Dynamics, geometry, and the moduli space of Riemann surfaces

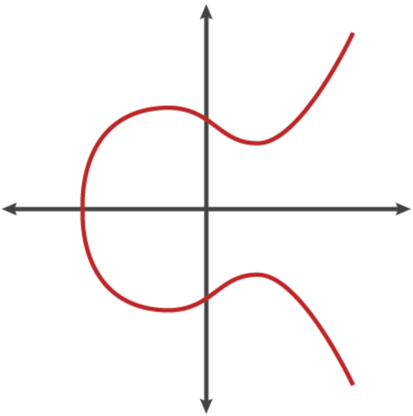
Alex Wright

Stanford University

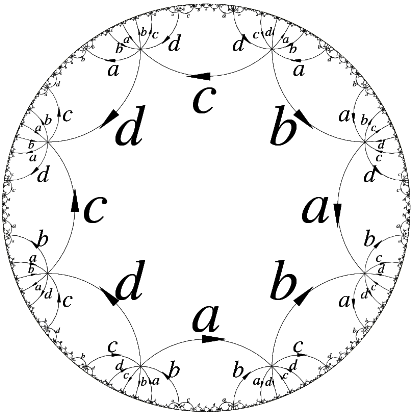
A Riemann surface is a surface with an atlas of charts to $\mathbb{C}$, so that the transition maps are biholomorphisms.

Can be specified in several transcendentally related ways.

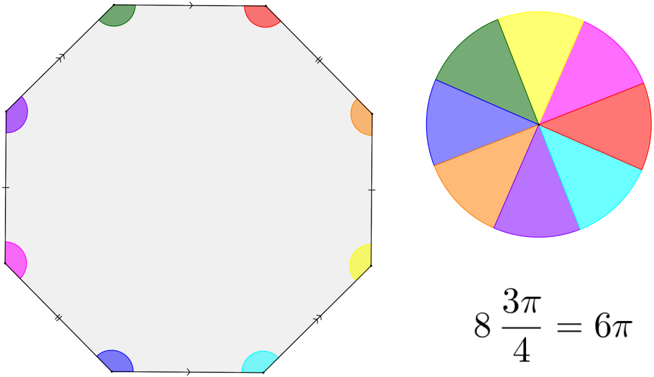
Ex: polynomial equations in $\mathbb{P}^n$.



Ex: $\mathbb{H}/\Gamma$, where $\Gamma \subset PSL(2, \mathbb{R})$.



Ex: Polygons in the plane with parallel edges identified.



$$8\,\frac{3\pi}{4} = 6\pi$$

Flat metric given by $q = (dz)^2$.

There are many Riemann surface of the same genus: Can vary polynomial equations / matrices in $PSL(2, \mathbb{R})$ / edges of polygons.

The moduli space $\mathcal{M}_g$ of all Riemann surfaces of genus g is a complex orbifold of dimension $3g - 3$.

$\mathcal{M}_g$ is also the moduli space of algebraic curves, and hyperbolic metrics.

It is the quotient of Teichmüller space $\mathcal{T}_g$ by the mapping class group.

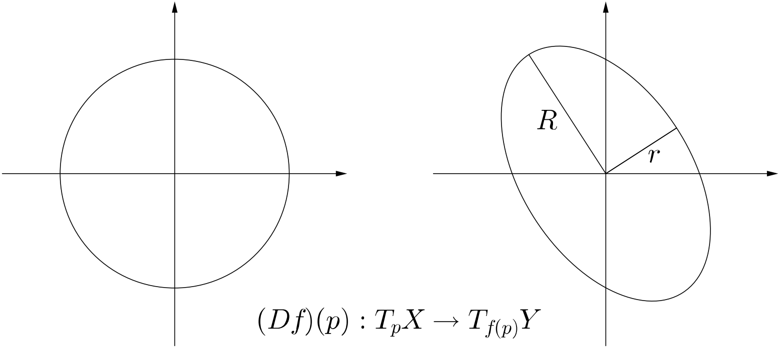
It is one of the most important algebraic varieties in mathematics.

