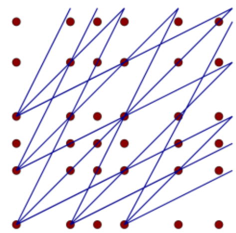
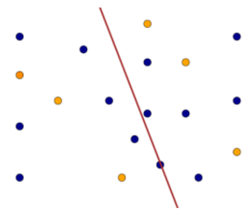
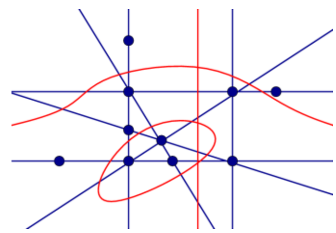
Geometric Incidences and the Polynomial Method





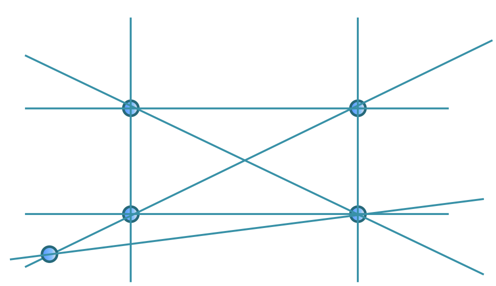


Adam Sheffer
Caltech

Incidences

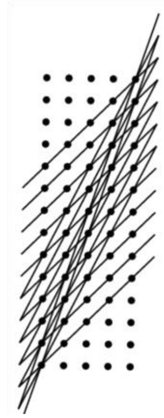
- P – a set of m points.
- L – a set of n lines.
- An incidence: $(p, \ell) \in P \times L$ such that $p \in L$.

15 incidences



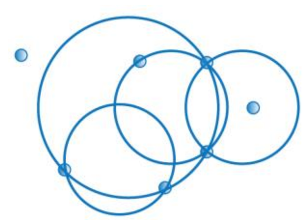
Incidences

- **Szemerédi and Trotter '83.** The number of incidences between any m points and n lines is $O(m^{2/3}n^{2/3} + m + n)$.



Incidences

- **Szemerédi and Trotter '83.** The maximum number of incidences between m points and n lines is $O(m^{2/3}n^{2/3} + m + n)$.
- Most of the other variants are still open:
  - Point-circle incidences.
  - Point-parabola incidences.



Namedropping

- Incidences have MANY applications.
- Examples from the last few years:
  - Guth and Katz used them to solve Erdős' distinct distances problem.
  - Brougain and Demeter used them to solve restriction problems in harmonic analysis.
  - Bombieri and Bourgain used them in a recent number theory paper.
  - Raz, Sharir, and Solymosi used them to study expanding polynomials.

More Namedropping

- More applications of incidences:
  - Many applications in additive combinatorics, including Elekes' Sum-Product bound.
  - Dvir, Saraf, Wigderson and others use them in a family of papers about coding theory.
  - Farber, Ray, and Smorodinsky used them to study minors of totally positive matrices.
  - Other uses involve extractors, point covering problems, range searching algorithms, and more.

Sumsets

- $A = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}$.
- $A + A = \{a + b \mid a, b \in A\}$.

- Can $A + A$ contain only $O(n)$ elements?

Product Sets

- $A = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}$.
- $AA = \{a \cdot b \mid a, b \in A\}$.

- Can AA contain only $O(n)$ elements?

Sum-Product

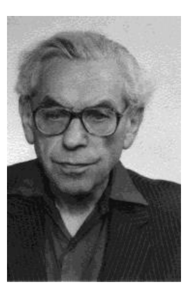
- $A = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}$.
- $A + A = \{a + b \mid a, b \in A\}$.
- $AA = \{a \cdot b \mid a, b \in A\}$.

- Can both $A + A$ and AA be small?

