The global nonlinear stability of Minkowski spacetime for self-gravitating massive fields

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AIM of this TALK

- **Global geometry of spacetimes**
- **Einstein equations** for self-gravitating matter
- **The $f(R)$-gravity theory**
- **Global nonlinear stability of Minkowski spacetime**
  - initial data prescribed on a spacelike hypersurface
  - small perturbation of an asymptotically flat slice in Minkowski space
- **Partial differential equations**
  - nonlinear wave equations
  - global-in-time solutions in weighted Sobolev spaces

CHALLENGES

- **Gravitational waves**
  - waves propagating in a curved space
  - Weyl curvature (vacuum), Ricci curvature (matter)
- **Nonlinear wave interactions**
  - exclude dynamical instabilities, self-gravitating massive modes
  - avoid gravitational collapse (trapped surfaces, black holes)
- **Global dynamics**
  - (small) perturbations disperse in timelike directions
  - asymptotic convergence to Minkowski spacetime
  - future timelike geodesically complete spacetime

OUTLINE

1. Einstein gravity versus f(R)-gravity
2. The wave-Klein-Gordon formulation
3. The global nonlinear stability
4. Stability theorems in wave coordinates

1. EINSTEIN GRAVITY versus F(R)-GRAVITY

- Lorentzian manifolds $(M, g_{\alpha\beta})$ with signature $(-,+,+,+)$
- In (local) coordinates $g = g_{\alpha\beta} dx^\alpha dx^\beta$
- Minkowski spacetime $M = \mathbb{R}^{3+1}$ and $g_M = -(dx^0)^2 + \sum_{a=1}^3 (dx^a)^2$
- Covariant derivative $\nabla_\alpha X = \partial_\alpha X + \Gamma \star X$ with $\Gamma \simeq \partial g$
- Ricci curvature $R_{\alpha\beta} = \partial^2 g + \partial g \star \partial g$
- scalar curvature $R := R_\alpha^\alpha = g^{\alpha\beta} R_{\alpha\beta}$

$\alpha, \beta = 0, 1, 2, 3$ (expressions in coordinates below)

Einstein equations for self-gravitating matter

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

- Einstein curvature $G_{\alpha\beta} = R_{\alpha\beta} - (R/2) g_{\alpha\beta}$
- $T_{\alpha\beta}$: energy-momentum of the matter field

Self-gravitating massive fields

Massive scalar field with potential $U(\phi)$, for instance $U(\phi) = \frac{c^2}{2}\phi^2$, described by the energy-momentum tensor

$$T_{\alpha\beta} := \nabla_\alpha\phi\nabla_\beta\phi - \Big(\frac{1}{2}g^{\alpha'\beta'}\nabla_{\alpha'}\phi\nabla_{\beta'}\phi + U(\phi)\Big)g_{\alpha\beta}$$

Einstein-Klein-Gordon system for the unknown $(M, g_{\alpha\beta}, \phi)$

$$\begin{aligned} R_{\alpha\beta} - 8\pi\Big(\nabla_\alpha\phi\nabla_\beta\phi + U(\phi)\, g_{\alpha\beta}\Big) &= 0 \\ \Box_g\phi - U'(\phi) &= 0 \end{aligned}$$

with $\Box_g = \nabla_\alpha\nabla^\alpha$

(geometric PDE's, gauge invariance)

Field equations of the $f(R)$-modified gravity

Generalized Hilbert-Einstein functional

- action functional $\int_M \left(f(R) + 16\pi L[\phi, g] \right) dV_g$
- $f(R) = R + \frac{\kappa}{2} R^2 + \kappa^2 \mathcal{O}(R^3)$ with $\kappa > 0$
- long history in physics: Weyl 1918, Pauli 1919, Eddington 1924, ...
- Local-in-time Cauchy developments

Alternative theories of gravity are relevant

- new observational data
  - accelerated expansion of the Universe
  - instabilities observed in galaxies
- gravitation mediated by additional fields
- without explicitly introducing notions of 'dark matter'