What does it mean for two Riemann surfaces X and Y to be close?

If $X = Y$, there is a map $X \to Y$ that is conformal, i.e. maps infinitesimal circles to circles.

A differentiable map $f : X \to Y$ maps infinitesimal circles to ellipses.

Measure failure to be conformal by max of ratio of major to minor axes, called dilatation.



$$(Df)(p) : T_pX \to T_{f(p)}Y$$

Dilatation $= \max_{p \in X} R/r$.

The Teichmüller metric on $\mathcal{M}_g$ is defined as

$$d(X, Y) = \inf_{f:X\to Y} \log K_f,$$

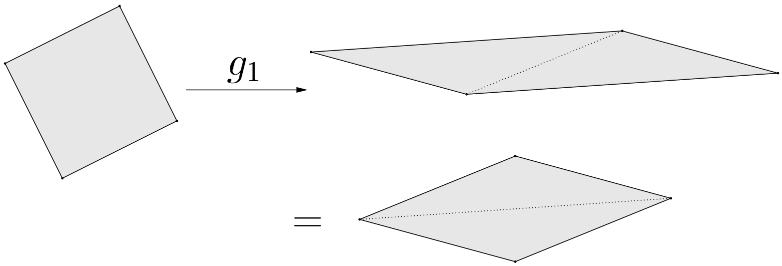
where K_f is the dilatation of f.

Finsler.

There is a unique geodesic through every pair of points in $\mathcal{T}_g$.

Geodesics through X obtained by presenting X with polygons, and acting by

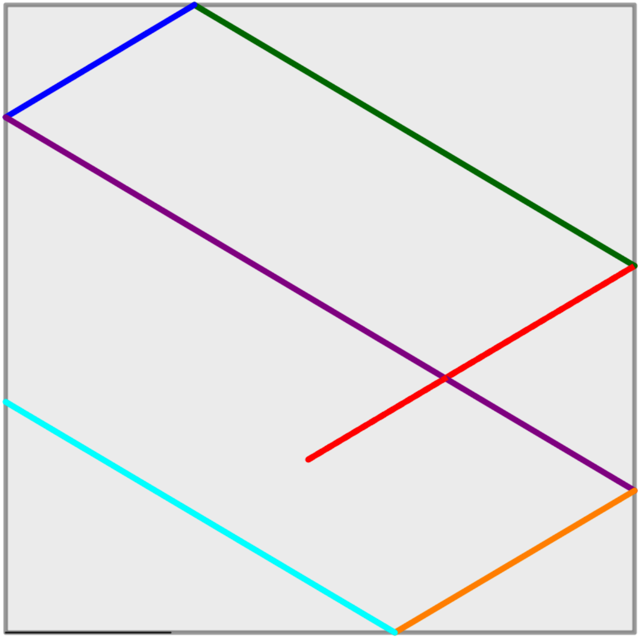
$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$


g_1
=

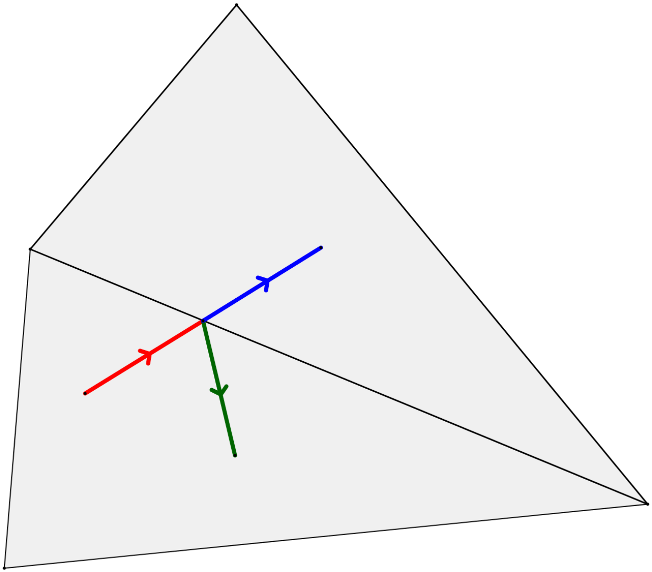
$\{g_t : t \in \mathbb{R}\}$ acts on the bundle QD of quadratic differentials over $\mathcal{M}_g$.