The Sum-Product Conjecture

- **Conjecture (Erdős and Szemerédi `83).** For any $\varepsilon > 0$, every sufficiently large set A satisfies

$$\max\{|A + A|, |AA|\} = \Omega(|A|^{2-\varepsilon}).$$



Paul Erdős

Endre Szemerédi

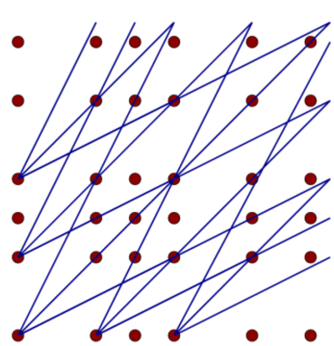
The Sum-Product Conjecture

- **Solymosi `09.**
  $$\max\{|A+A|,|AA|\} = \Omega^*\left(|A|^{4/3}\right).$$
- **Konyagin and Shkredov `16.**
  $$\max\{|A+A|,|AA|\} = \Omega^*\left(|A|^{\frac{4}{3}+\frac{5}{9813}}\right).$$
- We will prove an older bound of **Elekes**.
  $$\max\{|A+A|,|AA|\} = \Omega\left(|A|^{5/4}\right).$$

Elekes's Proof

- A – a set of n real numbers.

$$P = \{(a,b) \mid a \in A+A \quad b \in AA\}$$
$$L = \{y = c(x-d) \mid c,d \in A\}$$



Elekes's Proof (2)

- A – a set of n real numbers.

$$P = \{(a,b) \mid a \in A + A \quad b \in AA\}$$
$$L = \{y = c(x - d) \mid c, d \in A\}$$

- By the Szemerédi–Trotter theorem:

$$I(P,L) = O\left(|P|^{2/3}|L|^{2/3} + |P| + |L|\right)$$
$$= O\left(|A + A|^{2/3}|AA|^{2/3}n^{4/3}\right).$$

Elekes's Proof (3)

- A – a set of n real numbers.

$$P = \{(a,b) \mid a \in A + A \quad b \in AA\}$$
$$L = \{y = c(x - d) \mid c, d \in A\}$$

- Every line $y = c(x - d)$ contains exactly the n points of P of the form $(d + a', ca')$ where $a' \in A$.

$$I(P,L) = |A|^3 = n^3$$

Elekes's Proof (end)

- We obtained the two bounds:

$$I(P,L) = n^3,$$

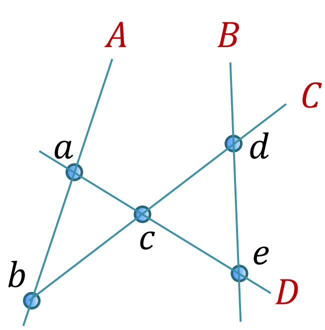
$$I(P,L) = O\left(|A+A|^{2/3}|AA|^{2/3}n^{4/3}\right).$$

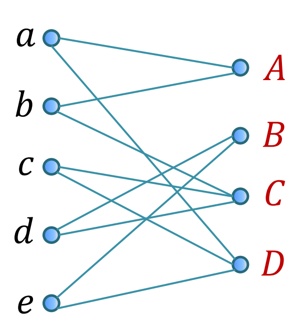
- Combining the two implies

$$|A+A||AA| = \Omega\left(n^{5/2}\right).$$

The Incidence Graph

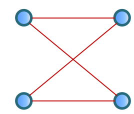
- A bipartite graph with a vertex for every point and for any "object".
- Every incidence yields an edge between the corresponding point and "object".

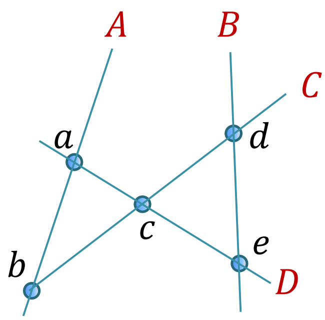


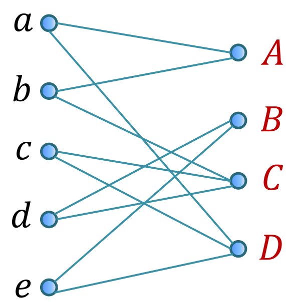


The Incidence Graph: Lines

- Two lines intersect in at most one point.
  - The incidence graph has no copy of $K_{2,2}$.