Critical point equation $N_{\alpha\beta} = 8\pi T_{\alpha\beta}$

$$N_{\alpha\beta} = f'(R)\, G_{\alpha\beta} - \frac{1}{2}\Big(f(R) - Rf'(R)\Big)g_{\alpha\beta} + \Big(g_{\alpha\beta}\Box_g - \nabla_\alpha\nabla_\beta\Big)\big(f'(R)\big)$$

- *Fourth-order* derivatives of g
- If f linear, $N_{\alpha\beta}$ reduces to $G_{\alpha\beta}$.
- Vacuum Einstein solutions are vacuum f(R)-solutions

Gravity/matter coupling

Energy-momentum tensor of a massive field (Jordan's coupling)

$$T_{\alpha\beta} := \nabla_\alpha\phi\nabla_\beta\phi - \Big(\frac{1}{2}\nabla_\gamma\phi\nabla^\gamma\phi + U(\phi)\Big)g_{\alpha\beta}$$

Bianchi identities (geometry) $\nabla^\alpha R_{\alpha\beta} = \frac{1}{2}\nabla_\beta R$

- imply $\nabla^\alpha G_{\alpha\beta} = 0$, but also $\nabla^\alpha N_{\alpha\beta} = 0$.
- Euler equations $\nabla^\alpha T_{\alpha\beta} = 0$
- wave equation $\Box_g\phi - U'(\phi) = 0$

Numerical evidence

Stability of asymptotically flat matter spacetimes

- Family of "oscillating soliton stars"
  - suggests a possible instability mechanism for small perturbations of massive fields
- Yet massive fields were finally conjectured to be stable in asymptotically flat spacetimes:
  - Initial phase: the matter *tends to collapse.*
  - Intermediate phase: below a certain threshold in the mass density, the *collapse slows down.*
  - Final phase: *dispersion is the main feature* of the evolution.

(H. Okawa, V. Cardoso, and P. Pani, 2014)

Asymptotically anti-Sitter (AdS) spacetimes

- such instabilities are observed ! the effect of gravity is dominant
- generic (even arbitrarily small) initial data lead to black hole formation
- in AdS spacetime, matter is confined and cannot disperse: the timelike boundary is reached in finite proper time

2. THE WAVE-KLEIN-GORDON FORMULATION

Field equations in coordinates

Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

- Second-order system with no specific PDE type
- *Wave gauge* $\Box_g x^\gamma = 0$

$$2\, g^{\alpha\beta}\, \partial_\beta g_{\alpha\gamma} - g^{\alpha\beta}\, \partial_\gamma g_{\alpha\beta} = 0, \qquad \gamma = 0, \ldots, 3$$

- $R_{\alpha\beta} \simeq \Box_g g_{\alpha\beta}$ Einstein,... Choquet-Bruhat,..., De Turck's trick for the Ricci flow
- Second-order system of 11 *nonlinear wave-Klein-Gordon equations*
- Hamiltonian-momentum Einstein's contraints

Einstein system for a self-gravitating massive field

$$\tilde{\Box}_g g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g) - 8\pi \left(2\partial_\alpha \phi \partial_\beta \phi + c^2 \phi^2\, g_{\alpha\beta}\right)$$

$$\tilde{\Box}_g \phi - c^2 \phi = 0$$

(see next slide)

- Quadratic nonlinearities $F_{\alpha\beta}(g, \partial g)$?
- Null terms: $g^{\alpha\beta} \partial_\alpha u \partial_\beta u$ and $\partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v$

Ricci curvature in wave gauge

$$R_{\alpha\beta} = \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta}\Gamma^\delta_{\lambda\delta} - \Gamma^\lambda_{\alpha\delta}\Gamma^\delta_{\beta\lambda}$$