Orbits project to geodesics.

QD is the cotangent bundle to $\mathcal{M}_g$.



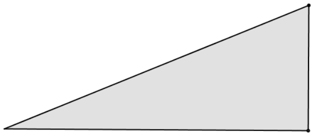
Continue straight into reflected copy.

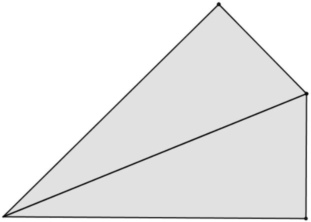


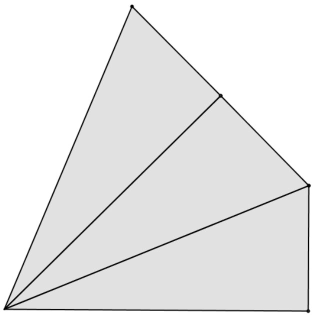
"Unfolding":

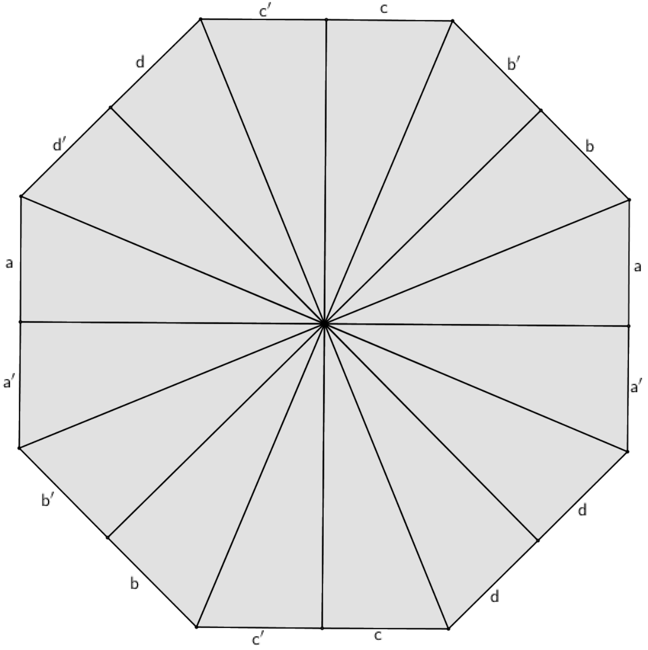
Keep reflecting polygons about edges.

If a reflection would give a translated copy, identify the edges instead.







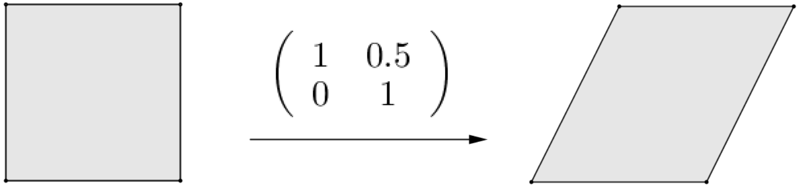


c′
c
d
b′
d′
b
a
a
a′
a′
b′
d
b
d
c′
c

Unfolding works when all angles are rational multiples of π.

The result is a Riemann surface with a quadratic differential.

g_t action can be extended to an action of $GL(2,\mathbb{R})$.



Masur, Veech: almost every (X, q) has dense $GL(2,\mathbb{R})$ orbit! Teichmüller geodesic flow is ergodic.