The Incidence Graph: Circles

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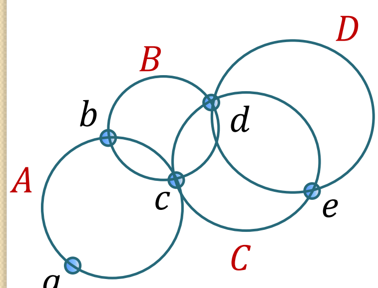
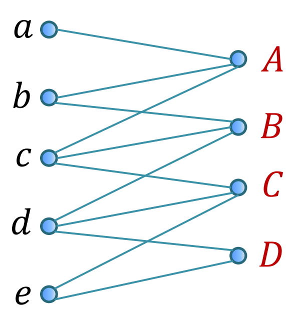




Incidence for Algebraic Curves

- **Pach and Sharir `92.**
  - P – set of m points in $\mathbb{R}^2$.
  - Γ – set of n constant-degree polynomial curves.
  - No $K_{s,t}$ in the incidence graph.

$$I(P,\Gamma) = O\left(m^{s/(2s-1)}n^{(2s-2)/(2s-1)} + m + n\right)$$

János Pach   *Micha Sharir*

The Case of $\mathbb{R}^3$

- **Zahl `13.**
  - P – set of m points in $\mathbb{R}^3$.
  - S – set of n constant-degree polynomial *surfaces* in $\mathbb{R}^3$.
  - No $K_{s,t}$ in the incidence graph.
  - Every three surfaces have a zero-dimensional intersection.

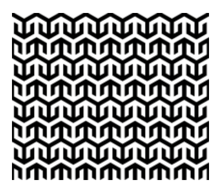
$$I(P,S) = O\left(m^{2s/(3s-1)}n^{(3s-3)/(3s-1)} + m + n\right)$$

The Case of $\mathbb{R}^4$


- **Basu and Sombra**.
  - P – set of m points in $\mathbb{R}^4$.
  - S – set of n constant-degree polynomial *hyper-surfaces* in $\mathbb{R}^4$.
  - No $K_{s,t}$ in the incidence graph.
  - Every four surfaces have a zero-dimensional intersection.

$$I(P,S) = O\left(m^{3s/(4s-1)}n^{(4s-4)/(4s-1)} + m + n\right)$$

Find the Pattern



- $\mathbb{R}^2$:

$$I(P,S) = O\left(m^{s/(2s-1)}n^{(2s-2)/(2s-1)} + m + n\right)$$

- $\mathbb{R}^3$:

$$I(P,S) = O\left(m^{2s/(3s-1)}n^{(3s-3)/(3s-1)} + m + n\right)$$

- $\mathbb{R}^4$:

$$I(P,S) = O\left(m^{3s/(4s-1)}n^{(4s-4)/(4s-1)} + m + n\right)$$

General Result

Fox, Pach, Suk, S', and Zahl:

- P – set of m points in $\mathbb{R}^d$.
- V – set of n constant-degree *varieties* in $\mathbb{R}^d$.
- No $K_{s,t}$ in the incidence graph.
- Any $\varepsilon > 0$.

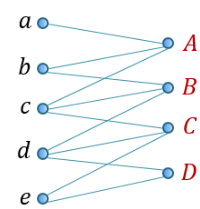
$$I(P,V) = O\left(m^{(d-1)s/(ds-1)+\varepsilon} n^{d(s-1)/(ds-1)} + m + n\right)$$

Lower Bounds

- **Theorem (S' 16).**
  - Matching lower bounds for up to an extra ε in the exponent for hypersurfaces in $\mathbb{R}^d$, where $d \geq 4$.
  - Works for many types of varieties but tight only for no $K_{2,t}$.
- Almost the first time that an incidence problem is nearly settled.
- Proof combines Fourier transform, basic number theory, and probability.