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\dagger\,\lambda\lambda'} \left(\partial_\alpha g^\dagger{}_{\beta\lambda'} + \partial_\beta g^\dagger{}_{\alpha\lambda'} - \partial_{\lambda'} g^\dagger{}_{\alpha\beta}\right)$$

$$\begin{aligned} 2\,R_{\alpha\beta} &= -\widetilde{\Box}_g g_{\alpha\beta} + Q_{\alpha\beta} + P_{\alpha\beta} \\ &= -g^{\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'} g_{\alpha\beta} + Q_{\alpha\beta} + P_{\alpha\beta} \end{aligned}$$

(i) terms satisfying Klainerman's null condition (good decay in time)

$$\begin{aligned} Q_{\alpha\beta} :&= g^{\lambda\lambda'} g^{\delta\delta'} \partial_\delta g_{\alpha\lambda'} \partial_{\delta'} g_{\beta\lambda} \\ &\quad - g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\delta g_{\alpha\lambda'} \partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\beta\delta'} \partial_\lambda g_{\alpha\lambda'}\right) \\ &\quad + g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta} \partial_\delta g_{\lambda'\delta'}\right) + \dots\dots\dots\dots \end{aligned}$$

(ii) "quasi-null terms" (need again the gauge conditions)

$$P_{\alpha\beta} := -\tfrac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\delta\lambda'} \partial_\beta g_{\lambda\delta'} + \tfrac{1}{4} g^{\delta\delta'} g^{\lambda\lambda'} \partial_\beta g_{\delta\delta'} \partial_\alpha g_{\lambda\lambda'}$$

Remark: Weak null condition by Lindblad-Rodnianski (2010)

Modified gravity equations

$$N_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

- Theory based on $f(R) = R + \frac{\kappa}{2}R^2 + \dots$
- Fourth-order system with no specific PDE type
- The augmented formulation in $({g^\dagger}_{\alpha\beta}, \rho)$
  - conformal transformation $\quad {g^\dagger}_{\alpha\beta} := f'(R_g) g_{\alpha\beta}$
    - set $\rho := \frac{1}{\kappa} \ln f'(R_g)$
    - relation between the Ricci curvature tensors of g and $g^\dagger$:

      $$ {R^\dagger}_{\alpha\beta} = R_{\alpha\beta} - 2\big(\nabla_\alpha\nabla_\beta\rho - \nabla_\alpha\rho\nabla_\beta\rho\big) - \big(\Box_g\rho + 2g(\nabla\rho,\nabla\rho)\big)g_{\alpha\beta}$$

    - leads to a third-order system
  - evolution equation for the scalar curvature taken as an *independent variable* (new degree of freedom)
- *Wave coordinates* $\Box_{g^\dagger} x^\alpha = 0$
- Second-order system of 12 *nonlinear wave-Klein-Gordon equations*
- More involved algebraic structure, and additional constraints

f(R)-gravity for a self-gravitating massive field

$$\tilde{\Box}_{g^\dagger} {g^\dagger}_{\alpha\beta} = F_{\alpha\beta}(g^\dagger, \partial g^\dagger) - 8\pi \left(2e^{-\kappa\rho}\partial_\alpha\phi\partial_\beta\phi + c^2\phi^2 e^{-2\kappa\rho}\, {g^\dagger}_{\alpha\beta}\right) - 3\kappa^2\partial_\alpha\rho\partial_\beta\rho + \kappa\,\mathcal{O}(\rho^2){g^\dagger}_{\alpha\beta}$$

$$\tilde{\Box}_{g^\dagger}\phi - c^2\phi = c^2\left(e^{-\kappa\rho} - 1\right)\phi + \kappa {g^\dagger}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\rho$$

$$3\kappa\,\tilde{\Box}_{g^\dagger}\rho - \rho = \kappa\,\mathcal{O}(\rho^2) - 8\pi\left({g^\dagger}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{c^2}{2}\,e^{-\kappa\rho}\phi^2\right)$$

- wave gauge conditions ${g^\dagger}^{\alpha\beta}{\Gamma^\dagger}^\lambda_{\alpha\beta} = 0$
- curvature compatibility $e^{\kappa\rho} = f'(R_{e^{-\kappa\rho}g^\dagger})$
- Hamiltonian and momentum constraints