To understand (X, q) from every possible perspective, understand its orbit:

$$\{g(X, q) : g \in GL(2, \mathbb{R})\}$$

Easier to understand orbit closure:

$$\overline{\{g(X, q) : g \in GL(2, \mathbb{R})\}}$$

This talk addresses the question:

Can we classify orbit closures? Can we compute orbit closures of unfoldings?

It will turn out that the orbit closures are intrinsically interesting and beautiful loci.

Veech 1989: Some unfoldings of triangles have closed $GL(2,\mathbb{R})$ orbit.

For every periodic billiard trajectory in the right-angled triangle with angle $\frac{\pi}{5}$, there is one golden ratio times larger or smaller!

Projection of closed orbit to $\mathcal{M}_g$ is a Teichmüller curve ($dim_{\mathbb{C}} = 1$).

Calta, McMullen, 2003: There are many interesting orbit closures in genus 2.

McMullen: They are varieties closely related to Hilbert modular surfaces. Classification in genus 2.

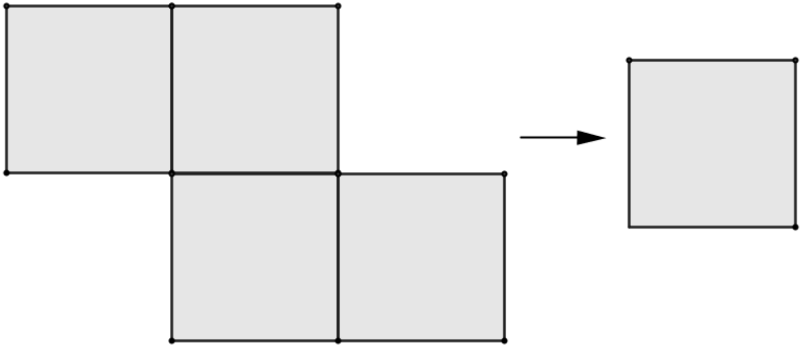
As of a few years ago, only very few additional orbit closures had been found, and all known examples where either “trivial” or “cousins of closed orbits”.

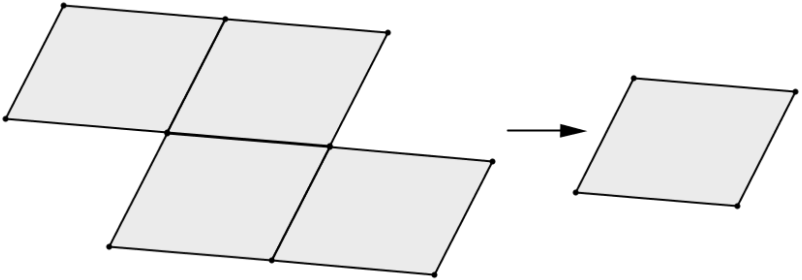
Furthermore, besides a few very special examples, almost no orbit closures of unfoldings were known. An enormous amount of machinery motivated by billiards wasn’t known to apply to any billiard tables!

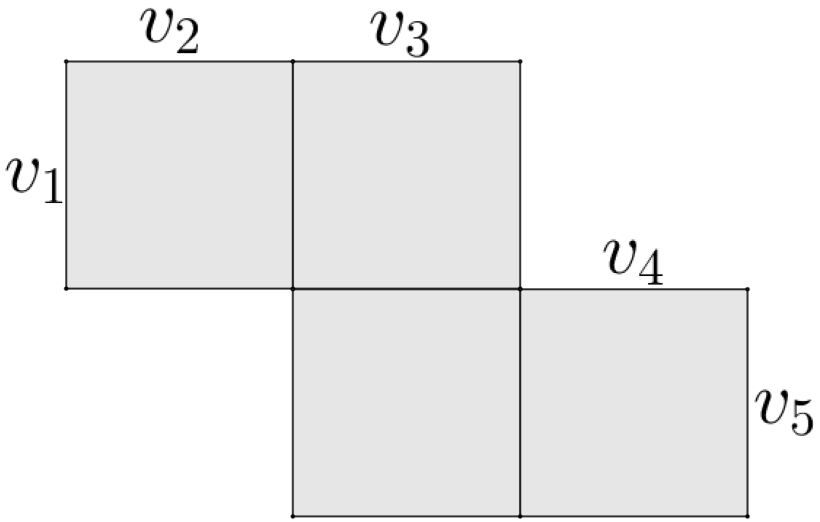
Theorem (Eskin-Mirzakhani-Mohammadi)

