Wolpert 4/19

Prime Geodesic Theorems

For a hyperbolic surface R of genus g, n punctures the count of closed geodesics of length at most L is $\frac{e^L}{L}$

In 2008 Mirzakhani gave a beautiful argument for counting simple closed geodesics

A *multi curve* is a finite positive weighted union of disjoint simple closed geodesics $\gamma = \sum_j a_j \gamma_j$

Choosing geodesics to represent free homotopy classes we have the length $\ell_\gamma = \sum_j a_j \ell_{\gamma_j}$

Write MCG for the mapping class group $Homeo^+(R)/Homeo_0(R)$
MCG acts on free homotopy classes of closed curves
The orbit MCG γ is the set of all multicurves topologically equivalent to γ

Introduce the counting function

$$S_R(L,\gamma) = \#\{\alpha \in MCG\,\gamma \mid \ell_\alpha(R) \le L\}$$

Theorem (Mirzakhani) For a rational multi-curve γ

$$S_R(L,\gamma) \sim \mu_{Th}(B_R)\,\frac{c(\gamma)}{b_{g,n}}\,L^{6g-6+2n} \qquad L \to \infty$$

μ_{Th} Thurston volume, B_R is a unit-length ball

$c(\gamma)$ is a sum of intersection numbers of the moduli space of R.s.

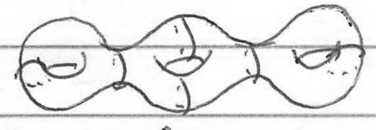
$$b_{g,n} = \int_{\mathcal{M}} \mu_{Th}(B_R)\, dV_{wp}$$

Formula provides the opportunity to give an exposition of the geometry of the moduli space and Thurston theory

There are follow up results of Mirzakhani, Erlandsson-Souto and Rafi-Souto

As part of describing Mirzakhani's approach I will describe two possible generalizations

Pants decompositions $\mathcal{P}$

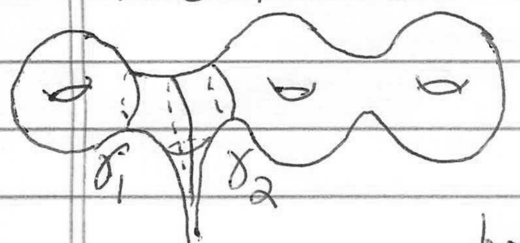


$\ell_1, \ldots, \ell_{3g-3+n}$

a collection of simple closed curves decomposing a surface into 3-holed spheres

Conjecture Given intervals $[a_j, b_j]$ $j = 1, \ldots, 3g-3+n$

pants decompositions in a MCG orbit $\ell_j \in L[a_j, b_j] \sim \dfrac{\mu_{Th}(B_R)}{b_{g,n}}\, c(\mathcal{P}) \prod_j \dfrac{b_j^2 - a_j^2}{2} L^{6g-6+2n}$



Consider biinfinite simple geodesics with both ends at a common cusp

Reduced length is length of segment outside of a horoball region

By hyperbolic trigonometry

$$\text{reduced length} = \max\{\ell_1, \ell_2\} + O(1)$$

Rather than counting weighted sum $a_1\ell_1 + a_2\ell_2$
want to count $\max\{\ell_1, \ell_2\}$ ~ not included in original method

Conjecture

$\#$ lariats in a MCG orbit w reduced length $\leq L$

$$\sim \frac{\mu_{Th}(B_R)}{b_{g,n}} \frac{c(\gamma_1+\gamma_2)}{4} L^{6g-6+2n}$$

Before getting into full argument, why do we get a power $L^{6g-6+2n}$ in place of an exponential $\frac{e^L}{L}$?

We will be counting integral lattice points in a Euclidean lattice.

Mirzakhani's approach has two components

1. Compute $\int_{\mathcal{M}} s_R(L,\gamma)\, dV_{WP}$ as a polynomial

   with coefficients intersection numbers for the moduli space

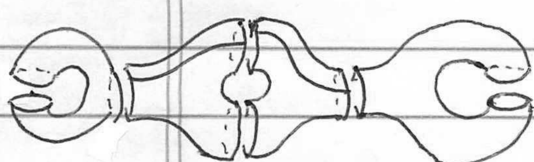
2. Show on the space of measured geodesic laminations $\mathcal{MGL}$ the counting is given by a multiple of the Thurston measure

start with

$$s_R(L,\gamma) = \sum_{h \in MCG/Stab(\gamma)} \chi_{[0,L]}(h(R))$$

$$\int_{\mathcal{M}} s_R(L,\gamma)\, dV_{wp} = \frac{1}{|Sym(\gamma)|} \int_{\mathcal{T}/Stab(\gamma)} \chi_{[0,L]}(R)\, dV_{wp}$$