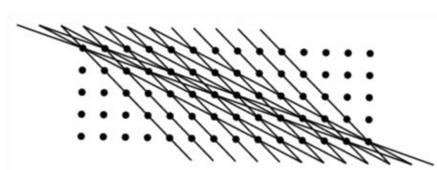
Szemerédi-Trotter: Proof Sketch

- Consider m points and n lines in $\mathbb{R}^2$.
  - The incidence graph contains no $K_{2,2}$.
  - A bipartite graph with vertex sets of size m and n and no $K_{2,2}$ contains $O(m\sqrt{n}+n)$ edges.
  - So $O(m\sqrt{n}+n)$ incidences.
  - Worse than the Szemerédi-Trotter $O\left(m^{2/3}n^{2/3}+m+n\right)$.
  -



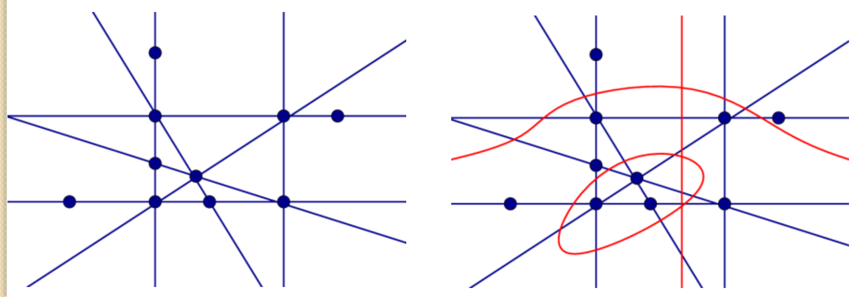
The Polynomial Method

- The polynomial method: Collections of objects that exhibit extremal behavior often have hidden algebraic structure.
  - Once this algebraic structure has been found, it can be exploited to gain a better understanding of the original problem.



Polynomial Partitioning

- P – a set of m points in $\mathbb{R}^d$.
- A polynomial $f \in \mathbb{R}[x, y]$ is an *r-partitioning polynomial* for P if no connected component of $\mathbb{R}^d \setminus \boldsymbol{Z}(f)$ contains more than m/r^d points of P.



Polynomial Partitioning Theorem

- **Theorem (Guth and Katz `10).** For every $r > 1$ and every set of points in $\mathbb{R}^d$, there exists an r-partitioning polynomial of degree $O(r)$.


Larry Guth

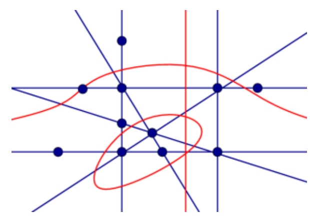

Nets Katz

Incidences in the Cells

- We apply the weak bound $O(m\sqrt{n}+n)$ separately in each cell:

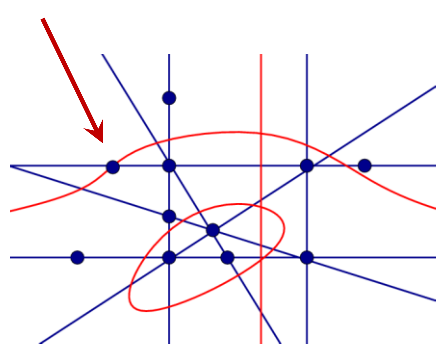
$$\sum_j O\left(m_j\sqrt{n_j}+n\right) = O\left(\frac{m}{r^2}\sum_j \sqrt{n_j} + \sum_j n_i\right).$$

- By setting $r = m^{2/3}/n^{1/3}$, we obtain $O(m^{2/3}n^{2/3}+m+n)$



Still not done…

- What is still missing in the proof?

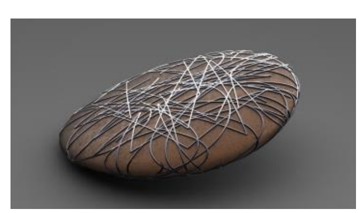


Recap: Incidences via Partitioning

- Obtain a weaker incidence bound.
  - Using a standard combinatorial trick.
- Partition the space into cells.
  - Using polynomial partitioning.
- "Amplify" the weaker bound by applying it in every cell.
- Bound the number of incidences on the partition itself.

A Problem

- When using polynomial partitioning in $\mathbb{R}^d$ with $d \geq 3$, how do we handle incidences on the partition?
  - Already in $\mathbb{R}^3$ we might get a complicated surface with many curves fully contained in it.