The $GL(2,\mathbb{R})$ *orbit closure of a quadratic differential is always a manifold, defined by linear equations on the edges of the polygons.*

Converse is easy.







$$v_1 = v_5, \qquad v_2 = v_3 = v_4$$

Spaces of quadratic differentials are very inhomogeneous,

but somewhat analogous to homogeneous manifolds like G/Γ,

$G = SL(3, \mathbb{R})$, $\Gamma = SL(3, \mathbb{Z})$.

Theorem (Ratner, Dani, Margulis)

If $G = SL(3,\mathbb{R})$, $\Gamma = SL(3,\mathbb{Z})$, *and*

$$h_t = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

then every h_t *orbit closure in* G/Γ *is a manifold.*

Theorem (Mirzakhani-W)

Infinitely many triangles unfold to surfaces with dense orbit.

In each genus only finitely many (X, q) are unfoldings of triangles, and from some points of view they all look likely to have small orbit closure.

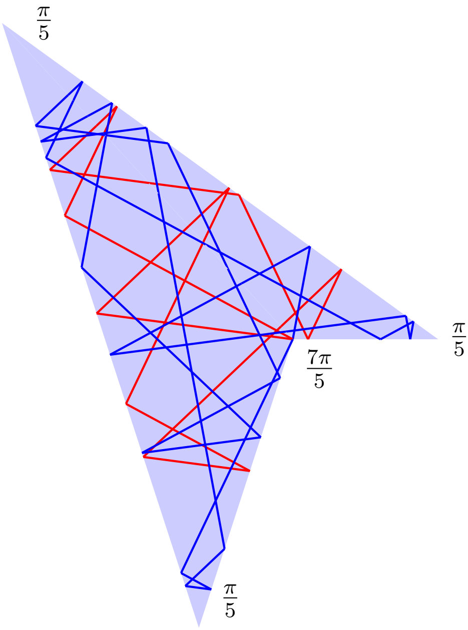
(1, 2, 8)	(1, 3, 7)	(2, 4, 5)			
(1, 4, 8)	(2, 3, 8)	(3, 4, 6)			
(4, 5, 6)					
(1, 2, 14)	(1, 4, 12)	(1, 5, 11)	(2, 3, 12)	(2, 4, 11)	(2, 6, 9)
(2, 7, 8)	(3, 4, 10)	(3, 6, 8)	(4, 6, 7)		
(1, 2, 16)	(1, 3, 15)	(1, 4, 14)	(1, 5, 13)	(1, 6, 12)	(1, 7, 11)
(2, 3, 14)	(2, 4, 13)	(2, 5, 12)	(2, 6, 11)	(2, 7, 10)	(2, 8, 9)
(3, 4, 12)	(3, 5, 11)	(3, 6, 10)	(3, 7, 9)	(4, 6, 9)	(4, 7, 8)
(5, 6, 8)					
(1, 4, 16)	(1, 5, 15)	(3, 5, 13)	(4, 7, 10)	(6, 7, 8)	
(1, 2, 20)	(1, 3, 19)	(1, 4, 18)	(1, 5, 17)	(1, 6, 16)	(1, 7, 15)
(1, 8, 14)	(1, 9, 13)	(2, 3, 18)	(2, 4, 17)	(2, 5, 16)	(2, 6, 15)
(2, 7, 14)	(2, 8, 13)	(2, 9, 12)	(2, 10, 11)	(3, 4, 16)	(3, 5, 15)
(3, 6, 14)	(3, 7, 13)	(3, 8, 12)	(3, 9, 11)	(4, 5, 14)	(4, 6, 13)
(4, 7, 12)	(4, 8, 11)	(4, 9, 10)	(5, 6, 12)	(5, 7, 11)	(5, 8, 10)
(6, 7, 10)	(6, 8, 9)				
(1, 3, 21)	(1, 4, 20)	(1, 6, 18)	(1, 7, 17)	(1, 8, 16)	(1, 10, 14)
(2, 3, 20)	(2, 4, 19)	(2, 5, 18)	(2, 7, 16)	(2, 8, 15)	(2, 10, 13)
(3, 4, 18)	(3, 5, 17)	(3, 6, 16)	(3, 7, 15)	(3, 8, 14)	(3, 9, 13)
(3, 10, 12)	(4, 5, 16)	(4, 6, 15)	(4, 8, 13)	(4, 9, 12)	(4, 10, 11)
(5, 6, 14)	(6, 7, 12)	(6, 8, 11)	(6, 9, 10)	(7, 8, 10)	