(propagating from a Cauchy hypersurface)

In the limit $\kappa \to 0$ one has $g^\dagger \to g$ and $\rho \to 8\pi\left(g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + \frac{c^2}{2}\phi^2\right)$

Einstein system for a self-gravitating massive field

$$\tilde{\Box}_g g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g) - 8\pi\left(2\partial_\alpha\phi\partial_\beta\phi + c^2\phi^2\, g_{\alpha\beta}\right)$$

$$\tilde{\Box}_g\phi - c^2\phi = 0$$

- This completes the formulation of the field equations in a PDE form
- Global existence problem for coupled nonlinear wave equations

Main challenges

- Not invariant by scaling
  - must rely on fewer symmetries
  - for instance in defining energy-like functionals
- Coupling wave equations and Klein-Gordon equations
  - drastically different time asymptotic behavior
  - $O(t^{-1})$ for wave equations and $O(t^{-3/2})$ for Klein-Gordon equations
- Dependence in f and *singular* limit $f(R) \to R$

Global existence theory

- Sufficient decay of interactions in time ?
- Nonlinear coupling between the geometry and massive matter
- Leads to strong interactions at the PDE level
  - (almost) sharp L^2 time-decay for the metric and matter field
  - (almost) sharp L^∞ time-decay for the metric and matter field
- Quasi-null structure of the Einstein equations
  - the null condition is violated
  - no amplification/feedback phenomena

3. THE GLOBAL NONLINEAR STABILITY

Initial value problem for the Einstein equations

- Initial data set
  - geometry of the initial hypersurface $(M_0 \simeq \mathbb{R}^3, g_0, k_0)$
  - matter fields ϕ_0, ϕ_1
  - initial data sets "close to Minkowski"
- Local existence theory by Choquet-Bruhat 1954
  - globally hyperbolic Cauchy developments
    (spacetime determined by the prescribed initial data)
  - existence of *maximal* developments

Works on vacuum spacetimes or massless matter

- Christodoulou-Klainerman 1993
  - fully geometric proof, Bianchi identities, geometry of null cones
  - null foliation, maximal foliation, *all Killing fields* of Minkowski
  - extensions to massless models (same asymptotics and Killing fields)
- Lindblad-Rodnianski 2010
  - first global existence result in coordinates
  - wave coordinates (despite an "instability" result by Choquet-Bruhat)
  - asymptotically flat foliation, *all Killing fields* of Minkowski

Recent work on the dynamics of self-gravitating massive matter

LeFloch-Ma 2015

- does not rely on Minkowski's scaling field $r\partial_r + t\partial_t$
- asymptotically hyperbolic foliation
- the Hyperboloidal Foliation Method (LeFloch-Ma, 2014): key of be able to tackle massive matter fields
- also provides a simpler proof for the case of massless fields

Decay conditions

- Positive mass theorem
  - no solution can be exactly Minkowski "at infinity"
  - exactly Schwarzschild outside a spatially compact region
  - more generally, approaching Schwarzschild near space infinity
- Initial slice: asymptotically Schwarzschild data with ADM mass m

$$g_{ab} = g_{S,ab} + O(r^{-1-\delta}) = \delta_{ab}\Big(1 + \frac{2m}{r}\Big) + O(r^{-1-\delta})$$
$$k_{ab} = k_{S,ab} + O(r^{-2-\delta}) = O(r^{-2-\delta})$$
$$\phi = O(r^{-1-\delta})$$

  - Compare the solution with an interpolation between Minkowki and Schwarzschild $g_{MS} = \big(1 + \chi(r/t)\chi(r)\big)g_M$ in which $\chi(r) = 0$ for $r \leqslant 1/2$ and $\chi(r) = 1$ for $r \geqslant 3/4$.

