem C. *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$. If*

$$W^+ \neq 0 \quad \text{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

*everywhere on M , then actually $\det(W^+) > 0$. Thus, after at worst passing to a double cover $\hat{M} \rightarrow M$, h becomes conformally Kähler, in the manner described by **Theorem B**. In particular, if (M, h) is a simply-connected Einstein manifold, it actually falls under the purview of **Theorem A**.*

Today's talk has focused on Einstein 4-manifolds
with $\lambda \geq 0$.

Today's talk has focused on Einstein 4-manifolds with $\lambda \geq 0$.

But the real moral is that **most** Einstein 4-manifolds have $\lambda < 0$!

Today's talk has focused on Einstein 4-manifolds with $\lambda \geq 0$.

But the real moral is that **most** Einstein 4-manifolds have $\lambda < 0$!

We have also learned a huge amount about this **negative case** in recent years.

Today's talk has focused on Einstein 4-manifolds with $\lambda \geq 0$.

But the real moral is that **most** Einstein 4-manifolds have $\lambda < 0$!

We have also learned a huge amount about this **negative case** in recent years.

In this setting, **Seiberg-Witten theory** plays the starring role.

Today's talk has focused on Einstein 4-manifolds with $\lambda \geq 0$.

But the real moral is that **most** Einstein 4-manifolds have $\lambda < 0$!

We have also learned a huge amount about this **negative case** in recent years.

In this setting, **Seiberg-Witten theory** plays the starring role.

But that would be the subject of an **entirely different** colloquium!