Theorem (Eskin-McMullen-Mukamel-W)

All quadrilaterals with these angles unfold to quadratic differentials with very special orbit closure.

$$\left(\frac{\pi}{5}, \frac{\pi}{5}, \frac{\pi}{5}, \frac{7\pi}{5}\right) \quad \left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{2}\right) \quad \left(\frac{\pi}{8}, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{2}\right)$$

$$\left(\frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{11\pi}{8}\right) \quad \left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{3}\right) \quad \left(\frac{\pi}{10}, \frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{2}\right)$$


$\frac{\pi}{5}$
$\frac{\pi}{5}$
$\frac{7\pi}{5}$
$\frac{\pi}{5}$

$\mathcal{M}_{g,n}$ = moduli space of genus g surfaces with n marked points.

A complex submanifold of $\mathcal{M}_{g,n}$ is totally geodesic if it contains a geodesic between any two of its points.

Theorem (Eskin-McMullen-Mukamel-W)

There are totally geodesic complex surfaces in $\mathcal{M}_{1,3}, \mathcal{M}_{1,4}, \mathcal{M}_{2,1}$.

Images of "exotic Teichmüller spaces" in $\mathcal{T}_{g,n}$.

Counterexamples to Mirzakhani's conjecture that all orbit closures are "trivial" or "cousins of closed orbits".

Theorem (Eskin-Filip-W)

There are only finite many orbit closures in each genus that are not "cousins of closed orbits".

Also have finiteness results for closed orbits, reminiscent of Manin-Mumford, André-Oort, etc.

It is helpful to assume that the quadratic differential is the square of an Abelian differential (holomorphic 1-form $f(z)dz$),

$$q = \omega^2.$$

This can always be arranged after passing to a double cover.

Locally $\omega = dz$, zeros of ω correspond to corners of the polygons.

$$\int_\gamma \omega$$

Absolute or relative: γ a closed loop or a path joining two zeros of ω.

Measures edges of polygons.

Are there natural/important examples of Riemann surfaces X with an Abelian differential ω whose periods satisfy some linear equations?

Yes!

They involve $\mathrm{Jac}(X) = H^{1,0}(X)^*/H_1(X, \mathbb{Z})$.
(An complex torus like $\mathbb{C}^g/\mathbb{Z}[i]^g$).

If $\mathrm{Jac}(X)$ has an endomorphism with ω as an eigenform, there exists $A : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$, $r \in \mathbb{R}$, so

$$\int_{A\gamma} \omega = r \int_{\gamma} \omega.$$

"Hidden symmetry" of X.

If two zeros p, q of ω have $n(p-q) = 0$ in $\mathrm{Jac}(X)$, then if $\gamma_{p,q}$ is a path from p to q, there exists $\gamma \in H_1(X, \mathbb{Z})$ so

$$\int_{\gamma_{p,q}} \omega = \frac{1}{n} \int_{\gamma} \omega.$$

McMullen 2003: The locus of eigenforms for real multiplication ($A^2 = D \cdot \mathrm{id}$) in genus 2 is $GL(2, \mathbb{R})$ invariant.