Theorem 1. Nonlinear stability of Minkowski spacetime with self-gravitating massive fields

Consider the Einstein-massive field system when the initial data set $(M_0 \simeq \mathbb{R}^3, g_0, k_0, \phi_0, \phi_1)$ is asymptotically Schwarzschild and sufficiently close to Minkowski data, and satisfies the Einstein constraint equations. Then, the initial value problem

- admits a globally hyperbolic Cauchy development,
- which is foliated by asymptotically hyperbolic hypersurfaces.
- Moreover, this spacetime is future causally geodesically complete and asymptotically approaches Minkowski spacetime.

Theorem 2. Nonlinear stability of Minkowski spacetime in f(R)-gravity

Consider the field equations of $f(R)$-modified gravity when the initial data set $(M_0 \simeq \mathbb{R}^3, g_0, k_0, R_0, R_1, \phi_0, \phi_1)$ is asymptotically Schwarzschild and sufficiently close to Minkowski data, and satisfies the constraint equations of modified gravity. Then, the initial value problem

- admits a globally hyperbolic Cauchy development,
- which is foliated by asymptotically hyperbolic hypersurfaces.
- Moreover, this spacetime is future causally geodesically complete and asymptotically approaches Minkowski spacetime.

Limit problem $\kappa \to 0$

- relaxation phenomena for the spacetime scalar curvature
- passing from the second-order wave equation

$$3\kappa \,\widetilde{\Box}_{g^\dagger} \rho - \rho = \kappa\, \mathcal{O}(\rho^2) - 8\pi \Big({g^\dagger}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{c^2}{2}\, e^{-\kappa\rho} \phi^2 \Big)$$

to the purely algebraic equation

$$\rho \to 8\pi \big(g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + \frac{c^2}{2} \phi^2 \big)$$

Theorem 3. f(R)-spacetimes converge toward Einstein spacetimes

The Cauchy developments of modified gravity in the limit $\kappa \to 0$

> when the nonlinear function $f = f(R)$ (the integrand in the Hilbert-Einstein action) approaches the scalar curvature function R

converge (in every bounded time interval, in a sense specified quantitatively in Sobolev norms) to Cauchy developments of Einstein's gravity theory.

4. STABILITY THEOREMS in WAVE COORDINATES

Foliations by asymptotically hyperboloidal hypersurfaces

(capture well the behavior near the light cone)

For simplicity in this presentation, we focus on the interior of the light cone and, in fact, on the *future of a hyperboloid*

Foliation expressed in wave coordinates

- hyperboloids $\mathcal{H}_s := \big\{(t,x) \,/\, t > 0;\ t^2 - |x|^2 = s^2\big\}$ parametrized by their hyperbolic radius $s \geqslant 1$
- tangent vectors: boosts $L_a := x^a \partial_t + t\partial_a$ for $a = 1, 2, 3$
- data prescribed on $\mathcal{H}_{s_0}$ for some $s_0 > 1$

Frames

Hyperboloidal frame	$\overline{\partial}_0 := \partial_s$	$\overline{\partial}_a = \dfrac{L_a}{t}$
Change of frame formulas	$\overline{\partial}_\alpha = \overline{\Phi}_\alpha^{\alpha'} \partial_{\alpha'}$	$\partial_\alpha = \overline{\Psi}_\alpha^{\alpha'} \overline{\partial}_{\alpha'}$
Tensor components	$\underline{T}_{\alpha\beta} = T_{\alpha'\beta'} \underline{\Phi}_\alpha^{\alpha'} \underline{\Phi}_\beta^{\beta'}$	

Weighted norms associated with the hyperboloidal foliation

Motivation

- Energy norms based on the translations ∂_α and Lorentzian boosts L_a
- For instance $\Box_{g_M} \phi = f$ implies $\Box_{g_M} L_a \phi = L_a f$
- Good commutator properties on the curved space

The norms

- On each hypersurface (using the boosts)

$$\left(\|u\|_{\mathcal{H}^n[s]} \right)^2 := \sup_{a=1,2,3} \sum_{|J| \leqslant n} \int_{\mathcal{H}_s \simeq \mathbb{R}^3} |L_a^J u|^2 \, dx$$