The Plan

- S_1 – our partition in $\mathbb{R}^d$.
- Still need to deal with incidences on the $(d-1)$-dimensional variety S_1.
- S_2 – a second partition.
  - r-partitioning of the points of $P \cap S_1$ but does not fully contain any components of S_1.

The Plan

- S_1 – our partition in $\mathbb{R}^d$.
- S_2 – a second partition.
  - r-partitioning of the points of $P \cap S_1$ but does not fully contain any components of S_1.
- Still need to deal with incidences on the $(d-2)$-dimensional variety $S_1 \cap S_2$.

The Plan

- S_1 – our partition in $\mathbb{R}^d$.
- S_2 – a second partition.
  - Partitions the points of $P \cap S_1$ but does not fully contain S_1.
- S_3 – a third partition.
  - r-partitioning of the points of $P \cap S_1 \cap S_2$ but does not fully contain any components of $S_1 \cap S_2$.
- …

Multiple Partitions

- After j partitionings, it remains to deal with points on a $(d-j)$-dimensional variety.

Multiple Partitions

- After j partitionings, it remains to deal with points on a $(d-j)$-dimensional variety.
- **Problem.** Given an irreducible $(d-j)$-dimensional variety V_j, find a polynomial f_{j+1} so that:
  - ? f_{j+1} is an r-partitioning for $P \cap V_j$.
  - ? f_{j+1} does not vanish identically on V_j.
  - ? The degree of f_{j+1} is not too large.

Polynomial Partitioning Theorem

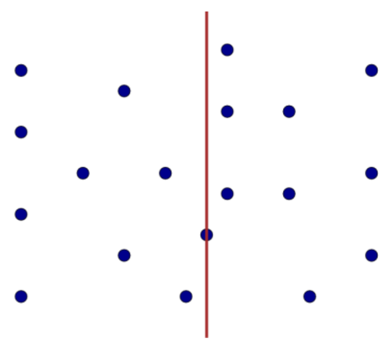
- **Theorem (Guth and Katz `10).** For every $r > 1$ and every set of points in $\mathbb{R}^d$, there exists an r-partitioning polynomial of degree $O(r)$.

Larry Guth

Nets Katz

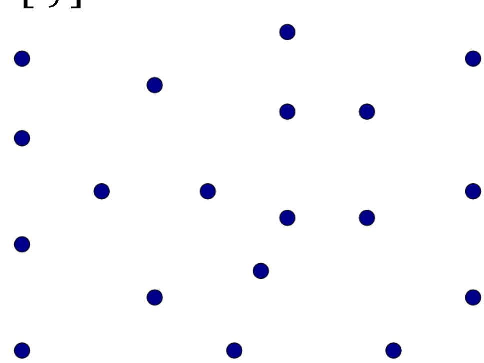
Bisecting Hyperplanes

- A hyperplane h ***bisects*** a finite set A if each of the open half-spaces defined by h contains at most $\lfloor |A|/2 \rfloor$ points of A.



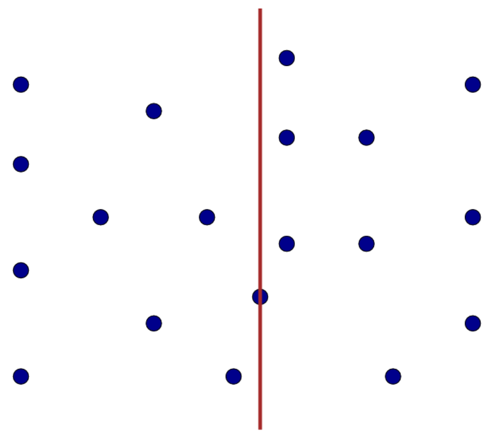
Finding a Polynomial Partition

- $m = 19$ points and $r = 3$.
- **Goal.** Every cell should contain at most $\left\lfloor \frac{m}{r^2} \right\rfloor = \left\lfloor \frac{19}{9} \right\rfloor = 2$ points.