Not true in bigger genus.

Möller 2006: All the linear equations defining a closed orbit come from $\mathrm{Jac}(X)$.

Filip 2015: True for all orbit closures. Hence, orbit closures are varieties.

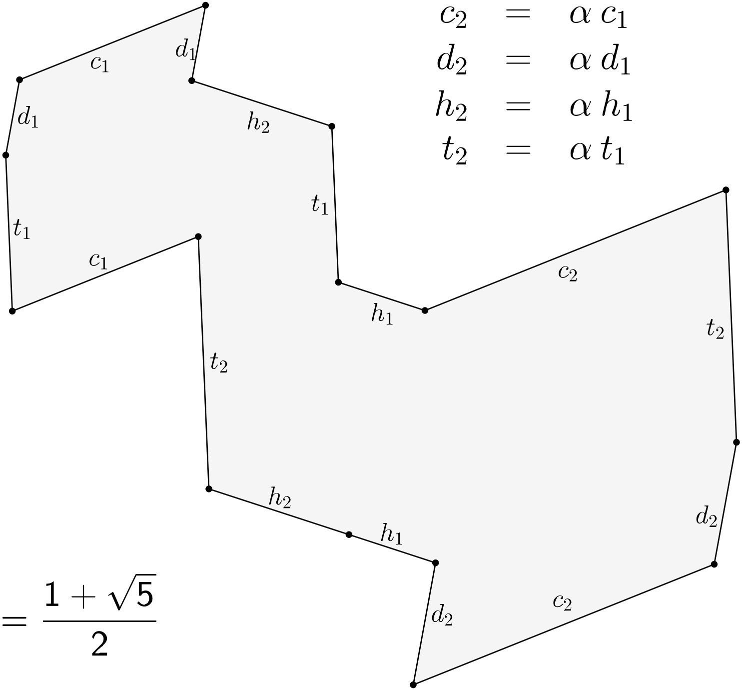
A d dimensional subvariety contained in a d dimensional linear subspace of $\mathbb{C}^n$ must be linear.

To build orbit closures, come up with loci "of maximal dimension" where $\text{Jac}(X)$ is special. Unlikely!

For six new orbit closures:

X is a cover of $\mathbb{P}^1$, endomorphisms of Jac(X) come from automorphisms of Galois closure (Hecke type).

$$
\begin{aligned}
c_2 &= \alpha\, c_1 \\
d_2 &= \alpha\, d_1 \\
h_2 &= \alpha\, h_1 \\
t_2 &= \alpha\, t_1
\end{aligned}
$$



$$\alpha = \frac{1+\sqrt{5}}{2}$$

To show certain unfoldings of polygons have dense orbit closure:

Jacobian of unfoldings is very special, but we can show that $GL(2,\mathbb{R})$ orbit is not tangent to loci where $\mathrm{Jac}(X)$ has endomorphisms.

Uses the Ahlfors variational formula for the derivative of the period matrix.

The period matrix is a way of recording the Hodge decomposition

$$H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

which varies as X changes, i.e. as the complex structure on the surface changes.

To show finiteness, use that the conditions of the Jacobian restrict the VHS over the orbit closure.

If there are infinitely many orbit closures, a subset must be dense in a larger orbit closure.

This would make the VHS of the larger orbit closure unexpectedly simple, however we show that to a certain extent the VHS over any orbit closure is always as complicated as possible.

In so doing, we calculate the algebraic hull of the Kontsevich-Zorich cocycle, which encodes the non-trivial part of the dynamics of the $GL(2, \mathbb{R})$ action.

In summary, the connection between orbit closures and Jacobians and VHS allow to build new orbit closures, calculate some orbit closures, and prove finiteness results.

There is also a related story where flat geometry (such as "cylinder deformations") is being successfully applied to the classification problem.

Dynamics, geometry, and the moduli space of Riemann surfaces

Alex Wright

Stanford University