- In spacetime (using the translations)

$$\|u\|_{\mathcal{H}^N[s_0,s_1]} := \sup_{s \in [s_0,s_1]} \sum_{|I|+n \leqslant N} \|\partial^I u\|_{\mathcal{H}^n[s]}$$

Global-in-time existence for the initial value problem

- Initial data prescribed on an asymptotically hyperbolic hypersurface, identified with $\mathcal{H}_{s_0}$ in our coordinates
- Energy balance laws between by two hyperboloids

Bootstrap argument

- total contribution of the interaction terms contributes only a finite amount to the growth of the total energy
- time-integrability of the source terms
- **sharp pointwise estimate** required to handle strong interaction terms
- Sobolev inequalities, Hardy inequalities adapted to the hyperboloidal foliation
- **hierarchy of energy bounds**
    - various order of differentiation / growth in s
    - successive improvements of the energy bounds
    - successive applications of sup-norm estimates

Theorem 1. Nonlinear stability of Minkowski spacetime for self-gravitating massive fields

Consider the Einstein-massive field system in wave coordinates. Given any sufficiently large integer N, there exist constants $\epsilon, \delta, C_0 > 0$ such that for any asymptotically hyperboloidal initial data set $(\mathbb{R}^3, \overline{g}_0, \overline{k}_0, \phi_0, \phi_1)$ satisfying Einstein's Hamiltonian and momentum constraints together with the smallness conditions

$$\|\overline{\partial}_c \left(\overline{g}_{0,ab} - \overline{g}_{MS,ab}\right)\|_{\mathcal{H}^N[1]} + \|\overline{k}_{0,ab} - \overline{k}_{MS,ab}\|_{\mathcal{H}^N[1]} \leqslant \epsilon$$

$$\|\overline{\partial}_a \phi_0, \phi_0, \phi_1\|_{\mathcal{H}^N[1]} \leqslant \epsilon$$

the solution to the Einstein equations exists for all times $s \geqslant 1$

$$\|\overline{\partial}_\gamma \left(g_{\alpha\beta} - g_{MS,\alpha\beta}\right)\|_{\mathcal{H}^N[1,s]} \leqslant C_0 \epsilon s^\delta$$

$$\|\overline{\partial}_\alpha \phi, \phi\|_{\mathcal{H}^N[1,s]} \leqslant C_0 \epsilon s^{\delta+1/2} \qquad \text{(high-order energy)}$$

$$\|\overline{\partial}_\alpha \phi, \phi\|_{\mathcal{H}^{N-4}[1,s]} \leqslant C_0 \epsilon s^\delta \qquad \text{(low-order energy)}$$

Observations

- sufficient decay so that the spacetime is future geodesically complete
- smallness conditions on both g, ϕ necessary (gravitational collapse)
- similar theorem for the theory of modified gravity

Notation: $\sigma := 8\pi\left(g^{\dagger\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{c^2}{2}\, e^{-\kappa\rho}\phi^2\right)$

Theorem 3. The singular limit problem for the modified gravity equations

Consider a sequence of initial data sets depending upon $\kappa \to 0$, as follows:

$$\|\overline{\partial}_c\left(\overline{g}^\dagger_{0,ab} - \overline{g}_{MS,ab}\right)\|_{\mathcal{H}^N[1]} + \|\overline{k}^\dagger_{0,ab} - \overline{k}_{MS,ab}\|_{\mathcal{H}^N[1]} \leqslant \epsilon$$

$$\|\kappa^{1/2}\,\rho_1, \kappa^{1/2}\,\overline{\partial}_a\rho_0, \rho_0\|_{\mathcal{H}^N[1]} + \|\overline{\partial}_a\phi_0, \phi_0, \phi_1\|_{\mathcal{H}^N[1]} \leqslant \epsilon$$

$$\|\rho_1 - \sigma_1, \overline{\partial}_a(\rho_0 - \sigma_0), \kappa^{-1/2}\,(\rho_0 - \sigma_0)\|_{\mathcal{H}^{N-2}[1]} \leqslant \epsilon.$$