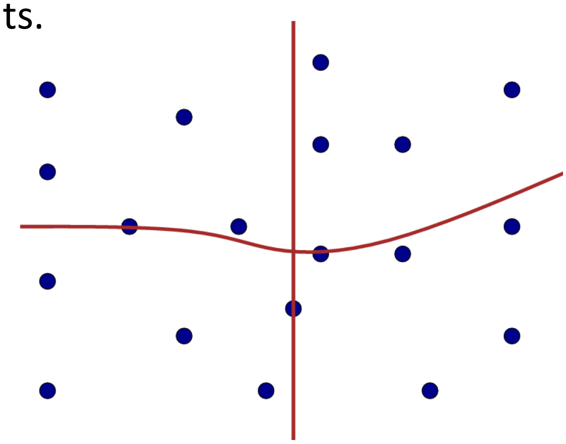
Finding a Polynomial Partition

- **Step 1.** Bisect the set into two sets, each with at most $\left\lfloor \frac{19}{2} \right\rfloor = 9$ points.



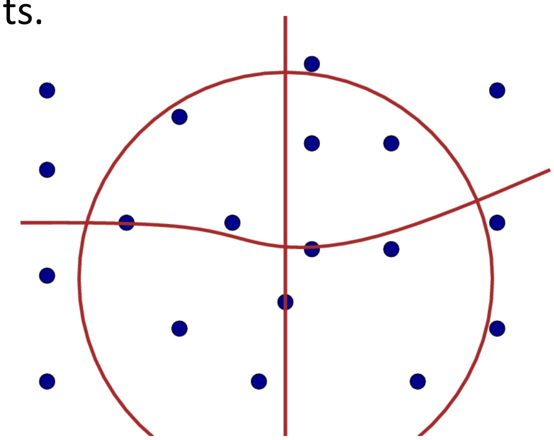
Finding a Polynomial Partition

- **Step 2.** Bisect each of the two sets into two subsets, each with at most $\left\lfloor \frac{19}{4} \right\rfloor = 4$ points.



Finding a Polynomial Partition

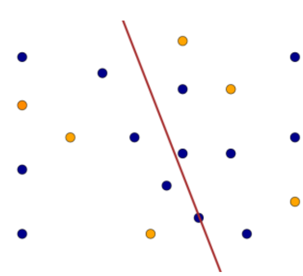
- **Step 3.** Bisect each of the four sets into two subsets, each with at most $\left\lfloor \frac{19}{8} \right\rfloor = 2$ points.



Discrete Ham Sandwich Theorem

- **Theorem.** Any d finite sets in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane.

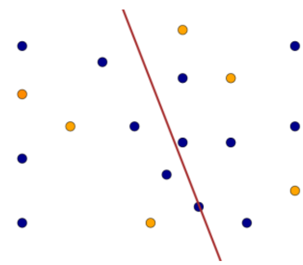
(Proved by using the *Borsuk–Ulam theorem*).



Using Discrete Ham Sandwich

- In $\mathbb{R}^d$, we can perform $\sim \log_2 d$ partitioning steps by using the discrete ham sandwich theorem.

- **Then what?**



Proof Outline

$\boxed{\text{Point sets } P_1, \ldots P_k \subset \mathbb{R}^d.}$

$\downarrow$

The Veronese Map

- *Veronese map* $\nu_D : \mathbb{R}^d \to \mathbb{R}^m$ is defined as

$$\nu_D(x_1, \ldots, x_d) = \left(x_1^{u_1} x_2^{u_2} \cdots x_d^{u_d}\right)_{u \in U_D}$$

where

$$U_D = \{(i_1, \ldots, i_d) \mid 1 \le i_1 + \cdots + i_d \le D\}.$$

- Consider the map $\nu_2 : \mathbb{R}^2 \to \mathbb{R}^5$:

$$\nu_2(x_1, x_2) = (x_1^2, x_2^2, x_1 x_2, x_1, x_2).$$

Veronese Map + Ham Sandwich

- If we need to bisect k sets, we choose D such that the number m_D of monomials of degree $\le D$ is at least k.
  - Every point set P_i is mapped to a point set P_i' in $\mathbb{R}^{m_D}$.
  - **Ham sandwich theorem:** there exists a hyperplane $h \subset \mathbb{R}^{m_D}$ that bisects each P_i'.

+

Proof Outline

Point sets $P_1, \ldots P_k \subset \mathbb{R}^d$.