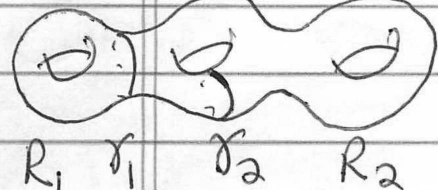
Recall about Fenchel-Nielsen coordinates and dV_{wp}



Define parameters at each seam
$\tau_j \in \mathbb{R}$ and $\ell_j \in \mathbb{R}^+$

FN (τ_j, ℓ_j) are global coords on Teichmüller space

$$dV_{wp} = \frac{1}{(3g-3+n)!} \bigwedge_j d\tau_j \wedge d\ell_j$$



Case: $\gamma = a_1\gamma_1 + a_2\gamma_2$

γ_1, γ_2 disect R into two pieces R_1, R_2
Stab$(\gamma_1 + \gamma_2)$ includes MCG of components

$\mathcal{T}/Stab(\gamma)$ = product of moduli spaces of R_1, R_2 modulo finite symmetries

$$\text{above integral} = \int_{a_1\ell_1 + a_2\ell_2 \leq L} \begin{matrix}\text{moduli}\\ \text{volume}\end{matrix}(R_1)(\ell_1) \cdot \begin{matrix}\text{moduli}\\ \text{volume}\end{matrix}(R_2)(\ell_1,\ell_2)\, \ell_1 d\ell_1\, \ell_2 d\ell_2$$

= polynomial in ℓ_1, ℓ_2 of total degree $6g-6+2n$

Theorem (Mirzakhani) Symplectic reduction. Volume of a moduli space of R.s. with geodesic boundaries is a polynomial of degree -2χ(R.s.) with coefficients ψ, κ_1 intersection numbers

2. Counting measure is a multiple of the Thurston measure

On a hyperbolic surface, geodesics intersect the minimal number of times for their homotopy classes
A simple closed geodesic α defines a functional on multicurves by considering weighted intersection number $\sum_j a_j \, \alpha \# \gamma_j$

The completion of the set of multicurves wrt weighted intersections is $\mathcal{MGL}$ Thurston's space of measured geodesic laminations

Properties of $\mathcal{MGL}$
* has a characterization independent of hyperbolic surface $\Rightarrow$ MCG acts
* is a dimension $6g-6+2n$ PL manifold
* has a natural volume form μ_{Th}

Theorem (Masur) MCG acts ergodically on $\mathcal{MGL}$ wrt μ_{Th}

Defn $B_R = \{\mu \in \mathcal{MGL} \mid l_\mu(R) \leq 1\}$

Consider $\mathcal{MGL}(\mathbb{Z})$ set of multicurves with integer weights

Given $\mathcal{U}$ open, convex in $\mathcal{MGL}$ define counting measures

$$\mu_{L,\mathbb{Z}}(\mathcal{U}) = \frac{\#\{L\mathcal{U} \cap \mathcal{MGL}(\mathbb{Z})\}}{L^{6g-6+2n}} \qquad \mu_{L,\gamma}(\mathcal{U}) = \frac{\#\{L\mathcal{U} \cap MCG\,\gamma\}}{L^{6g-6+2n}}$$

By counting integral lattice points in an open set

$$\mu_{L,\mathbb{Z}} \longrightarrow \mu_{Th} \quad \text{as } L \to \infty$$

A multiple $m\gamma$ is an integral multicurve

$$\mu_{L,\gamma} \le \frac{1}{m^{6g-6+2n}} \mu_{L,\mathbb{Z}}$$ and any limit of $\mu_{L,\gamma}$ is abs. cont wrt μ_{Th}

$\mu_{L,\gamma}$ is MCG invariant, so

$$\lim_{L} \mu_{L,\gamma} = c' \mu_{Th}$$

independent of surface

Evaluate measures on B_R

$$\mu_{L,\gamma}(B_R) = \frac{s_R(L,\gamma)}{L^{6g-6+2n}} \xrightarrow[L\to\infty]{} c'\mu_{Th}(B_R)$$

Now integrate over the moduli space

$$\left[\int_{\mathcal{M}} s_R(L,\gamma)\, dV\right]_{\substack{\text{leading}\\ \text{coeff}}} = c' \int_{\mathcal{M}} \mu_{Th}(B_R)\, dV$$

apply Theorem on coefficients