Then, the solutions exist for all times $s \geqslant 1$ and all $\kappa \to 0$, with a constant C_0 **independent** of κ

$$\|\overline{\partial}_\gamma\left(g^\dagger_{\alpha\beta} - g_{MS,\alpha\beta}\right)\|_{\mathcal{H}^N[1,s]} \leqslant C_0\epsilon s^\delta$$

$$\|\kappa^{1/2}\overline{\partial}_\alpha\rho, \rho\|_{\mathcal{H}^N[1,s]} + \|\overline{\partial}_\alpha\phi, \phi\|_{\mathcal{H}^N[1,s]} \leqslant C_0\epsilon s^{\delta+1/2}$$

$$\|\kappa^{1/2}\overline{\partial}_\alpha\rho, \rho\|_{\mathcal{H}^{N-4}[1,s]} + \|\overline{\partial}_\alpha\phi, \phi\|_{\mathcal{H}^{N-4}[1,s]} \leqslant C_0\epsilon s^\delta$$

$$\|\overline{\partial}_\alpha(\rho - \sigma), \kappa^{-1/2}(\rho - \sigma)\|_{\mathcal{H}^{N-2}[1,s]} \leqslant C_0\epsilon s^{\delta+1/2}$$

$$\|\overline{\partial}_\alpha(\rho - \sigma), \kappa^{-1/2}(\rho - \sigma)\|_{\mathcal{H}^{N-6}[1,s]} \leqslant C_0\epsilon s^\delta$$

Moreover, if

- the initial data set $\left(\overline{g}^{\dagger(\kappa)}, \overline{k}^{\dagger(\kappa)}, \rho_0^{\dagger(\kappa)}, \rho_1^{\dagger(\kappa)}, \phi_0^{\dagger(\kappa)}, \phi_1^{\dagger(\kappa)}\right)$ converges to some limit $\left(\overline{g}^{(0)}, \overline{k}^{(0)}, \rho_0^{(0)}, \rho_1^{(0)}, \phi_0^{(0)}, \phi_1^{(0)}\right)$
- in the norms associated with the uniform bounds above

then the solutions $\left(g^{\dagger(\kappa)}, \rho^{\dagger(\kappa)}, \phi^{\dagger(\kappa)}\right)$ to **the system of modified gravity converge to a solution** $\left(g^{(0)}, \phi^{(0)}\right)$ **of the Einstein-massive field system** with, in particular, in the $\mathcal{H}^{N-2}$ norm on each compact set in time

$$\rho^{(\kappa)} \to R^{(0)} := 8\pi \left(g^{(0)\alpha\beta}\partial_\alpha\phi^{(0)}\partial_\beta\phi^{(0)} + \frac{c^2}{2}(\phi^{(0)})^2\right) \quad \text{as } \kappa \to 0.$$

Remarks.

- *Fourth-order system* vs. *second-order system*
- The highest ($(N+1)$-th order) derivatives of the scalar curvature
    - are $O(\varepsilon\kappa^{-1/2})$ in L^2 and may blow-up when $\kappa \to 0$
- while the N–th order derivatives are solely bounded and need not converge in a strong sense.

ONGOING RESEARCH

Application of the Hyperboloidal Foliation Method

- A broad class of nonlinear wave-Klein-Gordon systems with *strong interactions* and *quasi-null nonlinearities*

Improve the growing energy estimate by $s^{1/2}$

- Sharper energy bounds a la Morawetz
- Fully geometric construction by Q. Wang in work in progress

Extension to other massive fields

- Kinetic models (density), Vlasov equa. (collisionless), Boltzmann equation: work in progress by Fajman, Joudioux, Smulevici

Penrose's peeling estimates

- Asymptotics for the spacetime curvature along timelike directions

  Penrose's conjecture, Christodoulou-Klainerman's theorem

- Open problem in wave coordinates

- Our *Hyperboloidal Foliation Method* provides a possible approach to establishing the peeling estimates in wave gauge.