$\downarrow$ $\nu_D : \mathbb{R}^d \to \mathbb{R}^{m_D}$

Point sets $P_1', \ldots, P_k' \subset \mathbb{R}^{m_D}$.

$\downarrow$ Ham sandwich in $\mathbb{R}^{m_D}$

Hyperplane h bisects $P_1', \ldots, P_k' \subset \mathbb{R}^{m_D}$.

Proof Outline

Point sets $P_1, \ldots P_k \subset \mathbb{R}^d$.

$\downarrow$ $\nu_D : \mathbb{R}^d \to \mathbb{R}^{m_D}$

Point sets $P_1', \ldots, P_k' \subset \mathbb{R}^{m_D}$.

$\downarrow$ Ham sandwich in $\mathbb{R}^{m_D}$

Hyperplane h bisects $P_1', \ldots, P_k' \subset \mathbb{R}^{m_D}$.

$\downarrow$ $\nu_D^{-1} : \mathbb{R}^{m_D} \to \mathbb{R}^d$

A variety $Z(f)$ of degree $\leq D$ in $\mathbb{R}^d$.

Proof Outline

Point sets $P_1, \ldots P_k \subset \mathbb{R}^d$.

$\downarrow$ $\nu_D : \mathbb{R}^d \to \mathbb{R}^{m_D}$

Point sets $P'_1, \ldots, P'_k \subset \mathbb{R}^{m_D}$.

$\downarrow$ Ham sandwich in $\mathbb{R}^{m_D}$

Hyperplane h bisects $P'_1, \ldots, P'_k \subset \mathbb{R}^{m_D}$.

$\downarrow$ $\nu_D^{-1} : \mathbb{R}^{m_D} \to \mathbb{R}^d$

The variety $Z(f)$ of degree $\leq D$ bisects $P_1, \ldots P_k \subset \mathbb{R}^d$.

Recall: Multiple Partitions

- **Problem.** Given an irreducible $(d-j)$-dimensional variety V_j in $\mathbb{R}^d$, find a polynomial f_{j+1} so that:
  - ? f_{j+1} is an r-partitioning for $P \cap V_j$.
  - ? f_{j+1} does not vanish identically on V_j.
  - ? The degree of f_{j+1} is not too large.

The Quotient Ring

- $\mathbb{R}[x_1, \dots, x_d]_{\leq D}$ – the set of polynomials in $x_1, \dots, x_d$ of degree $\leq D$.
- $I = \boldsymbol{I}(V_j)$ – the ideal of polynomials that vanish on V_j.
- $I_{\leq D}$ – the set of polynomials in I of degree $\leq D$.

$$R = \mathbb{R}[x_1, \dots, x_d]_{\leq D} / I_{\leq D}$$

- We consider only polynomials in R.

What We Already Know

- **Problem.** Given an irreducible $(d-j)$-dimensional variety V_j, find a polynomial f_{j+1} so that:
  - ? f_{j+1} is an r-partitioning for $P \cap V_j$.
  - ✓ f_{j+1} does not vanish identically on V_j.
  - ? The degree of f_{j+1} is not too large.

Quotient Ring + “Veronese” Map

$$R = \mathbb{R}[x_1, \dots, x_d]_{\leq D} / I_{\leq D}$$

- R is a vector space of dimension m_D.
- To bisect $P_1, \dots, P_k \subset V_i$:
  - Choose D such that $m_D \geq k$.
  - $b_1, \dots, b_{m_D}$ – a basis for R.
- Map $\nu_D^R \colon \mathbb{R}^d \to \mathbb{R}^{m_D}$ is defined as

$$\nu_D^R(x_1, \dots, x_d) = \left(b_1(x), \dots, b_{m_D}(x)\right)$$

Proof Outline

Point sets $P_1, \dots P_k \subset V_j \subset \mathbb{R}^d$.

$\downarrow$ $\nu_D^R \colon \mathbb{R}^d \to \mathbb{R}^{m_D}$

Point sets $P_1', \dots P_k' \subset \mathbb{R}^{m_D}$.

$\downarrow$ Ham sandwich in $\mathbb{R}^{m_D}$

Hyperplane h bisects $P_1', \dots P_k' \subset \mathbb{R}^{m_D}$.

$\downarrow$ $(\nu_D^R)^{-1} \colon \mathbb{R}^{m_D} \to \mathbb{R}^d$

The variety $Z(f)$ of degree $\leq D$ bisects $P_1, \dots P_k \subset V$.

What We Already Know

- **Problem.** Given an irreducible $(d-j)$-dimensional variety V_j, find a polynomial f_{j+1} so that:
  - ✓ f_{j+1} is an r-partitioning for $P \cap V_j$.
  - ✓ f_{j+1} does not vanish identically on V_j.
  - ? The degree of f_{j+1} is not too large.

The Hilbert Function

$$\boldsymbol{Z}(x^{25}y^{12} + 5x^{19}y^8 + 3.5x^{18}y^{11} + 39x^{11}y + x^9y^{20} + 3x^5y^{26}$$

The Hilbert Function (really!)

- An ideal $I = \boldsymbol{I}(V_j) \subset \mathbb{R}[x_1, \ldots x_d]$.
- Hilbert function of ideal I:

$$h_I(D) = \dim(\mathbb{R}[x_1, \ldots x_d]_{\leq D} / I_{\leq D})$$

- That is: $\boldsymbol{m_D = h_I(D)}$!
-

And That's It!

- **Problem.** Given an irreducible $(d-j)$-dimensional variety V_j, find a polynomial f_{j+1} so that:
  - ✓ f_{j+1} is an r-partitioning for $P \cap V_j$.
  - ✓ f_{j+1} does not vanish identically on V_j.
  - ✓ The degree of f_{j+1} is not too large.

Incidences in $\mathbb{C}^2$

- **Solymosi and Tao `12.** The number of incidences between m points and n lines in $\mathbb{C}^2$ is $O\left(m^{2/3+\varepsilon}n^{2/3}+m+n\right)$ for every $\varepsilon > 0$.
  - Holds for other types of curves, but under very strict restrictions.

Jozsef Solymosi

Terence Tao

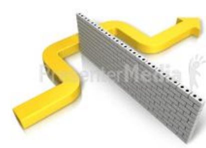
Incidences in $\mathbb{C}^2$

- **S', Szabo, and Zahl 16.**
  - P – set of m points in $\mathbb{C}^2$.
  - Γ – set of n constant-degree polynomial curves.
  - No $K_{s,t}$ in the incidence graph.
  - Any $\varepsilon > 0$.

$$I(P,\Gamma) = O\left(m^{s/(2s-1)+\varepsilon}n^{(2s-2)/(2s-1)}+m+n\right)$$

Incidences in $\mathbb{C}^2$

- In $\mathbb{C}^2$ this strategy fails.
  - The zero set of a polynomial does not divide $\mathbb{C}^2$ into connected components.
- Think of $\mathbb{C}^2$ as $\mathbb{R}^4$.



The Problem

- We are in $\mathbb{R}^4$.
  - The partition is a 3-dim variety V.
  - We need to handle the incidences between points and 2-dim varieties inside of V.

The Cauchy-Riemann Equations

- Consider the complex coordinates $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
- For $f \in \mathbb{C}[z_1, z_2]$ write $f = u + iv$ where $u, v \in \mathbb{R}[x_1, y_1, x_2, y_2]$.
- u and v satisfy the Cauchy-Riemann equations if

$$\frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k}, \qquad \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}, \qquad k \in \{1, 2\}.$$

The Problem

- We are in $\mathbb{R}^4$.
  - The partition is a 3-dim variety V.
  - We need to handle the incidences between points and 2-dim varieties inside of V.
- By the Cauchy-Riemann equations:
  - For a generic point $p \in V$, there is a 2-dim plane Π such that every 2-dim variety that is incident to p has Π as its tangent plane at p.

Completing the Proof Sketch

- We are in $\mathbb{R}^4$.
  - The partition is a 3-dim variety V.
  - For a generic point $p \in V$, there is a 2-dim plane Π associated with it.
- Finding a 2-dim variety in V that is incident to p is an initial value problem.
-

A Foliation

- Intuitively, the relevant parts of the partition are foliated: